

Factorizing the Rado graph and infinite complete graphs*

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Abstract

Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of infinite graphs, together with Λ . The Factorization Problem $FP(\mathcal{F}, \Lambda)$ asks whether \mathcal{F} can be realized as a factorization of Λ , namely, whether there is a factorization $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ of Λ such that each Γ_α is a copy of F_α .

We study this problem when Λ is either the Rado graph R or the complete graph K_{\aleph} of infinite order \aleph . When \mathcal{F} is a countably infinite family, we show that $FP(\mathcal{F}, R)$ is solvable if and only if each graph in \mathcal{F} has no finite dominating set. We also prove that $FP(\mathcal{F}, K_{\aleph})$ admits a solution whenever the cardinality of \mathcal{F} coincides with the order and the domination numbers of its graphs.

For countable complete graphs, we show some non existence results when the domination numbers of the graphs in \mathcal{F} are finite. More precisely, we show that there is no factorization of K_{\aleph} into copies of a k -star (that is, the vertex disjoint union of k countable stars) when $k = 1, 2$, whereas it exists when $k \geq 4$, leaving the problem open for $k = 3$.

Finally, we determine sufficient conditions for the graphs of a decomposition to be arranged into resolution classes.

Keywords: Factorization problem, resolution problem, Rado graph, infinite graphs.

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1 Introduction

We assume that the reader is familiar with the basic concepts in (infinite) graph theory, and refer to [10] for further details.

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In this paper all graphs will be simple, namely, without multiple edges or loops. As usual, we denote by $V(\Lambda)$ and $E(\Lambda)$ the vertex set and the edge set of a simple graph Λ , respectively. We say that Λ is finite (resp. infinite) if its vertex set is so, and refer to the cardinality of $V(\Lambda)$ and $E(\Lambda)$ as the order and the size of Λ , respectively. Note that in the finite case $|E(\Lambda)| \leq \binom{|V(\Lambda)|}{2}$, whereas if Λ is infinite, then its order, which is a cardinal number, is greater than or equal to its size. We use the notation K_v for any complete graph of order v , and denote by K_V the complete graph whose vertex set is V .

Given a subgraph Γ of a simple graph Λ , we denote by $\Lambda \setminus \Gamma$ the graph obtained from Λ by deleting the edges of Γ . If Γ contains all possible edges of Λ joining any two of its vertices, then Γ is called an induced subgraph of Λ (in other words, an induced subgraph is obtained by vertex deletions only). Instead, if $V(\Gamma) = V(\Lambda)$, then Γ is called a spanning subgraph or a factor of Λ (hence, a factor is obtained by edge deletions only). If Γ is also h -regular, then we speak of an h -factor. We recall that a set D of vertices of Λ is dominating if all other vertices of Λ are adjacent to some vertex of D . The minimum size of a dominating set of Λ is called the domination number of Λ . Finally, we say that Λ is locally finite if its vertex degrees are all finite.

A decomposition of Λ is a set $\mathcal{G} = \{\Gamma_1, \Gamma_2, \dots\}$ of subgraphs of Λ whose edge-sets partition $E(\Lambda)$. If the graphs Γ_i are all isomorphic to a given subgraph Γ of Λ , then we speak of a Γ -decomposition of Λ . When Γ and Λ are both complete graphs, we obtain 2-designs. More precisely, a K_k -decomposition of K_v is equivalent to a 2 - $(v, k, 1)$ design.

Classically, the graphs Γ_i and Λ are taken to be finite, and the same usually holds for the parameters v and k of a 2-design. However, there has been considerable interest in designs on an infinite set of v points, mainly when $k = 3$. In this case, we obtain infinite Steiner triple systems whose first explicit constructions were given in [12, 13]. Further results concerning the existence of rigid, sparse, and perfect countably Steiner triple systems can be found in [6, 7, 11]. Results showing that any Steiner system can be extended are given in [1, 15]. The existence of large sets of Steiner triple systems for every infinite v (and more generally, of infinite Steiner systems) can be found in [4]. Also, infinite versions of topics in finite geometry, including infinite Steiner triple systems and infinite perfect codes are considered in [3]. A more comprehensive list of results on infinite designs can be found in [9].

When each graph of a decomposition \mathcal{G} of Λ is a factor (resp. h -factor), we speak of a factorization (resp. h -factorization) of Λ . Also, when the factors of \mathcal{G} are all isomorphic to Γ , we speak of a Γ -factorization of Λ . A factorization of K_v into factors whose components are copies of K_k is equivalent to a resolvable 2 - $(v, k, 1)$ design.

In this paper, we consider the Factorization Problem for infinite graphs, which is here stated in its most general version.

Problem 1.1. Let Λ be a graph of order \aleph and let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of (non-empty) infinite graphs, not necessarily distinct, each of which has order \aleph , with $\aleph \geq |\mathcal{A}|$.

The Factorization Problem $FP(\mathcal{F}, \Lambda)$ asks for a factorization $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ of Λ such that Γ_α is isomorphic to F_α , for every $\alpha \in \mathcal{A}$. If Λ is the complete graph of order \aleph , we simply write $FP(\mathcal{F})$. If in addition to this each F_α is isomorphic to a given graph F and $|\mathcal{A}| = \aleph$, we write $FP(F)$.¹

¹Since in this case the factorization problem can be seen as a generalization of the Oberwolfach problem, in [8] the problem $FP(F)$ was denoted by $OP(F)$.

Note that the graphs of \mathcal{F} are allowed to have zero degree vertices. This means that if F_α contains isolated vertices, then Γ_α is a copy of F_α covering all vertices of K_\aleph (hence a factor of K_\aleph) and having the same number of isolated vertices as F_α . Otherwise, F_α has no zero degree vertices, hence Γ_α is a factor of K_α in the usual sense, that is, a spanning subgraph without isolated vertices.

As far as we know, there are only four papers dealing with the Factorization Problem for infinite complete graphs, and two of them, concern classic designs. In [14] it is shown that there exists a resolvable 2-design whenever $v = |\mathbb{N}|$ and k is finite; these designs have, in addition, a cyclic automorphism group G acting sharply transitively on the vertex set; briefly they are G -regular. In [9] it is shown that every infinite 2-design with $k < v$ is necessarily resolvable, and when $k = v$, both resolvable and non-resolvable designs exist. We point out that in [9, 14] both these results are proven more generally for t -designs whenever $t \geq 2$ is finite.

Furthermore, in [2] the authors construct a G -regular 1-factorization of a countably infinite complete graph for every finitely generated abelian infinite group G . Finally, [8] proves the following.

Theorem 1.2. *Let F be a graph whose order is the cardinal number \aleph . $FP(F)$ has a G -regular solution whenever the following two conditions hold:*

- (1) F is locally finite,
- (2) G is an involution free group of order \aleph .

Note that this result generalizes the one obtained in [14] to any complete graph of infinite order \aleph , blocks of any size less than \aleph , and groups G not necessarily cyclic. Furthermore, Theorem 1.2 can also be seen as a generalization of the result in [2] to complete graphs of any infinite order.

When dealing with infinite graphs, a central role is played by the Rado graph R (see [16]), named after Richard Rado who gave one of its first explicit constructions. Indeed, R is the unique countably infinite random graph, and it can be constructed as follows: $V(R) = \mathbb{N}$ and a pair $\{i, j\}$ with $i < j$ is an edge of R if and only if the i -th bit of the binary representation of j is one. R shows many interesting properties, such as the universal property: every finite or countable graph can be embedded as an induced subgraph of R .

When replacing the concept of induced subgraph with the dual one of factor, a weaker result holds. Indeed, in [5] it is pointed out that a countable graph F can be embedded as a factor of R if and only if the domination number of F is infinite. In the same paper, it is further shown that $FP(\mathcal{F}, R)$ has a solution whenever \mathcal{F} is infinite and each of its graphs is locally finite. Note that a locally finite countable graph has infinite domination number, but the converse is not true: for example, the Rado graph is not locally finite and it has no finite dominating set (indeed, for every $D = \{i_1, \dots, i_t\} \subset \mathbb{N}$, there exists an integer $j \in \mathbb{N}$ whose binary representation has 0 in positions i_1, \dots, i_t , which means that j is adjacent with no vertex of D).

In this paper, we extend this result to any countable family \mathcal{F} of admissible graphs. More precisely, we prove the following. We point out that throughout the paper, any countable family (or graph) is understood to be infinite.

Theorem 1.3. *Let \mathcal{F} be a countable family of countable graphs. Then, $FP(\mathcal{F}, R)$ has a solution if and only if the domination number of each graph of \mathcal{F} is infinite.*

Furthermore, we prove the solvability of $FP(\mathcal{F})$ whenever the size of \mathcal{F} coincides with the order and the domination number of its graphs.

Theorem 1.4. *Let \mathcal{F} be a family of graphs, each of which has order \aleph . $FP(\mathcal{F})$ has a solution whenever the following two conditions hold:*

- (1) $|\mathcal{F}| = \aleph$, and
- (2) the domination number of each graph in \mathcal{F} is \aleph .

When \mathcal{F} contains only copies of a given graph F satisfying condition (1) of Theorem 1.2 (i.e., F is locally finite), then \mathcal{F} satisfies both conditions (1) and (2) of Theorem 1.4. Therefore, Theorem 1.4 can be seen as a generalization of Theorem 1.2, even though it does not provide any information on the automorphisms of a solution to FP.

Note that if we just require that the domination number of each graph of \mathcal{F} is \aleph , there may exist factorizations with fewer factors than \aleph ; this means that the two conditions in Theorem 1.4 are independent. Indeed, the Rado graph R has no finite dominating set and Corollary 2.4 shows that for every $n \geq 2$ there exists a factorization of K_{\aleph} into n copies of R . We point out that Theorem 1.4 constructs instead factorizations of K_{\aleph} into infinite copies of R .

The paper is organized as follows. In Sections 2 and 3, we prove the main results of this paper, Theorems 1.3 and 1.4. In Section 4, we deal with F -factorizations of K_{\aleph} when F belongs to a special class of graphs with finite domination number (and hence not satisfying condition (2) of Theorem 1.4): the countable k -stars (briefly, S_k), that is, the vertex disjoint union of k countable stars. We prove that $FP(S_k)$ has a solution whenever $k > 3$, and there is no solution for $k \in \{1, 2\}$. This shows that there are families \mathcal{F} of graphs for which $FP(\mathcal{F})$ is not solvable. We leave open the problem when $k = 3$. In the last section, inspired by [9], we provide a sufficient condition for a decomposition \mathcal{F} of K_{\aleph} to be resolvable (i.e., the graphs of \mathcal{F} can be partitioned into factors of K_{\aleph}).

2 Factorizing the Rado graph

In this section, we prove Theorem 1.3. Also, since the Rado graph R is self-complementary, that is, $K_{\aleph} \setminus R$ is isomorphic to R , we obtain as a corollary the countable version of Theorem 1.4.

We start by recalling an important characterization of the Rado graph (see, for example, [5]).

Theorem 2.1. *A countable graph is isomorphic to the Rado graph if and only if it satisfies the following property:*

- ★ *for every disjoint finite sets of vertices U and W , there exists a vertex z adjacent to all the vertices of U and non-adjacent to all the vertices of W .*

Property ★ is usually referred to as the *existentially closed* property. Therefore, Theorem 2.1 states that, up to isomorphism, there is exactly one existentially closed countable graph: the Rado graph.

Now we slightly generalize the construction of the Rado graph given in the introduction.

Definition 2.2. Given a set $I \subset \{0, \dots, q-1\}$, with $1 \leq |I| < q$, we denote by R_I^q the following graph: $V(R_I^q) = \mathbb{N}$, and $\{x, y\}$, with $x < y$, is an edge of R_I^q whenever the x -th digit of y in the base q expansion of y belongs to I .

Cleraly, when $q = 2$ and $I = \{1\}$ we obtain our initial description of the Rado graph (i.e. $R = R_{\{1\}}^2$).

Proposition 2.3. *Every graph R_I^q is isomorphic to the Rado graph.*

Proof. By Theorem 2.1, it is enough to show R_I^q is existentially closed. We assume, without loss of generality, that $0 \in I$ while $1 \notin I$, and let U and V be two disjoint finite subsets of \mathbb{N} . Then there are infinitely many positive integers whose base q expansion has 0 in each position $u \in U$ and 1 in each position $v \in V$. Denoting by z one of these integers larger than $\max(U \cup V)$, we have that z is adjacent to all the vertices of U but to none in V . \square

Note that $K_{\mathbb{N}} = \bigcup_{i=0}^{q-1} R_{\{i\}}^q$ and $R_{\{0, \dots, q-2\}}^q = \bigcup_{i=0}^{q-2} R_i^q$. Considering that the $R_{\{i\}}^q$ s are pairwise edge-disjoint and isomorphic to the Rado graph, by taking $n = q-1$ we obtain the following.

Corollary 2.4. *For every positive integer n , the graphs R and $K_{\mathbb{N}}$ can be factorized into n and $n+1$ copies of R , respectively.*

The following result is crucial to prove Theorem 1.3. It strengthens a result given in [5] and allows us to suitably embed in the Rado graph R any countable graph with infinite domination number.

Proposition 2.5. *Let F be a countable graph with no finite dominating set. For every edge $e \in E(R)$, there exists an embedding σ_e of F in R such that:*

- (1) $\sigma_e(F)$ is a spanning subgraph of R containing the edge e ;
- (2) $R \setminus \sigma_e(F)$ is isomorphic to R .

Proof. By Proposition 2.3, the graphs $R_{\{0,1\}}^3$, $R_{\{0\}}^3$ and $R_{\{1\}}^3$ are isomorphic to R . Therefore, we can take $R = R_{\{0,1\}}^3$.

Let e be an edge of $R = R_{\{0\}}^3 \cup R_{\{1\}}^3$. We can assume without loss of generality that e lies in $R_{\{0\}}^3$. In [5, Proposition 8], it is shown that there exists an embedding σ_e of F into the Rado graph $R_{\{0\}}^3 \subset R$ satisfying condition (1). It is then left to prove that condition (2) holds. By Theorem 2.1, this is equivalent to saying that $R \setminus \sigma_e(F)$ satisfies \star .

Let U and V be two finite disjoint subsets of \mathbb{N} . Clearly, there are infinitely many positive integers whose base 3 expansion has 1 in each position $u \in U$ and 2 in each position $v \in V$. Let z be one of these integers larger than $\max(U \cup V)$. Since $R \setminus \sigma_e(F)$ contains $R_{\{1\}}^3$ and it is edge-disjoint from $R_{\{2\}}^3$, it follows that z is adjacent in $R \setminus \sigma_e(F)$ to all the vertices of U and is non-adjacent to all the vertices of V . This means that $R \setminus \sigma_e(F)$ is existentially closed. \square

We are now ready to prove Theorem 1.3, whose statement is recalled here, for clarity.

Theorem 1.3. *Let \mathcal{F} be a countable family of countable graphs. Then, $FP(\mathcal{F}, R)$ has a solution if and only if the domination number of each graph of \mathcal{F} is infinite.*

Proof. Since the Rado graph has no finite dominating set, the same holds for its spanning subgraphs. Hence, each graph of \mathcal{F} must have infinite domination number. Under this assumption, we are going to show that $FP(\mathcal{F}, R)$ has a solution.

By definition of Rado graph, it is easy to see that $|E(R)| = \aleph_0$, which is also the cardinality of \mathcal{F} . Let $E(R) = \{e_1, \dots, e_n, \dots\}$ and $\mathcal{F} = \{F_1, \dots, F_n, \dots\}$. By recursively applying Proposition 2.5, we obtain a sequence of isomorphisms $\sigma_{e_i} : F_i \rightarrow \Gamma_i$ satisfying for each $i \in \mathbb{N}$ the following properties:

- Γ_i is a spanning subgraph of R ;
- $R \setminus (\Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_{i-1})$ is isomorphic to R and contains Γ_i ;
- e_i lies in $\Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_i$.

It follows that the Γ_i s are pairwise edge-disjoint factors of R which partition $E(R)$. Therefore, $\{\Gamma_i : i \in \mathbb{N}\}$ is a solution to $FP(\mathcal{F}, R)$. □

The proof of Theorem 1.3 allows us to construct solutions to $FP(\mathcal{F}, R)$ even when the cardinality of \mathcal{F} is finite, provided that \mathcal{F} contains a copy of the Rado graph. In other words, we have the following.

Corollary 2.6. *Let \mathcal{F} be a finite family of countable graphs such that*

- (1) \mathcal{F} contains at least one graph isomorphic to the Rado graph;
- (2) the domination number of each graph in \mathcal{F} is infinite.

Then, $FP(\mathcal{F}, R)$ has a solution.

Recalling that R is self complementary, the countable version of Theorem 1.4 can be easily obtained as a corollary to Theorem 1.3.

Corollary 2.7. *Let \mathcal{F} be a countable family of countable graphs. $FP(\mathcal{F})$ has a solution whenever the domination number of each graph in \mathcal{F} is infinite.*

Proof. Recall that $R_{\{0\}}^2$ and $R_{\{1\}}^2$ are copies of R which together factorize $K_{\mathbb{N}}$. Therefore, it is enough to partition \mathcal{F} into two countable families \mathcal{F}_1 and \mathcal{F}_2 , and then apply Theorem 1.3 to get a solution \mathcal{G}_i to $FP(\mathcal{F}_i, R_{\{i\}}^2)$, for $i = 0, 1$. Clearly, $\mathcal{G}_1 \cup \mathcal{G}_2$ provides a solution to $FP(\mathcal{F})$. □

The natural generalization of property \star to a generic cardinality \aleph is the following one:

- \star_{\aleph} for every disjoint sets of vertices U and W whose cardinality is smaller than \aleph , there exists a vertex z adjacent to all the vertices of U and non-adjacent to all the vertices of W .

Then, using the transfinite induction (see Theorem 3.5 below), one could also prove the following generalization of Proposition 2.1:

Proposition 2.8. *Any two graphs of order \aleph that satisfy property \star_{\aleph} are pairwise isomorphic.*

Therefore, we can refer to any graph of order \aleph and satisfying property \star_{\aleph} as the \aleph -Rado graph R_{\aleph} . Its existence is guaranteed under the Generalized Continuum Hypothesis (GCH) which states that if $\aleph' \prec \aleph$ then $2^{\aleph'} \preceq \aleph$. Under GCH, one can see that the set \mathcal{S} of all q -ary sequences of length $\prec \aleph$ has size \aleph . Indeed, for every $\aleph' \prec \aleph$, the set of all q -ary

sequences of length \aleph' has cardinality $2^{\aleph'}$, and by GCH we have that $2^{\aleph'} \preceq \aleph$; therefore, $|\mathcal{S}|$ has size at most \aleph . Clearly, $\aleph' \prec 2^{\aleph'} \preceq |\mathcal{S}|$ for every $\aleph' \prec \aleph$. It then follows that $|\mathcal{S}| = \aleph$.

This means that the construction of the countable Rado graph (Definition 2.2) based on representing every natural number with a finite q -ary sequence (its base q expansion) can be generalized to any order.

By assuming that GCH holds, we will prove as a corollary to Theorem 1.4 the following generalization of Theorem 1.3.

Theorem 2.9. *Let \mathcal{F} be a family of graphs of order \aleph and assume that $|\mathcal{F}| = \aleph$. Then $FP(\mathcal{F}, R_\aleph)$ has a solution if and only if the domination number of each graph in \mathcal{F} is \aleph .*

3 Factorizing infinite complete graphs

We say that a graph or a set of vertices is \aleph -small (resp. \aleph -bounded) if their order or cardinality is smaller than \aleph (resp. smaller than or equal to \aleph). Given two graphs F and Λ of order \aleph , we denote by $\Sigma_\aleph(F, \Lambda)$ the set of all graph embeddings between an induced \aleph -small subgraph of F and a subgraph of Λ . A partial order on $\Sigma_\aleph(F, \Lambda)$ can be easily defined as follows: if $\sigma: G \rightarrow \Gamma$ and $\sigma': G' \rightarrow \Gamma'$ are embeddings of $\Sigma_\aleph(F, \Lambda)$, we say that $\sigma \leq \sigma'$ whenever σ' is an extension of σ , namely, G and Γ are subgraphs of G' and Γ' , respectively, and $\sigma'|_G = \sigma$ (where $\sigma'|_G$ is the restriction of σ' to G).

Lemma 3.1. *Let F be a graph of order \aleph and with no \aleph -small dominating set. Also, let Θ be an \aleph -small subgraph of K_\aleph , and let $\sigma \in \Sigma_\aleph(F, K_\aleph \setminus \Theta)$.*

(1) *If $v \in V(F)$, then there is an embedding $\sigma': G' \rightarrow \Gamma'$ in $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$ such that*

$$|V(G')| \leq |V(G)| + 1, \quad \sigma \leq \sigma' \quad \text{and} \quad v \in V(G');$$

(2) *If $x \in V(K_\aleph)$, then there is an embedding $\sigma'': G'' \rightarrow \Gamma''$ in $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$ such that*

$$|V(G'')| \leq |V(G)| + 1, \quad \sigma \leq \sigma'' \quad \text{and} \quad x \in V(\Gamma'').$$

Proof. Let $\sigma: G \rightarrow \Gamma$ be an embedding in $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$, and let $v \in V(F)$ and $x \in V(K_\aleph)$. Clearly, when $v \in V(G)$ or $x \in V(\Gamma)$, we can take $\sigma' = \sigma$ or $\sigma'' = \sigma$, respectively. Therefore, we can assume $v \notin V(G)$ and $x \notin V(\Gamma)$.

1. Let G' be the subgraph of F induced by v and $V(G)$. Since $V(\Theta)$ is \aleph -small, we can choose $a \in V(K_\aleph) \setminus V(\Theta)$ and let $\sigma': V(G) \cup \{v\} \rightarrow V(\Gamma) \cup \{a\}$ be the extension of σ such that $\sigma'(v) = a$. Setting $\Gamma' = \sigma'(G')$, we have that σ' is the required embedding of $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$.
2. Since F has no \aleph -small dominating set, $V(G)$ (which is an \aleph -small set) cannot be a dominating set for F . Hence, there is a vertex $a \in V(F)$ that is not adjacent to any of the vertices of G . We denote by G'' (resp., Γ'') the graph obtained by adding a to G (resp., x to Γ) as an isolated vertex. Clearly, G'' is an induced subgraph of F ; also, Γ'' and Θ have no edge in common, since $E(\Gamma'') = E(\Gamma)$. Therefore, the extension $\sigma'': G'' \rightarrow \Gamma''$ of σ such that $\sigma''(a) = x$ is the required embedding of $\Sigma_\aleph(F, K_\aleph \setminus \Theta)$. \square

From now on, we will work within the Zermelo-Frankel axiomatic system with the Axiom of Choice in the form of the Well-Ordering Theorem. We recall the definition of a well-order.

Definition 3.2. A well-order \prec on a set X is a total order on X with the property that every non-empty subset of X has a least element.

The following theorem is equivalent to the Axiom of Choice.

Theorem 3.3 (Well-Ordering). *Every set X admits a well-order \prec .*

Given an element $x \in X$, we define the section $X_{\prec x}$ associated to it:

$$X_{\prec x} = \{y \in X : y \prec x\}.$$

Corollary 3.4. *Every set X admits a well-order \prec such that the cardinality of any section is smaller than $|X|$.*

Proof. Let us consider a well-order \prec on X . Let x be the smallest element such that $X_{\prec x}$ has the same cardinality as X . The set $Y = X_{\prec x}$ is such that all its sections with respect to the order \prec have smaller cardinality. Since Y instead has the same cardinality as X , the order \prec on Y induces an order \prec' on X with the required property. \square

We recall now that well-orderings allow proofs by induction.

Theorem 3.5 (Transfinite induction). *Let X be a set with a well-order \prec and let P_x denote a property for each $x \in X$. Set $0 = \min X$ and assume that:*

- P_0 is true, and
- for every $x \in X$, if P_y holds for every $y \in X_{\prec x}$, then P_x holds.

Then P_x is true for every $x \in X$.

We are now ready to prove Theorem 1.4. The idea behind the proof can be better understood by restricting our attention to the countable case, $\aleph = \mathbb{N}$. To solve $FP(\{F_\alpha : \alpha \in \mathbb{N}\})$, we first order the edges of $K_{\mathbb{N}} : \{e_0, e_1, \dots, e_\gamma, \dots\}$. Then, we define embeddings $\sigma_\alpha^\beta : G_\alpha^\beta \rightarrow \Gamma_\alpha^\beta$ where G_α^β is an induced subgraph of F_α , and Γ_α^β is a subgraph of $K_{\mathbb{N}}$. These embeddings are obtained by recursively applying Lemma 3.1 which adds, at each step, a vertex to G_α^β and a vertex to Γ_α^β and makes sure that the vertex β belongs to both these graphs (this procedure can be seen as a variation of Cantor's "back-and-forth" method). We also make sure that, for every γ , the graphs $\Gamma_0^\gamma, \Gamma_1^\gamma, \dots, \Gamma_\gamma^\gamma$ are pairwise edge-disjoint and contain between them the edge e_γ . The solution to $FP(\{F_\alpha : \alpha \in \mathbb{N}\})$ will be represented by $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathbb{N}\}$ where $\Gamma_\alpha = \bigcup_\beta \Gamma_\alpha^\beta$.

Theorem 1.4 *Let \mathcal{F} be a family of graphs, each of which has order \aleph . $FP(\mathcal{F})$ has a solution whenever the following two conditions hold:*

1. $|\mathcal{F}| = \aleph$, and
2. the domination number of each graph in \mathcal{F} is \aleph .

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$. We consider a well-order \prec on \mathcal{A} satisfying Corollary 3.4. Since by assumption $|V(F_\alpha)| = |\mathcal{A}| = \aleph$, for every $\alpha \in \mathcal{A}$, we can take $V(F_\alpha) = V(K_\aleph) = \mathcal{A}$ and index the edges of K_\aleph over \mathcal{A} : $E(K_\aleph) = \{e_\alpha : \alpha \in \mathcal{A}\}$.

To prove the assertion, we construct a chain of families $(\mathcal{E}_\gamma)_{\gamma \in \mathcal{A}}$, where

$$\mathcal{E}_\gamma := \{\sigma_\alpha^\beta : G_\alpha^\beta \rightarrow \Gamma_\alpha^\beta \mid \sigma_\alpha^\beta \in \Sigma_\aleph(F_\alpha, K_\aleph), (\alpha, \beta) \in \mathcal{A}_{\preceq\gamma} \times \mathcal{A}_{\preceq\gamma}\},$$

which satisfy the ascending property, that is, $\mathcal{E}_{\gamma'} \subseteq \mathcal{E}_\gamma$ if $\gamma' \preceq \gamma$, and the following three conditions:

- (1 $_\gamma$) for every $(\alpha, \beta) \in \mathcal{A}_{\preceq\gamma} \times \mathcal{A}_{\preceq\gamma}$ and $\beta' \prec \beta$ we have that $\sigma_\alpha^{\beta'} \leq \sigma_\alpha^\beta$ and $\beta \in V(G_\alpha^{\beta'}) \cap V(\Gamma_\alpha^\beta)$;
- (2 $_\gamma$) for every $\beta \in \mathcal{A}_{\preceq\gamma}$, the graphs $\Gamma_\alpha^\beta : \alpha \preceq \beta$ are pairwise edge-disjoint, and the edge e_β belongs to their union;
- (3 $_\gamma$) for every $\alpha, \beta \in \mathcal{A}_{\preceq\gamma}$, the graph Γ_α^β is either finite or $|\mathcal{A}_{\preceq\gamma}|$ -bounded.

The desired factorization of K_\aleph is then $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$, where $\Gamma_\alpha = \bigcup_{\beta \in \mathcal{A}} \Gamma_\alpha^\beta$ for every $\alpha \in \mathcal{A}$. Indeed, property (1 $_\gamma$) guarantees that each Γ_α is a factor of K_\aleph isomorphic to F_α . Also, property (2 $_\gamma$) ensures that the Γ_α s are pairwise edge-disjoint and between them contain all the edges of K_\aleph .

We proceed by transfinite induction on γ .

BASE CASE. Let $0 = \min \mathcal{A}$, choose an edge $e \in E(F_0)$ and let $\sigma \in \Sigma_\aleph(F_0, K_\aleph)$ be the embedding that maps e to e_0 . By Lemma 3.1, there exists an embedding $\sigma_0^0 : G_0^0 \rightarrow \Gamma_0^0$ in $\Sigma_\aleph(F_0, K_\aleph)$ such that Γ_0^0 is a finite graph and

$$\sigma \leq \sigma_0^0 \text{ and } 0 \in V(G_0^0) \cap V(\Gamma_0^0).$$

Clearly, $\mathcal{E}_0 := \{\sigma_0^0\}$ satisfies properties (1 $_0$), (2 $_0$) and (3 $_0$).

TRANSFINITE INDUCTIVE STEP. We assume that, for any $\gamma' \prec \gamma$, there is a family $\mathcal{E}_{\gamma'}$ satisfying properties (1 $_{\gamma'}$), (2 $_{\gamma'}$) and (3 $_{\gamma'}$), and prove that it can be extended to a family \mathcal{E}_γ that satisfies properties (1 $_\gamma$), (2 $_\gamma$) and (3 $_\gamma$). Clearly it is enough to provide the maps σ_α^β where either $\alpha = \gamma$ or $\beta = \gamma$.

We start by constructing the maps σ_α^γ for every $\alpha \prec \gamma$. We proceed by transfinite induction on α .

- Base case. Set $\Theta_0 := \bigcup_{\alpha, \beta \prec \gamma} \Gamma_\alpha^\beta$ and note that, by property (3 $_{\gamma'}$), Θ_0 is \aleph -small. We also set $\sigma_0^{\prec\gamma} : \bigcup_{\beta \prec \gamma} G_0^\beta \rightarrow \bigcup_{\beta \prec \gamma} \Gamma_0^\beta$ to be the map of $\Sigma_\aleph(F_0, K_\aleph \setminus \Theta_0)$ whose restriction to G_0^β is σ_0^β . We note that property (3 $_{\gamma'}$) guarantees that the order of $\bigcup_{\beta \prec \gamma} G_0^\beta$ is either finite or $|\mathcal{A}_{\preceq\gamma}|$ -bounded, hence \aleph -small.

Therefore, we can apply Lemma 3.1 (with $\sigma = \sigma_0^{\prec\gamma}$) to obtain the map $\sigma_0^\gamma : G_0^\gamma \rightarrow \Gamma_0^\gamma$ in $\Sigma_\aleph(F_0, K_\aleph \setminus \Theta_0)$ such that $|V(\Gamma_0^\gamma)| \leq |V(\bigcup_{\beta \prec \gamma} \Gamma_0^\beta)| + 2$ and, for every $\gamma' \prec \gamma$,

$$\sigma_0^{\gamma'} \leq \sigma_0^\gamma \text{ and } \gamma \in V(G_0^\gamma) \cap V(\Gamma_0^\gamma).$$

- Inductive step. Assume we have defined the maps $\sigma_{\alpha'}^{\gamma}$, for every $\alpha' \prec \alpha$, and set

$$\Theta_{\alpha} := \bigcup_{\alpha' \prec \alpha} \Gamma_{\alpha'}^{\gamma} \cup \bigcup_{\alpha \prec \alpha' \prec \gamma, \beta \prec \gamma} \Gamma_{\alpha'}^{\beta}.$$

As before, by Lemma 3.1 there exists $\sigma_{\alpha}^{\gamma} : G_{\alpha}^{\gamma} \rightarrow \Gamma_{\alpha}^{\gamma}$ in $\Sigma_{\aleph}(F_{\alpha}, K_{\aleph} \setminus \Theta_{\alpha})$ such that $|V(\Gamma_{\alpha}^{\gamma})| \leq |V(\bigcup_{\beta \prec \gamma} \Gamma_{\alpha}^{\beta})| + 2$ and, for every $\gamma' \prec \gamma$,

$$\sigma_{\alpha}^{\gamma'} \leq \sigma_{\alpha}^{\gamma} \text{ and } \gamma \in V(G_{\alpha}^{\gamma}) \cap V(\Gamma_{\alpha}^{\gamma}).$$

Finally, we define the maps σ_{γ}^{β} when $\beta \preceq \gamma$. We set $\Theta := \bigcup_{\alpha \prec \gamma} \Gamma_{\alpha}^{\gamma}$ and proceed by transfinite induction on β .

- Base case. If $e_{\gamma} \in \Theta$, let σ be the empty map of $\Sigma_{\aleph}(F_{\gamma}, K_{\aleph} \setminus \Theta)$. Otherwise, chose an edge $e \in E(F_{\gamma})$, and let $\sigma \in \Sigma_{\aleph}(F_{\gamma}, K_{\aleph} \setminus \Theta)$ be the embedding that maps e to e_{γ} . By Lemma 3.1, there exists $\sigma_{\gamma}^0 : G_{\gamma}^0 \rightarrow \Gamma_{\gamma}^0$ in $\Sigma_{\aleph}(F_{\gamma}, K_{\aleph} \setminus \Theta)$ such that Γ_{γ}^0 is a finite graph and

$$\sigma \leq \sigma_{\gamma}^0 \text{ and } 0 \in V(G_{\gamma}^0) \cap V(\Gamma_{\gamma}^0).$$

- Inductive step. Assume we have defined the maps $\sigma_{\gamma}^{\beta'}$ for any $\beta' \prec \beta$. Again by Lemma 3.1, there exists $\sigma_{\gamma}^{\beta} : G_{\gamma}^{\beta} \rightarrow \Gamma_{\gamma}^{\beta}$ in $\Sigma_{\aleph}(F_{\gamma}, K_{\aleph} \setminus \Theta)$ such that $|V(\Gamma_{\gamma}^{\beta})| \leq |V(\bigcup_{\beta' \prec \beta} \Gamma_{\gamma}^{\beta'})| + 2$ and, for any $\beta' \prec \beta$,

$$\sigma_{\gamma}^{\beta'} \leq \sigma_{\gamma}^{\beta} \text{ and } \beta \in V(G_{\gamma}^{\beta}) \cap V(\Gamma_{\gamma}^{\beta}).$$

It follows from the construction that the family

$$\mathcal{E}_{\gamma} := \{\sigma_{\alpha}^{\beta} : G_{\alpha}^{\beta} \rightarrow \Gamma_{\alpha}^{\beta} \mid \sigma_{\alpha}^{\beta} \in \Sigma_{\aleph}(F_{\alpha}, K_{\aleph}), \alpha, \beta \preceq \gamma\}$$

satisfies properties (1_γ), (2_γ) and (3_γ). □

Assuming that GCH holds, we obtain Theorem 2.9 as a corollary to Theorem 1.4.

Theorem 2.9. *Let \mathcal{F} be a family of graphs of order \aleph and assume that $|\mathcal{F}| = \aleph$. Then $FP(\mathcal{F}, R_{\aleph})$ has a solution if and only if the domination number of each graph in \mathcal{F} is \aleph .*

Proof. By property \star_{\aleph} , one can easily check that R_{\aleph} is self-complementary, that is $K_{\aleph} \setminus R_{\aleph}$ is isomorphic to R_{\aleph} , and the domination number of the \aleph -Rado graph is \aleph . Therefore, the domination number of each graph of \mathcal{F} must be \aleph .

To prove sufficiency, note that $\mathcal{F}' := \mathcal{F} \cup \{R_{\aleph}\}$ satisfies the hypothesis of Theorem 1.4. Therefore $FP(\mathcal{F}')$ admits a solution. This means that \mathcal{F} factorizes $K_{\aleph} \setminus R_{\aleph} \simeq R_{\aleph}$. □

4 The factorization problem for k -stars

Theorem 1.4 does not provide solutions to $FP(F)$ whenever the graph F has a dominating set of cardinality less than its order. In particular, if F is countable with a finite dominating set, then the existence of a solution to $FP(F)$ is an open problem. In this section, we consider a special class of such graphs, the k -stars S_k . More precisely,

- the star S_1 , which we also call a 1-star, is the graph with vertex-set \mathbb{N} whose edges are of the form $\{0, i\}$ for every $i \in \mathbb{N} \setminus \{0\}$;
- the k -star S_k is the vertex-disjoint union of k stars.

Note that S_k contains exactly k vertices of infinite degree, which we call *centers* and form a finite dominating set of S_k .

In the following, we show that $FP(S_k)$ has no solution whenever $k \in \{1, 2\}$, while it admits a solution for every $k > 3$. Unfortunately, we leave open the problem for 3-stars.

4.1 The case $k \in \{1, 2\}$

Proposition 4.1. $FP(S_1)$ has no solution.

Proof. Assume for a contradiction that there is a factorization \mathcal{G} of $K_{\mathbb{N}}$ into 1-stars. Choose any star $\Gamma \in \mathcal{G}$ and let g denote its center. Note that all the edges of $K_{\mathbb{N}}$ incident with g belong to Γ . By recalling that \mathcal{G} is a factorization of $K_{\mathbb{N}}$ (and that a 1-star has no isolated vertices), it follows that g cannot be a vertex in any other star of \mathcal{G} . Therefore, every star of $\mathcal{G} \setminus \{\Gamma\}$ is not spanning, contradicting the assumption. \square

With essentially the same proof, one obtains the following.

Remark 4.2. Let F be the vertex-disjoint union of S_1 with a finite set of isolated vertices. Then $FP(F)$ has no solution.

To prove the non-existence of a solution to $FP(S_2)$ it will be useful the following lemma.

Lemma 4.3. *If \mathcal{G} is a factorization of $K_{\mathbb{N}}$ into k -stars, then there is at most one vertex of $K_{\mathbb{N}}$ that is never a center in any k -star of \mathcal{G} . It follows that $|\mathcal{G}| = |\mathbb{N}|$.*

Proof. It is enough to notice that every pair $\{a, b\}$ of vertices of $K_{\mathbb{N}}$ is the edge of some 2-star Γ of \mathcal{G} ; hence, either a or b is a center of Γ . \square

Proposition 4.4. $FP(S_2)$ has no solution.

Proof. Assume for a contradiction that there is a factorization \mathcal{G} of $K_{\mathbb{N}}$ into 2-stars. For every $\Gamma \in \mathcal{G}$, letting c be a center of Γ , we denote by $\Gamma(c)$ the set of vertices adjacent with c in Γ (i.e., the neighborhood of c in Γ).

Choose any 2-star $\Gamma \in \mathcal{G}$ and let a and b denote its centers. Also, let Γ' be the 2-star of $\mathcal{G} \setminus \{\Gamma\}$ containing the edge $\{a, b\}$. Without loss of generality, we can assume that a is a center of Γ' . Finally, by Lemma 4.3, we can choose $x \in \Gamma'(a) \setminus \{b\}$ such that there exists a 2-star $\Gamma'' \in \mathcal{G}$ having x as one of its centers.

Since Γ is a factor of $K_{\mathbb{N}}$, it follows that $x \in \Gamma(b)$. In other words, $\Gamma \cup \Gamma'$ contains the edges $\{x, a\}$ and $\{x, b\}$. Therefore, $a, b \notin \Gamma''(x)$. Since Γ'' is a factor of $K_{\mathbb{N}}$ and $\{a, b\}$ is an edge of Γ , it follows $a, b \in \Gamma''(y)$, where y is the other center of Γ'' . In other words, $\{y, a\}$ and $\{y, b\}$ belong to Γ'' , hence y cannot lie in Γ , contradicting the fact that Γ is a factor. \square

4.2 The case $k \geq 4$

In this section we prove the solvability of $FP(S_k)$ whenever $k \geq 4$. For our constructions we need to introduce the following notation.

Let \mathbb{D} be an integral domain and set $V = \mathbb{D} \times \{0, 1, \dots, h\}$, for $h \geq 0$. For the sake of brevity, we will denote each pair $(a, i) \in V$ by a_i . Given a graph Γ with vertices in V , for every $a, b \in \mathbb{D}$ we denote by $a\Gamma + b$ the graph obtained by replacing each vertex x_i of Γ with $(ax + b)_i$; further, if $\{x_i, y_i\}$ is an edge of Γ , then $\{(ax + b)_i, (ay + b)_i\}$ is an edge of $a\Gamma + b$. Also, we denote by $Orb_{\mathbb{D}}(\Gamma) = \{\Gamma + d : d \in \mathbb{D}\}$ the \mathbb{D} -orbit of Γ , that is, the set of all translates of Γ by the elements of \mathbb{D} .

Proposition 4.5. *For every $k \geq 4$, there exists a k -star Γ with vertex set $V = \mathbb{Z} \times \{0, 1\}$ such that $Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of K_V into k -stars.*

Proof. We first deal with the case $k = 4$. Set $\Gamma = \bigcup_{i=1}^4 \Gamma_i$, where each Γ_i is the 1-star with vertices in $V = \mathbb{Z} \times \{0, 1\}$ and center x_i defined as follows (see Figure 1):

- $x_1 = 0_0$ and $\Gamma_1(x_1) = \{i_0 : i \geq 1\}$;
- $x_2 = -1_1$ and $\Gamma_2(x_2) = \{i_1 : i \geq 0\} \cup \{-1_0\}$;
- $x_3 = -2_0$ and $\Gamma_3(x_3) = \{i_1 : i \leq -3\}$;
- $x_4 = -2_1$ and $\Gamma_4(x_4) = \{i_0 : i \leq -3\}$.

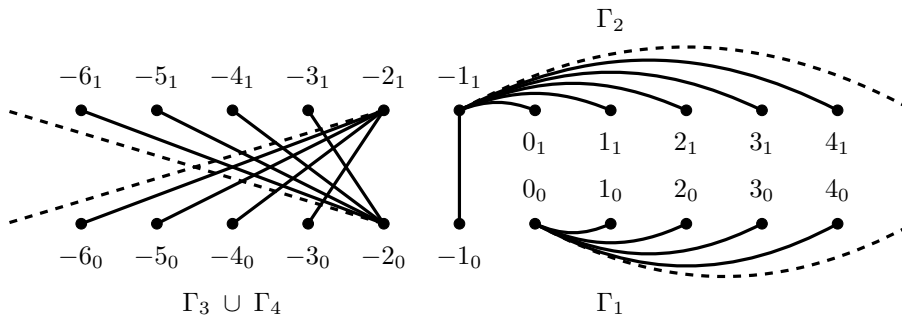


Figure 1: The graph Γ when $k = 4$.

We claim that $\mathcal{G} := Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of K_V into 4-stars. Denote by $K_{U,W}$ the complete bipartite graph whose parts are $U = \mathbb{Z} \times \{0\}$ and $W = \mathbb{Z} \times \{1\}$, and consider the 1-factor $I = \{\{i_0, i_1\} : i \in \mathbb{Z}\}$ of $K_{U,W}$. Clearly, K_V decomposed into $K_U, K_W \cup I$ and $K_{U,W} \setminus I$. One can check that

- $Orb_{\mathbb{Z}}(\Gamma_1)$ decomposes K_U ,
- $Orb_{\mathbb{Z}}(\Gamma_2)$ decomposes $K_W \cup I$, and
- $Orb_{\mathbb{Z}}(\Gamma_3 \cup \Gamma_4)$ decomposes $K_{U,W} \setminus I$.

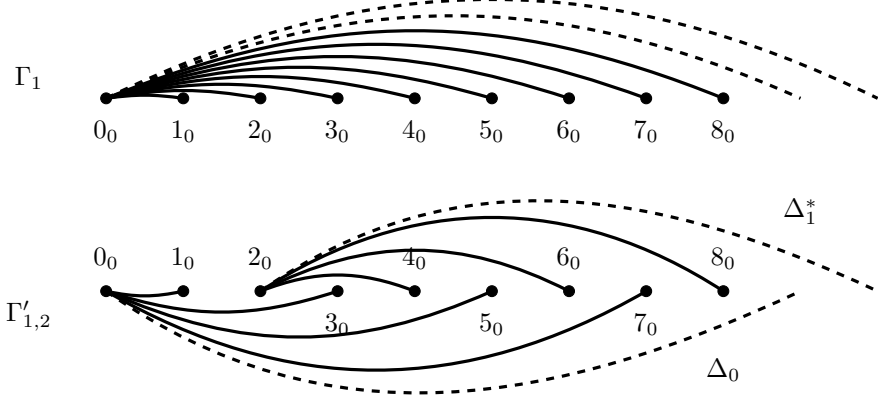


Figure 2: Replacing Γ_1 with $\Gamma'_{1,2}$ produces Γ when $k = 5$.

Hence, \mathcal{G} is a decomposition of K_V . Considering that the Γ_i s are pairwise vertex-disjoint and their vertex-sets partition V , we have that Γ and each of its translates (under the action of \mathbb{Z}) are factors of K_V isomorphic to a 4-star. Therefore, \mathcal{G} is a factorization of K_V into 4-stars.

To deal with the case $k \geq 5$, it is enough to replace the component Γ_1 of Γ with a $(k-3)$ -star Γ'_1 satisfying the following conditions:

$$V(\Gamma'_1) = V(\Gamma_1), \quad \text{and} \quad (4.1)$$

$$\text{Orb}_{\mathbb{Z}}(\Gamma'_1) \text{ decomposes } K_U. \quad (4.2)$$

Indeed, letting $\Gamma' = (\Gamma \setminus \Gamma_1) \cup \Gamma'_1$, by condition (4.1) we have that Γ' is a k -star with vertex-set V . Recalling that $\text{Orb}_{\mathbb{Z}}(\Gamma_1)$ decomposes K_U , by condition (4.2) it follows that $\text{Orb}_{\mathbb{Z}}(\Gamma')$ and $\text{Orb}_{\mathbb{Z}}(\Gamma)$ decompose the same graph, that is, K_V . Hence, $\text{Orb}_{\mathbb{Z}}(\Gamma')$ is a factorization of K_V into k -stars.

Let $k = h + 3$ with $h \geq 2$. It is left to construct an h -star $\Gamma'_{1,h}$ satisfying conditions (4.1) and (4.2), for every $h \geq 2$. For sake of clarity, in the rest of the proof we identify $U = \mathbb{Z} \times \{0\}$ with \mathbb{Z} . Therefore, Γ_1 is the 1-star centered in 0 with $\Gamma_1(0) = \{i : i \geq 1\}$.

Let Δ_j and Δ_j^* be the 1-stars centered in $c_j = 2(2^j - 1)$ such that

$$\Delta_j(c_j) = \{c_j + i : 0 < i \equiv 2^j \pmod{2^{j+1}}\},$$

$$\Delta_j^*(c_j) = \{c_j + i : 0 < i \equiv 0 \pmod{2^j}\},$$

for $j \geq 0$, and set $\Gamma'_{1,h} = \Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_{h-2} \cup \Delta_{h-1}^*$ for $h \geq 2$. It is not difficult to check that $\{\Delta_j - c_j : 0 \leq j \leq h-2\} \cup \{\Delta_{h-1}^* - c_{h-1}\}$ decomposes Γ_1 . Therefore, $\text{Orb}_{\mathbb{Z}}(\Gamma'_{1,h})$ and $\text{Orb}_{\mathbb{Z}}(\Gamma_1)$ decompose the same graph, that is, K_U . Hence, $\Gamma'_{1,h}$ satisfies condition (4.2).

We show that $\Gamma'_{1,h}$ is an h -star satisfying condition (4.1) by induction on h . If $h = 2$, then $V(\Delta_0) = \{0, 1, 3, 5, \dots\}$ and $V(\Delta_1^*) = \{2, 4, 6, \dots\}$. Therefore, $\Gamma'_{1,2} = \Delta_0 \cup \Delta_1^*$ is

a 2-star with the same vertex-set as Γ_1 . Now assume that $\Gamma'_{1,h}$ is an h -star satisfying condition (4.1) for some $h \geq 2$. Recalling the definition of $\Gamma'_{1,h}$ and $\Gamma'_{1,h+1}$, and considering that the vertex-sets of Δ_{h-1} and Δ_h^* partition $V(\Delta_{h-1}^*)$, we have that $\Gamma'_{1,h+1}$ is an $(h+1)$ -star with the same vertex-set as $\Gamma'_{1,h}$, that is, $V(\Gamma_1)$, and this concludes the proof. \square

Propositions 4.1, 4.4 and 4.5 leave open $FP(S_k)$ only when $k = 3$. In this case, an approach similar to Theorem 4.5 cannot work, as shown in the following.

Proposition 4.6. *There is no 3-star Γ with vertex-set $V = \mathbb{Z} \times \{0, 1, \dots, k\}$ such that the \mathbb{Z} -orbit of Γ is an S_3 -factorization of K_V .*

Proof. Assume for a contradiction that there exists a 3-star Γ with vertex-set $V = \mathbb{Z} \times \{0, 1, \dots, k\}$ such that $\mathcal{G} = Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of K_V .

We first notice that Γ must have at least a center in $\mathbb{Z} \times \{i\}$, for every $i \in \{0, 1, \dots, k\}$. Indeed, if Γ has no center in $\mathbb{Z} \times \{i\}$ for some $i \in \{0, 1, \dots, k\}$, then no edge of $K_{\mathbb{Z} \times \{i\}}$ can be covered by \mathcal{G} . Since Γ has 3 centers, it follows that $k \leq 2$. Note that if $k = 2$, the centers of Γ must be x_0, y_1, z_2 for some $x, y, z \in \mathbb{Z}$, but in this case the edge $\{x_0, y_1\}$ cannot lie in any translate of Γ . Therefore $k \leq 1$.

If $k = 1$, without loss of generality we can assume that the centers of Γ are $0_0, x_1$ and y_1 with $x \neq y$. Since the edge $\{0_0, x_1\}$ does not belong to Γ , it lies in some of its translates, say $\Gamma + z$ with $z \neq 0$. This is equivalent to saying that $\{(-z)_0, (x - z)_1\} \in \Gamma$. It follows that $x - z = y$, hence $\{(y - x)_0, y_1\} \in \Gamma$. Similarly, we can show that $\{(x - y)_0, x_1\} \in \Gamma$. It follows that Γ cannot contain the edges $\{0_0, (x - y)_0\}$ and $\{0_0, (y - x)_0\}$. This implies that no edge of the form $\{w_0, (x - y + w)_0\}$ lies in any translate of Γ , contradicting again the assumption that \mathcal{G} is a factorization of K_V . Therefore $k = 0$.

Let $V = \mathbb{Z}$ and denote by $\Delta\Gamma$ the multiset of all differences $y - x$ between any two adjacent vertices x and y of Γ , with $x < y$:

$$\Delta\Gamma = \{y - x : \{x, y\} \in E(\Gamma), x < y\}.$$

It is not difficult to see that $\mathcal{G} = Orb_{\mathbb{Z}}(\Gamma)$ is a factorization of $K_{\mathbb{Z}}$ if and only if $\Delta\Gamma = \mathbb{N} \setminus \{0\}$. Denoting by $\Gamma + i$ the translate of Γ obtained by replacing each vertex $x \in V(\Gamma)$ with $x + i$, one can easily see that $\Delta(\Gamma + i) = \Delta\Gamma$ for every $i \in \mathbb{Z}$. Therefore, up to a translation, we can assume that the centers of Γ are $0, x, n$ with $0 < x < n$. Now, for every $i \geq n$, denote by Γ_i the induced subgraph of Γ with vertex-set $\{0, 1, \dots, i\}$. Also, let Γ^* be the induced subgraph of Γ on the vertices $\{-3, -2, -1, 0, x, n\}$. Clearly, $|\Delta\Gamma^*| = 3$, $|\Delta\Gamma_i| = i - 2$ and $\Delta\Gamma_i \subset \{1, 2, \dots, i\}$. Also, since the multiset $\Delta\Gamma$ contains all positive integers with no repetition, it follows that $\Delta\Gamma^*$ and $\Delta\Gamma_i$ are disjoint, hence $\Delta\Gamma_i \subset \{1, 2, \dots, i\} \setminus \Delta\Gamma^*$ for every $i \geq n$. Then, for $i = \max(\Delta\Gamma^*)$, we obtain the following contradiction: $i - 2 = |\Delta\Gamma_i| \leq |\{1, 2, \dots, i\} \setminus \Delta\Gamma^*| = i - 3$. \square

5 The resolvability problem

Theorem 1.4 allows us to construct decompositions of K_{\aleph} into \aleph graphs of specified type. More precisely, we have the following.

Corollary 5.1. *Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be an infinite family of (non-empty) \aleph -bounded graphs, where $\aleph = |\mathcal{A}|$. Then there exists a decomposition $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ of K_{\aleph} such that each Γ_α is isomorphic to F_α .*

Furthermore, for any $\beta \in \mathcal{A}$ such that the domination number of F_β is less than \aleph , we have that $|V(K_\aleph) \setminus V(\Gamma_\beta)| = \aleph$. Otherwise, for every $0 \preceq \aleph' \preceq \aleph$, the decomposition \mathcal{G} can be constructed so that $|V(K_\aleph) \setminus V(\Gamma_\beta)| = \aleph'$.

Proof. For every $\alpha \in \mathcal{A}$, set $\aleph_\alpha = \aleph$ if the domination number of F_α is less than \aleph ; otherwise, let $0 \preceq \aleph_\alpha \preceq \aleph$. By adding to each graph F_α a set of \aleph_α isolated vertices we obtain a graph F'_α whose order and domination number are \aleph . Since the assumptions of Theorem 1.4 are satisfied, there exists a factorization $\mathcal{G}' = \{\Gamma'_\alpha : \alpha \in \mathcal{A}\}$ of K_\aleph such that each Γ'_α is isomorphic to F'_α . By replacing Γ'_α with the isomorphic copy of F_α , we obtain the desired decomposition \mathcal{G} . \square

Inspired by [9], we ask under which conditions a decomposition \mathcal{G} of K_\aleph is resolvable, namely, its graphs can be partitioned into factors of K_\aleph , also called *resolution classes*. It follows that a resolvable decomposition \mathcal{G} of K_\aleph must satisfy the following two conditions:

N1. if $\Gamma \in \mathcal{G}$ is not a factor of K_\aleph , then $|V(K_\aleph) \setminus V(\Gamma)| \geq \min\{|\Gamma| : \Gamma \in \mathcal{G}\}$;

N2.

$$\mathcal{G}(z) \subseteq \mathcal{G}(x) \cup \mathcal{G}(y) \Rightarrow \mathcal{G}(z) \supseteq \mathcal{G}(x) \cap \mathcal{G}(y),$$

where $\mathcal{G}(v) = \{\Gamma \in \mathcal{G} : v \in V(\Gamma)\}$ is the set of all graphs of \mathcal{G} passing through v .

In the following, we easily construct decompositions of K_\aleph that do not satisfy the above conditions, and therefore they are non-resolvable.

Example 5.2. Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be an infinite family of (non-empty) \aleph -bounded graphs, where $\aleph = |\mathcal{A}|$. Also, assume that the domination number of at least one of its graphs, say F_β , is \aleph . Then, by applying Corollary 5.1 with $\aleph' \prec \min\{|\Gamma_\alpha| : \alpha \in \mathcal{A}\}$, we construct a decomposition that does not satisfy condition N1.

For instance, if $\aleph = |\mathbb{N}|$, each F_α is a countable locally finite graph (hence, its domination number is \aleph) and $\aleph' = 1$ for every $\beta \in \mathbb{N}$, then we construct a decomposition $\mathcal{G} = \{G_\beta : \beta \in \mathbb{N}\}$ of K_\aleph into connected regular graphs where $V(G_\beta) = \mathbb{N} \setminus \{x_\beta\}$ for some $x_\beta \in \mathbb{N}$. Clearly, no graph of \mathcal{G} is a factor of K_\aleph , and any two graphs of \mathcal{G} have common vertices. Therefore, \mathcal{G} is not resolvable.

Example 5.3. Let \mathcal{G} be any decomposition of the infinite complete graph K_V (for example, one of those constructed by Corollary 5.1). Let y and z be vertices not belonging to K_V and set $W = V \cup \{y, z\}$. We can easily extend \mathcal{G} to a non-resolvable decomposition \mathcal{G}' of K_W in the following way.

Choose $x \in V$ and let \mathcal{C} be the following family of paths of length 1 or 2:

$$\mathcal{C} = \{[y, v, z] : v \in V \setminus \{x\}\} \cup \{[x, z, y], [x, y]\}.$$

Clearly, \mathcal{C} decomposes $K_W \setminus K_V$, hence $\mathcal{G}' = \mathcal{G} \cup \mathcal{C}$ is a decomposition of K_W . Also, x, y and z do not satisfy condition N2, since $\mathcal{G}'(z) \subseteq \mathcal{G}'(x) \cup \mathcal{G}'(y)$, while $[x, y]$ belongs to $\mathcal{G}'(x) \cap \mathcal{G}'(y)$, but not to $\mathcal{G}'(z)$. Therefore, \mathcal{G}' is non-resolvable. Indeed, any resolution class of \mathcal{G}' could cover the vertex z only with graphs passing through x or y . This means that the graph $[x, y]$ cannot belong to any resolution class of \mathcal{G}' .

The following result provides sufficient conditions for a decomposition \mathcal{G} to be resolvable.

Theorem 5.4. *Let \mathcal{G} be a decomposition of the infinite complete graph K_{\aleph} satisfying the following properties for some $\aleph' \prec \aleph$:*

- R1. *each graph in \mathcal{G} is \aleph' -bounded;*
- R2. *$|\mathcal{G}(x) \cap \mathcal{G}(y)| \preceq \aleph'$ for every distinct $x, y \in V(K_{\aleph})$.*

Then \mathcal{G} is resolvable.

Proof. Let $\mathcal{G} = \{G_{\alpha} : \alpha \in \mathcal{A}\}$. We consider a well-order \prec on \mathcal{A} satisfying Corollary 3.4. Since the graphs of \mathcal{G} are \aleph' -bounded, we have that $|\mathcal{A}| = \aleph$ and we can assume $V(K_{\aleph}) = \mathcal{A}$. Here we need to construct an ascending chain $(\mathcal{G}_{\gamma})_{\gamma \in \mathcal{A}}$ of families $\mathcal{G}_{\gamma} := \{G_{\alpha}^{\gamma} : \alpha \in \mathcal{A}_{\preceq \gamma}\}$ (where $\Gamma_{\alpha}^{\gamma'}$ is a subgraph of Γ_{α}^{γ} whenever $\gamma' \preceq \gamma$) that satisfy the following properties:

- (1 $_{\gamma}$) each Γ_{α}^{γ} is a vertex-disjoint union of graphs of \mathcal{G} ;
- (2 $_{\gamma}$) for every $\alpha \in \mathcal{A}_{\preceq \gamma}, \gamma \in V(\Gamma_{\alpha}^{\gamma})$;
- (3 $_{\gamma}$) G_{γ} is contained in exactly one Γ_{α}^{γ} where $\alpha \in \mathcal{A}_{\preceq \gamma}$;
- (4 $_{\gamma}$) for every $\alpha \in \mathcal{A}_{\preceq \gamma}, \Gamma_{\alpha}^{\gamma}$ is either a finite graph or $(\aleph' \cdot |\mathcal{A}_{\preceq \gamma}|)$ -bounded.

The desired resolution of K_{\aleph} is then $\mathcal{R} = \{\Gamma_{\alpha} : \alpha \in \mathcal{A}\}$, where $\Gamma_{\alpha} = \bigcup_{\gamma \in \mathcal{A}} \Gamma_{\alpha}^{\gamma}$ for every $\alpha \in \mathcal{A}$. Indeed, due to properties (1 $_{\gamma}$) and (2 $_{\gamma}$), each Γ_{α} is a resolution class of \mathcal{G} and, by property (3 $_{\gamma}$), \mathcal{R} is a partition of \mathcal{G} into resolution classes.

We proceed by transfinite induction on γ .

BASE CASE. Let $0 = \min X$. By condition R2, if 0 is not a vertex of G_0 , $|\mathcal{G}(0) \cap \mathcal{G}(x)| \preceq \aleph'$ for any $x \in V(G_0)$. Since, due to condition R1, $|\mathcal{G}(0)| = \aleph$, there exists $G \in \mathcal{G}(0)$ disjoint from G_0 . Therefore we can define $\mathcal{G}_0 = \{\Gamma_0^0\}$ where Γ_0^0 is either $G_0 \cup G$ or, if 0 belongs to $V(G_0)$, G_0 .

TRANSFINITE INDUCTIVE STEP. For every $\gamma' \prec \gamma$, we assume there is a family $\mathcal{G}_{\gamma'}$ satisfying $(i_{\gamma'})$ for $1 \leq i \leq 4$. We show that $\mathcal{G}_{\gamma'}$ can be extended to a family \mathcal{G}_{γ} that satisfies the same properties, (i_{γ}) for $1 \leq i \leq 4$.

We are going to define, recursively, the graphs Γ_{α}^{γ} whenever $\alpha \preceq \gamma$. First, we consider the case $\alpha \prec \gamma$. We start by setting $\Gamma_{\alpha}^{\prec \gamma} := \bigcup_{\gamma' \prec \gamma} \Gamma_{\alpha}^{\gamma'}$. Note that property (4 $_{\gamma'}$) guarantees that $\Gamma_{\alpha}^{\prec \gamma}$ is either finite or $|\Gamma_{\alpha}^{\prec \gamma}| \preceq \aleph' \cdot |\mathcal{A}_{\preceq \gamma}|$; hence, $\Gamma_{\alpha}^{\prec \gamma}$ is \aleph -small.

- **Base case.** If $\gamma \in V(\Gamma_0^{\prec \gamma})$, set $\Gamma_0^{\gamma} = \Gamma_0^{\prec \gamma}$.

If $\gamma \notin V(\Gamma_0^{\prec \gamma})$, by condition R2 we have $|\mathcal{G}(\gamma) \cap \mathcal{G}(x)| \preceq \aleph'$ for every $x \in V(\Gamma_0^{\prec \gamma})$. Since $\Gamma_0^{\prec \gamma}$ is \aleph -small, this means that the family of graphs of $\mathcal{G}(\gamma)$ that intersect $V(\Gamma_0^{\prec \gamma})$ is \aleph -small.

Moreover, any $\Gamma_{\alpha}^{\prec \gamma}$ is either finite or $(\aleph' \cdot |\mathcal{A}_{\preceq \gamma}|)$ -bounded (note that $\aleph' \cdot |\mathcal{A}_{\preceq \gamma}| \prec \aleph$, since $|\mathcal{A}_{\preceq \gamma}| \prec \aleph$). Hence, the set of graphs in $\mathcal{G}(\gamma)$ that are contained in some $\Gamma_{\alpha}^{\prec \gamma}$ is \aleph -small.

Finally, by condition R1, we have that $|\mathcal{G}(\gamma)| = \aleph$. Therefore, there exists a graph $G \in \mathcal{G}(\gamma)$ that is not contained in any $\Gamma_{\alpha}^{\prec \gamma}$ and such that $V(G) \cap V(\Gamma_0^{\prec \gamma}) = \emptyset$. Then, we set $\Gamma_0^{\gamma} = \Gamma_0^{\prec \gamma} \cup G$.

- **Recursive step.** Let $\alpha \prec \gamma$. If $\gamma \in V(\Gamma_\alpha^{\prec\gamma})$, set $\Gamma_\alpha^\gamma = \Gamma_\alpha^{\prec\gamma}$. Otherwise, by proceeding as in the previous case, we obtain the existence of a graph $G \in \mathcal{G}(\gamma)$ that is not in any $\Gamma_{\alpha'}^{\prec\gamma}$ or any $\Gamma_{\alpha''}^\gamma$ (where $\alpha' \prec \gamma$ and $\alpha'' \prec \alpha$), and such that $V(G) \cap V(\Gamma_\alpha^{\prec\gamma}) = \emptyset$. In this case, we set $\Gamma_\alpha^\gamma = \Gamma_\alpha^{\prec\gamma} \cup G$.

It is left to define Γ_γ^γ . We proceed by constructing, recursively, an ascending chain of graphs Γ_γ^α , for $\alpha \in \mathcal{A}_{\preceq\gamma}$, that are either finite or $(\aleph' \cdot |\mathcal{A}_{\preceq\gamma}|)$ -bounded.

- **Base case.** Let us first suppose that G_γ is not contained in any $\Gamma_{\alpha'}^\gamma$ (where $\alpha' \prec \gamma$). Again, by conditions **R1** and **R2**, there exists $G \in \mathcal{G}(0)$ that is also not contained in any $\Gamma_{\alpha'}^\gamma$ such that G is either G_γ or is disjoint from G_γ . We set Γ_γ^0 to be $G_\gamma \cup G$. Otherwise, we set Γ_γ^0 to be any graph G in $\mathcal{G}(0)$ that is not contained in any $\Gamma_{\alpha'}^\gamma$.
- **Recursive step.** Let us suppose that $\alpha \neq 0$ and that we have defined $\Gamma_\gamma^{\alpha'}$ for every $\alpha' \prec \alpha$. Here we set $\Gamma_\gamma^{\prec\alpha}$ to be $\bigcup_{\alpha' \prec \alpha} \Gamma_\gamma^{\alpha'}$. Note that, by construction, $\Gamma_\gamma^{\prec\alpha}$ is either a finite graph or $|\Gamma_\gamma^{\prec\alpha}| \preceq \aleph' \cdot |\mathcal{A}_{\preceq\gamma}|$. If α belongs to $V(\Gamma_\gamma^{\prec\alpha})$, we set Γ_γ^α to be $\Gamma_\gamma^{\prec\alpha}$. Otherwise, proceeding as in the previous case, we obtain that there exists $G \in \mathcal{G}(\alpha)$ disjoint from $\Gamma_\gamma^{\prec\alpha}$ that does not belong to any of the $\Gamma_{\alpha'}^\gamma$. Now we set Γ_γ^α to be $G \cup \Gamma_\gamma^{\prec\alpha}$.

Then the family $\mathcal{G}_\gamma = \{\Gamma_\alpha^\gamma : \alpha \in \mathcal{A}_{\preceq\gamma}\}$ satisfies the properties **(1_γ)**, **(2_γ)**, **(3_γ)** and **(4_γ)** by construction. \square

Remark 5.5. A cardinal \aleph is said to be *regular* if any \aleph -small union of \aleph -small sets (resp. graphs) is still an \aleph -small set (resp. graph) otherwise it is said to be *singular*. It is easy to see that, for regular cardinals, conditions **R1** and **R2** of Theorem 5.4 can be relaxed to:

R1'. each graph in \mathcal{G} is \aleph -small;

R2'. $|\mathcal{G}(x) \cap \mathcal{G}(y)| \prec \aleph$ for every distinct $x, y \in V(K_\aleph)$.

However, if \aleph is a singular cardinal, then conditions **R1'** and **R2'** are no longer sufficient. Indeed, we can construct a decomposition \mathcal{G} of K_\aleph into \aleph -small graphs such that

- $|\mathcal{G}|$ is \aleph -small,
- \mathcal{G} satisfies conditions **R1'** and **R2'**,
- there are two (possibly isolated) vertices x and y belonging to every graphs of \mathcal{G} , that is, $\mathcal{G} = \mathcal{G}(x) \cap \mathcal{G}(y)$.

Then, choosing any vertex z such that $\mathcal{G}(z) \neq \mathcal{G}$, we have that

$$\mathcal{G}(z) \subseteq \mathcal{G}(x) \cup \mathcal{G}(y) = \mathcal{G} \quad \text{but} \quad \mathcal{G}(z) \not\supseteq \mathcal{G}(x) \cap \mathcal{G}(y) = \mathcal{G}.$$

This means that condition **N2** does not hold, therefore the decomposition \mathcal{G} is not resolvable.

We conclude by showing that there is always a resolution for an ‘almost’ 2-design with blocks that are \aleph' -bounded for some $\aleph' \prec \aleph$, that is, a decomposition of K_\aleph whose graphs are almost all \aleph' -bounded complete graphs. This extends some results on the resolvability of 2-designs given in [9].

Proposition 5.6. *Let \mathcal{G} be a decomposition of the infinite complete graph K_{\aleph} into \aleph' -bounded graphs for some $\aleph' \prec \aleph$, where \aleph' is not necessarily infinite. If the subset of \mathcal{G} consisting of all non-complete graphs is \aleph' -bounded, then \mathcal{G} has a resolution.*

Proof. By assumption, condition **R1** of Theorem 5.4 holds. To prove that \mathcal{G} satisfies condition **R2** for some $\aleph'' \prec \aleph$, we assume for a contradiction the existence of vertices x and y such that $|\mathcal{G}(x) \cap \mathcal{G}(y)| \succ \aleph'' := (\aleph' + 1)$. It follows that there are at least two complete graphs in $\mathcal{G}(x) \cap \mathcal{G}(y)$, meaning that the edge $\{x, y\}$ is covered more than once by graphs in \mathcal{G} , and this is a contradiction. The assertion follows from Theorem 5.4. \square

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