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Decomposable subspaces, linear sections of Grassmann varieties, and higher weights of Grassmann codes

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ABSTRACT

We consider the question of determining the maximum number of points on sections of Grassmannians over finite fields by linear subvarieties of the Plücker projective space of a fixed codimension. This corresponds to a known open problem of determining the complete weight hierarchy of linear error correcting codes associated to Grassmann varieties. We recover most of the known results as well as prove some new results. A basic tool used is a characterization of decomposable subspaces of exterior powers, that is, subspaces in which every nonzero element is decomposable. Also, we use a generalization of the Griesmer–Wei bound that is proved here for arbitrary linear codes.

1. Introduction

Let V be an m-dimensional vector space over a field F. Given a positive integer ℓ with $\ell \leqslant m$, let $G_{\ell,m}$, denote the Grassmann variety consisting of all ℓ-dimensional subspaces of V. We identify $G_{\ell,m}$ with a subvariety of $\mathbb{P}(\bigwedge^\ell V)$ via the Plücker embedding. (See Section 5 for details.) Consider the linear section $L \cap G_{\ell,m}$ of $G_{\ell,m}$ by a linear subvariety L of $\mathbb{P}(\bigwedge^\ell V)$ of a given codimension, say r. It may be noted that the Schubert subvarieties of $G_{\ell,m}$ are among such linear sections, and also that, in general, the geometry of such linear sections is not particularly well understood. (See, for example, [5, Section 6] together with the references therein and [6].) We are interested in the following general question: For a fixed r, which of these linear sections are 'maximal'? Of course, the term 'maximal' can be interpreted in a variety of ways. But in one special case, all possible meanings of

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'maximal' would coincide. Namely, if there are linear subvarieties L of $\mathbb{P}(\bigwedge^\ell V)$ of codimension r such that $L \subseteq G_{\ell,m}$, then clearly, $L \cap G_{\ell,m} = L$, and these are evidently the maximal linear sections in every sense of the term 'maximal.' Thus, a special case of the above question is to determine the linear subvarieties of $G_{\ell,m}$. It turns out that this latter question is rather classical and an answer can be gleaned, for example, from treatises such as [9]. Determining the linear subvarieties of $G_{\ell,m}$ can also be viewed as a question of (multi)linear algebra where it corresponds to determining the *decomposable subspaces* of $\bigwedge^\ell V$, namely, subspaces in which every nonzero element ω is *decomposable*, that is, $\omega = v_1 \wedge \cdots \wedge v_\ell$ for some $v_1, \ldots, v_\ell \in V$. In Section 2 below, we outline a complete answer to the latter question in the setting of exterior algebras in a manner in which we had discovered it independently before we became aware of [9, §24.2]. In effect, it is seen that there are only two types of decomposable subspaces. Subsequently, in Section 3 we observe that a nice duality prevails among the two types of decomposable subspaces via the Hodge star operator, and that this can be particularly well understood when $\ell = 2$.

In general, for any $r \geqslant 0$, the Grassmannian $G_{\ell,m}$ need not contain a linear subvariety of $\mathbb{P}(\bigwedge^\ell V)$ of codimension r, and we consider the following precise version of the general question stated above. Let F be the finite field $\mathbb{F}_q$ with q elements. Now we ask: What is the maximum number of $\mathbb{F}_q$-rational points that a linear section of $G_{\ell,m}$ can have by a linear subvariety of $\mathbb{P}(\bigwedge^\ell V)$ of codimension r? This question admits an equivalent formulation in terms of linear error correcting codes, and as such, it has been considered by various authors. Indeed, if we let $C(\ell, m)$ denote the linear code associated to $G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}(\bigwedge^\ell V)$, then its rth higher weight (see Section 4 for definitions) is given by

$$d_r\big(C(\ell,m)\big) = n - \max_L \big|L \cap G_{\ell,m}(\mathbb{F}_q)\big|,$$

where the maximum is taken over projective linear subspaces L of $\mathbb{P}(\bigwedge^\ell \mathbb{F}_q^m)$ of codimension r, and where n denotes the Gaussian binomial coefficient defined by

$$n = \big|G_{\ell,m}(\mathbb{F}_q)\big| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{\ell-1})}{(q^\ell-1)(q^\ell-q)\cdots(q^\ell-q^{\ell-1})}.$$

With this in view, we shall now consider the equivalent question of determining $d_r = d_r(C(\ell,m))$ for any $r \geqslant 0$, where $d_0 := 0$, by convention. This question is open, in general, and the known results can be summarized as follows. From general facts in Coding Theory and the fact that the embedding $G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}(\bigwedge^\ell \mathbb{F}_q^m)$ is nondegenerate, one knows that

$$0 = d_0 < d_1 < d_2 < \cdots < d_k = n \quad \text{where } k := \binom{m}{\ell},$$

and also that

$$d_r\big(C(\ell,m)\big) \geqslant q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{where } \delta := \ell(m-\ell). \tag{1}$$

The latter is a consequence of the so-called Griesmer–Wei bound for linear codes and a result of Nogin [14] which says that $d_1 = q^\delta$. In fact, Nogin [14] showed that the Griesmer–Wei bound is sometimes attained, that is,

$$d_r\big(C(\ell,m)\big) = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{for } 0 \leqslant r \leqslant \mu, \tag{2}$$

where

$$\mu := \max\{\ell, m-\ell\} + 1.$$

Alternative proofs of Nogin's result for higher weights of $C(\ell,m)$ were given by Ghorpade and Lachaud [4] using the notion of a close family. Recently, Hansen, Johnsen and Ranestad [7,8] have observed that a dual result holds as well, namely,

$$d_{k-r}\big(C(\ell,m)\big) = n - \big(1+q+\cdots+q^{r-1}\big) \quad \text{for } 0 \leqslant r \leqslant \mu. \tag{3}$$

In general, the values of $d_r(C(\ell,m))$ for $\mu < r < k-\mu$ are not known. For example, if $\ell = 2$ and we assume (without loss of generality) that $m \geqslant 4$, then $\mu = m-1$, and $d_r(C(\ell,m))$ for $m \leqslant r < \binom{m-1}{2}$ are not known, except that in the first nontrivial case, Hansen, Johnsen and Ranestad [7] have shown by clever algebraic–geometric arguments that

$$d_5\big(C(2,5)\big) = q^6+q^5+2q^4+q^3 = d_4+q^4. \tag{4}$$

Notice that the Griesmer–Wei bound in (1) is not attained in this case, although the difference $d_r - d_{r-1}$ of consecutive higher weights of $C(2,5)$ continues to be a power of q for $r \geqslant 5$ as in the case of $r \leqslant 4$.

Our main results concerning the determination of $d_r(C(\ell,m))$ are as follows. First, we recover (2) and (3) as an immediate corollary of the general results in Sections 2 and 3 for decomposable subspaces. Next, we further analyze the structure of decomposable vectors in $\bigwedge^2 V$ to extend (3) by showing that

$$d_{k-\mu-1}\big(C(2,m)\big) = n - \big(1+q+\cdots+q^{\mu-1}+q^2\big) = d_{k-\mu} - q^2 \quad \text{for any } m \geqslant 4. \tag{5}$$

Finally, we use the abovementioned analysis of decomposable vectors in $\bigwedge^2 V$ and also exploit the Hodge star duality to prove the following generalization of (4) for any $m \geqslant 4$,

$$d_{\mu+1}\big(C(2,m)\big) = q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1} = d_\mu + q^{\delta-2}. \tag{6}$$

In the course of deriving these formulae, we use a mild generalization of the Griesmer–Wei bound, proved here in the general context of arbitrary linear codes, which may be of independent interest.

It is hoped that these results, and more so, the methods used in proving them, will pave the way for the solution of the problem of determination of the complete weight hierarchy of $C(\ell,m)$ at least in the case $\ell = 2$. To this end, we provide, toward the end of this paper, an initial tangible goal by stating conjectural formulae for $d_r(C(2,m))$ when $\mu+1 \leqslant r \leqslant 2\mu-3$, and also when $k-2\mu+3 \leqslant r \leqslant k-\mu-1$. It may be noted that these conjectural formulae, and of course both (5) and (6), corroborate a conjecture of Hansen, Johnsen and Ranestad [7,8] that in most cases, the differences of consecutive higher weights of Grassmann codes is a power of q.

2. Decomposable subspaces

Let us fix, in this as well as the next section, positive integers ℓ, m with $\ell \leqslant m$, a field F, and a vector space V of dimension m over F. Let

$$I(\ell,m) := \big\{\alpha = (\alpha_1,\ldots,\alpha_\ell) \in \mathbb{Z}^\ell\colon\ 1 \leqslant \alpha_1 < \cdots < \alpha_\ell \leqslant m\big\}.$$

If $\{v_1,\ldots,v_m\}$ is a basis of V, then $\{v_\alpha\colon\ \alpha \in I(\ell,m)\}$ is a basis of $\bigwedge^\ell V$, where $v_\alpha := v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_\ell}$. Given any $\omega \in \bigwedge^\ell V$, define

$$V_\omega := \{v \in V\colon\ v \wedge \omega = 0\}.$$

Clearly, V_ω is a subspace of V. It is evident that $\omega = 0$ if and only if $\dim V_\omega = m$. We begin by stating an elementary characterization of decomposability and an easy corollary of the same. Both the results are fairly well known (see, e.g., Exercise 17(a) in [2, p. 650] or Theorems 1.1 and 1.3 in [12, Section 4.1]), and the proofs are omitted. Here, and hereafter, it may be useful to keep in mind that for us, a decomposable vector is necessarily nonzero.

Lemma 1. *Assume that $\ell < m$ and let $\omega \in \bigwedge^\ell V$. Then*

$$\omega \text{ is decomposable} \quad \Leftrightarrow \quad \dim V_\omega = \ell.$$

Moreover, if $\dim V_\omega = \ell$ and $\{v_1, \dots, v_\ell\}$ is a basis of V_ω, then $\omega = c(v_1 \wedge \dots \wedge v_\ell)$ for some $c \in F$ with $c \neq 0$.

Corollary 2. *If $\ell = 1$ or $\ell = m - 1$, then the space $\bigwedge^\ell V$ is decomposable, that is, every nonzero element of $\bigwedge^\ell V$ is decomposable.*

The following elementary characterization of decomposability of sums will be useful in the sequel. This is an easy consequence of Exercise 17(c) in [2, p. 651], and we omit the proof.

Lemma 3. *Let $\omega_1, \omega_2 \in \bigwedge^\ell V$ be decomposable and linearly independent, and let $V_i = V_{\omega_i}$ for $i = 1, 2$. Then*

$$\omega_1 + \omega_2 \text{ is decomposable} \quad \Leftrightarrow \quad \dim V_1 \cap V_2 = \ell - 1 \quad \Leftrightarrow \quad \dim V_1 + V_2 = \ell + 1.$$

Corollary 4. *Let $v_1, v_2, v_3, v_4 \in V$ and suppose $\omega := (v_1 \wedge v_2) + (v_3 \wedge v_4) \in \bigwedge^2 V$ is nonzero. Then ω is decomposable if and only if $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.*

Proof. When $v_1 \wedge v_2$ and $v_3 \wedge v_4$ are linearly independent, the result follows from Lemma 3. The case when $v_1 \wedge v_2$ and $v_3 \wedge v_4$ are linearly dependent is easy. $\square$

Given a subspace E of $\bigwedge^\ell V$, let us define

$$V_E := \bigcap_{\omega \in E} V_\omega \quad \text{and} \quad V^E := \sum_{0 \neq \omega \in E} V_\omega.$$

Now, let $r = \dim E$. We say that the subspace E is *close of type I* if there are $\ell + r - 1$ linearly independent elements $f_1, \dots, f_{\ell-1}, g_1, \dots, g_r$ in V such that

$$E = \operatorname{span}\{f_1 \wedge \dots \wedge f_{\ell-1} \wedge g_i\colon\ i = 1, \dots, r\}.$$

And we say that E is *close of type II* if there are $\ell + 1$ linearly independent elements $u_1, \dots, u_{\ell-r+1}, g_1, \dots, g_r$ in V such that

$$E = \operatorname{span}\{u_1 \wedge \dots \wedge u_{\ell-r+1} \wedge g_1 \dots \wedge \check{g}_i \wedge \dots \wedge g_r\colon\ i = 1, \dots, r\},$$

where $\check{g}_i$ indicates that g_i is deleted. We say that E is a *close subspace* of $\bigwedge^\ell V$ if E is close of type I or close of type II.

Evidently, every one-dimensional subspace of $\bigwedge^\ell V$ is close of type I as well as of type II, whereas for two-dimensional subspaces, the notions of close subspaces of type I and type II are identical. A corollary of the following lemma is that in dimensions three or more, the two notions are distinct and mutually disjoint.

Lemma 5. *Let E be a close subspace of $\bigwedge^\ell V$ of dimension r. Then E is decomposable. Moreover, if $\{\omega_1, \dots, \omega_r\}$ is a basis of E, then $V_E = V_{\omega_1} \cap \dots \cap V_{\omega_r}$ and $V^E = V_{\omega_1} + \dots + V_{\omega_r}$. Further, assuming that $r > 1$, we have $\dim V_E = \ell - 1$ and $\dim V^E = \ell + r - 1$ if E is close of type I, whereas $\dim V_E = \ell - r + 1$ and $\dim V^E = \ell + 1$ if E is close of type II.*

Proof. Since E is close of dimension r, there is an r-dimensional subspace G of V [in fact, $G = \mathrm{span}\{g_1, \dots, g_r\}$, in the above notation] such that E is naturally isomorphic to $\bigwedge^1 G$ or to $\bigwedge^{r-1} G$ according as E is close of type I or of type II. Thus, in view of Corollary 2, we see that E is decomposable. Next, suppose $\{\omega_1, \dots, \omega_r\}$ is a basis of E. Then obviously, $V_E = V_{\omega_1} \cap \dots \cap V_{\omega_r}$. Moreover, in view of Lemmas 1 and 3, we see that $V_{\omega+\omega'} \subseteq V_\omega + V_{\omega'}$ for all nonzero $\omega, \omega' \in E$ such that $\omega + \omega' \neq 0$. Hence, by induction on r, we obtain $V^E = V_{\omega_1} + \dots + V_{\omega_r}$. Finally, suppose $r > 1$. In case E is close of type I, and $f_1, \dots, f_{\ell-1}, g_1, \dots, g_r$ are linearly independent elements of V as in the definition above, then in view of Lemma 1, we see that $V_E = \bigcap_{i=1}^r \mathrm{span}\{f_1, \dots, f_{\ell-1}, g_i\} = \mathrm{span}\{f_1, \dots, f_{\ell-1}\}$ and $V^E = \sum_{i=1}^r \mathrm{span}\{f_1, \dots, f_{\ell-1}, g_i\} = \mathrm{span}\{f_1, \dots, f_{\ell-1}, g_1, \dots, g_r\}$. On the other hand, if E is close of type II, and $u_1, \dots, u_{\ell-r+1}, g_1, \dots, g_r$ are linearly independent elements of V as in the definition above, then as before, in view of Lemma 1, we see that $V_E = \mathrm{span}\{u_1, \dots, u_{\ell-r+1}\}$ and $V^E = \mathrm{span}\{u_1, \dots, u_{\ell-r+1}, g_1, \dots, g_r\}$. This proves the desired assertions about $\dim V_E$ and $\dim V^E$. $\square$

In geometric terms, close subspaces of type I (respectively type II) correspond precisely to projective linear subspaces of $\mathbb{P}(\bigwedge^\ell V)$ comprising of (the Plücker coordinates of) those elements of $G_{\ell,m}$ containing a fixed $(\ell-1)$-dimensional subspace of V (respectively contained in a fixed $(\ell+1)$-dimensional subspace of V). With this in view, the following result can be obtained as a consequence of Theorems 24.2.9 and 24.2.11 of [9], and we omit the proof.[1] This result can also be viewed as an algebraic counterpart of the combinatorial structure theorem for the so-called close families of subsets of a finite set (cf. [5, Theorem 4.2]).

Theorem 6 *(Structure theorem for decomposable subspaces). A subspace of $\bigwedge^\ell V$ is decomposable if and only if it is close.*

Corollary 7. *Let $\mu := \max\{\ell, m-\ell\} + 1$ and r be any positive integer. Then $\bigwedge^\ell V$ has a decomposable subspace of dimension r if and only if $r \leqslant \mu$. Moreover, a close subspace of type I (respectively type II) of dimension r exists if and only if $r \leqslant m - \ell + 1$ (respectively $r \leqslant \ell + 1$).*

Proof. Let E be a subspace of $\bigwedge^\ell V$ of dimension r. By Lemma 5, if E is close of type I, then $\ell + r - 1 = \dim V^E \leqslant m$, that is, $r \leqslant m - \ell + 1$, whereas if E is close of type II, then $\ell - r + 1 = \dim V_E \geqslant 0$, that is, $r \leqslant \ell + 1$. Thus, Theorem 6 implies that if $\bigwedge^\ell V$ has a decomposable subspace of dimension r, then $r \leqslant \mu$. The converse is an immediate consequence of the definition of close subspaces and their decomposability. $\square$

Remark 8. Decomposable vectors in $\bigwedge^\ell V$ are variously known as *pure ℓ-vectors* (cf. [2, §11.13]), *extensors of step ℓ* (cf. [1, §3]), or *completely decomposable vectors* (cf. [14]). Characterizations of decomposable subspaces have been studied in the setting of symmetric algebras. Although one comes across subspaces of various types, including those similar to the ones considered in this section, the situation for subspaces of symmetric powers is rather different and the characteristic of the underlying field plays a role. We refer to the papers of Cummings [3] and Lim [11] for more on this topic. In the context of tensor algebras, the opposite of decomposable subspaces has been considered, namely, *completely entangled* subspaces wherein no nonzero element is decomposable. A neat formula for the maximum possible dimension of completely entangled subspaces of the tensor product of finite dimensional complex vector spaces is given by Parthasarathy [15]. As remarked earlier, determining the structure of decomposable subspaces corresponds to determining the linear subvarieties in the Grassmann variety $G_{\ell,m}$. A special case of this has been considered, in a similar, but more general, geometric setting by Tanao [16], where subvarieties of $G_{2,m}$ biregular to $\mathbb{P}^{m-1}$ over an algebraically closed field of characteristic zero are studied.

[1] An alternative and completely self-contained proof is given in an earlier and slightly longer version of this paper available on the arXiv [math.AG/0710.5161v1]. Detailed proofs of Lemmas 1 and 3 as well as Corollary 2 are also given there.

3. Duality and the Hodge star operator

We have seen in Section 2 that a decomposable subspace of $\bigwedge^{\ell} V$ is close of type I or of type II. It turns out that the two types are dual to each other. This is best described using the so-called Hodge star operator $\mathfrak{h} : \bigwedge^{\ell} V \to \bigwedge^{m-\ell} V$, which may be defined as follows. Fix an ordered basis $\{e_1, \dots, e_m\}$ of V and use it to identify $\bigwedge^{m} V$ with F so that $e_1 \wedge \cdots \wedge e_m = 1$. Let $I(\ell, m)$ and e_α for $\alpha \in I(\ell, m)$ be as in Section 2. Moreover, for $\alpha = (\alpha_1, \dots, \alpha_\ell) \in I(\ell, m)$, let $\alpha^c = (\alpha^c_1, \dots, \alpha^c_{m-\ell})$ denote the unique element of $I(m-\ell, m)$ such that $\{\alpha_1, \dots, \alpha_\ell\} \cup \{\alpha^c_1, \dots, \alpha^c_{m-\ell}\} = \{1, \dots, m\}$. Then $\mathfrak{h} : \bigwedge^{\ell} V \to \bigwedge^{m-\ell} V$ is the unique F-linear map satisfying

$$\mathfrak{h}(e_\alpha) = (-1)^{\alpha_1 + \cdots + \alpha_\ell + \ell(\ell+1)/2} e_{\alpha^c} \quad \text{for } \alpha \in I(\ell, m).$$

Clearly, $\mathfrak{h}$ is a vector space isomorphism. The key property of $\mathfrak{h}$ is that it is essentially independent of the choice of ordered basis of V, and as such, it maps decomposable elements in $\bigwedge^{\ell} V$ to decomposable elements in $\bigwedge^{m-\ell} V$. (See, for example, [1, Section 6] and [12, Section 4.1].) In particular, decomposable subspaces of $\bigwedge^{\ell} V$ are mapped to decomposable subspaces of $\bigwedge^{m-\ell} V$. Moreover, it is easy to see that via the Hodge star operator, close subspaces of type I are mapped to close subspaces of type II, whereas close subspaces of type II are mapped to close subspaces of type I. Thus, the two types are dual to each other.

In the case $\ell = 2$, both $\bigwedge^{\ell} V$ and $\bigwedge^{m-\ell} V$ are closely related to the space B_m of all $m \times m$ skew-symmetric matrices with entries in F, and the relation is compatible with the Hodge star operator. To state this a little more formally, we introduce some terminology below and make a few useful observations. In the remainder of this section we tacitly assume that $m > 2$.

Given any $u \in V$, let $\mathbf{u}$ denote the $m \times 1$ column vector whose entries are the coordinates of u with respect to the ordered basis $\{e_1, \dots, e_m\}$. In particular, $\mathbf{e}_i$ has 1 as its ith entry and all other entries are 0. Consider the F-linear maps

$$\sigma : \bigwedge^2 V \to B_m \quad \text{and} \quad \pi : \bigwedge^{m-2} V \to B_m$$

defined by

$$\sigma(e_r \wedge e_s) = \mathbf{e}_r \mathbf{e}_s^t - \mathbf{e}_s \mathbf{e}_r^t \quad \text{for } 1 \leqslant r < s \leqslant m \quad \text{and} \quad \pi(\omega) = A_\omega \quad \text{for } \omega \in \bigwedge^{m-2} V,$$

where $\mathbf{e}^t$ denotes the transpose of $\mathbf{e}$ and A_ω denotes the $m \times m$ matrix whose (i, j)th entry is (the unique scalar corresponding to) $e_i \wedge e_j \wedge \omega$.

Lemma 9. $\sigma = \pi \circ \mathfrak{h}$.

Proof. We have $\mathfrak{h}(e_r \wedge e_s) = (-1)^{r+s+1}(e_1 \wedge \cdots \wedge \check{e}_r \wedge \cdots \wedge \check{e}_s \wedge \cdots \wedge e_m)$ for $1 \leqslant r < s \leqslant m$, where $\check{}$ indicates that the corresponding entry is removed. Now,

$$e_i \wedge e_j \wedge (e_1 \wedge \cdots \wedge \check{e}_r \wedge \cdots \wedge \check{e}_s \wedge \cdots \wedge e_m) = \begin{cases} (-1)^{i+j-3} & \text{if } (r,s) = (i,j), \\ (-1)^{i+j-2} & \text{if } (r,s) = (j,i), \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leqslant i, j, r, s \leqslant m$ with $r < s$. It follows that $\pi \circ \mathfrak{h}(e_r \wedge e_s) = \mathbf{e}_r \mathbf{e}_s^t - \mathbf{e}_s \mathbf{e}_r^t = \sigma(e_r \wedge e_s)$ for $1 \leqslant r < s \leqslant m$. Since $\{e_r \wedge e_s\colon 1 \leqslant r < s \leqslant m\}$ is a basis of $\bigwedge^2 V$ and all the maps are linear, the lemma is proved. □

Given any $\omega' \in \bigwedge^2 V$ and $\omega \in \bigwedge^{m-2} V$, we refer to the rank of $\sigma(\omega')$ [respectively $\pi(\omega)$] as the *rank* of ω' [respectively ω], and denote it by $\operatorname{rank}(\omega')$ [respectively $\operatorname{rank}(\omega)$]. Note that if $\omega = \mathfrak{h}(\omega')$, then $\operatorname{rank}(\omega') = \operatorname{rank}(\omega)$, thanks to Lemma 9.

Corollary 10. *Both σ and π are vector space isomorphisms. Moreover,*

$$\omega' \text{ is decomposable} \quad \Leftrightarrow \quad \operatorname{rank}(\omega') = 2 \quad \text{for any } \omega' \in \bigwedge^2 V, \tag{7}$$

and

$$\omega \text{ is decomposable} \quad \Leftrightarrow \quad \operatorname{rank}(\omega) = 2 \quad \text{for any } \omega \in \bigwedge^{m-2} V. \tag{8}$$

Proof. It is evident that σ is an isomorphism. Hence by Lemma 9, so is π. Now, given any $\omega \in \bigwedge^{m-2} V$, the kernel of (the linear map from V to V corresponding to) $\pi(\omega) = A_\omega$ is the space V_ω. Hence (8) follows from Lemma 1. Next, if $\omega' \in \bigwedge^2 V$ is decomposable, then $\omega' = u \wedge v$ for some $u, v \in V$ and $\sigma(\omega') = \mathbf{u}\mathbf{v}^t - \mathbf{v}\mathbf{u}^t$. It follows that $\sigma(\omega')$ is of rank 2. This proves the implication $\Rightarrow$ in (7). The other implication follows from (8) together with Lemma 9 and the fact that $\mathfrak{h}$ gives a one-to-one correspondence between decomposable elements. □

Corollary 11. *Let $v_1, v_2, v_3, v_4 \in V$ and suppose $\omega := (v_1 \wedge v_2) + (v_3 \wedge v_4) \in \bigwedge^2 V$ is nonzero. Then the rank of $\sigma(\omega)$ is 2 or 4 according as the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent or linearly independent.*

Proof. Follows from (7) above and Corollary 4 in view of the fact that a skew-symmetric matrix is always of even rank. □

4. Griesmer–Wei bound and its generalization

Let us begin by reviewing some generalities about (linear, error correcting) codes. Fix integers k, n with $1 \leqslant k \leqslant n$ and a prime power q. Let C be a linear $[n,k]_q$-code, that is, let C be a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements. Given any $x = (x_1, \dots, x_n)$ in $\mathbb{F}_q^n$, let

$$\operatorname{supp}(x) := \{i\colon\ x_i \neq 0\} \quad \text{and} \quad \|x\| := \left|\operatorname{supp}(x)\right|$$

denote the *support* and the (*Hamming*) *norm* of x. More generally, for $D \subseteq \mathbb{F}_q^n$, let

$$\operatorname{supp}(D) := \left\{i\colon\ x_i \neq 0 \text{ for some } x = (x_1, \dots, x_n) \in D\right\} \quad \text{and} \quad \|D\| := \left|\operatorname{supp}(D)\right|$$

denote the *support* and the (*Hamming*) *norm* of D. The *minimum distance* of C is defined by $d(C) := \min\{\|x\|\colon\ x \in C \text{ with } x \neq 0\}$. More generally, for any positive integer r, the rth *higher weight* $d_r = d_r(C)$ of the code C is defined by

$$d_r(C) := \min\left\{\|D\|\colon\ D \text{ is a subspace of } C \text{ with } \dim D = r\right\}.$$

Note that $d_1(C) = d(C)$. If C is *nondegenerate*, that is, if C is not contained in a coordinate hyperplane of $\mathbb{F}_q^n$, then it is easy to see that

$$0 < d_1(C) < d_2(C) < \cdots < d_k(C) = n.$$

See, for example, [17] for a proof as well as a great deal of basic information about higher weights of codes. The set $\{d_r(C)\colon\ 1 \leqslant r \leqslant k\}$ is often referred to as the *weight hierarchy* of the code C. It is usually interesting, and difficult, to determine the weight hierarchy of a given code. Again, we refer to [17] for a variety of examples, such as affine and projective Reed–Muller codes, codes associated to Hermitian varieties or Del Pezzo surfaces, hyperelliptic curves, etc., where the weight hierarchy is completely or partially known.

The following elementary result will be useful in the sequel. It appears, for example, in [10, Lemma 2]. We include a proof for the sake of completeness.

Lemma 12. *Let D be an r-dimensional subcode of an $[n,k]_q$-code C. Then*

$$\|D\| = \frac{1}{q^r - q^{r-1}} \sum_{x \in D} \|x\|.$$

In particular,

$$d_r(C) = \frac{1}{q^r - q^{r-1}} \min\left\{ \sum_{x \in D} \|x\|:\ D \text{ is a subspace of } C \text{ with } \dim D = r \right\}.$$

Proof. Clearly, $(x,i) \mapsto (i,x)$ gives a bijection of $\{(x,i):\ x \in D \text{ and } i \in \operatorname{supp}(x)\}$ onto $\{(i,x):\ i \in \operatorname{supp}(D),\ x \in D \text{ and } x_i \neq 0\}$. Hence

$$\sum_{x \in D} \|x\| = \sum_{x \in D} \sum_{i \in \operatorname{supp}(x)} 1 = \sum_{i \in \operatorname{supp}(D)} \sum_{\substack{x \in D \\ x_i \neq 0}} 1 = \sum_{i \in \operatorname{supp}(D)} \left(q^r - q^{r-1}\right) = \left(q^r - q^{r-1}\right)\|D\|,$$

where the penultimate equality follows by noting that if $i \in \operatorname{supp}(D)$, then $x \mapsto x_i$ defines a nonzero linear map of $D \to \mathbb{F}_q$. $\square$

Let C be a linear $[n,k]_q$-code. Given any subspace D of C, we let

$$\Delta(D) := \left|\left\{x \in D:\ \|x\| = d(C)\right\}\right|.$$

Given any $r \in \mathbb{Z}$ with $1 \leqslant r \leqslant k$, we let

$$\Delta_r(C) := \max\left\{\Delta(D):\ D \text{ is a subspace of } C \text{ with } \dim D = r\right\}.$$

Further, upon letting $S_C := \{\|x\|:\ x \in C \text{ with } \|x\| > d(C)\}$, we define

$$e(C) := \begin{cases} \min S_C & \text{if } S_C \text{ is nonempty}, \\ d(C) & \text{if } S_C \text{ is the empty set}. \end{cases}$$

We now prove a simple, but useful generalization of the Griesmer–Wei bound [18].

Theorem 13. *Let C be a linear $[n,k]_q$-code and r be an integer with $1 \leqslant r \leqslant k$. Then*

$$d_r(C) \geqslant \frac{d(C)\Delta_r(C) + e(C)(q^r - 1 - \Delta_r(C))}{q^r - q^{r-1}}.$$

Proof. Let D_r be an r-dimensional subspace of C such that

$$\sum_{x \in D_r} \|x\| = \min\left\{ \sum_{x \in D} \|x\|:\ D \text{ is a subspace of } C \text{ with } \dim D = r \right\}.$$

Then D_r has $q^r - 1$ nonzero elements and so, in view of Lemma 12, we have

$$\begin{aligned}(q^r - q^{r-1})d_r(C) &= \sum_{\substack{x\in D_r\\ \|x\|=d(C)}} d(C) + \sum_{\substack{x\in D_r\\ \|x\|>d(C)}} \|x\| \\ &\geqslant d(C)\Delta(D_r) + e(C)\big(q^r - 1 - \Delta(D_r)\big) \\ &\geqslant e(C)\big(q^r - 1\big) - \Delta_r(C)\big(e(C) - d(C)\big),\end{aligned}$$

where the last inequality follows since $\Delta(D_r) \leqslant \Delta_r(C)$ and $d(C) \leqslant e(C)$. This yields the desired formula. □

Corollary 14 *(Griesmer–Wei bound). Given any linear $[n,k]_q$-code C, we have*

$$d_r(C) \geqslant \sum_{j=0}^{r-1} \frac{d(C)}{q^j} \quad \textit{for } 1 \leqslant r \leqslant k.$$

Proof. Using Theorem 13 and the fact that $e(C) \geqslant d(C)$, we see that

$$d_r(C) \geqslant \frac{d(C)(q^r-1)}{q^r - q^{r-1}} = \sum_{i=0}^{r-1} \frac{d(C)q^i}{q^{r-1}} = \sum_{j=0}^{r-1} \frac{d(C)}{q^j}$$

for any integer r with $1 \leqslant r \leqslant k$. □

Remark 15. The Griesmer–Wei bound [17, Corollary 3.3] is, in fact, a stronger inequality than the one in Corollary 14 wherein the fraction $d(C)/q^j$ is replaced by its ceiling $\lceil d(C)/q^j \rceil$. However, in the situations where we have used it, notably when C is the Grassmann code $C(\ell,m)$ and $r \leqslant \ell(m-\ell)+1$, the fraction $d(C)/q^j$ is always an integer, and the two versions are equivalent.

5. The Grassmann code $C(\ell, m)$

Let us fix, throughout this section, a prime power q and integers ℓ, m with $1 \leqslant \ell \leqslant m$, and let

$$n := \begin{bmatrix} m \\ \ell \end{bmatrix}_q, \qquad k := \binom{m}{\ell}, \quad \text{and} \quad \delta := \ell(m-\ell),$$

where $\begin{bmatrix} m \\ \ell \end{bmatrix}_q$ is the Gaussian binomial coefficient, which was defined in Section 1. It may be remarked that $\begin{bmatrix} m \\ \ell \end{bmatrix}_q$ is a polynomial in q of degree δ with positive integral coefficients. The Grassmann code $C(\ell,m)$ is the linear $[n,k]_q$-code associated to the projective system corresponding to the Plücker embedding of the $\mathbb{F}_q$-rational points of the Grassmannian $G_{\ell,m}$ in $\mathbb{P}^{k-1}_{\mathbb{F}_q} = \mathbb{P}(\bigwedge^\ell \mathbb{F}_q^m)$; see, for example, [4,14] for greater details. Alternatively, $C(\ell,m)$ may be defined as follows.

Let $V := \mathbb{F}_q^m$. Fix a basis $\{e_1, \dots, e_m\}$ of V. Then we can, and will, fix a corresponding basis of $\bigwedge^\ell V$ given, in the notations of Section 5, by $\{e_\alpha\colon \alpha \in I(\ell,m)\}$. Let $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ be the Grassmann variety consisting of all ℓ-dimensional subspaces of V. The Plücker embedding $G_{\ell,m} \hookrightarrow \mathbb{P}(\bigwedge^\ell V)$ simply maps an ℓ-dimensional subspace of V spanned by $v_1, \dots, v_\ell$ to the point of $\mathbb{P}(\bigwedge^\ell V)$ corresponding to $v_1 \wedge \cdots \wedge v_\ell$. It is well known that this embedding is well defined and nondegenerate. Fix representatives $\omega'_1, \dots, \omega'_n$ in $\bigwedge^\ell V$ corresponding to distinct points of $G_{\ell,m}(\mathbb{F}_q)$. We denote the subset $\{\omega'_1, \dots, \omega'_n\}$ of $\bigwedge^\ell V$ by $T(\ell,m)$. Having fixed a basis of V, we can identify each element of $\bigwedge^m V$ with a unique scalar in $\mathbb{F}_q$. With this in view, we obtain a linear map

$$\tau : \bigwedge^{m-\ell} V \to \mathbb{F}_q^n \quad \text{given by } \tau(\omega) := \big(\omega'_1 \wedge \omega, \omega'_2 \wedge \omega, \dots, \omega'_n \wedge \omega\big).$$

Since the Plücker embedding is nondegenerate, it follows that τ is injective. The Grassmann code $C(\ell, m)$ is defined as the image of the map τ. It is clear that $C(\ell, m)$ is a linear $[n, k]_q$-code. Given any codeword $c \in C(\ell, m)$, there is unique $\omega \in \bigwedge^{m-\ell} V$ such that $\tau(\omega) = c$; we denote this ω by ω_c.

Given any subspace $\mathcal{E}$ of $\bigwedge^\ell V$, we let $g(\mathcal{E}) := |\mathcal{E} \cap T(\ell, m)|$. Note that since $T(\ell, m)$ consists of nonzero elements, no two of which are proportional to each other, we always have

$$\left|g(\mathcal{E})\right| \leqslant \frac{q^r - 1}{q - 1} \quad \text{for any subspace } \mathcal{E} \text{ of } \textstyle\bigwedge^\ell V \text{ with } \dim \mathcal{E} = r. \tag{9}$$

Given any integer s with $1 \leqslant s \leqslant k$, we let

$$g_s(\ell, m) := \max\left\{g(\mathcal{E})\colon\ \mathcal{E} \text{ a subspace of } \textstyle\bigwedge^\ell V \text{ of codimension } s\right\}.$$

Note that as a consequence of (9), we have

$$g_s(\ell, m) \leqslant \frac{q^r - 1}{q - 1} \quad \text{where } r := k - s. \tag{10}$$

Lemma 16. *Let D be a subspace of $C(\ell, m)$ and $s = \dim D$. If $\mathcal{D} := \tau^{-1}(D)$, then $\mathcal{E} := \mathcal{D}^\perp := \{\omega' \in \bigwedge^\ell V\colon\ \omega' \wedge \omega = 0\}$ is a subspace of $\bigwedge^\ell V$ of codimension s and*

$$\|D\| = n - g(\mathcal{E}).$$

Proof. Since τ is an isomorphism of $\bigwedge^{m-\ell} V$ and $C(\ell, m)$, we have $\dim \mathcal{D} = s$. Also $(\omega', \omega) \mapsto \omega' \wedge \omega$ gives a nondegenerate bilinear map of $\bigwedge^\ell V \times \bigwedge^{m-\ell} V \to \mathbb{F}_q$, and so $\mathcal{E} := \mathcal{D}^\perp$ is a subspace of $\bigwedge^\ell V$ of codimension s. For $1 \leqslant i \leqslant n$, we have

$$i \notin \operatorname{supp}(D) \quad \Leftrightarrow \quad \omega_i' \wedge \omega = 0 \quad \text{for all } \omega \in \mathcal{D} \quad \Leftrightarrow \quad \omega_i' \in \mathcal{E}.$$

It follows that $\|D\| = n - g(\mathcal{E})$. $\square$

Corollary 17. *$d_s(C(\ell, m)) = n - g_s(\ell, m)$ for $s = 1, \dots, k$.*

Proof. Clearly, $\mathcal{E} \mapsto \tau(\mathcal{E}^\perp)$ sets up a one-to-one correspondence between subspaces of $\bigwedge^\ell V$ of codimension s and subspaces of $C(\ell, m)$ of dimension s. Hence the desired result follows from Lemma 16. $\square$

We now recall some important results of Nogin [14]. Combining Theorem 4.1, Proposition 4.4 and Corollary 4.5 of [14], we have the following.

Proposition 18. *The minimum distance of $C(\ell, m)$ is q^δ and the codewords c of $C(\ell, m)$ such that ω_c is decomposable attain the minimum weight q^δ. Moreover, the number of minimum weight codewords in $C(\ell, m)$ is $(q - 1)n$.*

A useful consequence is the following.

Corollary 19. *Given any $c \in C(\ell, m)$, we have*

$$\|c\| = q^\delta \quad \Leftrightarrow \quad \omega_c \text{ is decomposable.}$$

Moreover $\Delta(C(\ell, m)) = (q - 1)n$.

Proof. The implication $\Leftarrow$ follows from Proposition 18. The other implication also follows from Proposition 18 by noting that the number of decomposable elements of $\bigwedge^{m-\ell} V$ is equal to the number of decomposable elements of $\bigwedge^{\ell} V$, and that the latter is equal to $(q-1)n$. $\square$

In [14], Nogin goes on to determine some of the higher weights of $C(\ell,m)$ using Proposition 18 and some additional work. More precisely, he proves formula (2) in Section 1. As remarked in Section 1, alternative proofs of (2) are given in [4] as well as [7]. The latter also proves the dual version (3). We give below yet another proof of (2) and (3) as an application of Theorem 6 and Corollary 19.

Theorem 20. *Let $\mu := \max\{\ell, m-\ell\}+1$. Then for $0 \leqslant r \leqslant \mu$ we have*

$$d_r\big(C(\ell,m)\big) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{and} \quad d_{k-r}\big(C(\ell,m)\big) = n - \big(1+q+\cdots+q^{r-1}\big).$$

Proof. The case $r=0$ is trivial. Assume that $1 \leqslant r \leqslant \mu$. By Corollary 7, there is a decomposable subspace E of $\bigwedge^{\ell} V$ of dimension r. Then $\mathfrak{h}(E)$ is a decomposable subspace of $\bigwedge^{m-\ell} V$ and hence by Corollary 19, $D := \tau(\mathfrak{h}(E))$ is an r-dimensional subspace of $C(\ell,m)$ in which every nonzero vector is of minimal weight. Consequently, by Lemma 12, we have

$$\|D\| = \frac{1}{q^r - q^{r-1}} \sum_{c \in D} \|c\| = \frac{d(C(\ell,m))(q^r-1)}{q^r - q^{r-1}} = \sum_{j=0}^{r-1} \frac{d(C(\ell,m))}{q^j} = \sum_{j=0}^{r-1} q^{\delta-j}.$$

In other words, the Griesmer–Wei bound is attained. This proves the desired formula for $d_r(C(\ell,m))$. Next, E is a subspace of $\bigwedge^{\ell} V$ of codimension $k-r$, and since E is decomposable, every $\omega' \in E$ with $\omega' \neq 0$ can be uniquely written as $\omega' = \lambda \omega'_i$ where $\lambda \in \mathbb{F}_q \setminus \{0\}$ and $i \in \{1,\ldots,n\}$. It follows that $g(E) = (q^r-1)/(q-1) = 1+q+\cdots+q^{r-1}$, and so, in view of (10), we find $g_{k-r}(\ell,m) = 1+q+\cdots+q^{r-1}$. This, together with Corollary 17, yields the desired formula for $d_{k-r}(C(\ell,m))$. $\square$

6. Higher weights of the Grassmann code $C(2,m)$

The results on the higher weights of $C(2,m)$ mentioned in the Introduction will be proved in this section. Throughout, let q, ℓ, m, k, n, δ be as in Section 5, except we set $\ell = 2$. Also, we let $F := \mathbb{F}_q$ and $V := \mathbb{F}_q^m$. Note that the complete weight hierarchy of $C(2,m)$ is easily obtained from Theorem 20 if $m \leqslant 4$. With this in view, we shall assume that $m > 4$. In particular, $\mu := \max\{\ell, m-\ell\}+1 = m-1$.

We begin by recalling a result of Nogin concerning the spectrum of $C(2,m)$. To this end, given any nonnegative integer t, let $N(m,2t)$ denote the number of skew-symmetric bilinear forms of rank $2t$ on $\mathbb{F}_q^m$. We know from [13, §15.2] that

$$N(m,2t) = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-2t+1}-1)}{(q^{2t}-1)(q^{2t-2}-1)\cdots(q^2-1)} q^{t(t-1)}. \tag{11}$$

The said result of Nogin [14, Theorem 5.1] is the following.

Proposition 21. *Given any $i \geqslant 0$, let $A_i := |\{c \in C(2,m)\colon \|c\| = i\}|$. Then*

$$A_i = \begin{cases} N(m,2t) & \text{if } i = q^{2(m-t-1)} \frac{q^{2t}-1}{q^2-1} \text{ for } 0 \leqslant t \leqslant \lfloor m/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Moreover, for any $c \in C(2,m)$ and $0 \leqslant t \leqslant \lfloor m/2 \rfloor$, we have

$$\|c\| = q^{2(m-t-1)} \frac{q^{2t}-1}{q^2-1} \quad \Leftrightarrow \quad \operatorname{rank}(\omega_c) = 2t.$$

Corollary 22. *$d(C(2,m)) = q^\delta$ and $e(C(2,m)) = q^\delta + q^{\delta-2}$.*

Proof. The numbers $\theta_t := q^{2(m-t-1)} \frac{q^{2t}-1}{q^2-1}$ increase with t and the first two positive values of θ_t ($t \geqslant 0$) are q^δ and $q^\delta + q^{\delta-2}$. $\square$

We now prove a number of auxiliary results needed to prove the main theorem.

Lemma 23. *Let E and E_1 be subspaces of $\bigwedge^2 V$ such that $E \subset E_1$ and $\dim E_1 = \dim E + 1$. Assume that E is decomposable and E_1 is not decomposable. Then we have the following.*

(i) *The set $E_1 \setminus E$ contains at most $q^2(q-1)$ decomposable vectors.*
(ii) *If $E_1 \setminus E$ contains a decomposable vector ω such that $V_\omega \subseteq V^E$, then $E_1 \setminus E$ contains exactly $q^2(q-1)$ decomposable vectors.*

Proof. Both (i) and (ii) hold trivially if $E_1 \setminus E$ contains no decomposable vector. Now, suppose $E_1 \setminus E$ contains a decomposable vector, say ω. Then $E_1 = E + F\omega$. Write $\omega = u \wedge v$, where $u, v \in V$, and let $r := \dim E$. By Theorem 6, we are in either of the following two cases.

Case 1: E is close of type I.

In this case, there are linearly independent elements $f, g_1, \ldots, g_r \in V$ such that $E = \operatorname{span}\{f \wedge g_i: i = 1, \ldots, r\}$. Let $G := \operatorname{span}\{g_1, \ldots, g_r\}$. Elements ξ of E_1 are of the form $\xi = f \wedge g + \lambda(u \wedge v)$, where $g \in G$ and $\lambda \in \mathbb{F}_q$. Clearly, ξ and (g, λ) determine each other uniquely, and $\xi \in E_1 \setminus E$ if and only if $\lambda \neq 0$. Observe that $\{f, u, v\}$ is linearly independent, lest we can write $u \wedge v = f \wedge h$ for some $h \in V$, and consequently, E_1 becomes decomposable. Hence, by Corollary 4, we see that if $\lambda \neq 0$, then $\xi = f \wedge g + \lambda(u \wedge v)$ is decomposable if and only if $g \in \operatorname{span}\{f, u, v\}$. Further, in view of Lemmas 1 and 5, we have $V_\omega = \operatorname{span}\{u, v\}$ and $V^E = \operatorname{span}\{f, g_1, \ldots, g_r\}$. Thus, $g \in \operatorname{span}\{f, u, v\}$ if and only if $f \wedge g = f \wedge x$ for some $x \in V_\omega \cap V^E$. It follows that decomposable elements of $E_1 \setminus E$ are precisely of the form $f \wedge x + \lambda(u \wedge v)$, where $x \in V_\omega \cap V^E$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$. Since $|V_\omega \cap V^E| \leqslant |V_\omega| = q^2$ and $|\mathbb{F}_q \setminus \{0\}| = q - 1$, both (i) and (ii) are proved.

Case 2: E is close of type II, but not close of type I.

In this case, by Corollary 7, we must have $\dim E = 3$. Thus, there are linearly independent elements $g_1, g_2, g_3 \in V$ such that $E = \operatorname{span}\{g_2 \wedge g_3, g_1 \wedge g_3, g_1 \wedge g_2\}$. Let $G := \operatorname{span}\{g_1, g_2, g_3\}$. Note that since $G = V^E$ and $\omega = u \wedge v \notin E$, the possibility that $V_\omega \subseteq V^E$ does not arise in this case. Thus $\dim V_\omega \cap V^E \leqslant 1$ and (ii) holds vacuously. The elements of E_1 are of the form $\xi = g \wedge h + \lambda(u \wedge v)$, where $g, h \in G$ and $\lambda \in \mathbb{F}_q$. Clearly, ξ is a decomposable element of $E_1 \setminus E$ if $g \wedge h = 0$ and $\lambda \neq 0$. If, in addition, $\xi = g \wedge h + \lambda(u \wedge v)$ is decomposable for some $g, h \in G$ with $g \wedge h \neq 0$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$, then by Corollary 4, $\{g, h, u, v\}$ is linearly dependent, and hence $\dim V_\omega \cap V^E = 1$. So we may assume without loss of generality that $u = g_1$. Then it is clear that the elements of $E_1 \setminus E$ are precisely the (unique) linear combinations of the form $\lambda(u \wedge v) + \lambda_1(g_2 \wedge g_3) + \lambda_2(g_1 \wedge g_3) + \lambda_3(g_1 \wedge g_2)$, where $\lambda \in \mathbb{F}_q \setminus \{0\}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}_q$; moreover, by Corollary 4, such a linear combination is decomposable if and only if $\lambda_1 = 0$. It follows that $E_1 \setminus E$ contains at most $q^2(q-1)$ decomposable elements. $\square$

The bound $q^2(q-1)$ in Lemma 23 can be improved if the dimension of the decomposable subspace E is small.

Lemma 24. *Let E and E_1 be subspaces of $\bigwedge^2 V$ such that $E \subset E_1$ and $\dim E_1 = \dim E + 1$. Assume that E is decomposable of dimension $r \geqslant 1$ and E_1 is not decomposable. Then $E_1 \setminus E$ contains at most $q^{r-1}(q-1)$ decomposable elements.*

Proof. If $r \geqslant 3$, then the result is an immediate consequence of part (i) of Lemma 23. Also, the result holds trivially if $E_1 \setminus E$ contains no decomposable element. Thus, let us assume that $r \leqslant 2$ and $E_1 = E + F\omega$, where $\omega \in E_1 \setminus E$ is decomposable.

First, suppose $r=1$. Then $E=F\omega_0$ for some decomposable $\omega_0 \in \bigwedge^2 V$. Since E_1 is not decomposable and $\omega \notin E$, in view of Lemmas 1 and 3, we see that $\dim V_\omega \cap V_{\omega_0}=0$. Hence from Corollary 4, it follows that the only decomposable elements in $E_1 \setminus E$ are those of the form $\lambda\omega$ where $\lambda \in \mathbb{F}_q \setminus \{0\}$. Thus $E_1 \setminus E$ contains at most $(q-1)$ decomposable elements, as desired.

Next, suppose $r=2$. Then in view of Theorem 6, there are linearly independent elements $f, g_1, g_2 \in V$ such that $E=\operatorname{span}\{f \wedge g_1, f \wedge g_2\}$. As in the proof of Lemma 23, we can write $\omega = u \wedge v$, where $u, v \in V$ are such that $\{f, u, v\}$ is linearly independent. Further, if $\dim V_\omega \cap V^E = 2$, then $V_\omega \subseteq V^E$ and we may assume without loss of generality that $g_1, g_2 \in V_\omega$; hence $E_1 = \operatorname{span}\{f \wedge g_1, f \wedge g_2, g_1 \wedge g_2\}$, and so E_1 is close of type II, which is a contradiction. Thus $\dim V_\omega \cap V^E < 2$ and so $|V_\omega \cap V^E| \leqslant q$. Moreover, as in the proof of Lemma 23, decomposable elements of $E_1 \setminus E$ are precisely of the form $f \wedge x + \lambda(u \wedge v)$, where $x \in V_\omega \cap V^E$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$. Thus $E_1 \setminus E$ contains at most $q(q-1)$ decomposable elements, as desired. $\square$

Lemma 25. *There exists a $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$ containing exactly $(q^\mu - 1) + q^2(q-1)$ decomposable vectors. Moreover, the remaining $(q^\mu - q^2)(q-1)$ nonzero elements in this subspace are of rank 4.*

Proof. By Corollary 7, there exists a μ-dimensional decomposable subspace of $\bigwedge^2 V$, say E. Since $m > 4$, we have $\mu > 3$, and so by Theorem 6 and Corollary 7, E is close of type I. Thus there exist $\mu+1$ linearly independent elements $f, g_1, \ldots, g_\mu \in V$ such that $E = \operatorname{span}\{f \wedge g_i\colon\ i = 1, \ldots, \mu\}$. Now, consider $\omega := g_1 \wedge g_2$ and $E_1 := E + F\omega$. It is clear that $\omega \notin E$ and E_1 is not decomposable. Moreover, by Theorem 6 and Lemma 5, $\dim V^E = \mu + 1 = m$, and thus $V^E = V \supseteq V_\omega$. So it follows from part (ii) of Lemma 23 that $E_1 \setminus E$ contains exactly $q^2(q-1)$ decomposable elements. Since every nonzero element of E is decomposable, we see that E_1 is a $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$ containing exactly $(q^\mu - 1) + q^2(q-1)$ decomposable vectors. Since every element of E_1 is of the form $a(f \wedge g) + b(g_1 \wedge g_2)$ for some $a, b \in F$ and $g \in \operatorname{span}\{g_1, \ldots, g_\mu\}$, it follows from Corollaries 10 and 11 that the remaining $(q^{\mu+1} - 1) - (q^\mu - 1) - q^2(q-1)$ elements are of rank 4. $\square$

Lemma 26. *Every $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$ contains at most $(q^\mu - 1) + q^2(q-1)$ decomposable vectors.*

Proof. Let E^* be any $(\mu+1)$-dimensional subspace of $\bigwedge^2 V$. Let r be the maximum among the dimensions of all decomposable subspaces of E^*. If $r=0$, then E^* contains no decomposable element and the assertion holds trivially. Assume that $r \geqslant 1$. Let E_r be a decomposable r-dimensional subspace of E^*. Extending a basis of E_r to E^*, we obtain a subspace E' of E^* such that $E_r \cap E' = \{0\}$ and $E^* = E_r + E'$. Clearly,

$$E^* = \bigcup_{\omega \in E'} E_r + F\omega \quad \text{and} \quad E^* \setminus E_r = \bigcup_{0 \neq \omega \in E'} (E_r + F\omega) \setminus E_r. \tag{13}$$

Given any nonzero $\omega \in E'$, the space $E_r + F\omega$ is not decomposable, thanks to the maximality of r, and so by part (i) of Lemma 23, $(E_r + F\omega) \setminus E_r$ contains at most $q^2(q-1)$ decomposable elements. Moreover, for any nonzero $\omega, \omega' \in E'$, we have $E_r + F\omega = E_r + F\omega'$ if ω and ω' differ by a nonzero constant, whereas $(E_r + F\omega) \cap (E_r + F\omega') = E_r$ if ω and ω' do not differ by a nonzero constant. Thus the second decomposition in (13) is disjoint if we let ω vary over nonzero elements of E' that are not proportional to each other. It follows that $E^* \setminus E_r$ contains at most $\frac{q^2(q-1)|E' \setminus \{0\}|}{(q-1)} = q^2(q^{\mu+1-r} - 1)$ decomposable elements. In case $r \leqslant 2$, then using Lemma 24 instead of part (i) of Lemma 23, it follows that $E^* \setminus E_r$ contains at most $q^{r-1}(q^{\mu+1-r} - 1)$ decomposable elements. Thus, if we let $s := \min\{2, r-1\}$ and $N_r := (q^r - 1) + q^s(q^{\mu+1-r} - 1)$, then we see that E^* contains at most N_r decomposable elements. To complete the proof it suffices to observe that

$$\left(q^\mu - 1\right) + q^2(q-1) - N_r = \begin{cases} (q^r - q^3)(q^{\mu-r} - 1) & \text{if } r \geqslant 3, \\ (q^2 - q^{r-1})(q-1) & \text{if } 1 \leqslant r \leqslant 2 \end{cases}$$

is always nonnegative. $\square$

Corollary 27. *We have* $\Delta_{\mu+1}(C(2,m)) = (q^\mu - 1) + q^2(q-1)$ *and* $g_{k-\mu-1}(2,m) = 1+q+q^2+\cdots+q^\mu+q^2$.

Proof. The assertion about $\Delta_{\mu+1}(C(2,m))$ follows from Lemmas 25, 26, Proposition 18, and Corollary 19. Further, by Lemma 26, we see that if $\mathcal{E}$ is any subspace of $\bigwedge^2 V$ of codimension $k-\mu-1$, that is, of dimension $\mu+1$, then

$$g(\mathcal{E}) \leqslant \frac{(q^\mu - 1) + q^2(q-1)}{q-1} = 1 + q + q^2 + \cdots + q^{\mu-1} + q^2,$$

and by Lemma 25, we see that the bound is attained for some subspace of codimension $k-\mu-1$. This proves that $g_{k-\mu-1}(2,m) = 1 + q + q^2 + \cdots + q^{\mu-1} + q^2$. $\square$

Theorem 28. *For the Grassmann code* $C(2,m)$*, we have*

$$d_{k-\mu-1}\bigl(C(2,m)\bigr) = n - \bigl(1 + q + \cdots + q^{\mu-1} + q^2\bigr)$$

and

$$d_{\mu+1}\bigl(C(2,m)\bigr) = q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1}.$$

Proof. The formula for $d_{k-\mu-1}(C(2,m))$ is an immediate consequence of Corollaries 27 and 17. To prove the formula for $d_{\mu+1}(C(2,m))$, we use Corollaries 27 and 22 to observe that for $C(2,m)$, the generalized Griesmer–Wei bound in Theorem 13 can be written as

$$d_{\mu+1}\bigl(C(2,m)\bigr) \geqslant q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1}.$$

Moreover, by Lemma 25, there exists a $(\mu+1)$-dimensional subspace, say E_1, of $\bigwedge^2 V$ containing $(q^\mu - 1) + q^2(q-1)$ decomposable elements such that the remaining $(q^\mu - q^2)(q-1)$ nonzero elements are of rank 4. Thus, in view of Proposition 21, we see that $D_1 := \tau(\mathfrak{h}(E_1))$ is a $(\mu+1)$-dimensional subspace of $C(2,m)$ in which $(q^\mu - 1) + q^2(q-1)$ elements are of weight q^δ while the remaining $(q^\mu - q^2)(q-1)$ nonzero elements are of weight $q^\delta + q^{\delta-2}$. Consequently, by Lemma 12, we have

$$\begin{aligned}
\|D_1\| &= \frac{1}{q^{\mu+1} - q^\mu} \sum_{c \in D} \|c\| \\
&= \frac{q^\delta[(q^\mu - 1) + q^2(q-1)]}{q^{\mu+1} - q^\mu} + \frac{(q^\delta + q^{\delta-2})[(q^\mu - q^2)(q-1)]}{q^{\mu+1} - q^\mu} \\
&= q^{\delta-\mu}\bigl(q^\mu + q^{\mu-1} + \cdots + q + 1\bigr) + q^{\delta-2} - q^{\delta-\mu}.
\end{aligned}$$

This proves that $d_{\mu+1}(C(2,m)) = q^\delta + q^{\delta-1} + 2q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-\mu+1}$. $\square$

Remark 29. It appears quite plausible that the new pattern which emerges with $d_{\mu+1}(C(2,m))$ continues for the next several values of $d_r(C(2,m))$. More precisely, we conjecture that for $\mu+1 < r \leqslant 2\mu - 3$, one has

$$d_r\bigl(C(2,m)\bigr) = \bigl(q^\delta + q^{\delta-1} + \cdots + q^{\delta-\mu+1}\bigr) + \bigl(q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-r+\mu-1}\bigr)$$

and

$$d_{k-r}\bigl(C(2,m)\bigr) = n - \bigl(1 + q + \cdots + q^{\mu-1}\bigr) - \bigl(q^2 + q^3 + \cdots + q^{r-\mu+1}\bigr).$$

These conjectural formulae yield the complete weight hierarchy of $C(2,6)$. In general, we believe that, as predicted in most cases by Hansen, Johnsen and Ranestad [7,8], the difference $d_r(C(\ell,m)) - d_{r-1}(C(\ell,m))$ is always a power of q, and that determining $d_r(C(\ell,m))$ from $d_{r-1}(C(\ell,m))$ is a matter of deciphering the correct term of the Gaussian binomial coefficient (which can be written as a finite sum of powers of q) that gets added to $d_{r-1}(C(\ell,m))$.

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