UNIVERSITÀ DEGLI STUDI DI BRESCIA

DOTTORATO DI RICERCA IN METODI E MODELLI PER L'ECONOMIA E IL MANAGEMENT

SECS-S/06 METODI MATEMATICI DELL'ECONOMIA E DELLE SCIENZE ATTUARIALI E FINANZIARIE

CICLO XXXIV

Essays on Fixed Income. Variance risk premium and target based optimal portfolio

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Preface

This Ph.D. thesis consists of two research papers and one chapter of literature review which are the result of my studies at the University of Brescia. This work was accomplished during my working period as Proprietary Trader at Banca IMI and later as European Government Bond Trader at UniCredit. The three chapters are linked by one common subject: innovations in Fixed Income instruments. This preface introduces the content of the three chapters and briefly explains how the research questions have been addressed. The first chapter of this thesis provides an introduction to the topic and presents the instruments. In particular, we will review the literature for bonds and related portfolio problem while later we will look at Credit Default Swap (CDS) and related payoff description.

The second chapter proposes a method for quantifying the variance risk premium in the Credit Default Swap Index market. We derive theoretically and we show numerically that the expected risk-neutral variance can be inferred from the market prices of the credit index options. The variance risk premium is defined as the difference between the realized variance and the synthetic simple variance swap rate. Using a novel data set, we calculate the risk neutral variance and we empirically test the historical variance risk premia magnitude on three European Credit Default Swap Indices.

The second paper, proposed in Chapter 3, jointly written with Francesco Menoncin derives a closed-form solution to the optimal investment target problem for the bond market. We show that the target problem is equivalent to the mean-variance one. We model the bond market using a Dynamic Nelson-Siegel-Svensson model starting from the yield to maturity curve that we enrich trough the addition of the liquidity spreads that lead to an incomplete market representation. Finally, we have presented numerical results of the optimal portfolio dynamics.

The three chapters, although not being conclusive, highlight the innovations in Fixed Income. Further research is needed to assess the optimal trading strategy that can be used to harvest the variance risk premium and to backtest the hedging performance of the target based bond portfolio.

CHAPTER 1

Bond and Credit Default Swap: a literature survey

1.1. Introduction

In this chapter, the literature on bonds and credit default swaps will be analyzed. In particular, we will look at the standard literature related to the modeling of these instruments that will be explored in Chapters 2 and 3 which show new frameworks that are more in line with the current market practice and the literature evolution. Subsequently, we will present an introduction and review of the literature for variance risk premia and mean variance portfolio problem that are studied in this thesis.

1.2. An overview on interest rate term structure

The modeling of interest rates and their term structure is at the very base of most of financial models. The following list summarizes a taxonomy, proposed by Rebonato (2018), of the different types of term-structure models:.

(1) Structural no-arbitrage models. These models start with the works of Vasicek (1977), Cox et al. (1985a), and Cox et al. (1985b). They model the instantaneous risk-less interest rate and they ensure that the no-arbitrage condition is satisfied. In this framework the three components that drive the yield curve (expectations, risk premia and convexity) can be derived. More recently, Cochrane and Piazzesi (2005); Piazzesi and Cochrane (2009) and Adrian et al. (2014) have used multiple risk factors to explain the risk premia.

(2) Statistical models. A part of the literature, summarized in Piazzesi and Cochrane (2009), describes how the yield curve moves using the Vector Auto-Regressive (VAR) models. In this framework the rates are modeled as discrete time stochastic processes and the risk premium is estimated as the difference between the forward and the yield forecasted by the model.. Given the quasi-unit-root nature of rates, the estimations based of times-series models have some statistical pitfalls and they cannot guarantee the absence of arbitrage.

(3) Snapshot models. The first models by Nelson and Siegel (1987), Fisher et al. (1995), and Svensson (1994) use cross-sectional functions that interpolate prices or yields of bonds that cannot be observed. Actually, these models are used by practitioners in order to find whether a given bond is cheap/rich with respect

to the model yield. Recent developments, proposed by Diebold and Rudebusch (2013), have shown a dynamic interpretation to their parameters making a comparison with the Principal Component Analysis factors. So, these latest developments combine features of structural, statistical and snapshot models. Moreover, Fontaine and Garcia (2009) have used a snapshot model to analyze the impact of liquidity on the yield curve by adding another risk factor, called liquidity premium, that drives the common dynamics of the bonds but with factor loadings varying with the maturity and the age of each bond. The model proposed in Chapter 3 is an evolution of the Diebold and Rudebusch (2013) Snapshot model that incorporated a liquidity correction for each bond of the curve.

(4) Derivatives models. They have been designed for derivatives pricing in such a way to ensure the absence of arbitrage. After the first works of Vasicek (1977), Cox et al. (1985a), and Cox et al. (1985b) the second generation of works by Hull and White (1990), Heath et al. (1992) and Brace et al. (1997) only represent the interest rate dynamics under the risk-neutral measure without estimating the risk premia. In fact, they are designed to build the riskless portfolio composed by the derivative and the hedging instrument therefore they do not provide a representation of the underlying under the physical measure.

1.3. An overview on Credit Default Swaps

Another instrument analyzed in this thesis that has great relevance in the financial literature is the Credit Default Swap (CDS). A CDS is a cleared derivative contract on the credit risk of a reference entity and it is equivalent to an insurance contract. In fact an investor called protection buyer can transfer the credit risk associated with the reference entity to another investor, called the protection seller, by paying a quarterly annual coupon called as CDS spread. The reference entity can be a corporation, a bank or a Government and if becomes insolvent before the maturity of the CDS the protection seller must pay the loss given default to the protection buyer. The failure of the reference entity is only one of the credit events that could trigger the payments; other events are obligation default, bankruptcy, repudiation, acceleration, and restructuring. The approaches used to model and price a CDS can be divided into two groups: the reduced-form models and the structural models.

(1) The structural approach was developed in Black and Scholes (1973) and Merton (1974). The total value of a firm's assets is modeled as a stochastic variable and the default is triggered when this variable falls below a given threshold. The asset value is often modeled by a geometric Brownian motion, while either the credit

spread or the risk neutral probability are calibrated on the firm's characteristics as asset volatility and leverage.

(2) Reduced-Form Models were introduced by Jarrow and Turnbull (1995) and they assume that the default time occurs randomly and follows a Poisson process. In this framework a CDS is priced under the risk neutral probability and it is possible to find the CDS spread that makes the contract fair at inception, as can be seen in Duffie (1999).

More recently, Jarrow and Protter (2012) have shown that if an incomplete information set is applied to a structural model it is possible to derive the equivalent reduced form model.

In the second chapter the Credit Default Swap Index, that is a standard CDS written on a fixed portfolio of several reference entities, will be described. The literature on these instruments is particularly limited and furthermore it does not take into account the market conventions currently in place. The main references are Morini and Brigo (2008) and Armstrong and Rutkowski (2009) where options on Credit Default Swap Index are presented too. These options are priced through an approximation of the payoff that does not respect market conventions. Our novelty is that in 2 we will show how to correctly represent the options.

1.4. Portfolio problem

1.4.1. Bond portfolio problem. Modern portfolio theory, proposed by Markowitz (1952), has been having an enormous impact on the literature and on investor's decisions. However, its application is mainly related to equity markets while for the fixed income market more suitable tools and models are missing. In fact, bond portfolios are nowadays mainly managed by a comparison of portfolio risk measures with respect to a benchmark.

Generally, the portfolio manager's views about the future evolution of the term structure of interest rates is simply applied by taking a relative value positions with respect to a benchmark, over weighting or under weighting pillars of the term structure, without using a theoretical portfolio optimization framework.

The very well known models about the evolution of interest rates are not commonly used in the industry because of the characteristics of the bond variance-covariance matrix. In fact, its estimation of the term structure is almost impossible, given the collinearity of the interest rates .

One of the first work about bond portfolio selection using a dynamic term structure model was proposed by Wilhelm (1992) that derives an optimal portfolio in a static mean-variance framework using a CIR model while Fabozzi and Fong (1994) identify the variance covariance matrix estimation as the main problem of the static mean variance portfolio.

Two other works related to the modern portfolio theory for bonds can be found in Elton (2003), that proposed a multifactor model for bond modeling, and in Korn and Koziol (2006) that analyze the problem of investing in zero-coupon bonds of different maturities.

Another branch of literature instead analyses continuous-time portfolio starting from seminal work of Merton (1971). Using a term structure driven by a Vasicek model Sørensen (1999) proposes a portfolio optimization for an investor that maximizes a constant relative risk aversion utility and can invest into a stock index, a zero-coupon bond, and a money market account. Korn and Kraft (2002) solve the problem of a portfolio selection with only bonds by using the stochastic control approach and affine process for the interest rates, while Kraft (2004) extends this model to different term structure models. Another important contribution is given by Munk and Sorensen (2004) who use the martingale approach to solve the bond portfolio selection problem in a general Heath et al. (1992) term structure framework.

1.4.2. CDS portfolio problem. The previous literature about portfolio optimization was focused on default-free fixed income securities, but there are works that use also structural and reduced form models. An optimal portfolio problem with defaultable assets was proposed by Korn and Kraft (2003) in a Merton structural default framework, while Steffensen and Kraft (2006) extended the analysis by defining the default as the first passage time of an economic state variable below a given threshold.

Furthermore, using a reduced form approach, Walder (2002) studies the optimal portfolio problem for an agent that can invest in a treasury bond and a portfolio of corporate zero-coupon bonds and Bielecki and Jang (2006) derive optimal investment strategies for a constant relative risk aversion (CRRA) investor. In a similar framework, Bo et al. (2010) study an infinite horizon portfolio optimization problem. Finally, Ambrosini and Menoncin (2018) derive the optimal investment strategy for an agent with hyperbolic absolute risk aversion (HARA) who can invest in a risk-free asset, a defaultable bond, and a CDS written on the bond.

1.4.3. Target based and mean variance optimization. The literature reviewed in the previous section divides portfolio problems into two classes: static mean-variance optimization and continuous-time optimization that maximizes a utility function. This is due to the fact that the mean-variance problem is time inconsistent given that it contains a non-linear function of the expectation (actually, the variance contains the square of an expected value).

The definition of time inconsistency was given by Strotz (1956) and occurs when an optimal strategy at some time t is no longer optimal at another time $s > t$. In this case, the Bellman's principle does not hold and the dynamic programming cannot be applied. Vigna (2020) presents a review of the literature regarding the possible approaches to deal with a time consistent version of the mean-variance portfolio problem. These approaches are summarized in the following list.

(1) Precommitment approach. In this case the optimal control at an initial time is computed and then, the agent precommits himself to follow this initial strategy despite the fact that it could not be optimal at a future date. Solutions to mean-variance problem using the precommitment approach can be found in Richardson (1989), Zhou and Li (2000), and Li and Ng (2000).
(2) Consistent planning or Nash equilibrium. As pointed by Strotz (1956), this approach consists in searching the *"best plan among those that will be actually followed"* and translates it into the search of a Nash subgame perfect equilibrium. Basak and Chabakauri (2010) adopt this technique in a mean-variance framework, while Bjork et al. (2017) extends it to a more general class of time-inconsistent problems.
(3) Dynamical optimality. The most recent approach, called dynamically optimal strategy, has been proposed by Pedersen and Peskir (2017) for the mean-variance portfolio selection problem. The strategy is time-consistent in the sense that it does not depend on initial time and initial state variable. In particular, it consists in representing the behavior of an optimizer who continuously reevaluates his position and solves infinitely many problems in an instantaneously optimal way. In fact, for each time the investor solves the precommitment problem *forgetting about his past and ignoring his future* (Vigna, 2020). Menoncin and Vigna (2020) present a comparison between the precommitment approach and the dynamical optimal strategy for a mean-variance problem for a DC pension scheme.

In 3 we propose a time consistent bond portfolio optimization using the approach proposed by Zhou and Li (2000) that transforms the mean variance problem into an equivalent linear quadratic target based problem, which can be solved by using the dynamic programming approach.

1.5. Variance risk premium

The Variance Risk Premium (VRP) is the premium that can be accrued by bearing the variance risk using a portfolio of options or a variance swap and it is defined as the difference between expected risk neutral variance and the expectation of the realized variance under the physical measure.

In their seminal work, Carr and Wu (2008) propose to measure the VRP by using the difference between a synthetic variance swap and the realized variance. In particular, they compute the variance swap rate as a linear combination of options prices. They also show that traditional risk factors fail to explain the VRP. In fact, later works as Bollerslev et al. (2009), Rosenberg and Engle (2002), Bakshi and Madan (2006), and Bollerslev et al. (2011) have interpreted the VRP as a measure of the aggregate risk aversion. Instead, Drechsler and Yaron (2011) interpret it as a measure of economic uncertainty.

Finally, Martin (2017) shows how to link the expected return of the equity market and the risk neutral variance (again calculated from option prices).

In the second chapter we show how to calculate the expected return for a generic asset. Then, we will consider the specific case of CDS Indices in order to show the relationship between the credit expected return and the risk neutral variance.

CHAPTER 2

Credit Variance Risk Premium

2.1. Introduction

The study of the variance risk premium is one of the topics that have received the most attention since Carr and Wu (2008) who showed the presence of VRP in the equity market. Subsequently, the same focus on risk premiums has been applied to: (i) the treasury market by Choi et al. (2017), (ii) the interest rate swap by Trolle and Schwartz (2014), (iii) commodities by Trolle and Schwartz (2010), and (iv) foreign exchange markets by Ammann and Buesser (2013).

However, there is a lack of works formalizing and estimating VRP, mainly because of the lack of data and the great complexity of the payoff of CDS indices and options.

Despite the literature on CDS Index and Options pricing, to the best of our knowledge, our thesis is the first work that is aligned to the market convention which involves the exchange of the upfront and not the payment of the running premium. This feature complicates both the pricing of CDS Index and Options, and the risk-neutral variance calculations, which are performed through approximation in Armstrong and Rutkowski (2009) Ammann and Moerke (2022) .

The results and the mathematical formulas that we present below are newly developed precisely to account for the proper modeling of the upfront amount.

In Section 2.2 we show how to to correctly represent the payoff of both the CDS Index and Options. In Section 2.3 we theoretically present the risk neutral variance and a decomposition that will allow us to numerically analyze the variance risk premium. In Section 2.4 we expose the novel data set and the methodology used to compute the variance risk premium and to test its magnitude. Section 2.5 describes and discusses the empirical findings, while Section 2.6 concludes.

2.2. Credit Default Swap index and options

2.2.1. The financial market. In this section we describe the Credit Default Swap Index and the CDS Index Option, and we introduce the variables involved in their pricing. Our main references for the description of the contracts are Bloomberg Quantitiative Analytics (2012); Pedersen (2003); Armstrong and Rutkowski (2009); ISDA (2014). Throughout this document, we work with a complete filtered probability space

$(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t))$, with $t \in [0, T]$, where the filtration $\mathcal{F}_t$ satisfies the usual hypotheses as defined in Pascucci (2011). We refer to $\mathcal{F}_t$ as the *reference filtration* sufficiently rich to contain all the information available about the financial market but the default times. We also assume that a riskless asset exists, whose price G_t solves the ordinary differential equation

$$\frac{dG_t}{G_t} = r_t dt,$$

where r_t is the stochastic instantaneously riskless interest rate and whose (unique) solution is

$$G_t = G_{t_0} e^{\int_{t_0}^{t} r_s ds},$$

given the boundary condition $G_{t_0} = 1$.

We assume the absence of arbitrage which ensures the existence of an equivalent martingale measure $\mathbb{Q}$ using G_t as *numéraire* of the economy, with $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$, the so called historical probability measure.

Using the Fundamental Theorem of Asset Pricing the value of a zero coupon bond $P(t, T)$, that pays 1 for sure in T, is given by

$$B(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{G_t}{G_T} \right| \mathcal{F}_t \right].$$

Following Armstrong and Rutkowski (2009), we define the default time τ as a strictly positive random variable defined on a filtered probability space $((\Omega, \mathcal{H}, \mathbb{Q}), (\mathcal{H}_t))$. However, in this work we are interested in a particular derivative on the default time (the Credit Default Index Swaps) which is written on a fixed portfolio of reference entities. Thus, we have to generalize the previous notation in order to take into account the multi-name default times.

A credit index is a fixed portfolio of n underlying reference entities and for every possible name we define a sequence of ordered default times, $\tau_1 \leq ... \leq \tau_n$, and $\mathcal{H}_t^i$ is the filtration generated by τ_i which, thus, contains only the information regarding the credit risk. We define $\hat{\mathcal{F}}_t \doteq \mathcal{F}_t \vee \mathcal{H}_t^1 \vee ...\mathcal{H}_t^{n-1}$ as the filtration that does not contain the last default and the enlarged filtration $\mathcal{G}_t \doteq \hat{\mathcal{F}}_t \vee \mathcal{H}_t^n$ given $\hat{\tau} = \tau_n$ the last default time when all firms are defaulted. In fact, in the next sections we will use directly $\hat{\mathcal{F}}_t$ since we will not analyze the behavior of the CDS Index or the CDS Index option at $\hat{\tau}$. Clearly, the default of all underlying credit names is not impossible but it is very unlikely and out of our scope since this particular case has been exposed in greater detail in Morini and Brigo (2008).

2.2.2. CDS index payoff. A Credit Default Swap Index (CDSI) is a standard CDS written on a credit index, in which the protection buyer pays quarterly a fixed annual coupon c (either 100 or 500 basis points). There are two kinds of CDSIs: the CDX, relative to the US market, and the iTraxx, relative to the European market.

At each time t, the “spread” of a CDSI (S_t) is the fixed coupon that should be paid from t up to the maturity of the contract for equating the value of the fixed leg to the value of the floating leg. Nevertheless, on the actual contract, the fixed payment is already set to c and, accordingly, any agent who enters the contract (and should pay a constant spread S_t) must pay (or receive) an upfront whenever $S_t \neq c$.

We will show how to compute both the spread S_t and the upfront, since the former is a quoting convention used by the market operators in order to compute the latter. In the next section we will analyze in detail the market conventions regarding the upfront.

When a new CDSI series is issued, it contains n reference entities called “names”, that are equally weighted and, accordingly, the notional of each name is $1/n$. A new series is established approximately every six months with a new underlying portfolio of names and maturity date. The last series of the CDSI index is called “on-the-run”, while the previous series are called “off-the-run”. Every CDSI series starts as “version 1” with n names. When a name is removed from the index because of a default, the CDSI begins trading on its “version 2” with $n-1$ names and so on.

The two legs of the swap, without upfront and with maturity M, are described below:

- Default (floating) Leg: when the i^{th} name defaults before maturity the protection buyer receives

$$\frac{1}{n}\left(1-\phi_{\tau_i}\right),$$

  where ϕ_{τ_i} is the name's recovery rate. We point out that although this amount is stochastic, it is a market convention to set ϕ_{τ_i} as a constant, and such a constant is assumed to be the same for all i, so we can write $\phi_{\tau_i} = \phi, \forall i \in \{1, 2, ..., n\}$.
- Premium (fixed) Leg: in exchange for the Default Leg, the protection buyer pays a constant amount of money on the names which have not defaulted yet. After a default: (i) a new version of the index without the defaulted name is created, and (ii) the protection buyer pays the same spread but on a face value that is multiplied by the ratio between the number of survived names and the original number of constituents, until the maturity or until the next default. Hence, the cash-flow of the Premium Leg can be written as

$$\int_t^M S_t \sum_{i=1}^{n_t} \mathbb{1}_{\tau_i > u} \frac{1}{n} du$$

at each instant [1] $u \in]t, M]$ where n_t is the number of survived names at time t, which is the date when the CDSI is traded.

Recalling the Fundamental Theorem of Asset Pricing, we can write the value of the Default Leg as

$$D_t \doteq \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n_t} \mathbb{1}_{t<\tau_i<M} \frac{1}{n}\left(1-\phi_{\tau_i}\right) \frac{G_t}{G_{\tau_i}} \middle| \hat{\mathcal{F}}_t\right], \tag{2.2.1}$$

and the value of the Premium Leg, given by $S_t A_t$, as

$$A_t \doteq \mathbb{E}^{\mathbb{Q}}\left[\int_t^M \sum_{i=1}^{n_t} \mathbb{1}_{\tau_i>u} \frac{1}{n} \frac{G_t}{G_u} du \middle| \hat{\mathcal{F}}_t\right]. \tag{2.2.2}$$

In particular, the term A_t is called risky annuity. At the time of subscription (t), the equivalence $D_t = S_t A_t$ must hold, and the theoretical spread S_t is given by:

$$S_t = \frac{D_t}{A_t} = \frac{\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n_t} \mathbb{1}_{t<\tau_i<M}\left(1-\phi_{\tau_i}\right) \frac{G_t}{G_{\tau_i}} \middle| \hat{\mathcal{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\int_t^M \sum_{i=1}^{n_t} \mathbb{1}_{\tau_i>u} \frac{G_t}{G_u} du \middle| \hat{\mathcal{F}}_t\right]}. \tag{2.2.3}$$

As mentioned before, the protection buyer does not actually pay S_t. Instead, he pays a constant $c \neq S_t$. As a consequence, an upfront fee is transferred at inception and is calculated as the difference between the premium leg, calculated by using c, and the premium leg of the standard running CDS, exchanged at zero, with the spread S_t. Since the upfront $U_t(S_t)$ is the amount that makes the trade fair, we can write

$$0 = D_t - cA_t - U_t(S_t) = D_t - S_t A_t,$$

from which

$$U_t(S_t) = A_t(S_t - c). \tag{2.2.4}$$

2.2.3. Market conventions for valuing a CDS index with upfront . The risky annuity A_t, defined in the previous section and necessary to obtain the upfront, is not directly observable on the market and it is calculated by using the following conventions described in the ISDA CDS Standard Model ISDA (2009):

(1) Homogeneous Portfolio: all the firms in the index are identical and the CDSI is calculated like a single name CDS with notional proportional to $\frac{n_t}{n}$ where n_t is the number of survived names at time t. Moreover, the intensity of default λ^{S_t} of a CDSI traded at spread level S_t is assumed to be constant, therefore under these hypotheses $\lambda_i = \lambda^{S_t}, \forall \lambda_i$. We derive λ^{S_t} under the continuous-time

[1] Here we have used the continuous time approximation. In reality, the coupon is paid quarterly and the accrual should be calculated.

approximation of the premium leg, while for the most realistic description of the contract a numerical root search is required. The precise convention used for the discrete time payment can be found in Bloomberg (2015).

(2) The cash flows are discounted by using a conventional discount curve, LIBOR based, where the interest rate is assumed to be independent of λ^{S_t}.

Given the first two assumptions, we can simplify equation (2.2.3) as follows

$$S_t = \frac{\mathbb{E}^{\mathbb{Q}}\left[\int_t^M (1-\phi)\,\lambda^{S_t} e^{-\int_t^s \left(\lambda^{S_t}+r_u\right)du} ds \,\middle|\, \hat{\mathcal{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\int_t^M e^{-\int_t^s \left(\lambda^{S_t}+r_u\right)du} ds \,\middle|\, \hat{\mathcal{F}}_t\right]}, \tag{2.2.5}$$

and obtain the value of the constant default intensity:

$$\lambda^{S_t} = \frac{S_t}{1-\phi}.$$

Using this result and the discount curve, LIBOR based, we can compute the market annuity, $A_t\left(\lambda^{S_t}, n_t\right)$, like a single name annuity with whose notional is $\frac{n_t}{n}$:

$$A_t\left(\lambda^{S_t}, n_t\right) = \int_t^M \frac{n_t}{n} e^{-\lambda^{S_t}(s-t)} B\left(t,s\right) ds, \tag{2.2.6}$$

given the Homogeneous Portfolio hypothesis. In fact, the summation of default times $\sum_{i=1}^{n_t} \mathbb{1}_{\tau_i>u}$ can be rewritten as $n_t \mathbb{1}_{\tau>u}$. Finally we obtain the upfront from (2.2.4):

$$U_t\left(S_t, n_t\right) = A_t\left(\lambda^{S_t}, n_t\right)\left(S_t - c\right). \tag{2.2.7}$$

We notice that the annuity $A_t\left(\lambda^{S_t}, n_t\right)$, calculated under the ISDA CDS Standard Model, is a particular case of the annuity A_t. In fact, while the latter does not depend on the spread, the former is a function of the spread itself since it is calculated by using the flat default intensity λ^{S_t}.

2.2.4. CDS index option. A CDS Index Option is a contract that provides buyers the right, at the exercise date $T < M$, to enter a CDS at the strike spread K. If the option buyer has the right to buy (sell) protection, we call the contract a payer (receiver) swaption. The main characteristics of these options are the following:

- they call for physical delivery rather than cash settlement,
- they have a European-style expiry,
- they are quoted in basis points upfront,
- their standard maturities are 1, 2, 3, or 6 months,
- they have a so called ‘no knockout’ clause: while the credit default swaption on a single name cancels out if default occurs before swaption maturity, this Index option can be exercised on the remaining names. Additional details can be found in ISDA (2014).

Finally, a payer swaption has the following payoff:

- Front-end protection: a cash amount equal to the accumulated loss on the defaults occurred since the option subscription inception t,

$$F(T, n_t) = \sum_{i=1}^{n_t} \mathbb{1}_{\tau_i \leq T} \frac{1-\phi_{\tau_i}}{n}. \tag{2.2.8}$$

- A CDS on the non-defaulted names with strike spread K, where the buyer of protection will receive a default leg, will pay a premium leg and will exchange an upfront fee calculated using K as the equilibrium spread. We notice that in case of defaults, if the auction occurred before the option expiry T, the upfront resulting from the exercise will be computed on the running version at the option inception, with reduced notional equal to $\frac{n_t}{n}$. On the other hand, the default leg and the premium leg will be on the next version with reduced notional equal to $\frac{n_T}{n}$.

Hence, the mark to market of the long protection CDSI, $C_T(K, n_t)$, resulting from the option exercise will be

$$C_T(K, n_t) = D_T - cA_T\left(\lambda^{S_T}, n_T\right) - U_T(K, n_t),$$

with $U_T(K, n_t) = A_T\left(\lambda^K, n_t\right)(S_T - c)$ and $A_T\left(\lambda^K, n_t\right) = \int_T^M \frac{n_t}{n} e^{-\lambda^K (s-T)} B(T, s)\, ds$.

Then, the price of a payer credit default index swaption at time $s \in [t, T]$ can be written as

$$Pa(s, T, K, n_t) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{G_s}{G_T}\left(F(T, n_t) + C_T(K, n_t)\right)^+\right| \hat{\mathcal{F}}_s\right],$$

where $(x)^+$ is either 0 is $x < 0$ or x if $x \geq 0$. On the contrary, the price of a receiver credit default index swaption satisfies

$$Re(s, T, K, n_t) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{G_s}{G_T}\left(-F(T, n_t) - C_T(K, n_t)\right)^+\right| \hat{\mathcal{F}}_s\right],$$

since it is an option to enter the opposite contract. In case of exercise, the option buyer pays the front-end protection and enters the CDS on the side of the protection seller.

By combining the two equations we can write

$$Pa(s, T, K, n_t) - Re(s, T, K, n_t) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{G_s}{G_T}\left(F(T, n_t) + C_T(K, n_t)\right)\right| \hat{\mathcal{F}}_s\right]. \tag{2.2.9}$$

The right side is the value of a forward CDSI Index, with strike K, plus the discounted expected value of the front-end protection. Then, we can write the put-call parity for the CDSI Options as:

(2.2.10)

$$Pa(s, T, K, n_t) - Re(s, T, K, n_t) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{G_s}{G_T} F(T, n_t)\right| \hat{\mathcal{F}}_s\right] + \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{G_s}{G_T} C_T(K, n_t)\right| \hat{\mathcal{F}}_s\right].$$

Moreover, since $C_T(S_T, n_T) = 0$, we can rewrite the mark to market of $C_T(K, n_t)$ in the following way

$$\begin{aligned} C_T(K, n_t) &= C_T(K, n_t) - C_T(S_T, n_T) \\ &= D_T - cA_T\left(\lambda^{S_T}, n_T\right) - U_T(K, n_t) - \left(D_T - cA_T\left(\lambda^{S_T}, n_T\right) - U_T(S_T, n_T)\right), \end{aligned}$$

that can be written as an upfront fee difference

(2.2.11)
$$C_T(K, n_t) = U_T(S_T, n_T) - U_T(K, n_t) = A_T\left(\lambda^{S_T}, n_T\right)(S_T - c) - A_T\left(\lambda^K, n_t\right)(K - c).$$

Then, the price of a payer can be written as

(2.2.12)
$$Pa(s, T, K, n_t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{G_s}{G_T}\left(F(T, n_t) + A_T\left(\lambda^{S_T}, n_T\right)(S_T - c) - A_T\left(\lambda^K, n_t\right)(K - c)\right)^+ \middle| \hat{\mathcal{F}}_s\right],$$

whereas the price of a receiver becomes

(2.2.13)
$$Re(s, T, K, n_t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{G_s}{G_T}\left(-F(T, n_t) - A_T\left(\lambda^{S_T}, n_T\right)(S_T - c) + A_T\left(\lambda^K, n_t\right)(K - c)\right)^+ \middle| \hat{\mathcal{F}}_s\right],$$

2.2.5. Loss-adjusted forward CDSI. In order to simplify the payoff of a credit default index swaption we can define a new "theoretical" instrument that has the same payoff described in the previous section. A loss-adjusted forward CDSI is a forward CDSI starting at time $f \geq t$ where the protection buyer will receive a forward protection leg plus the front end protection, $F(f, n_t)$, paid at f in case of defaults between subscription date, t, and the forward start of the protection leg f. Moreover, also the premium leg will start at f. Then, we can define the forward protection leg $\forall\, t$ with $t \leq f < M$:

$$D_t^f \doteq \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n_t} \mathbb{1}_{f<\tau_i<M} \frac{1}{n}\left(1 - \phi_{\tau_i}\right) \frac{G_t}{G_{\tau_i}} \middle| \hat{\mathcal{F}}_t\right], \tag{2.2.14}$$

and the forward annuity

$$A_t^f\left(\lambda^{S_t}, n_f\right) \doteq \mathbb{E}^{\mathbb{Q}}\left[\int_f^M \sum_{i=1}^{n_f} \mathbb{1}_{\tau_i>u} \frac{1}{n} \frac{G_t}{G_u} du \middle| \hat{\mathcal{F}}_t\right]. \tag{2.2.15}$$

Here, the subscript denotes the subscription date and the superscript denotes the forward date. Given the no arbitrage condition we can find $\hat{S}_t^f$, the so called loss-adjusted forward equilibrium spread, such that the value of the loss-adjusted forward CDSI, $\hat{C}_t^f\left(\hat{S}_t^f, n_t\right)$, is equal to zero:

$$\hat{S}_t^f = \frac{D_t^f + \mathbb{E}^{\mathbb{Q}}\left[F(f, n_t)\frac{G_t}{G_f}\middle|\hat{\mathcal{F}}_t\right]}{A_t^f\left(\lambda^{S_t}, n_f\right)} \tag{2.2.16}$$

$$= \frac{\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n_t}\mathbb{1}_{f<\tau_i<M}\frac{1-\phi_{\tau_i}}{n}\frac{G_t}{G_{\tau_i}}\middle|\hat{\mathcal{F}}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n_t}\mathbb{1}_{\tau_i\le f}\frac{1-\phi_{\tau_i}}{n}\frac{G_t}{G_f}\middle|\hat{\mathcal{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\int_f^M\sum_{i=1}^{n_f}\mathbb{1}_{\tau_i>u}\frac{1}{n}\frac{G_t}{G_u}du\middle|\hat{\mathcal{F}}_t\right]}.$$

Of course, for $f = t$, when the forward date coincides with the inception date, the forward CDSI becomes the spot CDSI, and accordingly $\hat{S}_f^f = \hat{S}_f$, so that $A_f^f\left(\lambda^{S_f}, n_f\right) = A_f\left(\lambda^{S_f}, n_f\right)$, $\hat{C}_f^f\left(\hat{S}_t^f, n_t\right) = \hat{C}_f\left(\hat{S}_t^f, n_t\right)$, and $D_f^f = D_f$. Moreover, we assume the ISDA model and the standard coupon payment also for the loss-adjusted forward CDSI, so we have to find the upfront fee $U_f\left(\hat{S}_t^f, n_f\right)$, exchanged at f, that makes the trade fair at t. Then we can write, recalling that interest rate is assumed to be independent of λ^{S_t},

$$\begin{aligned} 0 &= D_t^f + \mathbb{E}^{\mathbb{Q}}\left[F(f, n_t)\frac{G_t}{G_f}\middle|\hat{\mathcal{F}}_t\right] - cA_t^f\left(\lambda^{S_f}, n_f\right) - B(t, f)\,U_f\left(\hat{S}_t^f, n_f\right) \\ &= D_f^f + \mathbb{E}^{\mathbb{Q}}\left[F(f, n_t)\frac{G_t}{G_f}\middle|\hat{\mathcal{F}}_t\right] - \hat{S}_t^f A_t^f\left(\lambda^{S_f}, n_f\right), \end{aligned}$$

and therefore

$$U_f\left(\hat{S}_t^f, n_f\right) = \frac{A_t^f\left(\lambda^{S_f}, n_f\right)}{B(t, f)}\left(\hat{S}_t^f - c\right) = A_f^f\left(\lambda^{S_f}, n_f\right)\left(\hat{S}_t^f - c\right). \tag{2.2.17}$$

Since the payoff of a loss-adjusted forward CDSI, given the fixed strike K, is exactly the payoff of the CDSI swaption, we can simplify it without the front end protection but with a different upfront difference. In fact, at T the protection buyer has the following cash flows, considering n_t the number of survived names at time t:

$$\hat{C}_T^T(K, n_t) = D_T^T + F(T, n_t) - cA_T^T\left(\lambda^{S_T}, n_T\right) - U_T(K, n_t). \tag{2.2.18}$$

Following the same passages as in (2.2.11) we can show that the mark to market is equal to

$$\begin{aligned} \hat{C}_T(K, n_t) &= \hat{C}_T(K, n_t) - \hat{C}_T\left(\hat{S}_T, n_t\right) \\ &= D_T + F(T, n_t) - cA_T\left(\lambda^{S_T}, n_T\right) - U_T(K, n_t) \\ &- \left(D_T + F(T, n_t) - cA_T\left(\lambda^{S_T}, n_T\right) - U_T\left(\hat{S}_T, n_T\right)\right) \\ &= U_T\left(\hat{S}_T, n_T\right) - U_T(K, n_t) = A_T\left(\lambda^{S_T}, n_T\right)\left(\hat{S}_T - c\right) - A_T\left(\lambda^K, n_t\right)(K - c). \end{aligned} \tag{2.2.19}$$

So, the price of a payer (2.2.12) can be rewritten as
(2.2.20)

$$Pa\left(s,T,K,n_t\right)=\mathbb{E}^{\mathbb{Q}}\left[\frac{G_s}{G_T}\left(A_T\left(\lambda^{S_T},n_T\right)\left(\hat{S}_T-c\right)-A_T\left(\lambda^K,n_t\right)\left(K-c\right)\right)^+\bigg|\hat{\mathcal{F}}_t\right],$$

whereas the price of a receiver becomes
(2.2.21)

$$Re\left(s,T,K,n_t\right)=\mathbb{E}^{\mathbb{Q}}\left[\frac{G_s}{G_T}\left(-A_T\left(\lambda^{S_T},n_T\right)\left(\hat{S}_T-c\right)+A_T\left(\lambda^K,n_t\right)\left(K-c\right)\right)^+\bigg|\hat{\mathcal{F}}_s\right].$$

Moreover, we notice that the loss adjusted spread, at time T,

$$\hat{S}_T^T=\frac{D_T^T+\sum_{i=1}^{n_t}\mathbb{1}_{\tau_i\leq T}\dfrac{1-\phi_{\tau_i}}{n}}{A_T^T\left(\lambda^{S_T},n_T\right)}$$

coincides with the future spot spread S_T only if there were no defaults during the life of the option and the front end protection is worth zero.

Finally, we can observe that the payoff is the difference between the two upfront fees, calculated by using two different default intensities as noted in (2.2.6). Armstrong and Rutkowski (2009) derive the payoff as if the running spread S_t were effectively payed, and under this hypothesis the value of the CDSI can be written as

$$C_T\left(K\right)=\left(\hat{S}_T-K\right)A_T\left(\lambda^{S_T},n_T\right). \tag{2.2.22}$$

However, this simplification is inaccurate as it assumes that the premium legs of two CDSI differ only for the spread. However, in (2.2.6) we have shown that also the risky annuity $A_T\left(\lambda^{S_t},n_t\right)$ is different since it is function of the spread itself. Moreover, also the number of names in the upfront calculation differs in case of default. In order to write correctly the payoff of a CDS Index Option we have used the proper formula (2.2.11) despite the previous literature has used (2.2.22) as an approximation. We point out that neglecting how the upfront is calculated can lead to significant difference, especially when options are exercised at market spread far away from the strike, that generally happens in condition of deep credit market stress.

2.3. Expected returns and risk neutral variance

Following the work of Martin (2017), we can define the expected return for a generic asset. In particular, we are interested in obtaining a relationship between the expected return of iTraxx index and his risk neutral variance. First, we recall that the price at time t of a claim X_T at time T can be written also using the stochastic discount factor, (SDF), $M\left(t,T\right)$:

$$X_t = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{G_t}{G_T}X_T\right] = \mathbb{E}_t\left[M\left(t,T\right)X_T\right], \tag{2.3.1}$$

where $\mathbb{E}_t\left[\,\right]$ is a shorthand notation for $\mathbb{E}\left[\,|\,\mathcal{F}_t\right]$ and $\mathbb{E}$ is the expected value under the historical probability measure $\mathbb{P}$. We define the return of an asset as $R\left(t,T\right) = \dfrac{X_T}{X_t}$ while $R_f\left(t,T\right) = \dfrac{G_T}{G_t}$ is the return of the risk free bond that we assume to be deterministic. Hence, we can write the risk neutral variance of $R\left(t,T\right)$, by using the SDF, as

(2.3.2)

$$\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right] = \mathbb{E}_t^{\mathbb{Q}}\left[R\left(t,T\right)^2\right] - \left(\mathbb{E}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]\right)^2 = R_f\left(t,T\right)\mathbb{E}_t\left[M_T R\left(t,T\right)^2\right] - \left(R_f\left(t,T\right)\right)^2.$$

The expected return over the risk free rate and the risk neutral variance relationship can be written in the following way:

$$\begin{aligned}
\mathbb{E}_t\left[R\left(t,T\right)\right] - R_f\left(t,T\right) &= \left[\mathbb{E}_t\left[M_T R\left(t,T\right)^2\right] - R_f\left(t,T\right)\right] - \left[\mathbb{E}_t\left[M_T R\left(t,T\right)^2\right] - \mathbb{E}_t\left[R\left(t,T\right)\right]\right] \\
&= \frac{\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]}{R_f\left(t,T\right)} - \mathbb{C}_t\left[M_T R\left(t,T\right), R\left(t,T\right)\right],
\end{aligned}$$

recalling that $\mathbb{E}_t\left[M_T R\left(t,T\right)\right] = 1$. Now, we want to compute the risk neutral variance of $R\left(t,T\right)$

$$\frac{\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]}{R_f\left(t,T\right)} = \frac{1}{X_t^2}\left[\frac{\mathbb{E}_t^{\mathbb{Q}}\left[X_T^2\right]}{R_f\left(t,T\right)} - \frac{\left(\mathbb{E}_t^{\mathbb{Q}}\left[X_T\right]\right)^2}{R_f\left(t,T\right)}\right] = \frac{1}{X_t^2}\left[\frac{\mathbb{E}_t^{\mathbb{Q}}\left[X_T^2\right]}{R_f\left(t,T\right)} - \frac{\left(F_{t,T}\right)^2}{R_f\left(t,T\right)}\right],$$

where the forward price satisfies $F_{t,T} = \mathbb{E}_t^{\mathbb{Q}}\left[X_T\right]$ under the hypothesis of deterministic interest rate. We recall that $x^2 = 2\int_0^\infty \max\left\{0, x-K\right\}dK$ for any $x \geq 0$, so with $x = X_T$, taking risk neutral expectations we can write

$$\frac{\mathbb{E}_t^{\mathbb{Q}}\left[X_T^2\right]}{R_f\left(t,T\right)} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[2\int_0^\infty \max\left\{0, X_T-K\right\}dK\right]}{R_f\left(t,T\right)} = 2\int_0^\infty Call\left(t,T,K\right)dK.$$

Using the put call parity formula $Call\left(t,T,K\right) = Put\left(t,T,K\right) + \dfrac{1}{R_f\left(t,T\right)}\left(F_{t,T}-K\right)$, we can write :

$$\begin{aligned}
\int_0^\infty Call\left(t,T,K\right)dK &= \int_0^{F_{t,T}}\left(Put\left(t,T,K\right) + \frac{1}{R_f\left(t,T\right)}\left(F_{t,T}-K\right)\right)dK + \int_{F_{t,T}}^\infty Call_{t,T}\left(K\right)dK \\
&= \int_0^{F_{t,T}} Put\left(t,T,K\right) + \frac{F_{t,T}^2}{2R_f\left(t,T\right)} + \int_{F_{t,T}}^\infty Call\left(t,T,K\right)dK.
\end{aligned}$$

Hence, the risk neutral variance can be written as

$$\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right] = \frac{2R_f\left(t,T\right)}{X_t^2}\left[\int_0^{F_{t,T}} Put\left(t,T,K\right)dK + \int_{F_{t,T}}^{\infty} Call\left(t,T,K\right)dK\right],$$

and the expected return becomes:

$$\mathbb{E}\left[R\left(t,T\right)\right] - R_f\left(t,T\right) = \frac{2}{X_t^2}\left[\int_0^{F_{t,T}} Put\left(K\right)dK + \int_{F_{t,T}}^{\infty} Call_{t,T}\left(K\right)dK\right] - \mathbb{C}_t\left[M_T R\left(t,T\right), R\left(t,T\right)\right].$$

Now we want to use this formula to derive the expected return of a CDS Index that represent the credit market. We recall that given the upfront we can compute the CDSI equivalent bond price P_t[2] , that is always positive while the upfront could be also negative,

$$P_t\left(S_t\right) = 1 - U_t\left(S_t\right) = 1 - A_t\left(\lambda^{S_t}\right)\left(S_t - c\right). \tag{2.3.3}$$

The bond price of the forward loss adjusted spread becomes

$$P_t\left(\hat{S}_t^T\right) = 1 - U_t^T\left(\hat{S}_t^T\right) = 1 - A_t^T\left(\lambda^{S_t}\right)\left(\hat{S}_t^T - c\right).$$

Adding and subtracting 1 into the receiver payer formula we can rewrite the receiver as

$$\begin{aligned} Re\left(t,T,K,n_t\right) &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{G_t}{G_T}\left(-1 + A_T\left(\lambda^K, n_t\right)\left(K-c\right) + 1 - A_T\left(\lambda^{S_T}, n_T\right)\left(\hat{S}_T - c\right)\right)^+\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{G_t}{G_T}\left(P_T\left(\hat{S}_T\right) - P_T\left(K\right)\right)^+\right], \end{aligned}$$

and the price of a payer becomes

$$Pa\left(s,T,K,n_t\right) = \mathbb{E}^{\mathbb{Q}}\left[\frac{G_t}{G_T}\left(P_T\left(K\right) - P_T\left(\hat{S}_T\right)\right)^+\right].$$

Moreover, the put call parity formula, with the receiver as call and the payer as put, becomes

$$Re\left(t,T,K,n_t\right) - Pa\left(t,T,K,n_t\right) = \left(\mathbb{E}_t^{\mathbb{Q}}\left[P_T\left(\hat{S}_T\right)\right] - P_T\left(K\right)\right)\frac{1}{R_f\left(t,T\right)},$$

[2] In the CDX HY market the CDSI protection is quoted in bond price and not in spread, through the previous formula multiplied by 100 since the spread is quoted in basis points while the bond price is quoted in cents.

and defining $R(t,T) = \dfrac{P_T\left(\hat{S}_T^T\right)}{P_t\left(\hat{S}_t^T\right)}$, $X_T = P_T\left(\hat{S}_T\right)$ and $F_{t,T} = \mathbb{E}_t^{\mathbb{Q}}\left[P_T\left(\hat{S}_T\right)\right]$ we can write

$$\mathbb{E}\left[R(t,T)\right] - R_f(t,T) = \frac{2}{P_t\left(\hat{S}_t\right)^2}\left[\int_0^{F_{t,T}} Pa(t,T,K,n_t)\,dP_t(K) + \int_{F_{t,T}}^{\infty} Re(t,T,K,n_t)\,dP_t(K)\right]$$
$$- \mathbb{C}_t\left[M_T R(t,T), R(t,T)\right], \tag{2.3.4}$$

while the risk neutral variance can be written as

$$\mathbb{V}_t^{\mathbb{Q}}\left[R(t,T)\right] = \frac{2R_f(t,T)}{P_t\left(\hat{S}_t\right)^2}\left[\int_0^{F_{t,T}} Pa(t,T,K,n_t)\,dP_t(K) + \int_{F_{t,T}}^{\infty} Re(t,T,K,n_t)\,dP_t(K)\right]. \tag{2.3.5}$$

2.3.1. Decomposition of variance risk premium: upside and downside variance. In what follows, we decompose the variance into upside and downside variance, following the approach of Feunou et al. (2017) and Carr and Wu (2008). First we define the ex-post realized variance, $RV(t,T)$,[3] as the sum of the daily squared returns, $R(t,T)$, of the bond price of the forward loss adjusted spread $P_t\left(\hat{S}_t^T\right)$. The calculation of $RV(t,T)$ requires the calculation of the bond prices of the forward loss adjusted spread $P_t\left(\hat{S}_t^T\right)$ for each time step $t = t_0, t_1, ..., t_n = T$ as follows

$$RV(t_0,t_n) = \sum_{i=1}^{n} R(t_i,t_{i-1};T)^2,$$

$$R(t_i,t_{i-1};T) = \frac{P_t\left(\hat{S}_{t_i}^T\right)}{P_t\left(\hat{S}_{t_{i-1}}^T\right)} = \frac{1 - A_{t_i}^T\left(\lambda^{S_{t_i}}\right)\left(\hat{S}_{t_i}^T - c\right)}{1 - A_{t_{i-1}}^T\left(\lambda^{S_{t_{i-1}}}\right)\left(\hat{S}_{t_{i-1}}^T - c\right)},$$

where n is the number of days between t and T. For each time step, both the Forward CDSI and the forward Annuity are calculated as described in 2.2.3 for the same time horizons (45, 75 and 105 days) by linearly interpolating between adjacent maturities. In case of missing data for some time step, we keep the same value of the previous one. When dealing with historical variance, it may happen that the current on-the_run CDS series alive at time t ceases at time $t_k \in (t,T]$, hence becoming off-the-run. In this case, the ratio $R(t_{k+1},t_k;T)$ involves two bond price belonging to two different and consecutive on-the-run series. Such inconsistency may lead to jumps in Historical Variance which are not caused by market movements. We avoid such behavior by using the on-the-run

[3] In order to obtain the annualized variance we multiply the variance of period $T-t$ by $\frac{252}{T-t}$.

series until t_k (included) and the relative off-the-run series from t_{k+1} to $t_n = T$. We recall that $RV(t,T)$ can be seen as a measure of the variance under the physical probability as denoted in Andersen et al. (2003). In fact, conditionally on the observed price path over $[t,T]$, realized variance provides an ex-post unbiased estimator of the quadratic variation of the return which is the risk neutral variance 2.3.5 in our framework. Consequently, the conditional expectation at time $t=0$ of the future quadratic return variation, denoted as $\mathbb{V}_t^{\mathbb{P}}[R(t,T)]$, will also equal the conditional expectation of future realized variance:

$$\mathbb{V}_t^{\mathbb{P}}[R(t,T)] = \mathbb{E}_t^{\mathbb{P}}[RV(t,T)].$$

We can decompose the ex-post realized variance into ex-post upside, $RV_U(t,T)$, and downside, $RV_D(t,T)$, realized variance by choosing 0 as threshold:

$$RV_U(t,T) = \sum_{i=1}^{n} R(t_i,t_{i-1};T)^2\, \mathbb{1}_{R(t_i,t_{i-1};T)>0},$$
$$RV_D(t,T) = \sum_{i=1}^{n} R(t_i,t_{i-1};T)^2\, \mathbb{1}_{R(t_i,t_{i-1};T)<0},$$

and we can see that the $RV_U(t,T)$ is the variance of the positive returns, while $V_D(t,T)$ is the variance of the negative returns. Following Martin (2017), we can also decompose the risk neutral variance into downside and upside variance:

$$\mathbb{V}_t^{\mathbb{Q}}[R(t,T)] = \mathbb{V}_{t,U}^{\mathbb{Q}}[R(t,T)] + \mathbb{V}_{t,D}^{\mathbb{Q}}[R(t,T)] = \underbrace{\frac{2R_f(t,T)}{P_t\left(\hat{S}_t\right)^2}\left(\int_0^{F_{t,T}} Pa(t,T,K,n_t)\,dP_t(K)\right)}_{downside\ variance} +$$

$$\underbrace{\frac{2R_f(t,T)}{P_t\left(\hat{S}_t\right)^2}\left(\int_{F_{t,T}}^{\infty} Re(t,T,K,n_t)\,dP_t(K)\right)}_{upside\ variance}. \tag{2.3.6}$$

In the empirical section we will show the comparison between the ex-ante risk neutral variance and the ex-post realized variance in order to understand if the investors price correctly the total, the upside, and the downside variance risk.

2.3.2. Variance and skewness risk premium. We define the Variance Risk Premium (VRP) as the premium accrued by bearing variance risk and can be written as the difference between expectation of the risk neutral variance and the expectation of the realized variance under the physical measure, i.e. $\mathbb{E}_t^{\mathbb{P}}[RV(t,T)] = \mathbb{V}_t^{\mathbb{P}}[R(t,T)]$. Hence, we can write

$$VRP(t,T) = \mathbb{V}_t^{\mathbb{Q}}[R(t,T)] - \mathbb{V}_t^{\mathbb{P}}[R(t,T)]. \tag{2.3.7}$$

Moreover, given the previous decomposition we can write also the upside variance risk premium, $VRP_U\,(t,T)$, and the downside variance risk premium $VRP_D\,(t,T)$:

$$\begin{aligned} VRP\,(t,T) &= \left(\mathbb{V}^{\mathbb{Q}}_{t,D}\left[R\,(t,T)\right] + \mathbb{V}^{\mathbb{Q}}_{t,D}\left[R\,(t,T)\right]\right) - \left(\mathbb{V}^{\mathbb{P}}_{t,U}\left[R\,(t,T)\right] + \mathbb{V}^{\mathbb{P}}_{t,U}\left[R\,(t,T)\right]\right), \\ &= \left(\mathbb{V}^{\mathbb{Q}}_{t,U}\left[R\,(t,T)\right] - \mathbb{V}^{\mathbb{P}}_{t,U}\left[R\,(t,T)\right]\right) + \left(\mathbb{V}^{\mathbb{Q}}_{t,D}\left[R\,(t,T)\right] - \mathbb{V}^{\mathbb{P}}_{t,D}\left[R\,(t,T)\right]\right), \\ &\doteq VRP_U\,(t,T) + VRP_D\,(t,T)\,. \end{aligned}$$

Finally, we are interested in defining also the skewness risk premium and we start noticing that the difference between realized upside and downside variance can be seen as a measure of the realized skewness, denoted as

$$RSV\,(t,T) = RV_U\,(t,T) - RV_D\,(t,T)\,,$$

and we recall that distribution of $R\,(t,T)$ is left-skewed if $RSV\,(t,T) < 0$ otherwise it is right-skewed. The theoretical justification for using $RSV\,(t,T)$ as measure of skewness can be found in Feunou et al. (2017). Following the same definition of risk premium as we did for the variance, we define the skewness risk premium, $SRP\,(t,T)$, as the difference between the risk neutral skewness and the physical expectation of the realized skewness

$$\begin{aligned} SRP\,(t,T) &= \left(\mathbb{V}^{\mathbb{Q}}_{t,U}\left[R\,(t,T)\right] - \mathbb{V}^{\mathbb{Q}}_{t,D}\left[R\,(t,T)\right]\right) - \left(\mathbb{V}^{\mathbb{P}}_{t,U}\left[R\,(t,T)\right] - \mathbb{V}^{\mathbb{P}}_{t,D}\left[R\,(t,T)\right]\right) \\ &= VRP_U\,(t,T) - VRP_D\,(t,T)\,. \end{aligned}$$

If $SRP\,(t,T) < 0$, there is a skewness premium on the market as a compensation for an agent who bears downside risk. Alternatively, if $SRP\,(t,T) > 0$ there is a skewness discount that represents the amount that the agent is willing to pay to secure a positive return on an investment.

We notice that all the measures that we have presented are non-parametric and model-free. We will show the empirical results, based only on the price of the options and forwards effectively quoted on the market, in the next session. We have used equation 2.3.7 as ex-ante risk neutral variance while we have used the approach of Feunou et al. (2017), Bollerslev et al. (2014), and Martin (2017) for the expectation of future realized variance, where they use the past realized variance as proxy of the forward-looking real-world variance. This implies the following:

$$\mathbb{E}^{\mathbb{P}}_t\left[RV\,(t,T)\right] = \mathbb{V}^{\mathbb{P}}_t\left[R\,(t,T)\right] = RV\,(t-T,t)\,,$$

where the daily realized variance of the return, $RV\,(t-T,t)$, is computed at time t by looking backward over the same horizon length, $T-t$. Comparing the ex-ante risk neutral variance and the expectation of the realized variance we can test if the variance risk premia are good predictor of the excess returns.

2.4. Data and methodologies

In the present section we describe in detail the methodologies adopted to calculate the main quantities introduced in the previous sections.

2.4.1. Risk neutral variance. The crucial element of equation 2.3.5 is the integral of the credit swaption in the space of the CDSI equivalent bond price (equation 2.3.3).

Credit swaptions data (payer and receiver for several maturities and strikes) are obtained from Intesa Sanpaolo proprietary database and we have used daily frequency data available from April 2015 to December 2020 for two indices: iTraxx Main and iTraxx Financial Senior. Spot Annuity, which is necessary to remap the strike K in $P_t(K)$, is calculated through the discretization of the formula 2.2.6, assuming zero defaults and EURIBOR discounting. The coefficient $P_t\left(\hat{S}_t\right)^2$ of 2.3.5 is calculated remapping the Forward CDSI $\hat{S}_t$ in $P_t(\hat{S}_t)$ and using the forward Annuity $A_t^T\left(\lambda^{\hat{S}_t}\right)$, both retrieved from the same Database of the Credit Swaption, in correspondence with the same maturities; the coefficient $R_f\,(t,T)$ is calculated using the OIS curve as proxy of the risk free rate.

The integral is computed via the trapezoidal method, using the set of available strikes for each maturity. Therefore, for each day we calculate the risk neutral variance $\mathbb{V}_t^{\mathbb{Q}}\left[R\,(t,T)\right]$ of the bond price returns for fixed time to maturity, in order to compare the market implied variance with the the realized one. We have chosen 45, 75 and 105 days. In particular, we linear interpolate between the risk neutral variance for maturities which are adjacent to the reference fixed time to maturity. In case of missing lower bound or upper bound maturity, we simply extrapolate .

The calculation of downside and upside variance is performed as well, by splitting the integral as shown in 2.3.6.

2.5. Empirical results

In this section, we start by studying the existence, the sign and the average magnitude of the premia, and we will propose some explanation for this evidence. Then, we investigate the dynamic properties of the variance risk premia. Finally, we analyze the credit variance as predictor variable for the future excess returns of the underlying.

2.5.1. Which risks do investors price? If investors price the variance risk, the sample average of the risk neutral variance should be greater than the sample average of the expected realized variance. In fact, the VRP is the premium that an agent on the financial market is willing to pay to hedge against future realized volatility.

In Tables 1 and 2 we report the summary statistics for the excess return, the risk neutral variance, and the expected realized variance on the iTraxx Main and the iTraxx

Financial Senior, respectively. All the values are annualized and we report the mean and the median in basis points. The standard deviation is reported in percentage.

We can see that the VRP is positive and, in particular, the largest contributor is the downside Variance $\mathbb{V}_{t,D}^{\mathbb{Q}}\left[R\left(t,T\right)\right]$. As documented in Feunou et al. (2017) for the equity market, we can see a positive value for $VRP_D\left(t,T\right)$ and a negative value for $VRP_U\left(t,T\right)$. This leads the skewness risk premium to be negative as confirmed for both the iTraxx Main and the iTraxx Financial Senior.

These empirical evidence confirms our expectations given the microstructure of the market. Indeed, in the credit market, the agents are typically long on bonds and use synthetic derivatives to hedge their positions. The demand for payers is therefore much higher than the demand for receivers and leads to the downside risk premium being considerably higher than the upside risk premium. Moreover, given the illiquidity of options on credit indices, there is a lack of subjects who systematically sell options and volatility although downside realized volatility is lower than the downside risk neutral volatility priced in the market.

By following Carr and Wu (2008), we test the difference between the ex-post realized variance over the period $[t,T]$ and the ex-ante $\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]$ risk neutral variance. In particular, we compare the risk-neutral variance with the variance actually realized over the period, $RV\left(t,T\right)$, and not with the expectation $\mathbb{E}_t^{\mathbb{P}}\left[RV\left(t,T\right)\right]$ in order to see if the magnitude and the signs of the variance risk premia are confirmed.

In fact, we test the basic form of the expectation hypothesis that assumes zero variance risk premium. Therefore, the null hypothesis is that $\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]-RV\left(t,T\right)$ is equal to zero. In Tables 3 and 4 we gather the summary statistics for iTraxx Main and iTraxx Financial Senior of the realized VRP, respectively.

Moreover, we have calculated the log excess return, generated by a long position on realized volatility over the period $[t,T]$. We stress that in our framework the log return $\ln\left(RV\left(t,T\right)/\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]\right)$ is similar but not equal to a variance swap written on an iTraxx Index. In fact, these instruments are not traded in the market and, above all, our definition of variance does not coincide with the classic variance swap pricing formula (see Carr and Wu, 2008). Finally, we also plot the time series of the realized VRP in figures 2.6.1, 2.6.2, 2.6.3, 2.6.4, 2.6.5, and 2.6.6.

We can see that the mean of the realized risk premium is very close to the mean of the expected risk premium but is more volatile. The signs and the magnitude of the VRP are confirmed also in the ex-post comparison. In particular, looking at the log return, we can see that the investors pay the downside variance much more of what they have realized. In fact,we see that realized variance is typically lower than risk neutral except for a few spikes due to unexpected events such as the Brexit referendum or the COVID-19 pandemic. The excess return is above -60% for 45 days, in line with the results of -50% per month estimated by Carr and Wu (2008) for the equity variance risk premium. Also, in this case, the main contributor is the downside variance risk premium in line with the results showed by downside variance risk premia. Finally, we can see from the t-statistics that the results are strongly significant.

2.5.2. Dynamic behavior of the variance risk premia. A weaker form of the expectations hypothesis is to assume that the variance risk premia are constant or independent of the risk neutral variance. Then, we analyze the behavior of the variance risk premia by running the following regressions:

$$
\begin{aligned}
RV\left(t,T\right) &= \alpha + \beta \mathbb{V}_t^{\mathbb{Q}}\left[R_T\right] + \epsilon_t, \\
RV_U\left(t,T\right) &= \alpha + \beta \mathbb{V}_{t,U}^{\mathbb{Q}}\left[R_T\right] + \epsilon_t, \\
RV_D\left(t,T\right) &= \alpha + \beta \mathbb{V}_{t,D}^{\mathbb{Q}}\left[R_T\right] + \epsilon_t, \\
\left(RV_U\left(t,T\right) - RV_D\left(t,T\right)\right) &= \alpha + \beta\left(\mathbb{V}_{t,U}^{\mathbb{Q}}\left[R_T\right] - \mathbb{V}_{t,D}^{\mathbb{Q}}\left[R_T\right]\right) + \epsilon_t,
\end{aligned}
\tag{2.5.1}
$$

where ϵ_t is a zero-mean error term. Under the null hypothesis that VRP are constant or independent of the risk neutral variances we expect $\beta = 1$ and also $\alpha = 0$.

A positive slope coefficient would imply that the risk neutral variance is informative of the future realized variance. Moreover if β should be less than one, it might imply a time-variation in variance risk premia.

We use OLS estimator with Newey–West estimator of the covariance matrix in order to have estimates robust to autocorrelation and heteroscedasticity. Table 5 reports the estimates, the t-statistics and the R^2 of the regressions. All the estimated slope coefficients are positive but the skew's, in line with the previous results where we have seen that $SRP\left(t,T\right)$ is negative.

We can see that for the upside variance risk premium we cannot reject the null hypothesis of zero variance risk premium, in fact both α and β are not significantly different from 0 and 1. This implies that the upside variance risk premium is closer to a constant or is independent on both the variance risk premium and the downside risk premium. In fact, we reject the null hypothesis of $\alpha = 0$ for the downside variance that shows also a β significantly lower than one.

These results support our evidence that the downside variance risk premium for the credit market is the primary source of excess return. Finally, we highlight that our results are in line with the findings of Carr and Wu (2008) about credit variance risk premiums.

2.5.3. Term structure of variance risk premia. In the previous section we presented the results using 45 calendar days as interpolation period. Now, we show in Table

6 the realized variance risk premium also for the longer interpolation period of both 75 and 105 days.

We can see that total variance risk premium decreases over the interpolation period and remains highly statistically significant. This result holds also for both the downside variance risk premium and the upside variance risk premium, while the skewness risk premium remains stable.

These results can be explained by analyzing the liquidity of options on the iTraxx. Options with a maturity of 1 or 2 months are traded much more actively, while options with a longer maturity are relatively more illiquid. Since most option purchases are concentrated on payers for hedging purposes this raises the volatility skew and consequently the short-term variance risk premium.

2.5.4. Credit variance as predictor variable. Finally, we run a simple linear regression of the excess return of CDS index by using as predictor a set of variables that includes $\mathbb{V}_t^{\mathbb{Q}}[R_T]$, $\mathbb{V}_{t,U}^{\mathbb{Q}}[R(t,T)]$, $\mathbb{V}_{t,D}^{\mathbb{Q}}[R(t,T)]$, $RV(t,T)$, $RV_U(t,T)$, $RV_D(t,T)$, $VRP(t,T)$, $VRP_U(t,T)$, $VRP_D(t,T)$, and $SRP(t,T)$.

By following the methodology of Feunou et al. (2017) the model used for our analysis is

$$R_T - R_f(t,T) = \alpha + \beta x_i(t,T) + \epsilon_t, \tag{2.5.2}$$

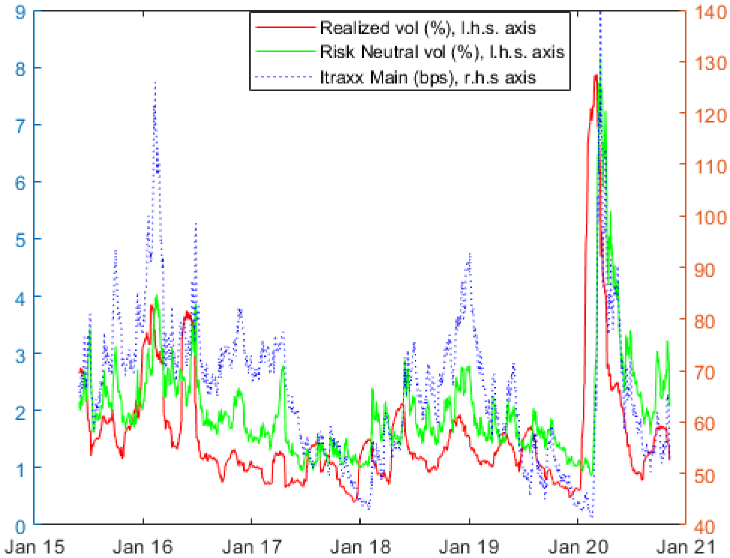
where $x_i(t,T)$ is one of the predictor variables and $R_T - R_f(t,T)$ is the excess return of the forward CDSI. We report the t-statistics, under the null hypothesis of $\beta = 0$, based on heteroscedasticity and serial correlations consistent standard errors using the Newey–West estimator and also we report the predictive ability of regressions, measured by the corresponding R^2s.

In Tables 8 and 7 we show the result for iTraxx Main and iTraxx Financial Senior, respectively.

We can see that the main source of predictability is driven by the risk neutral variance that shows significant coefficient β and higher R^2s with respect to the realized variance predictors.

Moreover, we see similar levels of predictive ability for all the risk neutral variances without significant differences between upside and downside variance. The predictability results increase as a function of the maturity T reaching the maximum value for $T = 105$ days. Finally, we notice that the VRP is not a statistically significant predictors of the excess return for both iTraxx Main and iTraxx Financial Senior. The R^2s are very low and all the coefficients are not significant. These results is the opposite of what discovered

FIGURE 2.6.1. Risk Neutral Volatility, Realized Volatility and iTraxx Main.



in Feunou et al. (2017) where the VRP is the main driver of the excess returns for the equity market.

2.6. Conclusions

In this paper we have described the payoff of CDSI and options using the market convention of the upfront amount exchange. We showed that the upfront amount cannot be neglected as it impacts both the payoff of the instruments and the definition of variance. Using a novel data set we have derived, from option's prices, the expected return and the risk neutral variance of European credit market. The empirical results shows that the VRP is mostly generated by the downside variance risk premium given the demand of hedging in the credit market and the absence of volatility seller. Future research will focus on developing an Equilibrium Model for VRP and on backtesting the optimal trading strategy to harvest the VRP.

FIGURE 2.6.2. Upside Risk Neutral Volatility, Upside Realized Volatility and iTraxx Main.

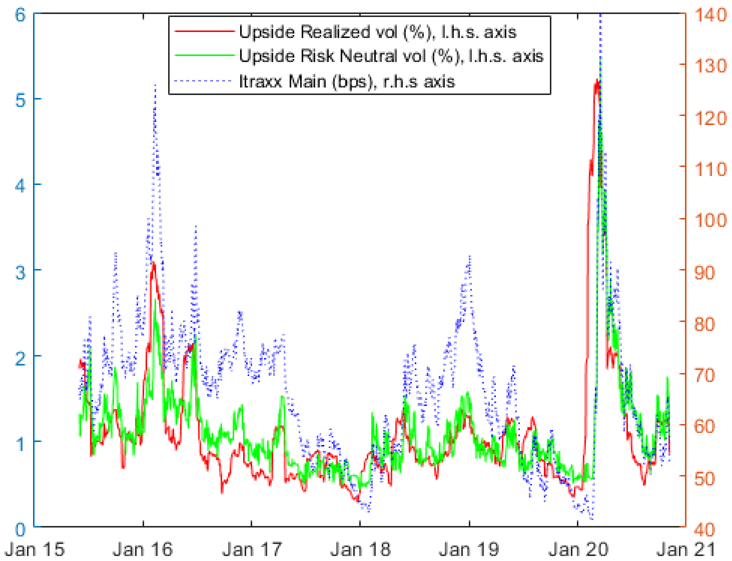


FIGURE 2.6.3. Downside Risk Neutral Volatility, Downside Realized Volatility and iTraxx Main.

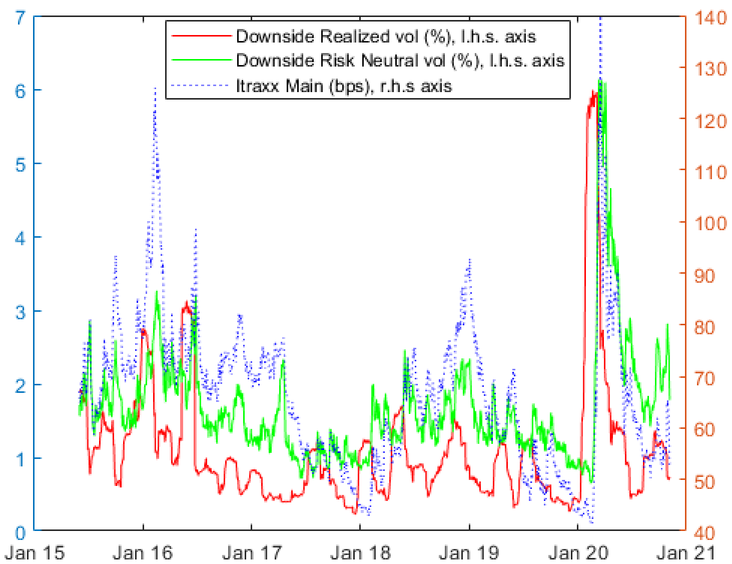


FIGURE 2.6.4. Risk Neutral Volatility, Realized Volatility and iTraxx Financial Senior.

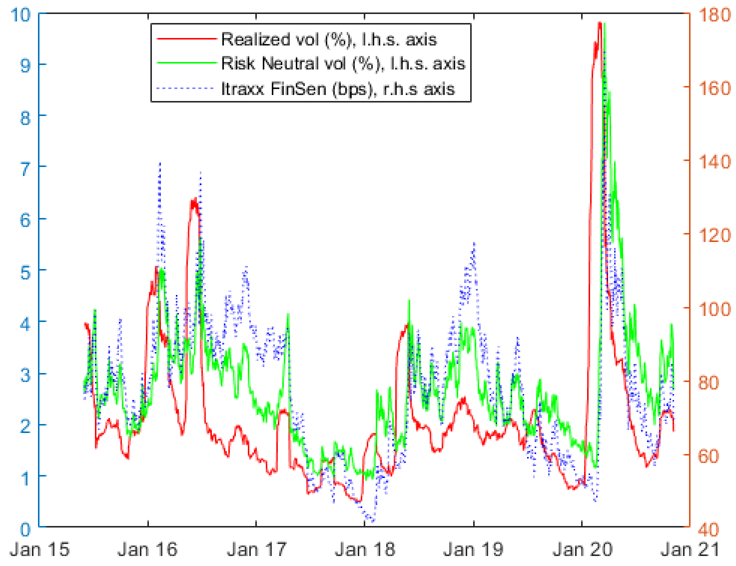


FIGURE 2.6.5. Upside Risk Neutral Volatility, Upside Realized Volatility and iTraxx Financial Senior.

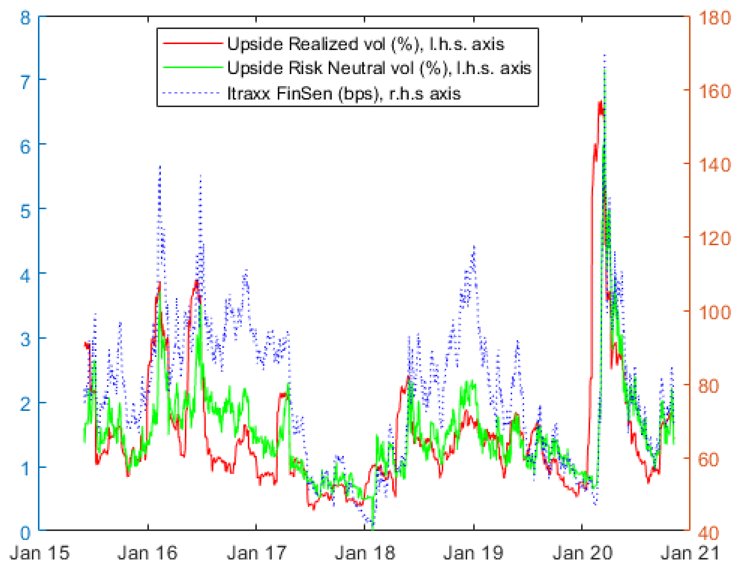


Table 1. **iTraxx Main Summary Statistics:** This table reports the summary statistics for the annualized quantities of interest. Mean, median are exposed in basis point while standard deviation value is exposed in percentage. The full sample is from April 2015 to December 2020.

iTraxx Main	Mean	Median	Std.Dev.	Skewness	Kurtosis
Panel A:Excess Return					
Itrxx Main	116.67	146.31	535.74	-1.28	10.15
Itrxx Main ex Covid	121.39	118.40	420.15	0.35	5.50
Panel B: Risk Neutral					
Variance	4.97	3.28	6.62	5.25	36.21
Upside Variance	1.54	1.00	2.23	6.28	54.49
Downside Variance	3.43	2.29	4.46	4.92	31.49
Skewness	-1.90	-1.26	2.41	-4.51	27.48
Panel C: Expected Realized					
Variance	3.82	1.59	7.92	5.05	30.86
Upside Variance	1.77	0.79	3.49	4.87	29.36
Downside Variance	2.05	0.79	4.83	5.29	32.86
Skewness	-0.27	0.02	2.88	-3.56	28.24
Panel D: Risk Premium					
Variance	1.15	1.25	4.96	-1.89	32.94
Upside Variance	-0.24	0.08	3.00	-1.96	29.76
Downside Variance	1.39	1.31	2.54	-1.61	27.17
Skewness	-1.62	-1.25	2.52	-2.84	23.59

TABLE 2. **iTraxx Financial Senior Summary Statistics:** This table reports the summary statistics for the quantities of interest. Mean, median and standard deviation value are annualized and in basis point. The full sample is from April 2015 to December 2020.

iTraxx Fin Sen	Mean	Median	Std.Dev.	Skewness	Kurtosis
Panel A:Excess Return					
Itrxx Fin Sen	131.54	178.34	674.11	-1.23	7.67
Itrxx Fin Sen ex Covid	140.86	142.82	547.51	-0.25	4.12
Panel B: Risk Neutral					
Variance	8.69	6.39	9.90	4.31	26.75
Upside Variance	2.88	2.01	3.73	5.70	49.08
Downside Variance	5.81	4.36	6.34	3.90	21.81
Skewness	-2.94	-2.18	3.19	-3.56	19.32
Panel C: Expected Realized					
Variance	7.05	3.27	13.37	4.47	25.18
Upside Variance	3.32	1.51	5.97	4.64	27.36
Downside Variance	3.73	1.57	7.89	4.37	23.66
Skewness	-0.42	0.07	4.14	-3.17	24.63
Panel C: Risk Premium					
Variance	1.64	2.04	8.30	-2.28	20.01
Upside Variance	-0.44	0.16	4.96	-1.43	27.69
Downside Variance	2.08	2.12	4.40	-2.35	14.79
Skewness	-2.52	-2.15	4.36	-0.63	24.29

TABLE 3. **iTraxx Main Realized Summary Statistics:** This table reports the summary statistics for the realized VRP and the log returns. Mean, median and standard deviation value are annualized and in basis point. The full sample is from April 2015 to December 2020.

iTraxx Main	Mean	Median	Std.Dev.	Skewness	Kurtosis	t
Panel A :Realized Risk Premium						
Variance	1.15	1.52	8.10	-3.91	30.50	5.21
Upside Variance	-0.27	0.19	2.78	-6.21	46.67	-3.48
Downside Variance	1.42	1.36	5.88	-2.03	22.50	8.84
Skewness	-1.68	-1.12	4.35	-1.21	19.38	-14.17
$ln\left(RV\left(t,T\right)/\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]\right)$						
Variance	-0.66	-0.81	0.89	2.05	10.56	-27.25
Upside Variance	-0.16	-0.28	0.74	2.04	11.54	-8.01
Downside Variance	-1.12	-1.23	1.15	1.25	6.61	-35.39
Skewness	0.96	0.95	0.80	-0.05	3.22	43.88

TABLE 4. **iTraxx Financial Senior Realized Summary Statistics:** This table reports the summary statistics for the realized VRP and the log returns. Mean, median and standard deviation value are annualized and in basis point. The full sample is from April 2015 to December 2020

iTraxx Financial Senior	Mean	Median	Std.Dev.	Skewness	Kurtosis	t
Panel A :Realized Risk Premium						
Variance	1.64	3.05	13.77	-3.54	24.46	4.35
Upside Variance	-0.50	0.37	5.15	-5.40	37.36	-3.52
Downside Variance	2.14	2.65	9.23	-2.15	17.48	8.45
Skewness	-2.64	-2.21	5.80	-0.33	13.65	-16.56
$ln\left(RV\left(t,T\right)/\mathbb{V}_t^{\mathbb{Q}}\left[R\left(t,T\right)\right]\right)$						
Variance	-0.61	-0.83	0.93	2.03	8.72	-24.17
Upside Variance	-0.17	-0.33	0.82	2.05	10.40	-7.76
Downside Variance	-1.06	-1.27	1.17	1.34	5.61	-32.84
Skewness	0.88	0.88	0.80	-0.06	3.16	40.01

TABLE 5. **Expectation Hypothesis Regression :** This table shows results for predictive regressions 2.5.1. The coefficent α is reported in basis points. T-statistics are under the null hypothesis of $\alpha = 0$ and $\beta = 1$ and are adjusted for heteroscedasticity and autocorrelation using Newey and West (1987).*,** and *** denote the significance at the 10%, 5% and 1% levels respectively. The columns under "R^2" report the unadjusted R-squared of the regression. The full sample is from April 2015 to December 2020.

	α	t	β	t	R^2
iTraxx Main					
Variance	0.17	1.43	0.48	-8.80^{***}	0.15
Upside Variance	0.03	0.85	1.00	-0.01	0.39
Downside Variance	0.16	1.83^{*}	0.21	-15.98^{***}	0.04
Skewness	-0.13	-2.52^{**}	-0.42	-18.24^{***}	0.13
iTraxx Financial Senior					
Variance	0.37	1.66^{*}	0.45	-7.34^{***}	0.11
Upside Variance	0.10	1.34	0.87	-1.57	0.28
Downside Variance	0.30	1.97^{**}	0.21	-11.04^{***}	0.03
Skewness	-0.15	-3.03^{***}	-0.30	-23.20^{***}	0.06

TABLE 6. **Credit Variance Risk Premium term structure:** This table reports the realized credit variance risk premium for periods of 45,75 and 105 days. Mean and standard deviation value are annualized and in basis point. The full sample is from April 2015 to December 2020.

Realized Risk Premium	45 Days			75 Days			75 Days		
iTraxx Main	Mean	Std.Dev	t	Mean	Std.Dev	t	Mean	Std.Dev	t
Variance	1.15	8.10	5.21	1.03	7.56	4.90	0.86	6.97	4.35
Upside Variance	-0.27	2.78	-3.48	-0.34	2.69	-4.54	-0.42	2.54	-5.92
Downside Variance	1.42	5.88	8.84	1.37	5.19	9.49	1.28	4.67	9.71
Skewness	-1.68	4.35	-14.17	-1.71	3.33	-18.44	-1.70	2.81	-21.48
iTraxx Financial Senior	45 Days			75 Days			75 Days		
Variance	1.64	13.77	4.35	1.31	12.52	4.35	0.77	11.43	2.39
Upside Variance	-0.50	5.15	-3.52	-0.63	4.78	-3.52	-0.84	4.47	-6.65
Downside Variance	2.14	9.23	8.45	1.94	8.10	8.45	1.62	7.24	7.88
Skewness	-2.64	5.80	-16.56	-2.57	4.49	-16.56	-2.46	3.74	-23.19

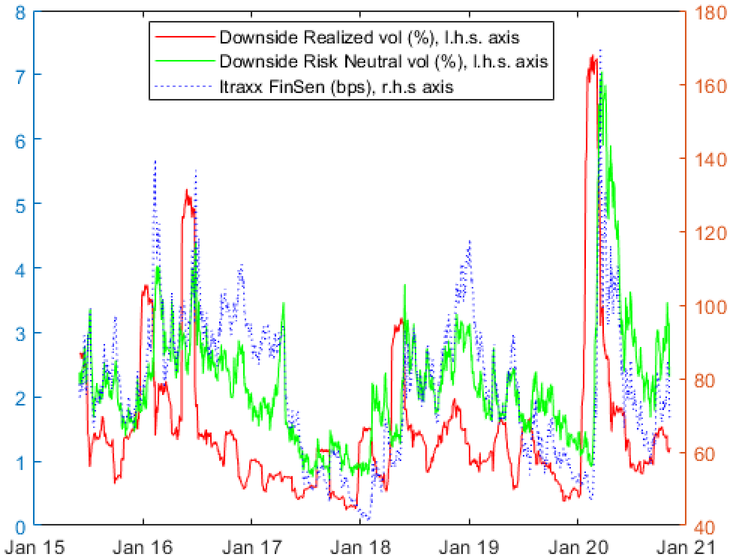
TABLE 7. **iTraxx Main Predictive Regression:** This table shows results for predictive regressions 2.5.2 The reported t-statistics for the beta parameters, under the null hypothesis of $\beta_1 = 0$, are constructed from heteroscedasticity and serial correlation consistent standard errors, following Newey and West (1987).*,** and *** denote the significance at the 10%, 5% and 1% levels respectively. The columns under "R^2" report the unadjusted R-squared of the regression. The full sample is from April 2015 to December 2020.

	45		75		105	
iTraxx Main	t	R^2	t	R^2	t	R^2
			Risk Premium			
Variance	0.73	0.82%	0.63	1.29%	0.64	1.50%
Upside Variance	0.26	0.13%	0.28	0.25%	0.18	0.09%
Downside Variance	1.34	1.80%	1.01	2.85%	1.11	4.24%
Skewness	-1.00	0.92%	-2.06^{**}	2.47%	-2.90^{**}	6.81%
			Risk Neutral			
Variance	4.73^{***}	19.02%	6.36^{***}	29.16%	6.08^{***}	33.96%
Upside Variance	3.96^{***}	19.20%	5.00^{***}	28.44%	5.65^{***}	32.97%
Downside Variance	4.47^{***}	18.38%	5.73^{***}	28.54%	5.85^{***}	33.02%
Skewness	-4.40^{***}	14.98%	-5.93^{***}	23.69%	-6.45^{***}	26.49%
			Realized			
Variance	4.27^{***}	9.49%	4.69^{***}	12.95%	3.46^{***}	13.32%
Upside Variance	4.97^{***}	6.33%	4.24^{***}	8.71%	3.31^{***}	10.03%
Downside Variance	4.09^{***}	10.48%	5.00^{***}	15.04%	3.67^{***}	14.91%
Skewness	-2.38^{**}	5.91%	-1.94^{*}	11.77%	-1.55^{*}	10.92%

TABLE 8. **iTraxx Financial Senior Predictive Regression:** This table shows results for predictive regressions 2.5.2 The reported t-statistics for the beta parameters, under the null hypothesis of $\beta_1 = 0$, are constructed from heteroscedasticity and serial correlation consistent standard errors, following Newey and West (1987).*,** and *** denote the significance at the 10%, 5% and 1% levels respectively. The columns under "R^2" report the unadjusted R-squared of the regression. The full sample is from April 2015 to December 2020.

	45		75		105	
iTraxx Financial Senior	t	R^2	t	R^2	t	R^2
Risk Premium						
Variance	0.32	0.11%	0.51	0.82%	0.82	1.84%
Upside Variance	0.00	0.00%	0.26	0.22%	0.45	0.48%
Downside Variance	0.69	0.40%	0.87	1.64%	1.25	3.81%
Skewness	-0.52	0.44%	-0.77	0.76%	-1.24	2.39%
Risk Neutral						
Variance	4.27^{***}	17.08%	6.43^{***}	27.91%	6.44^{***}	30.98
Upside Variance	3.68^{***}	16.47%	5.36^{***}	27.90%	5.67^{***}	31.26%
Downside Variance	3.92^{***}	16.55%	6.25^{***}	26.09%	6.67^{***}	28.45%
Skewness	-4.13^{***}	11.20%	-7.35^{***}	14.80%	-6.72^{***}	14.20%
Realized						
Variance	4.83^{***}	8.13%	3.69^{***}	10.84%	2.43^{**}	9.11%
Upside Variance	4.82^{***}	6.51%	3.52^{***}	8.43%	2.43^{**}	7.75%
Downside Variance	4.92^{***}	8.44%	3.75^{***}	11.90%	2.40^{**}	9.59%
Skewness	-1.89^{*}	3.66%	-1.75^{*}	7.61%	-1.22^{*}	4.97%

FIGURE 2.6.6. Downside Risk Neutral Volatility, Downside Realized Volatility and iTraxx Financial Senior.



CHAPTER 3

Mean-variance target-based optimization in bond market

3.1. Introduction

The main aim of this work is to obtain a mean-variance target based optimal bond portfolio. In Section 3.2 we present and enrich the Dynamic Term Structure Model of Diebold and Rudebusch (2013) deriving the dynamics of a bond starting from the yield curve. The bond market proposed here has two innovative elements compared to the literature. First, we directly model and use the yield curve to obtain the bond dynamics instead of modeling the short rate dynamics. This approach also allows us to add the second innovative element, namely, obtaining a liquidity spread for each bond. Next, the model was extended to the multifactor case to account for the different risk factors that move the yield curve. Moreover, to the best of our knowledge, this is the first work that derives the analytical formula of the so called roll-down. In Section 3.3 we analyze a mean-variance target based problem that can be used for hedging or trading purposes and we derive a closed form solution of the optimal portfolio. Finally, Section 3.6 concludes.

3.2. The Bond Market

Throughout this section, we work with a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, F_t)$ with $t \in [0, T]$. We also assume that a riskless asset exists, whose price G_t solves the ordinary differential equation

$$\frac{dG_t}{G_t} = r_t dt, \tag{3.2.1}$$

where r_t is the stochastic instantaneously riskless interest rate. We assume absence of arbitrage which ensures the existence of (at least) a martingale measure $\mathbb{Q}$ equivalent to the so-called historical probability $\mathbb{P}$, using G_t as *numéraire* of the economy.

Using the Fundamental Theorem of Asset Pricing the value of a zero coupon bond $B(t, T)$, that pays 1 for sure in T, is given by

$$B(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{G_t}{G_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right] = e^{-r(t,T)(T-t)},$$

where $r(t, T)$ is the spot rate known at time t. We decompose the price of a coupon bond in two components, the value of the bond $V(t, T)$ and the residual or liquidity spread $R(t, T)$, such as $P(t, T) = V(t, T) + R(t, T)$.

At time t we call $V(t,T)$ the value of a bond expiring in T, with a continuous stream of fixed payments δ, and that repays V_T at maturity:

$$V(t,T)=\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \delta\frac{G(t)}{G(s)}ds+V_T\frac{G(t)}{G(T)}\right]=\delta\int_t^T B(t,s)\,ds+V_T B(t,T).$$

The yield to maturity (YTM) of this bond is the constant interest rate $y_v(t,T)$, over the period $[t,T]$, that satisfies the following equation

$$V(t,T)=\delta\int_t^T e^{-y_v(t,T)(s-t)}ds+V_T e^{-y_v(t,T)(T-t)}. \tag{3.2.2}$$

Moreover, we recall that the YTM of a zero-coupon bond is equal to its spot rate since the following holds

$$B(t,T)=e^{-r(t,T)(T-t)}=e^{-y_v(t,T)(T-t)}. \tag{3.2.3}$$

Then, in a market with friction the price of a bond is given by the following equation

$$P(t,T)=\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \delta e^{-(y_v(t,T)+y_r(t,T))(s-t)}ds+V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}\right] \tag{3.2.4}$$

where $y_r(t,T)$, which measures the yield to maturity of the liquidity spread, is a bond specific constant, that incorporates the idiosyncratic adjustments to the price of each bond, driven by investor preference and market liquidity. Here, we assume that this component cannot be explained by curve movements. We define the yield to maturity of the price of a bond as $y(t,T)=y_v(t,T)+y_r(t,T)$. We assume that the YTM $y_v(t,T)$, which represents the level of the yield curve in this one factor model, has a stochastic dynamics

$$dy_v(t,T)=\mu(t,T)\,dt+\sigma(t,T)\,dW(t), \tag{3.2.5}$$

in which $dW(t)$ is a Wiener process under $\mathbb{P}$, with zero mean and variance dt and liquidity spread $y_r(t,T)$ evolves over time by the following mean-reverting stochastic differential equation

$$dy_r(t,T)=\kappa_r(\theta_r-y_r(t,T))\,dt+\sigma_r(t,T)\,dW_r(t), \tag{3.2.6}$$

with $dW_r(t)$ independent of $dW(t)$[1].

[1]this hypothesis will be relaxed in the next section.

Clearly, if $y_r(t,T)=0$ the price of the bond is equal to the value. In this toy model we could set $\theta_r=0$ since we assume that the liquidity spread should be temporary. In further analysis we could model the fact that a bond can have a permanent liquidity premium. We could add and analyze other aspects like presence of a bond in the deliverable basked of bond futures or the cost of funding using the prices available in the repo market.

Starting from the previous equation we can derive the evolution of the bond price by using the Ito's Lemma,

$$\begin{aligned} dP(t,T) &= \frac{\partial P(t,T)}{\partial t}dt + \frac{\partial P(t,T)}{\partial y_v(t,T)}dy_v(t,T) + \frac{\partial P(t,T)}{\partial y_r(t,T)}dy_r(t,T) \\ &+ \frac{1}{2}\frac{\partial^2 P(t,T)}{\partial y_v(t,T)^2}\left(dy_v(t,T)\right)^2 + \frac{1}{2}\frac{\partial^2 P(t,T)}{\partial y_r(t,T)^2}\left(dy_r(t,T)\right)^2 \\ &+ \frac{\partial^2 P(t,T)}{\partial y_v(t,T)\,\partial y_r(t,T)}\left(dy_v(t,T)\,dy_r(t,T)\right) \end{aligned}$$

whose derivatives are

$$\begin{aligned} \frac{\partial P(t,T)}{\partial t} &= -\delta + \int_t^T \delta \frac{\partial\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}\right)}{\partial t}ds + V_T\left(\frac{\partial\left(e^{-(y_v(t,T)+y_r(t,T))(T-t)}\right)}{\partial t}\right) \\ &= -\delta + \left(y_v(t,T)+y_r(t,T)\right)\int_t^T \delta e^{-(y_v(t,T)+y_r(t,T))(s-t)}ds \\ &+ V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}\left(y_v(t,T)+y_r(t,T)\right) \\ &= -\delta + \left(y_v(t,T)+y_r(t,T)\right)P(t,T), \\ \frac{\partial P(t,T)}{\partial y_v(t,T)} &= -\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)\right)ds - V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t), \\ \frac{\partial P(t,T)}{\partial y_r(t,T)} &= -\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)\right)ds - V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t), \\ \frac{\partial^2 P(t,T)}{\partial y_v(t,T)^2} &= -\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)^2\right)ds - V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t)^2, \\ \frac{\partial^2 P(t,T)}{\partial y_r(t,T)} &= -\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)^2\right)ds - V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t)^2, \end{aligned}$$

$$\frac{\partial^2 P(t,T)}{\partial y_v(t,T)\,y_r(t,T)} = -\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)^2\right)ds - V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t)^2.$$

We can define the duration, $\mathcal{D}(t,T)$, and the convexity, $\mathcal{C}(t,T)$, of the bond price as

$$\mathcal{D}(t,T) \doteq \frac{\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)\right)ds + V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t)}{P(t,T)}, \tag{3.2.7}$$

$$\mathcal{C}(t,T) \doteq \frac{\int_t^T \delta\left(e^{-(y_v(t,T)+y_r(t,T))(s-t)}(s-t)^2\right)ds + V_T e^{-(y_v(t,T)+y_r(t,T))(T-t)}(T-t)^2}{P(t,T)} \tag{3.2.8}$$

and finally we can rewrite the dynamics of the bond as

$$\begin{aligned}\frac{dP(t,T)}{P(t,T)} &= \left(y_v(t,T) + y_r(t,T) - \frac{\delta}{P(t,T)}\right)dt - \mathcal{D}(t,T)\left(dy_v(t,T) + dy_r(t,T)\right)\\ &+ \frac{1}{2}\mathcal{C}(t,T)\left(dy_v(t,T)^2 + dy_r(t,T)^2\right),\\ &= \left(y_v(t,T) + y_r(t,T) - \frac{\delta}{P(t,T)} - \left(\mu(t,T) + \kappa_r(\theta_r - y_r(t,T))\right)\mathcal{D}(t,T)\right)dt \qquad (3.2.9)\\ &+ \frac{1}{2}\left(\sigma(t,T)^2 + \sigma_r(t,T)^2\right)\mathcal{C}(t,T)\,dt\\ &- \mathcal{D}(t,T)\,\sigma(t,T)\,dW(t) - \mathcal{D}(t,T)\,\sigma_r(t,T)\,dW_r(t),\end{aligned}$$

recalling that $dy_v(t,T)\,dy_r(t,T) = 0$ given the independence of $dW_r(t)$ and $dW(t)$. We notice that both the duration and the convexity can be solved analytically as showed in appendix. The addition of the liquidity spread is particularly important for bonds with very close maturity as those belonging to the bond futures basket. We will explain the rationale of this addiction in the subsection 3.2.3.

.

3.2.1. Dynamic Nelson-Siegel-Svensson Curve. In the previous section we have presented a YTM evolution in order to derive the bond price evolution as a function of $y_v(t,T)$ and of $y_r(t,T)$. Now, we enrich the yield curve model by moving from one factor to a multifactor setting to obtain a more realistic description of the risk factors that drive the yield curve. We use the dynamic Nelson-Siegel-Svensson (DNSS) proposed by Diebold and Rudebusch (2013) but we model the yield to maturity instead of the short rate. The YTM of the value of a bond is given by

$$\begin{aligned}y_v(t,T) &= L_t + S_t\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)}\right) + C_t^1\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)} - e^{-\lambda_1(T-t)}\right)\\ &+ C_t^2\left(\frac{1-e^{-\lambda_2(T-t)}}{\lambda_2(T-t)} - e^{-\lambda_2(T-t)}\right),\end{aligned} \tag{3.2.10}$$

where L_t, S_t, C_t^1, C_t^2 are stochastic variables and λ_1 and λ_2 are constants.

With this formulation, the DNSS models the YTM curve of bond value as a four factor model, with latent factors L_t, S_t, C_t^1, C_t^2, the dynamics of which determines the dynamic of $y_v(t,T)$ for any T. Moreover, at any t, the cross section of $y_v(t,T)$ is determined by the coefficients $1, \left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)}\right), \left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)} - e^{-\lambda_1(T-t)}\right), \left(\frac{1-e^{-\lambda_2(T-t)}}{\lambda_2(T-t)} - e^{-\lambda_2(T-t)}\right)$ that we will call also factor loadings. Another interpretation to the four latent factors is given by Diebold and Rudebusch (2013). In fact, we note that L_t, the level, moves yields in a parallel fashion while S_t changes the slope of the yield curve. In fact $\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)}\right)$ represents an exponential decay and allows the term structure to slope upwards (with $\beta_{1t} < 0$) or downwards (with $\beta_{1t} > 0$). Finally, the last two factor loadings, the curvatures, produce a butterfly effect; both C_t^1 and C_2^1 create a hump (if > 0) or a trough in the yield curve (if < 0).

The addition of C_2^1 is justified by the fact that it allows better calibration for long maturity bonds. In fact, C_t^1 is generally used to model the curvature of the short end while C_2^1 is used for the curvature of the extra long part of the curve. Furthermore, the signs of the two coefficients may differ depending on the shape of the curve.

The YTM of the value of a bond and its dynamic are given by:

(3.2.11)

$$\begin{aligned} dy_v(t,T) = {} & \frac{\partial y_v(t,T)}{\partial t}dt + dL_t + dS_t\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)}\right) + dC_t^1\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)} - e^{-\lambda_1(T-t)}\right) \\ & + dC_t^2\left(\frac{1-e^{-\lambda_2(T-t)}}{\lambda_2(T-t)} - e^{-\lambda_2(T-t)}\right), \end{aligned}$$

where $\frac{\partial y_v(t,T)}{\partial t}$ is is the 'roll-down' $\mathcal{R}(t,T)$, i.e., the change in yield as t changes by a small amount. In our model we can derive an analytical formula for the roll-down as follows

$$\begin{aligned} \frac{\partial y_v(t,T)}{\partial t} &= \left(\frac{\text{-}e^{-\lambda_1(T-t)}}{(T-t)} + \frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)^2}\right)S_t + \left(\frac{\text{-}e^{-\lambda_1(T-t)}}{(T-t)} + \frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)^2} + -\lambda_1 e^{-\lambda_1(T-t)}\right)C_t^1 \\ &+ \left(\frac{\text{-}e^{-\lambda_2(T-t)}}{(T-t)} + \frac{1-e^{-\lambda_2(T-t)}}{\lambda_2(T-t)^2} - \lambda_2 e^{-\lambda_2(T-t)}\right)C_t^2, \\ &= \left(\frac{\text{-}e^{-\lambda_1(T-t)}}{(T-t)} + \frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)^2}\right)\left(S_t + C_t^1\right) - \lambda_1 e^{-\lambda_1(T-t)}C_t^1 \end{aligned}$$

(3.2.12)

$$+ \left(\frac{\text{-}e^{-\lambda_2(T-t)}}{(T-t)} + \frac{1-e^{-\lambda_2(T-t)}}{\lambda_2(T-t)^2} - \lambda_2 e^{-\lambda_2(T-t)}\right)C_t^2.$$

Here, we assume that the factor dynamics follows a system of independent mean reverting SDE under historical probability $\mathbb{P}$:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t^1 \\ dC_t^2 \end{pmatrix} = \begin{pmatrix} \kappa_l & 0 & 0 & 0 \\ 0 & \kappa_s & 0 & 0 \\ 0 & 0 & \kappa_{c^1} & 0 \\ 0 & 0 & 0 & \kappa_{c^2} \end{pmatrix} \left[\begin{pmatrix} \theta_l \\ \theta_s \\ \theta_{c^1} \\ \theta_{c^2} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t^1 \\ C_t^2 \end{pmatrix} \right] dt + $$

$$+ \begin{pmatrix} \sigma_l & 0 & 0 & 0 \\ 0 & \sigma_s & 0 & 0 \\ 0 & 0 & \sigma_{c^1} & 0 \\ 0 & 0 & 0 & \sigma_{c^2} \end{pmatrix} \begin{pmatrix} dW_l\,(t) \\ dW_s\,(t) \\ dW_{c^1}\,(t) \\ dW_{c^2}\,(t) \end{pmatrix}. \tag{3.2.13}$$

where for each state variable follows a Vasicek process with κ, speed of mean reversion, θ, long term mean and σ, instantaneous volatility, constant parameters. Moreover, the four state variables may interact dynamically or their shocks may be correlated and in that case the matrix of the speed of mean reversion and the matrix of diffusion coefficients could have non zero off-diagonal elements.

3.2.2. Bond market with correlated state variables. We can enrich the previous model by adding correlations between the state variables. In fact, if the state variables in the system 3.2.13 are correlated, then we can use the Cholesky Decomposition of the covariance matrix in order to use independent Wiener processes. In particular, there exists a $C\,(t)$ sub-triangular matrix with elements $c_{i,j}$ such that

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t^1 \\ dC_t^2 \\ y_r\,(t, T_i) \end{pmatrix} = \begin{pmatrix} \kappa_l & 0 & 0 & 0 & 0 \\ 0 & \kappa_s & 0 & 0 & 0 \\ 0 & 0 & \kappa_{c^1} & 0 & 0 \\ 0 & 0 & 0 & \kappa_{c^2} & 0 \\ 0 & 0 & 0 & 0 & \kappa_{r^i} \end{pmatrix} \left[\begin{pmatrix} \theta_l \\ \theta_s \\ \theta_{c^1} \\ \theta_{c^2} \\ \theta_{r^i} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t^1 \\ C_t^2 \\ y_r\,(t, T_i) \end{pmatrix} \right] dt$$

$$+ \begin{pmatrix} \sigma_l & 0 & 0 & 0 & 0 \\ 0 & \sigma_s & 0 & 0 & 0 \\ 0 & 0 & \sigma_{c^1} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{c^2} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{r^i} \end{pmatrix} \begin{pmatrix} c_{1,1} & 0 & 0 & \dots & 0 \\ c_{2,1} & c_{2,2} & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & c_{n,2} & \dots & \dots & c_{n,n} \end{pmatrix} \begin{pmatrix} dW_l\,(t) \\ dW_s\,(t) \\ dW_{c^1}\,(t) \\ dW_{c^2}\,(t) \\ dW_{r^i}\,(t) \end{pmatrix}. \tag{3.2.14}$$

This system can be written in the following form

$$\underset{5\times1}{dz\,(t)} = \underset{5\times1}{\mu_z\,(t, z)}dt + \underset{5\times5}{\Omega^\intercal\,(t, z)}\underset{5\times1}{dW\,(t)}, \tag{3.2.15}$$

and then we can use the Ito Lemma to obtain the dynamics of $P(t,T_i)$:

and then we can use the multidimensional Ito Lemma in order to obtain the dynamics of $P(t,T_i)$.

$$dP(t,T_i) = \left(\frac{\partial P(t,T_i)}{\partial t} + \left(\underset{1\times 5}{\frac{\partial P(t,T_i)}{\partial z}}\right)^{\intercal} \underset{5\times 1}{\mu_z(t,z)} + \frac{1}{2}tr\left(\underset{5\times 5}{\Omega^{\intercal}(t,z)}\underset{5\times 5}{\Omega(t,z)}\underset{5\times 5}{\frac{\partial^2 P(t,T_i)}{\partial z'\partial z}}\right)\right)dt + \left(\underset{1\times 5}{\frac{\partial P(t,T_i)}{\partial z}}\right)^{\intercal}\underset{5\times 5}{\Omega^{\intercal}(t,z)}dW.$$

The YTM of a bond can be rewritten as $y(t,T_i) = f(t,T_i)'z_t$ where for the sake of simplicity, we define the following vector of factor loadings $f(t,T_i)$

$$\begin{aligned} f_1(t,T_i) &= 1 \\ f_2(t,T_i) &= \tfrac{1-e^{-\lambda_1(T_i-t)}}{\lambda_1(T_i-t)} \\ f_3(t,T_i) &= \tfrac{1-e^{-\lambda_1(T_i-t)}}{\lambda_1(T_i-t)} - e^{-\lambda_1(T_i-t)} \\ f_4(t,T_i) &= \tfrac{1-e^{-\lambda_2(T_i-t)}}{\lambda_2(T_i-t)} - e^{-\lambda_2(T_i-t)} \\ f_5(t,T_i) &= 1. \end{aligned}$$

The derivative with respect to the state variables becomes

$$\frac{\partial P(t,T_i)}{\partial z_t} = -\left(\int_t^{T_i}\delta\left(e^{-(y(t,T_i))(s-t)}\frac{\partial(y(t,T_i))(s-t)}{\partial z_t}\right)ds + (V_{T_i})\,e^{-(y(t,T_i))(T_i-t)}\frac{\partial(y(t,T_i))(T_i-t)}{\partial z_t}\right),$$

and noticing that $\frac{\partial y(t,T_i)}{\partial z_t} = f$ we can write

$$\begin{aligned}
\frac{\partial P(t,T_i)}{\partial L_t} &= -\left(\int_t^{T_i}\delta\left(e^{-(y(t,T_i))(s-t)}(s-t)\right)ds + V_{T_i}e^{-(y(t,T_i))(T_i-t)}(T_i-t)\right), \\
\frac{\partial P(t,T_i)}{\partial S_t} &= -\left(\int_t^{T_i}\delta\left(e^{-(y(t,T_i))(s-t)}(s-t)\right)ds + V_{T_i}e^{-(y(t,T_i))(T_i-t)}(T_i-t)\right)\left(\frac{1-e^{-\lambda_1(T_i-t)}}{\lambda_1(T_i-t)}\right), \\
\frac{\partial P(t,T_i)}{\partial C_t^1} &= -\left(\int_t^{T_i}\delta\left(e^{-(y(t,T_i))(s-t)}(s-t)\right)ds + V_{T_i}e^{-(y(t,T_i))(T_i-t)}(T_i-t)\right) \\
&\quad \left(\frac{1-e^{-\lambda_1(T_i-t)}}{\lambda_1(T_i-t)} - e^{-\lambda_1(T_i-t)}\right), \\
\frac{\partial P(t,T_i)}{\partial C_t^2} &= -\left(\int_t^{T_i}\delta\left(e^{-(y(t,T_i))(s-t)}(s-t)\right)ds + V_{T_i}e^{-(y(t,T_i))(T_i-t)}(T_i-t)\right) \\
&\quad \left(\frac{1-e^{-\lambda_2(T_i-t)}}{\lambda_2(T_i-t)} - e^{-\lambda_2(T_i-t)}\right), \\
\frac{\partial P(t,T_i)}{\partial y_r(t,T_i)} &= -\left(\int_t^{T_i}\delta\left(e^{-(y(t,T_i))(s-t)}(s-t)\right)ds + V_{T_i}e^{-(y(t,T_i))(T_i-t)}(T_i-t)\right).
\end{aligned}$$

The derivative with respect to t is

$$\begin{aligned}
\frac{\partial P(t,T_i)}{\partial t} =& -\delta + \int_t^{T_i} \delta \frac{\partial \left(e^{-y(t,T_i)(s-t)}\right)}{\partial t} ds + V_{T_i}\left(\frac{\partial \left(e^{-y(t,T_i)(T_i-t)}\right)}{\partial t}\right), \\
=& -\delta + y(t,T_i)\left(\int_t^{T_i} \delta e^{-y(t,T_i)(s-t)} ds + V_{T_i} e^{-y(t,T_i)(T_i-t)} y(t,T_i)\right) \\
& - \frac{\partial y(t,T_i)}{\partial t}\left(\int_t^{T_i} \delta e^{-y(t,T_i)(s-t)}(t-s)\, ds + V_{T_i} e^{-y(t,T_i)(T_i-t)}(T_i-s)\right) \\
=& -\delta + y(t,T_i) P(t,T_i) \\
& - \frac{\partial y(t,T_i)}{\partial t}\left(\int_t^{T_i} \delta e^{-y(t,T_i)(s-t)}(t-s)\, ds + V_{T_i} e^{-y(t,T_i)(T_i-t)}(T_i-s)\right).
\end{aligned}$$

Finally, we have to compute $\frac{\partial^2 P(t,T_i)}{\partial z' \partial z}$ that can be written as

$$\int_t^{T_i} \delta \left(e^{-(y(t,T_i))(s-t)} f(t,T_i) f(t,T_i)^\intercal (s-t)^2\right) ds + (V_{T_i})\, e^{-(y(t,T_i))(T_i-t)} f(t,T_i) f(t,T_i)^\intercal (T_i-t)^2.$$

The evolution of the price of a bond becomes

$$\begin{aligned}
\frac{dP(t,T_i)}{P(t,T_i)} =& \left(y(t,T_i) - \frac{\delta}{P(t,T_i)} - \mathcal{D}(t,T_i)\left(\mathcal{R}(t,T_i) + f(t,T_i)^\intercal \mu_z(t,z)\right)\right) dt \\
& + \mathcal{C}(t,T_i)\frac{1}{2} tr\left(\Omega(t,z)\,\Omega(t,z)^\intercal f(t,T_i) f(t,T_i)^\intercal\right) dt \\
& - \mathcal{D}(t,T_i) f(t,T_i)^\intercal \Omega(t,z)\, dW.
\end{aligned} \tag{3.2.16}$$

Using the DNSS framework to model the YTM curve, the liquidity spread yield of a bond is immediately recovered. In fact, in a cross-sectional environment we have n bonds, with n different maturities, that shares the yield-curve factors but each bond has its own idiosyncratic factor given by its liquidity. Firstly, given n prices of the bonds, we calculate the n YTM and we calibrate the YTM of value of the bonds using equation 3.2.10 and, secondly, we calculate the yield spread as the difference between the yield of the bond and the yield of the fitted curve. We notice that this representation generates incompleteness in the market since we have $n+4$ risk factors and n bonds. The market price of risk $\xi \in \mathbb{R}^k$ can be written as

$$\underset{n\times n+4}{\Sigma(t,z)^\intercal}\ \underset{n+4\times 1}{\xi(t)} = \underset{n\times 1}{\mu_V(t,z)} + \underset{n\times 1}{I_V^{-1}\delta_V} - r \underset{n\times 1}{\mathbf{1}}, \tag{3.2.17}$$

with $\mathcal{D}(t,T)\, f(t,T)^{\intercal}\, \Omega(t,z) = \Sigma(t,z)^{\intercal}$. If we add the liquidity spread to the model the market is incomplete and there exist infinitely many market prices of risk that eliminate any arbitrage opportunity from the market.

3.2.3. Futures price and delivery option. One application of the previous model is the valuation of the delivery option embedded in the bond future contract. In the following subsection we will describe the problem and why we need to use a multifactor model with liquidity spreads.

The seller of a bond futures has the choice about the bond to be delivered. We define the deliverable basket $\Psi(t,D)$ as the set, at time t, of n bonds deliverable at the expiry, D, of the futures contract . The flow received by the seller is the futures settlement price multiplied by a conversion factor, plus the accrued interest.

The conversion factor is used to homogenize the deliverable bonds that have different coupons and maturities. The conversion factor (CF) is a constant, different for each bond and defined by the exchange at the beginning of the trading period of each futures. The CF roughly corresponds to one-hundredth of the price of the deliverable bond at the yield level of the notional coupon of the bond future contract, that usually is at 6%.

At maturity, the seller receives the payoff

$$F(D,D)\, CF_i - P(D,T_i),$$

where $F(D,D)$ is the settlement price of the futures, CF_i is the conversion factor of i^{th} bond whose value is $P(D,T_i) \in \Psi(t,D)$.

The deliverable bond with the best payoff for the seller of the futures is called the *cheapest-to-deliver* (CTD).

The futures contract is settled at each time so that its mark-to-market is constantly zero. Since the settlement at any time s is worth the change in its market value $dF(s,D)$, then this market value must satisfy

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^D \frac{G(t_0)}{G(s)} dF(s,D)\right] = 0.$$

This equation can hold only if $dF(s,D)$ is a martingale under $\mathbb{Q}$, i.e.

$$dF(s,D) = \sigma_F dW^{\mathbb{Q}}(s),$$

and if we integrate both sides and we take the expected value we obtain

$$F(t,D) = \mathbb{E}_t^{\mathbb{Q}}\left[F(D,D)\right].$$

The price of the future contract at delivery must satisfy the following boundary condition, in order to prevent arbitrage,

$$\prod_{i=1}^{n} \left(F\left(D,D\right)CF_i - P\left(D,T_i\right)\right)\mathbb{1}\left(T_i = \tau_D\right) = 0, \tag{3.2.18}$$

where τ_D is the maturity of the CTD bond at time D, since the equation is equal to zero only if the delivered bond is the CTD. Thus, we can rearrange the future price by defining the CTD at t as the n^{th} in the deliverable basket such as $T_n = \tau_t$:

(3.2.19)

$$F\left(t,D\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P\left(D,T_{\tau_t}\right)}{CF_{\tau_t}}\mathbb{1}\left(T_n = \tau_t = \tau_D\right) + \prod_{i=1}^{n-1}\frac{P\left(D,T_i\right)}{CF_i}\mathbb{1}\left(T_i = \tau_D\right)\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P\left(D,T_{\tau_t}\right)}{CF_{\tau_t}}\right] - DO\left(t,D\right)$$

where $DO\left(t,D\right)$, the value of the delivery option at time t, can be written as

$$DO\left(t,D\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P\left(D,T_{\tau_t}\right)}{CF_{\tau_t}}\mathbb{1}\left(T_n = \tau_t \neq \tau_D\right) - \prod_{i=1}^{n-1}\frac{P\left(D,T_i\right)}{CF_i}\mathbb{1}\left(T_i = \tau_D\right)\right],$$

which is the different of the ratio between the price of bonds in the deliverable basket and their conversion factor multiplied by the indicator function of the event that each bond of the basket will become the *cheapest-to-deliver* at D. In this way we can see that the price of the futures is equal to the forward price of the current CTD minus the delivery option. We define the CTD switch as the event such that during the life of the futures the CTD changes. We notice that the delivery options cannot be negative and could be zero only if the probability of a CTD switch occurring during the life of the contract is also zero.

3.2.4. Gross and net basis. The no-arbitrage relationship at delivery, given by 3.2.18, is the key for basis trade between the deliverable bonds and futures, with the following conventions

$$GB\left(t,D,T_i\right) = P\left(D,T_i\right) - F\left(t,D\right)CF_i \tag{3.2.20}$$

$$NB\left(t,D,T_i\right) = \mathbb{E}_t^{\mathbb{Q}}\left[P\left(D,T_i\right)\right] - F\left(t,D\right)CF_i \tag{3.2.21}$$

where $GB\left(t,D,T_i\right)$ is the gross basis at t of the bond with maturity T_i over the futures with delivery at D. We define the net basis as the same relationship but using the forward bond price $P\left(t,D,T_i\right) = \mathbb{E}_t^{\mathbb{Q}}\left[P\left(D,T_i\right)\right]$ that represent the price of buying the bond with maturity T_i at t until D through a repo transaction.

If at t a trader enters in a long basis trade he will buy the CTD bond, with maturity τ_t, and he will profit if before D another bond will become the CTD. In fact, he will sell the previous CTD and he will buy the new one at a lower price with the following payoff, obtained by

$$NB\left(t,D,T_{i}\right)=\mathbb{E}_{t}^{\mathbb{Q}}\left[P\left(D,T_{\tau_{t}}\right)\right]-\left(\frac{\mathbb{E}_{t}^{\mathbb{Q}}\left[P\left(D,T_{\tau_{t}}\right)\right]}{CF_{\tau_{t}}}-DO\left(t,D\right)\right)CF_{\tau_{t}}.$$

This profile of risk makes the long net basis position like an out of the money options written on the delivery options. In fact, if at D there is no switch the indicator function $\mathbb{1}\left(T_{n}=\tau_{t}=\tau_{D}\right)$ is equal to one and the net value of the net basis is zero. Otherwise, if there was a CTD switch the net basis would have the following payoff greater than zero

$$P\left(t,D,T_{\tau_{t}}\right)-\prod_{i=1}^{n-1}\frac{P\left(D,T_{i}\right)}{CF_{i}}\mathbb{1}\left(T_{i}=\tau_{D}\right)CF_{\tau_{t}}.$$

To determine the value of the delivery option, we should calculate the probability that one of the bonds in the basket could become the new cheapest to deliver. To model the market realistically, we propose the use of the model exposed in the previous section since both the change in the 4 risk factors of the bond value and the change in the liquidity spread can determine the switch of the CTD.

Given the complexity of the calculation, the numerical example will be made in future research by considering whether there is a closed-form solution to the problem or whether it should be obtained through numerical simulations.

3.3. Target based portfolio optimization

3.3.1. Multifactor market . In this section we introduce the financial market, based on the modeling of the previous section that we will use to solve a portfolio optimization of a market maker who has to hedge a bond using other bonds on the same curve but with different maturities. We use the value of the bond, without the liquidity spreads, in order to have a complete market in our numerical results. We will also show how to solve the portfolio optimization problem in a incomplete market.

The framework is defined as follows:

- A set of k state variables exist, whose values z_t solve the stochastic differential equation:

$$\underset{k\times 1}{dz\left(t\right)}=\underset{k\times 1}{\mu_{z}\left(t,z\right)}dt+\underset{k\times k}{\Omega^{\intercal}\left(t,z\right)}\underset{k\times 1}{dW\left(t\right)},\tag{3.3.1}$$

  where $dW\left(t\right)$ is a vector of independent Wiener processes with zero mean and volatility dt. The set of the state variables could contain the factor dynamics of the DNSS curve as 3.2.13.

- $n = k$ bonds whose values $V(t, T_i)$ are continuously traded on a financial market and solves the matrix stochastic differential equation

$$\underset{n\times n}{I_V^{-1}} \underset{n\times 1}{dV(t,z)} = \underset{n\times 1}{\mu_V(t,z)}dt + \underset{n\times k}{\Sigma^{\intercal}(t,z)} \underset{k\times 1}{dW(t)}, \tag{3.3.2}$$

in which I_V is a diagonal matrix gathering the prices $V(t, T_i)$, T_i is the $n \times 1$ vector of maturity of the bonds, $\mu_V(t, z)$ is the vector of the instantaneous expected returns of the bond and $\Sigma(t, z)^{\intercal}$ is their volatility matrix as defined from equation 3.2.16.

This market is arbitrage free if and only if there exists (at least a) vector $\xi \in \mathbb{R}^k$ such that

$$\underset{n\times n}{\Sigma(t,z)^{\intercal}} \underset{n\times 1}{\xi(t)} = \underset{n\times 1}{\mu_V(t,z)} + \underset{n\times 1}{I_V^{-1}\delta_V} - r \underset{n\times 1}{\mathbf{1}}, \tag{3.3.3}$$

in which $\mathbf{1}$ is a vector of ones and δ_V is a vector of the bond coupons. Furthermore, the market is complete if the vector ξ is unique. The vector ξ is called "market price of risk" – MPR. By using the MPR we can define a new probability $\mathbb{Q}$ under which the original Wiener processes are transformed as follows (Girsanov's theorem):

$$dW^{\mathbb{Q}}(t) = \xi(t)\,dt + dW(t)\,.$$

3.3.2. The wealth process. We assume that the trader is a market maker who has to buy or sell one the specific bond $V(t, T_x)$, with maturity $T_x \in [T_1, T_n]$ on a forced basis in order to satisfy customer demands. We assume that the curve is composed by n bonds with different maturity such as the vector for the maturities of the curve, $T_c, \in [T_1, T_2, ..., T_n]$ with $T_1 > 0$.

Once the quantity of the bond is known, the trader then has to choose one or more bonds of the curve, different from $V(t, T_x)$, in order to reduce the variance of the total wealth, maximizing the return, at a given time horizon T, with $T \in [T_0, T_n]$ and $0 < T_0 \leq T_1$.

We denote with R_t the total wealth at time t. The coupon paid or received from the bond $V(t, T_x)$ is constant and equal to c unit per time, with $c = \delta_x w_x$ where w_x could be either 1 or -1 if the bond is bought or sold, respectively. Let us call ω_t the dollar amount invested at time t into the portfolio used to hedge w_x the constant dollar amount of the bond $V(t, T_x)$. Then, the wealth dynamics is

$$\begin{aligned} dR_t &= \left(R_t - \omega_t^{\intercal}\mathbf{1}\right)\frac{dG_t}{G_t} + \omega_t^{\intercal} I_V^{-1}\left(dV(t) + \delta_V\right) - cdt. \\ &= \left(R_t r_t - c + \omega_t^{\intercal}\left(\mu(t,z) - r\mathbf{1}\right)\right)dt + \omega_t^{\intercal}\left(\Sigma(t,z)^{\intercal}\,dW(t)\right), \end{aligned} \tag{3.3.4}$$

with $\mu(t, z) = \mu_V(t, z) + I_V^{-1}\delta_V$. Furthermore, we set $R_0 = w_x$.

3.3.3. The target hedging problem. We assume that the trader chooses the mean-variance approach to solve his portfolio problem.

DEFINITION 1. The mean-variance optimization problem is defined as

$$(P_\alpha) \quad \min J(\omega(\cdot)) \doteq \alpha \mathbb{V}_0 [R_T - X_T] - \mathbb{E}_0 [R_T - X_T], \tag{3.3.5}$$

with $\alpha > 0$ and the scalar X_T equal to $w_x V(T, T_x)$ over a set of admissible strategies, that can be introduced as follows:

DEFINITION 2. A portfolio $\omega(\cdot)$ is said to be *admissible* if $\omega(\cdot) \in L^2_{\mathcal{F}}(0, T, \mathbb{R}^2)$ and $\mathbb{E}\left(\int_0^T |\omega_t|^2\right) < \infty$.

If $\omega^*(\cdot)$ solves(P_α) for some $\alpha > 0$ and $R_T - X_T$ is the associated wealth level, the set $(\mathbb{V}_0 [R_T - X_T], \mathbb{E}_0 [R_T - X_T])$ is called the efficient frontier.

Moreover, we define

$$\Pi_\alpha(\cdot) = \{\omega(\cdot) \,| \omega(\cdot) \text{ is an optimal control of } P_\alpha\}.$$

We know from Zhou and Li (2000) that the problem can be approached by solving the following auxiliary problem

$$(P_{\alpha,\beta}) \quad \min J(\omega(\cdot)) \doteq \mathbb{E}_0 \left[\alpha (R_T - X_T)^2 - \beta (R_T - X_T)\right], \tag{3.3.6}$$

with $-\infty < \beta < \infty$ and we also define $\Pi_{\alpha,\beta}(\cdot)$ as the optimal control of auxiliary problem $P_{\alpha,\beta}$. Then, the following result shows the relationship between problems P_α and $P_{\alpha,\beta}$.

THEOREM 3. *For any $\alpha > 0$, one has*

$$\Pi_\alpha(\cdot) \subseteq \bigcup_{-\infty<\beta<+\infty} \Pi_{\alpha,\beta}(\cdot).$$

Moreover, if $\omega^*(\cdot) \in \Pi_\alpha(\cdot)$, then $\omega^*(\cdot) \in \Pi_{\alpha,\beta}(\cdot)$ with $\beta = 1 + 2\alpha\mathbb{E}_0 [R_T^* - X_T]$, where $R^*(\cdot)$ is the wealth under optimal control.

PROOF. See Appendix .2 □

3.3.4. The optimal portfolio. The implication of 3 is that if $\omega^*(\cdot)$ is a solution to 3.3.5 than it is also a solution to 3.3.6 with $\beta = 1 + 2\alpha\mathbb{E}_0 [R_T^* - X_T]$. Finally, we could set $\gamma = \frac{\beta}{2\alpha}$ and it turns out that 3.3.6 is equivalent to

$$\min_\omega J(w_t) \doteq \mathbb{E}_0 \left[\frac{\alpha}{2} ((R_T - X_T) - \gamma)^2\right],$$

over a set of admissible strategies. Since the actual value of $J(w_t)$ is mainly immaterial, we rewrite the problem as the target problem P_γ

$$(P_\gamma) \quad \min J(\omega(\cdot)) \doteq \mathbb{E}_0\left[\frac{1}{2}\left(R_T - (X_T+\gamma)\right)^2\right]. \tag{3.3.7}$$

The Problem (3.3.7) can be solved by backward induction as shown in Appendix .3.

PROPOSITION 4. *The optimal portfolio solving Problem (3.3.7) given the state variables (3.3.4) and (3.3.1) is*

(3.3.8)

$$\omega_t^* = \left(\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T +ce^{-\int_t^s r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)e^{-\int_t^T r_u du}\right] - R_t\right)\Sigma^\intercal\Sigma^{-1}(\mu - r\mathbf{1})$$
$$+\Sigma^{-1}\Omega\frac{\left(\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T +ce^{-\int_t^s r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)e^{-\int_t^T r_u du}\right]\right)}{\partial z(t)}$$
$$-\Sigma^{-1}\Omega\frac{\left(\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T +ce^{-\int_t^s r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)e^{-\int_t^T r_u du}\right] - R_t\right)}{\mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T\left(2r_u - \xi_u^\intercal\xi_u\right)du}\right]}\frac{\partial\mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T\left(2r_u - \xi_u^\intercal\xi_u\right)du}\right]}{\partial_z}.$$

in which the Wiener processes under the new probability $\mathbb{Q}_2$ are defined as

$$dW^{\mathbb{Q}_2}(t) = 2\xi dt + dW(t). \tag{3.3.9}$$

PROOF. See Appendix .3. □

3.3.5. The auxiliaries functions. The optimal portfolio can be rewritten using two auxiliaries functions $H(t,z_t)$ and F_z as follow:

$$\omega_t^* = (H - R_t)\left(\Sigma^\intercal\Sigma\right)^{-1}(\mu - r\mathbf{1}) + (\Sigma)^{-1}\Omega.\left(H_z - \frac{(H-R_t)F_z}{F}\right),$$

with $H(t,z_t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T +ce^{-\int_t^s r_u du}ds\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)e^{-\int_t^T r_u du}\right]$ and $F(t,z_t) = \mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T\left(2r_u-\xi_u^\intercal\xi_u\right)du}\right]$. First, we notice that $H(t,z_t)$ can be simplified by using the Fundamental Theorem of Finance

$$H(t,z_t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T ce^{-\int_t^s r_u du}ds\right] + \mathbb{E}_t^{\mathbb{Q}}\left[\gamma e^{-\int_t^T r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[X_T e^{-\int_t^T r_u du}\right]$$
$$= \gamma B(t,T) + \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \delta_x e^{-\int_t^s r_u du}ds\right] + \mathbb{E}_t^{\mathbb{Q}}\left[V(T,T_x)e^{-\int_t^T r_u du}\right],$$

with $\omega_x = 1$ for the sake of simplicity. We notice that $\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \delta_x e^{-\int_t^s r_u du}ds\right] + \mathbb{E}_t^{\mathbb{Q}}\left[V(T,T_x)e^{-\int_t^T r_u du}\right]$ is equal to $V(t,T_x)$ then we can write the following vector of derivatives of $V(t,T_x)$ with respect to the state variables that drives $y_v(t,T_x)$:

$$\begin{aligned}
\frac{\partial V(t,T_x)}{\partial L_t} =& -\int_t^{T_x} \delta\left(e^{-(y_v(t,T_x))(s-t)}(s-t)\right)ds - (V_{T_x})\,e^{-(y_v(t,T_x))(T_x-t)}(T_x-t),\\
\frac{\partial V(t,T_x)}{\partial S_t} =& -\left(\frac{1-e^{-\lambda_1(T_x-t)}}{\lambda_1(T_x-t)}\right)\int_t^{T_x} \delta\left(e^{-(y_v(t,T_x))(s-t)}(s-t)\right)ds\\
& - (V_{T_x})\left(\frac{1-e^{-\lambda_1(T_x-t)}}{\lambda_1(T_x-t)}\right)e^{-(y_v(t,T_x))(T_x-t)}(T_x-t)\\
\frac{\partial V(t,T_x)}{\partial C_t^1} =& -\left(\frac{1-e^{-\lambda_1(T_x-t)}}{\lambda_1(T_x-t)} - e^{-\lambda_1(T_x-t)}\right)\int_t^{T_x} \delta\left(e^{-(y_v(t,T_x))(s-t)}(s-t)\right)ds\\
& - (V_{T_x})\left(\frac{1-e^{-\lambda_1(T_x-t)}}{\lambda_1(T_x-t)} - e^{-\lambda_1(T_x-t)}\right)e^{-(y_v(t,T_x))(T_x-t)}(T_x-t),\\
\frac{\partial V(t,T_x)}{\partial C_t^2} =& -\left(\frac{1-e^{-\lambda_2(T_x-t)}}{\lambda_2(T_x-t)} - e^{-\lambda_2(T_x-t)}\right)\int_t^{T_x} \delta\left(e^{-(y_v(t,T_x))(s-t)}(s-t)\right)ds\\
& - (V_{T_x})\left(\frac{1-e^{-\lambda_2(T_x-t)}}{\lambda_2(T_x-t)} - e^{-\lambda_2(T_x-t)}\right)e^{-(y_v(t,T_x))(T_x-t)}(T_x-t).
\end{aligned} \tag{3.3.10}$$

Now, we still have to compute

$$\frac{\partial \mathbb{E}_t^{\mathbb{Q}}\left[\gamma e^{-\int_t^T r_u du}\right]}{\partial_z},$$

that can be written and computed as

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\gamma e^{-\int_t^T r_u du}\right] =& \gamma B(t,T) = e^{-y_v(t,T)(T-t)}\\
\frac{\partial \gamma B(t,T)}{\partial L_t} =& -\gamma e^{-(y_v(t,T))(T-t)}(T-t),\\
\frac{\partial \gamma B(t,T)}{\partial S_t} =& -\gamma\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)}\right)e^{-(y_v(t,T))(T-t)}(T-t)\\
\frac{\partial \gamma B(t,T)}{\partial C_t^1} =& -\gamma\left(\frac{1-e^{-\lambda_1(T-t)}}{\lambda_1(T-t)} - e^{-\lambda_1(T-t)}\right)e^{-(y_v(t,T))(T-t)}(T-t),\\
\frac{\partial \gamma B(t,T)}{\partial C_t^2} =& -\gamma\left(\frac{1-e^{-\lambda_2(T-t)}}{\lambda_2(T-t)} - e^{-\lambda_2(T-t)}\right)e^{-(y_v(t,T))(T-t)}(T-t).
\end{aligned}$$

Finally, we can write $\frac{\partial H(t,z_t)}{\partial z_t}$ as the sum of the two vector derivatives $\frac{\partial V(t,T_x)}{\partial z_t}$ and $\frac{\partial \gamma B(t,T)}{\partial z_t}$.

3.3.6. Computation of F_z. In order to compute F_z , under the assumption of constant MPR, we have to rewrite the dynamics of $z_r(\cdot) = 2r(\cdot)$ under the new probability $\mathbb{Q}_2$. But first we note, from 3.2.10, that the short rate r_t is the limit of y_v for infinitely short maturity

$$r_t = \lim_{T\to t} y_v = L_t + S_t.$$

Thereby, the dynamics of the short rate are given by the limits of $T \to t$ of 3.2.11:

$$\lim_{T\to t} dy_v = dr_t = \left(\frac{\lambda_1}{2}\left(S_t - C_t^1\right) - \frac{\lambda_2}{2}C_t^2\right) dt + dL_t + dS_t, \tag{3.3.11}$$

where we set the limit of the instantaneous roll-down equal to zero:

$$\lim_{T\to t} \frac{\partial y_v\left(t,T\right)}{\partial t} = \left(\frac{\lambda_1}{2}\left(S_t - C_t^1\right) - \frac{\lambda_2}{2}C_t^2\right) = 0.$$

Under the assumptions of equal speed of mean reversion, $\theta_l = \theta_s$, the sum of two Vasicek processes is still a Vasicek process so we can obtain the dynamics of r_t

$$dr\left(t\right) = \kappa_r\left(\theta_r - r\left(t\right)\right) dt + \sigma_r dW_r\left(t\right), \tag{3.3.12}$$

where $\kappa_r = \kappa_l + \kappa_s$, $\theta_r = \theta_l = \theta_s$, and $\sigma_r = [(\sigma_l^2 + \sigma_s^2)]^{\frac{1}{2}}$. We define also

$$W_r\left(t\right) = \int_0^t \frac{\sigma_l}{\sigma_r} dW_l\left(u\right) + \int_0^t \frac{\sigma_s}{\sigma_r} dW_s\left(u\right).$$

We can notice that $W_r\left(t\right)$ is a local martingale with

$$\left(dW_r\left(t\right)\right)^2 = \left(\int_0^t \frac{\sigma_l}{\sigma_r} dW_l\left(u\right) + \int_0^t \frac{\sigma_s}{\sigma_r} dW_l\left(u\right)\right)^2 = \left(\frac{\left(\sigma_l^2 + \sigma_s^2\right)}{\sigma_r^2} dt\right) = dt.$$

Hence, we can see that $W_r\left(t\right)$ is a Brownian motion in the sense of the following theorem.

Theorem 5. ***Lévy Theorem***.

Let $M\left(t\right)$, $t \geq 0$, be a martingale relative to a filtration $\mathcal{F}_t, t \geq 0$. Assume that $M\left(0\right) = 0$, $M\left(t\right)$ has continuous paths, and $\left[M, M\right]\left(t\right) = t$ for all $t \geq 0$. Then $M\left(t\right)$ is a Brownian motion.

Given the SDE 3.3.11 the price of a zero-coupon bond can be written as

$$B\left(t,T\right) = e^{f(t,T) - g(t,T) r_t}, \tag{3.3.13}$$

where

$$f\left(t,T\right) = \left(\frac{1 - e^{\kappa_r(T-t)}}{\kappa_r} - \left(T - t\right)\right)\left(\theta_r - \frac{\sigma_r \xi_r}{\kappa_r} - \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\right) - \frac{\sigma_r^2}{4\kappa_r^3}\left(1 - e^{\kappa_r(T-t)}\right)^2,$$

$$g\left(t,T\right) = \frac{1 - e^{\kappa_r(T-t)}}{\kappa_r}.$$

The dynamic of z_r can be written as

$$\begin{aligned}
dz_r(t) =& 2\kappa_r\left(\theta_r - \frac{1}{2}z_r(t)\right)dt + 2\sigma_r dW_r(t)\\
=& 2\kappa_r\left(\theta_r - \frac{1}{2}z_r(t)\right)dt + 2\sigma_r\left(dW^{\mathbb{Q}_2}(t) - 2\xi_r dt\right)\\
=& \kappa_r\left(2\theta_r - 4\frac{\xi_r\sigma_r}{\kappa_r} - z_r(t)\right)dt + 2\sigma_r dW^{\mathbb{Q}_2}(t),
\end{aligned}$$

and finally we obtain

$$F(t, z_t) = e^{\xi^\intercal\xi}\mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T z_r(u)du}\right] = e^{\xi^\intercal\xi}e^{f_2(t,T)-2g(t,T)r(t)},$$

$$\begin{aligned}
F(t, z_t) =& e^{\xi^\intercal\xi}\mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T z_r(u)du}\right] = e^{\xi^\intercal\xi}e^{f_2(t,T)-2g(t,T)L_t+S_t}\\
\frac{\partial F(t, z_t)}{\partial L_t} =& -2g(t,T)\, e^{\xi^\intercal\xi}e^{f_2(t,T)-2g(t,T)(L_t+S_t)}\\
\frac{\partial F(t, z_t)}{\partial S_t} =& -2g(t,T)\, e^{\xi^\intercal\xi}e^{f_2(t,T)-2g(t,T)(L_t+S_t)}
\end{aligned}$$

and then

$$\begin{array}{rcc}
\frac{\partial F(t,z_t)}{\partial L_t F} & = & -2g(t,T)\\
\frac{\partial F(t,z_t)}{\partial S_t F} & = & -2g(t,T)\\
\frac{\partial F(t,z_t)}{\partial C_t^1 F} & = & 0\\
\frac{\partial F(t,z_t)}{\partial C_t^2 F} & = & 0
\end{array}$$

3.4. Incomplete market: martingale method

In this section, we work in an incomplete market where we have 4 state variables that drives the value of the bonds and one state variable, the liquidity spread, for each bonds. Since there are n bonds and $n+4$ risk factors there is no arbitrage but market incompleteness. We solve this problem by using the martingale approach in order to find the optimal wealth. Because of the incompleteness, we will minimize the mean square distance between the diffusion terms of the optimal wealth and of the wealth of the portfolio instead of replicating the optimal wealth. First, we solve the following problem

$$\min_{\omega} J(w_t) \doteq \mathbb{E}_0\left[\frac{1}{2}\left(R_T - (X_T + \gamma)\right)^2\right], \tag{3.4.1}$$

under the constraint

$$R_0 = \mathbb{E}_0^{\mathbb{Q}}\left[\int_0^T c e^{-\int_0^s r_u du} ds + R_T e^{-\int_0^T r_u du}\right]. \tag{3.4.2}$$

The Lagrangian function of 3.4.1 is

$$\mathcal{L} = \mathbb{E}_0\left[\frac{1}{2}\left(R_T - (X_T+\gamma)\right)^2\right] - \phi\left(R_0 - \mathbb{E}_t^{\mathbb{Q}}\left[\int_0^T ce^{-\int_0^s r_u du}ds + R_T e^{-\int_0^T r_u du}\right]\right),$$

where ϕ is the (constant) Lagrangian multiplier. This Lagrangian function cannot be directly simplified since the two expected values are written under two different probabilities. Thus, we can rewrite the second expected value $\mathbb{E}$ under the historical probability as follows:

$$\begin{aligned}\mathcal{L} = &\mathbb{E}_0\left[\frac{1}{2}\left(R_T - (X_T+\gamma)\right)^2 + \phi R_T e^{-\int_0^T r_u du} m(0,T)\right]\\ &- \phi\left(R_0 - \mathbb{E}_0\left[\int_0^T m(0,s)\, ce^{-\int_0^s r_u du}ds + m(0,T)\, R_T e^{-\int_0^T r_u du}\right]\right)\end{aligned}$$

where $e^{-\int_0^T r_u du}m(0,T)$ is the stochastic discount factor. The derivative of $\mathcal{L}$ with respect to $R(T)$ must be set to zero for each state of the world, i.e.

$$R_T^* = (X_T+\gamma) - \phi e^{-\int_0^T r_u du} m(0,T)\,. \tag{3.4.3}$$

Now, the optimal final wealth can be plugged into the budged constraint 3.4.2

$$\phi = \frac{(X_T+\gamma)\,B(0,T) + \int_t^T +cB(0,s)\,ds - R_0}{\mathbb{E}_0\left[e^{-2\int_0^T r_u du} m^2(0,T)\right]},$$

and the final weal can be rewritten as

$$R_T^* = (X_T+\gamma) - \frac{(X_T+\gamma-\chi_T)\,B(0,T)}{\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]} e^{\Phi(0,T)}, \tag{3.4.4}$$

with $\Phi(t,T) = -\int_t^T r_u du - \frac{1}{2}\int_t^T \xi_u^\intercal \xi_u du - \int_t^T \xi_u^\intercal dW_u$ and $\chi_T = \frac{R_0 - \int_t^T +cB(0,s)ds}{B(0,T)}$.

In the optimal solution, the constraint 3.4.2 must hold at any instant in time:

$$R_t^* = \mathbb{E}_t\left[\int_t^T m(t,s)\, ce^{-\int_t^s r_u du}ds + R_T^* e^{-\int_t^T r_u du} m(t,T)\right]$$

and if the optimal final wealth 3.4.3 is plugged into this equation we have:

$$
\begin{aligned}
R_t^* &= \mathbb{E}_t\left[\int_t^T m(t,s)\, ce^{-\int_t^s r_u du} ds + \Big((X_T+\gamma) - \phi e^{-\int_0^T r_u du} m(0,T)\Big) e^{-\int_t^T r_u du} m(t,T)\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T ce^{-\int_t^s r_u du} ds + (X_T+\gamma)\, e^{-\int_t^T r_u du}\right] - \mathbb{E}_t\left[\phi e^{-\int_0^T r_u du} m(0,T)\, e^{-\int_t^T r_u du} m(t,T)\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T ce^{-\int_t^s r_u du} ds + (X_T+\gamma)\, e^{-\int_t^T r_u du}\right] - \phi m(0,t)\, e^{-\int_0^t r_u du} \mathbb{E}_t\left[e^{2\Phi(t,T)}\right].
\end{aligned}
$$

Now we can define the following two functions

$$
H(t,z_t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T +ce^{-\int_t^s r_u du} ds\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)\, e^{-\int_t^T r_u du}\right], \tag{3.4.5}
$$

$$
F(t,z_t) = \mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T \left(2r_u - \xi_u^\intercal \xi_u\right) du}\right], \tag{3.4.6}
$$

where the new probability $\mathbb{Q}_2$ is defined using the following Girsanov's kernel

$$
dW^{\mathbb{Q}_2}(t) = 2\xi dt + dW(t).
$$

Then, the optimal wealth becomes

$$
R_t^* = H(t,z_t) - \phi m(0,t)\, e^{-\int_0^t r_u du} F(t,z_t).
$$

Now we can compute the dR_t^* through Ito's lemma differentiating with respect to $m(0,t)$ and the state variables 3.3.1 and since we are interested in replicating it we just need to compute the diffusion term.

$$
\begin{aligned}
R_t^* &= (\ldots)\, dt + \frac{\partial R_t^*}{\partial z_t} \Omega^\intercal(t,z)\, dW(t) - \frac{\partial R_t^*}{\partial m(0,t)} m(0,t)\, \xi^\intercal dW(t) \\
&= (\ldots)\, dt + \left(\frac{\partial H(t,z_t)^\intercal}{\partial z_t} - \phi m(0,t)\, e^{-\int_0^t r_u du}\left(\frac{\partial F(t,z_t)^\intercal}{\partial z_t}\right)\right) \Omega^\intercal(t,z)\, dW(t) \\
&\quad + \phi e^{-\int_0^t r_u du} F(t,z_t)\, m(0,t)\, \xi^\intercal dW(t).
\end{aligned}
$$

We can substitute $\phi m(0,t)\, e^{-\int_0^t r_u du} F(t,z_t) = H(t,z_t) - R_t^*$ and we obtain

$$
\begin{aligned}
R_t^* &= (\ldots)\, dt + \frac{\partial R_t^*}{\partial z_t} \Omega^\intercal(t,z)\, dW(t) - \frac{\partial R_t^*}{\partial m(0,t)} m(0,t)\, \xi^\intercal dW(t) \\
&= (\ldots)\, dt + \left(\frac{\partial H(t,z_t)^\intercal}{\partial z_t} - \left(\frac{H(t,z_t) - R_t^*}{F(t,z_t)}\right)\left(\frac{\partial F(t,z_t)^\intercal}{\partial z_t}\right)\right) \Omega^\intercal(t,z)\, dW(t) \\
&\quad + (H(t,z_t) - R_t^*)\, \xi^\intercal dW(t).
\end{aligned} \tag{3.4.7}
$$

3.4.1. The case of an incomplete market. If market is complete the diffusion term of 3.4.7 is set equal to the diffusion term of 3.3.4 and we find the portfolio which replicates the optimal wealth that coincides with the portfolio 3.3.8 found via the dynamic programming.

However, we are interested in the case of the incomplete market generated by liquidity spreads. Also, for practical reasons, it is possible for the market maker to use only one or two bonds to hedge its position on $V(t, T_x)$, and in this case, even without the liquidity spreads, the market would be incomplete. In this two cases of market incompleteness, the matrix $\Sigma(t,z)^\intercal$ cannot be inverted and then the replicating portfolio cannot be found. Nevertheless, we can find the portfolio that minimizes the square of the distance between the diffusion terms of the optimal wealth and of the wealth in 3.3.4, where the diffusion term of the optimal wealth is

$$\Sigma^*(t,z) = \left(H(t,z_t) - R_t^*\right)\xi^\intercal + \left(\frac{\partial H(t,z_t)^\intercal}{\partial z_t} - \left(\frac{H(t,z_t) - R_t^*}{F(t,z_t)}\right)\left(\frac{\partial F(t,z_t)^\intercal}{\partial z_t}\right)\right)\Omega^\intercal(t,z).$$

We can write the problem as

$$\min_{\omega_t}\left(\omega_t^\intercal \Sigma(t,z)^\intercal - \Sigma^*(t,z)\right)\left(\omega_t \Sigma(t,z) - \Sigma^*(t,z)^\intercal\right)$$

whose solution is

$$\omega_t^{**} = \left(\Sigma(t,z)^\intercal \Sigma(t,z)\right)^{-1}\Sigma(t,z)^\intercal \Sigma^*(t,z)^\intercal.$$

The optimal portfolio in case of market incompleteness can thus be written as

(3.4.8)

$$\begin{aligned}
\omega_t^{**} = & \left(\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T + ce^{-\int_t^s r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)\, e^{-\int_t^T r_u du}\right] - R_t\right)\Sigma^{**}\xi \\
& + \Sigma^{**}\Omega \frac{\left(\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T + ce^{-\int_t^s r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)\, e^{-\int_t^T r_u du}\right]\right)}{\partial z(t)} \\
& - \Sigma^{**}\Omega \frac{\left(\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T + ce^{-\int_t^s r_u du}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[(X_T+\gamma)\, e^{-\int_t^T r_u du}\right] - R_t\right)}{\mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T \left(2r_u - \xi_u^\intercal \xi_u\right) du}\right]} \frac{\partial \mathbb{E}_t^{\mathbb{Q}_2}\left[e^{-\int_t^T \left(2r_u - \xi_u^\intercal \xi_u\right) du}\right]}{\partial_z}
\end{aligned}$$

with $\Sigma^{**} = \left(\Sigma(t,z)^\intercal \Sigma(t,z)\right)^{-1}\Sigma(t,z)^\intercal$ $n \times k$ matrix.

3.5. Numerical Application

In this section, we show a numerical application of the model presented. First, we calibrate the various parameters and then we will show the dynamics of the optimal portfolio.

The yield curve is calibrated by using the Dynamic Nelson-Siegel-Svensson model with only three parameters. Actually, the fourth parameter generates instability in the optimal portfolio since hedging bonds are too correlated among themselves, as we will show in details later. We have used daily data, from 04/October/2019 to 04/Oct/2021, on all the Italian Government bond curve .

We set λ_1 equal to 0.3587 which is the value that maximizes the curvature factor at the 5 year pillar. The details of this procedure can be found in Diebold and Rudebusch (2013).

The evolution of the parameters can be seen in figure 3.6.1, while figure 3.6.2 shows an example of calibration of the Yield Curve.

By using these time series we have calibrated a VAR model of order one in order to obtain the continuous time parameters for the equation 3.2.14 without the C_t^2 random variable. The Tables 3.6.3 and 3.6.4 show the results of the econometric estimation while table 3.6.1 show the continuous time parameters.

We have computed the optimal portfolio evolution by using 3, 7, and 50 year bonds in order to hedge a 10 year bond position over a period of 1 year with target equal to 1%. In particular, we have used BTP EUR 2.500 01-Dec-2024, BTP EUR 2.800 01-Dec-2028, BTP EUR 1.700 01-Sep-2051 as hedge for a short position on BTP EUR 0.950 01-Dec-2031.

For each time step we have simulated 5000 path of the state variables and then we have computed ω_t^*. In figures 3.6.5, 3.6.6, and 3.6.7 we show the results for the total portfolio, for the speculative component and for the hedging component, calculated as the median of the portfolio weight given the paths of the state variables. Moreover, we have fixed $R_0 = V(0, T_{10})$.

As we can see, the weights of the hedging portfolio are much more stable than the speculative one. In fact, we see that the speculative portfolio has significant variations in the initial period and is extremely related to the mean reversion of the state variables.

Once the state variables converge to the long-term mean, the speculative component also becomes more stable. Finally, we show the evolution of the total gain of the trader $R_t^* - V\left(t, T_{10}\right)$ in figure 3.6.8 where we have also calculated the upper and lower percentile evolution.

The gain process converges quite fast to the target, as expected. We have seen that the convergence of the gain to the target is highly dependent on the mean reversion of the state

TABLE 1. **Continuous Time Parameters:** This table reports the continuous time parameters estimation of the state variables calibrated between 04/October/2019 to 04/Oct/2021.

State Variable	Kappa	Theta	Sigma		C	
L	3.723	0.024	0.007	1.000	0.000	0.000
S	3.252	-0.028	0.008	-0.628	0.778	0.000
C	9.625	-0.026	0.022	0.430	0.308	0.849

variables. Changing the time horizon of the problem could lead to a slower convergence. In fact, in figure 3.6.9 we show the same gain process, computed by changing the initial calibration period from 04/October/2019 to 05/October/2018, that doesn't reach the target during the period. In particular, as shown in Table 2, varying the calibration horizon yields to similar values for theta and sigma but to a significantly lower mean-reversion speed.

3.6. Conclusion

This study propose a new Dynamic Term Structure model that describes the bond market using the YTMs instead of the affine short rates framework. We find that the model can archive a good fit and stability over time and also the parameters have a clear economic interpretation. Moreover, in this framework we have used the target based approach to solve a mean-variance problem . Once a closed-form solution of the problem is found, we calibrate our model to market data and we find the portfolio weights for the complete market case. We have seen that the portfolio reaches the target and the stability of the speculative part is highly dependent on the speed of mean reversion of the state variables. Future research will focus on studying the portfolio problem in a incomplete market adding the liquidity spread to better describe the bond market and the valuation of the delivery option embedded in the bond futures contract.

FIGURE 3.6.1. Dynamic Nelson-Siegel parameters evolution, calibrated on the BTPs Curve (between 04/October/2019 to 04/Oct/2021).

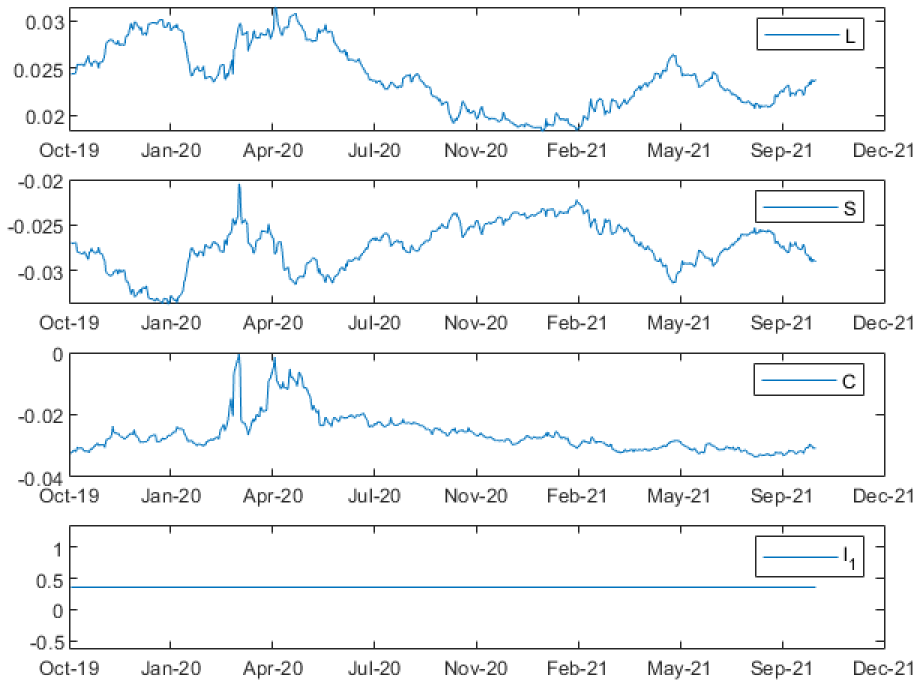


TABLE 2. **Continuous Time Parameters:** This table reports the continuous time parameters estimation calibrated between 04/October/2019 to 04/Oct/2021.

State Variable	Kappa	Theta	Sigma		C	
L	1.246	0.024	0.008	1.000	0.000	0.000
S	0.953	-0.028	0.008	-0.722	0.692	0.000
C	3.887	-0.025	0.026	0.296	0.332	0.896

FIGURE 3.6.2. Nelson-Siegel Curve calibration as 01-Dec-2020.

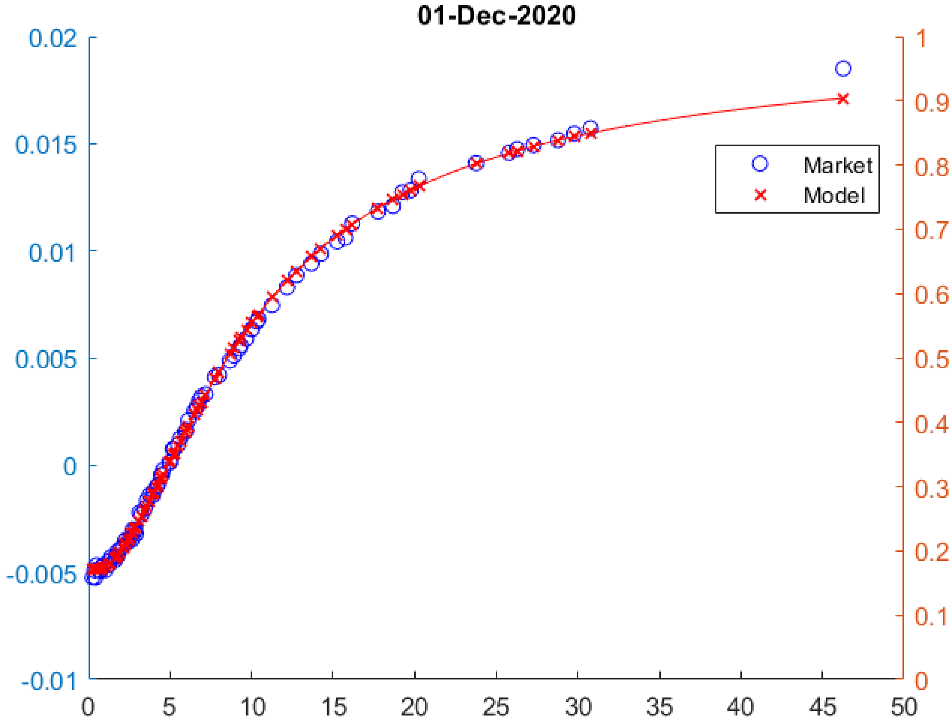


FIGURE 3.6.3. Vector Autoregressive estimation results (between 04/October/2019 to 04/Oct/2021).

AR-Stationary 3-Dimensional VAR(1) Model

```
Effective Sample Size: 496
Number of Estimated Parameters: 6
LogLikelihood: 8950.57
AIC: -17889.1
BIC: -17863.9
```

	Value	StandardError	TStatistic	PValue
Constant(1)	0.00035702	0.00012203	2.9257	0.0034363
Constant(2)	-0.00036028	0.00020058	-1.7961	0.072471
Constant(3)	-0.0010011	0.00025186	-3.975	7.0391e-05
AR{1}(1,1)	0.98511	0.0050008	196.99	0
AR{1}(2,1)	0	0	NaN	NaN
AR{1}(3,1)	0	0	NaN	NaN
AR{1}(1,2)	0	0	NaN	NaN
AR{1}(2,2)	0.98699	0.0072852	135.48	0
AR{1}(3,2)	0	0	NaN	NaN
AR{1}(1,3)	0	0	NaN	NaN
AR{1}(2,3)	0	0	NaN	NaN
AR{1}(3,3)	0.9615	0.0093384	102.96	0

FIGURE 3.6.4. Variance Covariance Matrix and Correlation Matrix.

Innovations Covariance Matrix:

```
1.0e-05 *

 0.0209   -0.0138    0.0279
-0.0138    0.0230   -0.0021
 0.0279   -0.0021    0.2023
```

Innovations Correlation Matrix:

```
 1.0000   -0.6283    0.4295
-0.6283    1.0000   -0.0304
 0.4295   -0.0304    1.0000
```

FIGURE 3.6.5. Total Portfolio Weights evolution over one year.

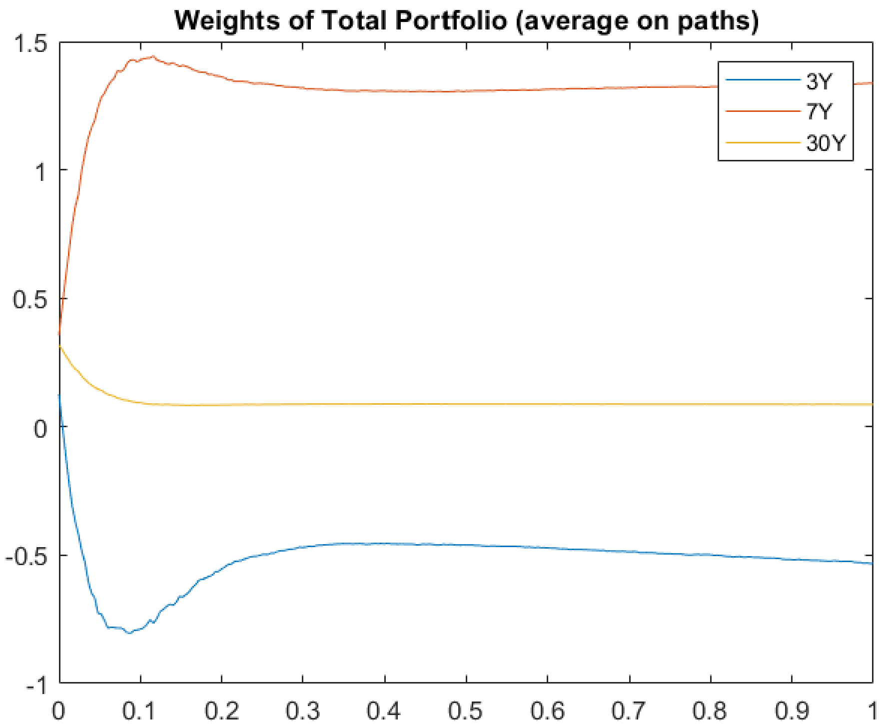


FIGURE 3.6.6. Speculative Portfolio Weights evolution over one year.

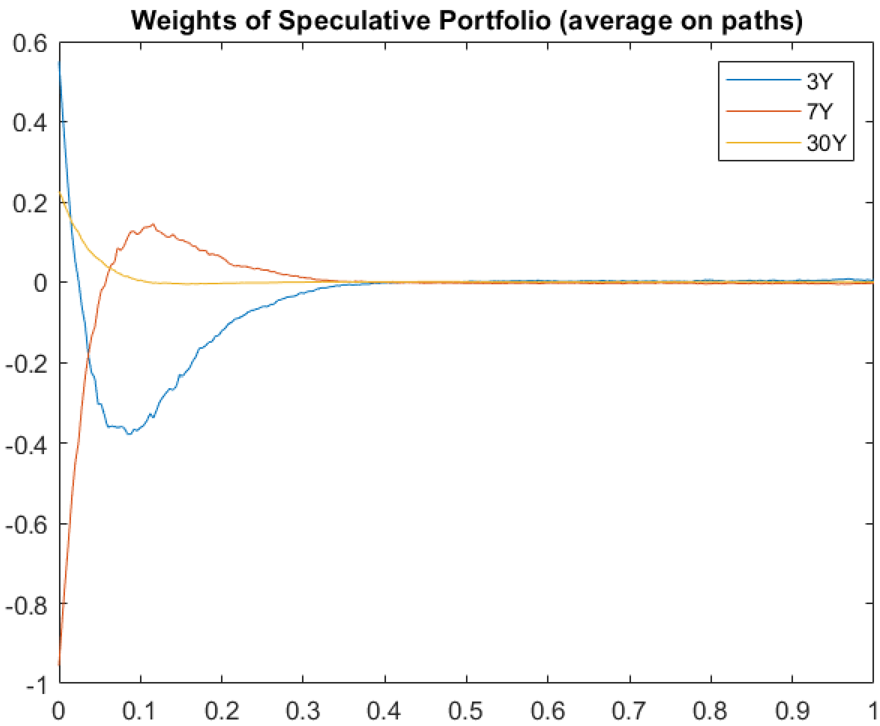


FIGURE 3.6.7. Hedging Portfolio Weights evolution over one year.

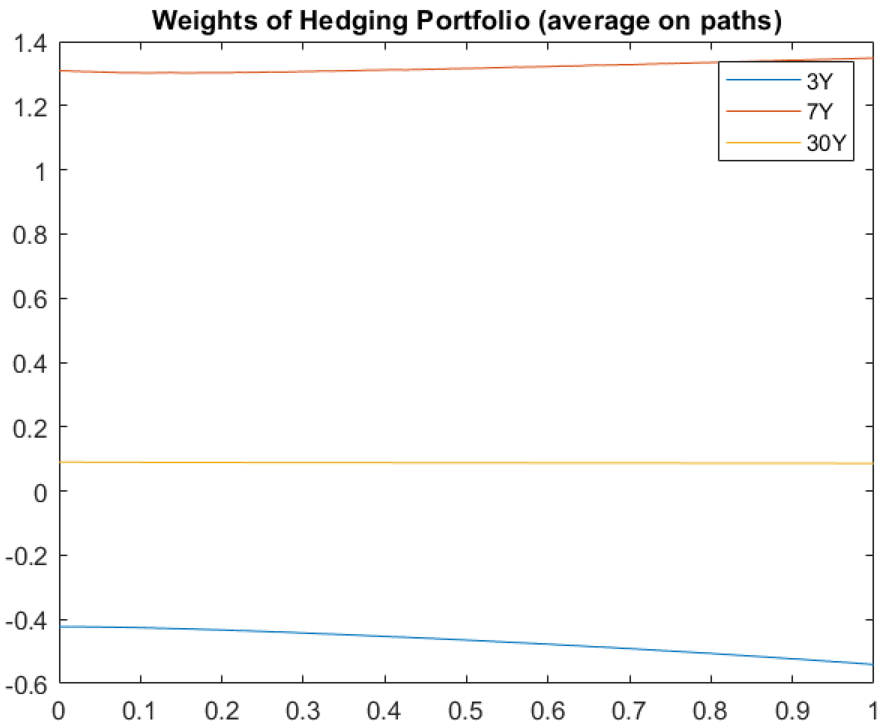


FIGURE 3.6.8. Gain Process median and percentiles using state variables calibrated between 04/October/2019 to 04/Oct/2021.

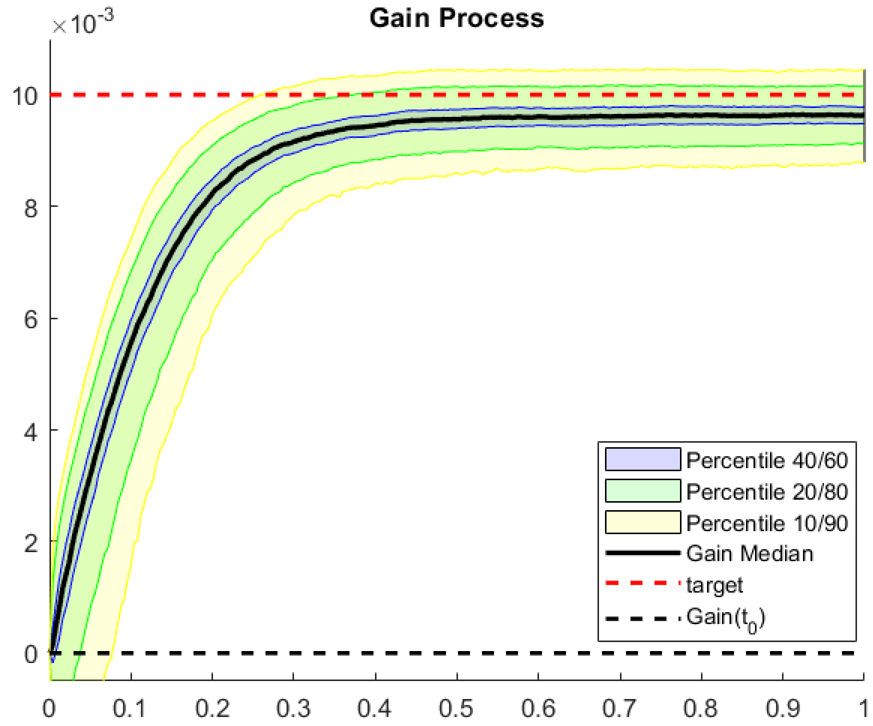
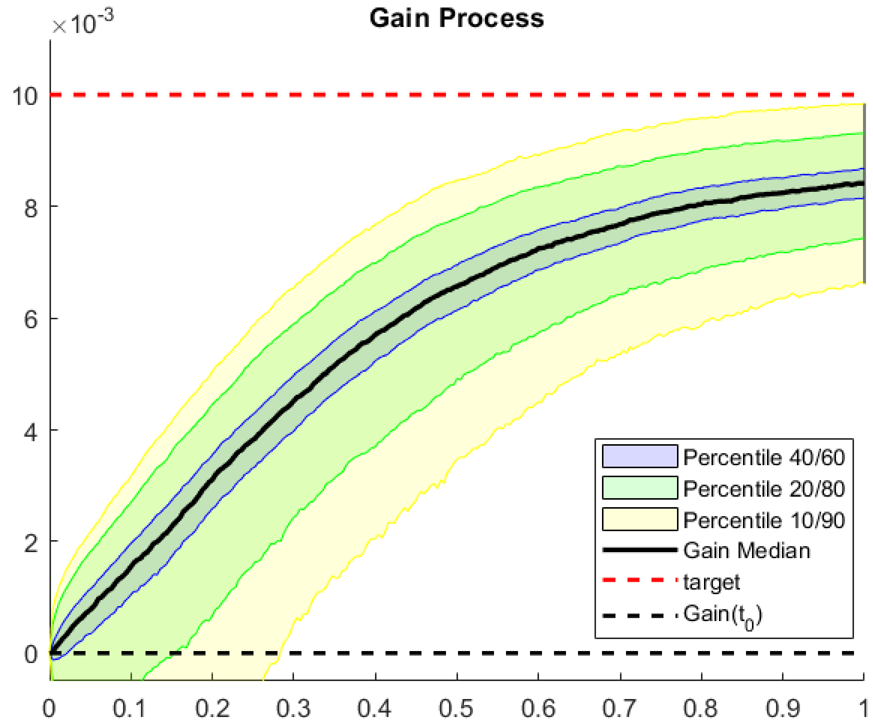


FIGURE 3.6.9. Gain Process median and percentiles using state variables calibrated between 04/October/2019 to 04/Oct/2021.



Bibliography

Adrian, T., Crump, R. K., Mills, B., and Moench, E. (2014). Treasury Term Premia: 1961-Present. 1

Ambrosini, G. and Menoncin, F. (2018). Optimal portfolios with credit default swaps. *Journal of Financial Services Research*, 54. 1.4.2

Ammann, M. and Buesser, R. (2013). Variance risk premiums in foreign exchange markets. *Journal of Empirical Finance*, 23(C):16–32. 2.1

Ammann, M. and Moerke, M. (2022). Credit variance risk premiums. *European Financial Management*. 2.1

Andersen, T., Bollerslev, T., Francis, D., and Labys, P. (2003). Modeling and forecasting realized volatility. *Econometrica*. 2.3.1

Armstrong, A. and Rutkowski, M. (2009). Valuation of credit default swaptions and credit default index swaptions. *International Journal of Theoretical and Applied Finance*. 1.3, 2.1, 2.2.1, 2.2.5

Bakshi, G. and Madan, D. (2006). A theory of volatility spreads. *Management Science*, 52(12). 1.5

Basak, S. and Chabakauri, G. (2010). Dynamic mean-variance asset allocation. *Review of Financial Studies*, 23(8):2970–3016. 2

Bielecki, T. and Jang, I. (2006). Portfolio optimization with a defaultable security. *Asia-Pacific Financial Markets*, 13(2). 1.4.2

Bjork, T., Khapko, M., and Murgoci, A. (2017). On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21(2):331–360. 2

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*, 81(3):637–654. 1

Bloomberg (2015). The bloomberg cds model. 1

Bloomberg Quantitiative Analytics (2012). Pricing credit index options. Technical report, Bloomberg. 2.2.1

Bo, L., Wang, Y., and Yang, X. (2010). An optimal portfolio problem in a defaultable market. *Advances in Applied Probability*, 42(3). 1.4.2

Bollerslev, T., Gibson, M., and Zhou, H. (2011). Dynamic estimation of volatility risk premia and investor risk aversion from option-implied and realized volatilities. *Journal* *of Econometrics*, 160. 1.5

Bollerslev, T., Tauchen, G., and Zhou, H. (2009). Expected stock returns and variance risk premia. *Review of Financial Studies*, 22(11). 1.5

Bollerslev, T., Tauchen, G., and Zhou, H. (2014). Expected stock returns and variance risk premia. *The Review of Financial Studies*. 2.3.2

Brace, A., Gatarek, D., and Musiela, M. (1997). The market model of interest rate dynamics. *Mathematical Finance*, 7. 4

Carr, P. and Wu, L. (2008). Variance risk premiums. *The Review of Financial Studies*. 1.5, 2.1, 2.3.1, 2.5.1, 2.5.2

Choi, H., Mueller, P., and Vedolin, A. (2017). Bond variance risk premiums. *Review of Finance*, 21(3):987–1022. 2.1

Cochrane, J. and Piazzesi, M. (2005). Bond risk premia. *American Economic Review*, 95. 1

Cox, J., Ingersoll, J., and Ross, S. (1985a). An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–84. 1, 4

Cox, J., Ingersoll, J., and Ross, S. (1985b). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407. 1, 4

Diebold, F. X. and Rudebusch, G. D. (2013). *Yield Curve Modeling and Forecasting: The Dynamic Nelson-Siegel Approach*. Princeton University press. 3, 3.1, 3.2.1, 3.2.1, 3.5

Drechsler, I. and Yaron, A. (2011). What's Vol Got to Do with It. *Review of Financial Studies*, 24. 1.5

Duffie, D. (1999). Credit swap valuation. *Financial Analysts Journal*, 55(1). 2

Elton, E. J. e. a. (2003). *Modern Portfolio Theory and Investment Analysis*. John Wiley. 1.4.1

Fabozzi, F. J. and Fong, H. G. (1994). *Advanced Fixed Income Portfolio Management: The State of the Art*. 1.4.1

Feunou, B., Jahan-Parvar, M. R., and Okou, C. (2017). Downside variance risk premium. *Journal of Financial Econometrics*. 2.3.1, 2.3.2, 2.5.1, 2.5.4, 2.5.4

Fisher, M., Nychka, D., and Zervos, D. (1995). Fitting the term structure of interest rates with smoothing splines. 3

Fontaine, J.-S. and Garcia, R. (2009). Bond Liquidity Premia. 3

Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60. 4, 1.4.1

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities. *Review of Financial Studies*, 3(4). 4

ISDA (2009). Standard European Corporate CDS Contract Specification1. 2.2.3

ISDA (2014). CDX Untranched Transactions Swaptions Standard Terms Supplement and Confirmation. 2.2.1, 2.2.4

Jarrow, R. and Protter, P. (2012). *Structural versus Reduced-Form Models: A New Information-Based Perspective*. Wiley. 1.3

Jarrow, R. and Turnbull, S. M. (1995). Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50. 2

Korn, O. and Koziol, C. (2006). Bond portfolio optimization. *The Journal of Fixed Income*, 15(4). 1.4.1

Korn, R. and Kraft, H. (2002). A stochastic control approach to portfolio problems with stochastic interest rates. *SIAM J. Control. Optim.*, 40:1250–1269. 1.4.1

Korn, R. and Kraft, H. (2003). Optimal portfolios with defaultable securities: a firm value approach. *International Journal of Theoretical and Applied Finance*, 06(08). 1.4.2

Kraft, H. (2004). *Optimal Portfolios with Stochastic Interest Rates and Defaultable Assets*. Lecture Notes in Economics and Mathematical Systems. 1.4.1

Li, D. and Ng, W.-L. (2000). Optimal dynamic portfolio selection: Multiperiod mean-variance formulation. *Mathematical Finance*, 10(3):387–406. 1

Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, 7(1). 1.4.1

Martin, I. (2017). What is the expected return on the market? *Quarterly Journal of Economics*. 1.5, 2.3, 2.3.1, 2.3.2

Menoncin, F. and Vigna, E. (2020). Mean-variance dynamic optimality for dc pension schemes. Carlo Alberto Notebooks 587, European Actuarial Journal. 3

Merton, R. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29(2). 1

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4). 1.4.1

Morini, M. and Brigo, D. (2008). Arbitrage-free pricing of credit index options: The no-armageddon pricing measure and the role of correlation after the subprime crisis. 1.3, 2.2.1

Munk, C. and Sorensen, C. (2004). Optimal consumption and investment strategies with stochastic interest rates. *Journal of Banking & Finance*, 28. 1.4.1

Nelson, C. R. and Siegel, A. F. (1987). Parsimonious modeling of yield curves. *Journal of Business*, 60:473–489. 3

Pascucci, A. (2011). *PDE and Martingale Methods in Option Pricing*. Springer Finance. 2.2.1

Pedersen, C. M. (2003). Valuation of portfolio credit default swaptions. Technical report, Lehman Brothers. 2.2.1

Pedersen, J. L. and Peskir, G. (2017). Optimal mean-variance portfolio selection. *Mathematics and Financial Economics*, 11:137–160. 3

Piazzesi, M. and Cochrane, J. (2009). Decomposing the yield curve. (18). 1, 2

Rebonato, R. (2018). *Bond Pricing and Yield Curve Modeling*. Cambridge University Press. 1.2

Richardson, H. R. (1989). A minimum variance result in continuous trading portfolio optimization. *Management Science*, 35(9):1045–1055. 1

Rosenberg, J. and Engle, R. (2002). Empirical pricing kernels. *Journal of Financial Economics*. 1.5

Steffensen, M. and Kraft, H. (2006). Portfolio problems stopping at first hitting time with applications to default risk. *Mathematical Methods of Operations Research*, 63(1). 1.4.2

Strotz, R. H. (1956). Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies*, 23(3). 1.4.3, 2

Svensson, L. E. (1994). Estimating and interpreting forward rates: Sweden 1992-4. *National Bureau of Economic Research*, Working Paper #4871. 3

Sørensen, C. (1999). Dynamic asset allocation and fixed income management. *Journal of Financial and Quantitative Analysis*, 34(4). 1.4.1

Trolle, A. and Schwartz, E. (2010). Variance risk premia in energy commodities. *Journal of Derivatives*, 17(3):15–32. 2.1

Trolle, A. and Schwartz, E. (2014). The swaption cube. *Review of Financial Studies*, 27(8):2307–2353. 2.1

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188. 1, 4

Vigna, E. (2020). On time consistency for mean-variance portfolio selection. *International Journal of Theoretical and Applied Finance*. 1.4.3, 3

Walder, R. (2002). Dynamic Allocation of Treasury and Corporate Bond Portfolios. Fame research paper series, International Center for Financial Asset Management and Engineering. 1.4.2

Wilhelm, J. (1992). Fristigkeitsstruktur und zinsaanderungsrisiko- vorauberlegungen zu einer markowitz-theorie des bond-portfolio-managements. *Zeitschrift fur betriebswirtschaftliche*. 1.4.1

Zhou, X. and Li, X. (2000). Continuous-time mean-variance portfolio selection: A stochastic lq framework. *Applied Mathematics and Optimization*. 1, 1.4.3, 2, .2

.1. Integrals of duration and convexity

The integral into the equation 3.2.7 can be solved in the following way. First we rewrite, the integral as

$$\int_t^T \delta\left(e^{-y(s-t)}(s-t)\right)ds = \delta e^{ty}\int_t^T \left(e^{-ys}(s-t)\right)ds$$

where the functional dependence of $y(t,T)$ with respect to time is omitted for the sake of simplicity. We integrate by parts the integral noticing that $\frac{\partial(s-t)}{\partial s} = 1$ and the primitive of e^{-ys} is equal to $\frac{-e^{-ys}}{y}$. Then

$$\int_t^T \left(e^{-ys}(s-t)\right)ds = \frac{-(s-t)\,e^{-ys}}{y} + \frac{\int_t^T e^{-ys}ds}{y} = \frac{-(s-t)\,e^{-ys}}{y} - \frac{e^{-ys}}{y^2}.$$

By substituting the the solved integral we can write

$$\begin{aligned}\delta e^{ty}\int_t^T \left(e^{-ys}(s-t)\right)ds &= \delta\left(\frac{-(s-t)\,e^{-y(s-t)}}{y} - \frac{e^{-y(s-t)}}{y^2}\right)\Big|_t^T,\\ &= \delta\left(\frac{-(T-t)\,e^{-y(T-t)}}{y} - \frac{e^{-y(T-t)}}{y^2} + \frac{1}{y^2}\right)\\ &\delta\left(\frac{(-(T-t)\,y-1)\,e^{-y(T-t)}+1}{y^2}\right).\end{aligned}$$

Repeating the same steps, the integral into the convexity equation 3.2.8 can be solved as

$$\int_t^T \delta\left(e^{-y(s-t)}(s-t)^2\right)ds = \delta\left(\frac{2-((t^2-2Tt+T^2)\,y^2+(2T-2t)\,y+2)\,e^{-y(T-t)}}{y^3}\right).$$

Finally, the duration can be rewritten as

$$\begin{aligned}\mathcal{D}(t,T) = \delta&\left(\frac{(-(T-t)\,y-1)\,e^{-y(T-t)}+1}{y^2P(t,T)}\right)\\ &+\frac{V_T}{P(t,T)}e^{-y(T-t)}(T-t),\end{aligned}$$

while the convexity becomes

$$\begin{aligned}\mathcal{C}(t,T) = \delta&\left(\frac{2-\left((T-t)^2y^2+2(T-t)\,y+2\right)e^{-y(T-t)}}{y^3P(t,T)}\right)\\ &+\frac{V_T}{P(t,T)}e^{-y(T-t)}(T-t)^2.\end{aligned}$$

.2. Proof of Theorem 3 3

Proof. We need to prove the second assertion given that the first one is a direct consequence of the second one, as showed in Zhou and Li (2000). Let $\omega^*(\cdot) \in \Pi_{P_\alpha}$. If $\omega^*(\cdot) \notin \Pi_{A(\mu,\bar{\beta})}$, then there exist $\omega(\cdot)$ and the corresponding $R(\cdot) - X(\cdot)$ such that

(.2.1)
$$\alpha\left(\mathbb{E}_0\left[(R_T - X_T)^2\right] - \mathbb{E}_0\left[(R_T^* - X_T)^2\right]\right) - \beta(\mathbb{E}_0\left[(R_T - X_T)\right] - \mathbb{E}_0\left[(R_T^* - X_T)\right]) < 0.$$

Set a function

$$\pi(x,y) = \alpha x - \alpha y^2 - y.$$

It is a concave function in (x,y) and

$$\pi\left(\mathbb{E}_0\left[(R_T - X_T)^2\right], \mathbb{E}_0\left[(R_T - X_T)\right]\right) = -\mathbb{E}_0\left[(R_T - X_T)\right] + \alpha\mathbb{V}_0\left[R_T - X_T\right]$$

which is exactly the objective function of problem P_α. Given the concavity of π we can write, recalling that $\frac{\partial\pi(x,y)}{x} = \alpha$, $\frac{\partial\pi(x,y)}{y} = -(1+2\alpha y)$,

(.2.2)
$$\begin{aligned}
\pi\left(\mathbb{E}_0\left[(R_T - X_T)^2\right], \mathbb{E}_0\left[(R_T - X_T)\right]\right) \leq & \pi\left(\mathbb{E}_0\left[(R_T^* - X_T)^2\right], \mathbb{E}_0\left[(R_T^* - X_T)\right]\right) \\
& + \alpha\left(\mathbb{E}_0\left[(R_T - X_T)^2\right] - \mathbb{E}_0\left[(R_T^* - X_T)^2\right]\right) \\
& - (1 + 2\alpha\mathbb{E}_0\left[(R_T^* - X_T)\right])(\mathbb{E}_0\left[(R_T - X_T)\right] - \mathbb{E}_0\left[(R_T^* - X_T)\right]) \\
< & \pi\left(\mathbb{E}_0\left[(R_T^* - X_T)^2\right], \mathbb{E}_0\left[(R_T^* - X_T)\right]\right),
\end{aligned}$$

where the last inequality is due to .2.1. By .2.2, $\omega^*(\cdot)$ is not optimal for problem P_α, leading to a contradiction. □

.3. Proof of Proposition 4

We tackle the standard stochastic optimal control problem with the dynamic programming approach. We define the value function as

$$\max_{\{\omega_t\}_{t\in[0,T]}} \mathbb{E}_0\left[\frac{1}{2}\left(R_T - (X_T + \gamma)\right)^2\right],$$

with the boundary condition

$$J(T, X_T, R_T) = \frac{\left(R_T - (X_T + \gamma)\right)^2}{2}. \tag{.3.1}$$

If we define $J(t, z_t, R_t)$ the value function that solves this problem, it must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} 0 = &\frac{\partial J}{\partial t} + \frac{\partial J}{\partial R_t}(R_t r - c) + \mu_z^\intercal \frac{\partial J}{\partial z_t} + \frac{1}{2} tr\left(\Omega^\intercal \Omega J_{zz}\right) \\ &+ \max_{\omega_t} \left\{ \frac{\partial J}{\partial R_t} \omega_t^\intercal (\mu - r\mathbf{1}) + \frac{1}{2}\frac{\partial^2 J}{\partial R_t^2} \omega_t^\intercal \Sigma^\intercal \Sigma \omega_t + \omega_t^\intercal \Sigma^\intercal \Omega \frac{\partial^2 J}{\partial R_t \partial z_t} \right\}, \end{aligned}$$

The First Order Condition (FOC) on the portfolio is

$$\omega_t^* = -\frac{\frac{\partial J}{\partial R_t}}{\frac{\partial^2 J}{\partial R_t^2}} \left(\Sigma^\intercal \Sigma\right)^{-1} (\mu - r\mathbf{1}) - \frac{1}{\frac{\partial^2 J}{\partial R_t^2}} \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega \frac{\partial^2 J}{\partial R_t \partial z_t},$$

and after substituting this optimal portfolio into the HJB equation we obtain

$$\begin{aligned} 0 = &\frac{\partial J}{\partial t} + \frac{\partial J}{\partial R_t}(R_t r - c) + \mu_z^\intercal \frac{\partial J}{\partial z_t} + \frac{1}{2} tr\left(\Omega^\intercal \Omega J_{zz}\right) \\ &- \frac{1}{2} \frac{\left(\frac{\partial J}{\partial R_t}\right)^2}{\frac{\partial^2 J}{\partial R_t^2}} (\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} (\mu - r\mathbf{1}) - \frac{\frac{\partial J}{\partial R_t}}{\frac{\partial^2 J}{\partial R_t^2}} (\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega \frac{\partial^2 J}{\partial R_t \partial z_t} \\ &- \frac{1}{2} \frac{1}{\frac{\partial^2 J}{\partial R_t^2}} \left(\frac{\partial^2 J}{\partial R_t \partial z_t}\right)^\intercal \Omega^\intercal \Sigma \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega \frac{\partial^2 J}{\partial R_t \partial z_t}. \end{aligned}$$

Given the boundary condition, a suitable guess function for solving the HJB equation should have the following form

$$J(t, z_t, R_t) = \frac{\left(R_t - H(t, z)\right)^2}{2F(t, z_t)},$$

in which the functions $F(t, z_t)$ and $H(t, z_t)$ must be computed in order to satisfy the HJB equation. The boundary condition (.3.1) can then be split into two boundary conditions

$$F(T, z_T) = 1,$$

$$H(T, z_T) = X_T + \gamma.$$

If we substitute this value function into the HJB we get

$$\begin{aligned} 0 = &J_t + J_R (R_t r - c) + \mu_z^\intercal J_z + \frac{1}{2} tr\left(\Omega^\intercal \Omega J_{zz}\right) \\ &- \frac{1}{2} \frac{J_R^2}{J_{RR}} (\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} (\mu - r\mathbf{1}) - \frac{J_R}{J_{RR}} (\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega J_{Rz} \\ &- \frac{1}{2} \frac{1}{J_{RR}} J_{Rz}^\intercal \Omega^\intercal \Sigma \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega J_{Rz} \end{aligned}$$

The partial derivatives are

$$J = \frac{(R - H)^2}{2F}$$

$$J_t = -\frac{(R-H)^2 F_t}{2F^2} - \frac{(R-H) H_t}{F}$$

$$J_R = \frac{(R-H)}{F}$$

$$J_{RR} = \frac{1}{F}$$

$$J_{Rz} = -\frac{(R-H) F_z}{F^2} + -\frac{H_z}{F}$$

$$J_z = -\frac{(R-H)^2}{2F^2} F_z - \frac{(R-H) H_z}{F}$$

$$\begin{aligned} J_{zz} = &+ \frac{(R-H)^2}{F^3} F_z F_z^\intercal - \frac{(R-H) F_z H_z^\intercal}{F^2} - \frac{(R-H)^2}{2F^2} F_{zz} \\ &+ \frac{(R-H) H_z F_z^\intercal}{F^2} + \frac{H_z H_z^\intercal}{F} - \frac{(R-H) H_{zz}}{F} \end{aligned}$$

Substitute into the HJB

$$\begin{aligned} 0 = &- \frac{(R-H)^2}{2F^2} F_t + \frac{(R-H)^2 r}{F} - \frac{(R-H)^2}{2F^2} \mu_z^\intercal F_z \\ &+ \frac{(R-H)^2}{2F^3} F_z^\intercal \Omega^\intercal \Omega F_z - \frac{(R-H)^2}{4F^2} tr\left(\Omega^\intercal \Omega F_{zz}\right) \\ &- \frac{1}{2} \frac{(R-H)^2}{F} (\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} (\mu - r\mathbf{1}) + \frac{(R-H)^2 (\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega F_z}{F^2} \\ &- \frac{(R-H)^2 F_z^\intercal \Omega^\intercal \Sigma \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega F_z}{2F^3} \\ \\ &- \frac{(R-H) H_t}{F} + \frac{(R-H)(Hr - c)}{F} - \frac{(R-H) \mu_z^\intercal H_z}{F} \\ &+ \frac{1}{2} \frac{(R-H) H_z^\intercal \Omega^\intercal \Omega F_z}{F^2} + \frac{1}{2} \frac{(R-H) F_z^\intercal \Omega^\intercal \Omega H_z}{F^2} \\ &- \frac{1}{2} \frac{(R-H) tr\left(\Omega^\intercal \Omega H_{zz}\right)}{F} + \frac{(R-H)(\mu - r\mathbf{1})^\intercal \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega H_z}{F} \\ &- \frac{(R-H) F_z^\intercal \Omega^\intercal \Sigma \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega H_z}{F^2} \\ \\ &+ \frac{1}{2} \frac{H_z^\intercal \Omega^\intercal \Omega H_z}{F} \\ &- \frac{1}{2} \frac{H_z^\intercal \Omega^\intercal \Sigma \left(\Sigma^\intercal \Sigma\right)^{-1} \Sigma^\intercal \Omega H_z}{F} \end{aligned}$$

where we have three terms, one containing $(R-H)^2$, one containing $(R-H)$, and one that do not contain $(R-H)$. The last terms disappear only if he market is complete. In this case, in fact,

$$\Sigma\left(\Sigma^\intercal\Sigma\right)^{-1}\Sigma^\intercal = I$$

and so

$$+\frac{1}{2}\frac{H_z^\intercal\Omega^\intercal\Omega H_z}{F} - \frac{1}{2}\frac{H_z^\intercal\Omega^\intercal\Sigma\left(\Sigma^\intercal\Sigma\right)^{-1}\Sigma^\intercal\Omega H_z}{F} = 0.$$

The HJB becomes

$$\begin{aligned}0 =& F_t + \left(\mu_z^\intercal - 2\left(\mu - r\mathbf{1}\right)^\intercal\left(\Sigma^\intercal\Sigma\right)^{-1}\Sigma^\intercal\Omega\right)F_z + \frac{1}{2}tr\left(\Omega^\intercal\Omega F_{zz}\right)\\ &- F\left(2r - \left(\mu - r\mathbf{1}\right)^\intercal\left(\Sigma^\intercal\Sigma\right)^{-1}\left(\mu - r\mathbf{1}\right)\right)\end{aligned}$$

$$0 = H_t + \left(\mu_z^\intercal - \left(\mu - r\mathbf{1}\right)^\intercal\left(\Sigma^\intercal\Sigma\right)^{-1}\Sigma^\intercal\Omega\right)H_z + \frac{1}{2}tr\left(\Omega^\intercal\Omega H_{zz}\right) - Hr - c$$

that can be rewritten using 3.3.3

$$0 = F_t + \left(\mu_z^\intercal - 2\xi^\intercal\Omega\right)F_z + \frac{1}{2}tr\left(\Omega^\intercal\Omega F_{zz}\right) - F\left(2r - \xi^\intercal\xi\right) \tag{.3.2}$$

$$0 = H_t + \left(\mu_z^\intercal - \xi^\intercal\Omega\right)H_z + \frac{1}{2}tr\left(\Omega^\intercal\Omega H_{zz}\right) - Hr - c. \tag{.3.3}$$

Remark 6. If Ito's lemma is applied to $H\left(t,z_t\right)$, it's differential can be written as

$$dH = \left(H_t + \left(\mu_z^\intercal - \xi^\intercal\Omega\right)H_z + \frac{1}{2}tr\left(\Omega^\intercal\Omega H_{zz}\right)\right)dt + H_z^\intercal\Omega^\intercal dW^{\mathbb{Q}},$$

and because of .3.3 , we get

$$\begin{aligned}dH &= \left(Hr - c\right)dt + H_z^\intercal\Omega^\intercal dW^{\mathbb{Q}}\left(t\right)\\ &= \left(Hr - c + H_z^\intercal\Omega^\intercal\xi\right)dt + H_z^\intercal\Omega^\intercal dW\left(t\right).\end{aligned} \tag{.3.4}$$

We can express the solutions of both PDEs .3.2 and .3.3 through the Feynman Kac representation theorem, but under two different probability. While $H\left(t,z_t\right)$ can be written as an expected value under $\mathbb{Q}$, the function $F\left(t,z_t\right)$ can be expressed under a probability such that the drift of the state variable 3.3.1 is $\mu_z^\intercal - 2\xi^\intercal\Omega$. Then, this new probability $\mathbb{Q}_2$ is defined as following:

$$\begin{aligned}\mu_z\left(t,z\right)dt + \Omega\left(t,z\right)dW\left(t\right) &= \left(\mu_z^\intercal - 2\xi^\intercal\Omega\right)dt + \Omega\left(t,z\right)dW^{\mathbb{Q}_2}\left(t\right)\\ dW^{\mathbb{Q}_2}\left(t\right) &= 2\xi dt + dW\left(t\right).\end{aligned}$$

Thus, the two function $F\left(t,z_t\right)$ and $H\left(t,z_t\right)$ can be written as the following expected values

$$H(t, z_t) = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^T + c e^{-\int_t^s r_u du} ds \right] + \mathbb{E}_t^{\mathbb{Q}} \left[(X_T + \gamma) e^{-\int_t^T r_u du} \right], \tag{.3.5}$$

$$F(t, z_t) = \mathbb{E}_t^{\mathbb{Q}_2} \left[e^{-\int_t^T \left(2r_u - \xi_u^{\intercal} \xi_u \right) du} \right]. \tag{.3.6}$$

Once the guess function is substituted into the optimal portfolio, we get

$$\omega_t^* = (H - R_t) \left(\Sigma^{\intercal} \Sigma \right)^{-1} (\mu - r\mathbf{1}) + (\Sigma)^{-1} \Omega. \left(H_z - \frac{(H - R_t) F_z}{F} \right).$$