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Projected solutions of generalized quasivariational problems in Banach spaces

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ABSTRACT

This paper focuses on the analysis of generalized quasivariational inequalities with non-self map. In Aussel et al., (2016), introduced the concept of the projected solution to study such problems. Subsequently, in the literature, this concept has attracted great attention and has been developed from different perspectives. The main contribution of this paper is to prove new existence results of the projected solution for generalized quasivariational inequality problems with non-self map in suitable infinite dimensional spaces. As an application, a quasiconvex quasioptimization problem is studied through a normal cone approach.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed space, $(X^*, \|\cdot\|_*)$ its topological dual, and $\langle\cdot,\cdot\rangle$ the duality pairing. The closed unit balls in X and X^* are denoted by B and B^*, respectively; $x+\eta B$ identifies the closed ball of radius $\eta>0$ around $x\in X$. Let $\Phi : X \rightrightarrows X^*$ be a set-valued map and $C \subseteq X$ be a nonempty set: a generalized variational inequality $GVI(\Phi, C)$ consists in finding

$$\bar{x} \in C \text{ such that } \exists x^* \in \Phi(\bar{x}) \text{ with } \langle x^*, x-\bar{x}\rangle \geq 0, \quad \forall x \in C.$$

This problem has its origins with Stampacchia and Fichera and it provides a broad unifying setting for the study of optimization and complementarity problems and more in general equilibrium problems. In 1982, Chan and Pang [1] introduced the generalized quasivariational inequality: it is a generalized variational

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inequality where the constraint set is subject to modifications depending on the considered point. In particular, let $K : C \rightrightarrows X$ be a set-valued map, the generalized quasivariational inequality $GQVI(\Phi, K)$ consists in finding

$$\bar{x} \in K(\bar{x}) \text{ such that } \exists x^* \in \Phi(\bar{x}) \text{ with } \langle x^*, x - \bar{x}\rangle \geq 0, \quad \forall x \in K(\bar{x}).$$

The classical way to get the existence of solutions consists in requiring that the feasibility set-valued map K is a self map, that is, it maps C to itself. However, the condition $K(C) \subseteq C$ is quite strong and it is not satisfied by some applications. The study of $GQVI(\Phi, K)$ with $K(C) \nsubseteq C$ was firstly investigated in [1] and, subsequently, in [2]. Anyway, in this situation, it may then be asking too much to expect that the solution $\bar{x}$ is a fixed point of K. For this reason, following the approach recently introduced in [3], we focus on the study of the existence of the so-called projected solution of a generalized quasivariational inequality with non-self map: a vector $\bar{x} \in C$ is said to be a projected solution of $GQVI(\Phi, K)$ if there exists $\bar{y} \in X$ such that

- ▷ $\bar{x}$ is a metric projection of $\bar{y}$ on C,
- ▷ $\bar{y}$ is a solution of $GVI(\Phi, K(\bar{x}))$.

Clearly, each solution of $GQVI(\Phi, K)$ is a projected solution and the two concepts coincide if K is a self map. When K is a non-self map, two different scenarios occur: either $K(C) \cap C = \emptyset$ or $K(C) \cap C \neq \emptyset$.

Example 1.1. Let X be a Hilbert space, with $X^* = X$ (Riesz representation theorem), and $C = B \subseteq X$. Fixed $u \in X$ with $\|u\| = 1$, let $K : C \rightrightarrows X$ and $\Phi : X \rightrightarrows X^*$ be defined as

$$K(x) = 4u + (2 - \|x\|)B \quad \text{and} \quad \Phi(x) = [-x, -u]\,,$$

where $[-x, -u]$ is the segment joining $-x$ and $-u$. Clearly, $K(C) \cap C = \emptyset$ and $GQVI(\Phi, K)$ has no classic solution; instead, $\bar{x} = u$ is the projected solution associated to $\bar{y} = 5u$.

When $K(C) \cap C \neq \emptyset$, the following example shows two different cases: either the classical does not exist or the classical solutions set is nonempty but, in general, it does not coincide with the set of projected solutions.

Example 1.2. Let X be a Hilbert space and $C = B \subseteq X$. Fixed $u \in X$ with $\|u\| = 1$, let Φ be defined as in Example 1.1.

(i) If $K : C \rightrightarrows X$ is defined as

$$K(x) = 3u + (2 - 2\|x\|)B\,,$$

$K(C) \cap C \neq \emptyset$, but $\operatorname{fix} K = \emptyset$. In this case, $GQVI(\Phi, K)$ has no classic solution; instead, $\bar{x} = u$ is the projected solution associated to $\bar{y} = 3u$.

(ii) If $K : C \rightrightarrows X$ is defined as

$$K(x) = 3u + (3 - \|x\|)B\,,$$

$K(C) \cap C \neq \emptyset$ and $\operatorname{fix} K = [0, u]$. In this case, $\bar{x}_c = 0$ is the classical solution to $GQVI(\Phi, K)$; instead, $\bar{x}_p = u$ is the projected solution associated to $\bar{y} = 5u$.

In [3], the existence of a projected solution was proved by using techniques inspired by [2]. Thereafter, the study of the existence of the projected solutions has been extended to general Ky Fan quasiequilibrium problems ([4,5] in a finite dimensional setting and [6] in Banach spaces) and to multistage stochastic quasivariational inequality problems [7].

Here, our main contribution is to prove new existence results for projected solutions of $GQVI(\Phi, K)$ exploiting suitable approximation techniques and continuous selection results. In particular, no monotonicity assumptions are required on the principal operator Φ which is assumed to be norm-to-weak* upper semicontinuous with nonempty weak*-compact convex values. This fact paves the way to a wide range of applications. Indeed, several optimization and equilibrium problems (for instance, generalized Nash equilibrium problems and economic equilibrium problems) are studied through operators having the quoted properties. In this light, we show the applicability of our techniques proposing the study of a quasiconvex quasioptimization problem by using a normal cone approach.

The paper is organized as follows. Section 2 is devoted to the introduction of some preliminary notations and definitions that we will be used in the work. On this basis, in Section 3, our main results are proved: the existence of the projected solutions of $GQVI(\Phi, K)$ is get under different assumptions and a comparison with the results available in the literature is provided. In order to support such results, in Section 4, a quasiconvex quasioptimization problem is analyzed. Finally, a section with the conclusions is given.

2. Notations and definitions

For the convenience of the reader, we recall some preliminary notations, definitions, and tools of set-valued analysis.[1]

Given any nonempty $A \subseteq X$ and $y \in X$, we denote by $\operatorname{dist}(y, A) = \inf\{\|y - x\| : x \in A\}$ the distance of y from A. Moreover, we denote by $\operatorname{co} A$, $\operatorname{cl} A$, and $\operatorname{int} A$ the convex hull, the closure, and the topological interior of A, respectively. The set A is relatively compact if its closure is compact.

Let $\Gamma : X \rightrightarrows Y$ be a set-valued map with X and Y two Hausdorff topological spaces. The domain of Γ is $\operatorname{dom} \Gamma = \{x \in X : \Gamma(x) \neq \emptyset\}$ and its graph is $\operatorname{gph} \Gamma = \{(x, y) \in \operatorname{dom} \Gamma \times Y : y \in \Gamma(x)\}$. Moreover, we denote by $\operatorname{co} \Gamma : X \rightrightarrows Y$ the convex hull of Γ such that, for all $x \in X$, $\operatorname{co} \Gamma(x) = \operatorname{co}(\Gamma(x))$. Given the set-valued maps $\Gamma_1 : X \rightrightarrows Y$ and $\Gamma_2 : Y \rightrightarrows Z$, the composition $\Gamma_2 \circ \Gamma_1 : X \rightrightarrows Z$ is defined as

$$(\Gamma_2 \circ \Gamma_1)(x) = \bigcup_{y \in \Gamma_1(x)} \Gamma_2(y).$$

The map Γ is lower semicontinuous at x if for each open set Ω such that $\Gamma(x) \cap \Omega \neq \emptyset$ there exists a neighborhood U_x of x such that $\Gamma(x') \cap \Omega \neq \emptyset$ for every $x' \in U_x$; instead, it is upper semicontinuous at x if for each open set Ω such that $\Gamma(x) \subseteq \Omega$ there exists a neighborhood U_x of x such that $\Gamma(x') \subseteq \Omega$ for every $x' \in U_x$. The map Γ is continuous at x if it is both upper and lower semicontinuous at x. If X and Y are two metric spaces the upper and lower semicontinuity at $x \in X$ may be characterized by means of sequences:

- ▹ Γ is upper semicontinuous at x and $\Gamma(x)$ is compact if and only if for each sequence $\{(x_n, y_n)\} \subseteq \operatorname{gph} \Gamma$ with $x_n \to x$, then $\{y_n\}$ has a limit point in $\Gamma(x)$;
- ▹ Γ is lower semicontinuous at x if and only if for each $x_n \to x$ and $y \in \Gamma(x)$ there exist a subsequence $\{x_{n_k}\}$ and elements $y_k \in \Gamma(x_{n_k})$ such that $y_k \to y$.

The set-valued map Γ is closed if $\operatorname{gph} \Gamma$ is closed in $X \times Y$. We recall that the closed graph theorem affirms that, when $\Gamma(X)$ is relatively compact, Γ is closed if and only if it is upper semicontinuous with closed values. A fixed point of a set-valued map $\Gamma : X \rightrightarrows X$ is a point $x \in X$ satisfying $x \in \Gamma(x)$; we denote by $\operatorname{fix} \Gamma$ the set of the fixed points of Γ.

Let A be a closed set: the set-valued map $P_A : X \rightrightarrows A$ defined as

$$P_A(y) = \{x \in A : \|x - y\| = \operatorname{dist}(y, A)\}$$

is the metric projection onto A. Clearly $P_A(y)$ is a closed, eventually empty, set.

[1] For further details, the interested reader can refer to [8,9] and the references therein.

The set A is called approximatively compact if for every $y \in X$ and every sequence $\{x_n\} \subseteq A$ such that $\|y - x_n\| \to \operatorname{dist}(y, A)$ there exists a subsequence $\{x_{n_k}\}$ that converges to an element $x \in A$. The word "approximatively" is occasionally misspelled "approximately" in the literature and this notion was introduced by Efimov and Stechkin in [10]. It is worth noting that a compact set is approximatively compact; instead, the converse may not hold [11, Example 4.1]. One of the reasons for the importance of approximative compactness in approximation theory is that the limit point x belongs to $P_A(y)$ and hence every approximatively compact set A is a proximinal set, that is, $P_A(y) \neq \emptyset$ for every $y \in X$. Moreover the set-valued map P_A is upper semicontinuous. Lastly, the approximative compactness of sets is inherited by the product.

Lemma 2.1. *Let $(X_i, \|\cdot\|_i)$ be normed spaces and $A_i \subseteq X_i$ approximatively compact for each $i \in \mathcal{I} = \{1, \ldots, I\}$. Then, $A = \prod_{i\in\mathcal{I}} A_i$ is approximatively compact in $X = \prod_{i\in\mathcal{I}} X_i$ with norm $\|\cdot\| = \sum_{i\in\mathcal{I}} \|\cdot\|_i$.*

Proof. Let $y = (y_1, \ldots, y_I) \in X$ and $\{x_n\} = \{(x_{1,n}, \ldots, x_{I,n})\} \subseteq A$ such that

$$\|x_n - y\| = \sum_{i\in\mathcal{I}} \|x_{i,n} - y_i\|_i \to \operatorname{dist}(y, A).$$

Since it results

$$\operatorname{dist}(y, A) = \inf_{x\in A} \|y - x\| = \inf_{x_1\in A_1,\ldots,x_I\in A_I} \sum_{i\in\mathcal{I}} \|y_i - x_i\|_i = \sum_{i\in\mathcal{I}} \operatorname{dist}_i(y_i, A_i),$$

where dist_i denotes the distance induced by the norm $\|\cdot\|_i$, then $\|x_{i,n} - y_i\|_i \to \operatorname{dist}_i(y_i, A_i)$ for each $i \in \mathcal{I}$. Hence, there exist a subsequence $\{x_{1,n_k}\}$ and $x_1 \in A_1$ such that $x_{1,n_k} \to x_1$. Again, there exist a subsequence $\{x_{2,n_{k_h}}\}$ of $\{x_{2,n_k}\}$ and $x_2 \in A_2$ such that $x_{2,n_{k_h}} \to x_2$. Clearly, $\{x_{1,n_{k_h}}\}$ converges to x_1. Then, continuing iteratively by repeating the process until the last element, we get a subsequence of $\{x_n\}$ which tends to $x = (x_1, \ldots, x_I) \in A$ as required. □

3. Existence results for $GQVI(\Phi, K)$

In this section, we aim to prove some results on the existence of the projected solution to a generalized quasivariational inequality with non-self constraint map by using suitable approximating techniques and opportune continuous selection theorems.

Theorem 3.1. *Let X be a Banach space and $C \subseteq X$ approximatively compact convex. Assume that $K(C)$ is relatively compact. Then, $GQVI(\Phi, K)$ admits a projected solution if the following properties hold:*

(i) *K is lower semicontinuous and closed with nonempty convex values;*

(ii) *Φ is norm-to-weak* upper semicontinuous on $K(C)$ with nonempty weak*-compact convex values.*

Proof. For every $n \in \mathbb{N}$, let $P_{C,n} : X \rightrightarrows C$ be defined as

$$P_{C,n}(y) = \left\{x \in C : \|x - y\| \leq \operatorname{dist}(y, C) + \frac{1}{n}\right\}.$$

Clearly $P_{C,n}(y)$ is a nonempty, closed and convex set and $P_{C,n}(y) = \operatorname{cl} P^{<}_{C,n}(y)$ for each $y \in X$, where

$$P^{<}_{C,n}(y) = \left\{x \in C : \|x - y\| < \operatorname{dist}(y, C) + \frac{1}{n}\right\}.$$

Moreover, $P^{<}_{C,n}$ has open graph on $X \times C$ and then it is lower semicontinuous [8, Lemma 17.12]. Consequently, $P_{C,n}$ is lower semicontinuous by [8, Lemma 17.22] and, thanks to a famous Michael selection result [12, Theorem 3.2″], it admits a continuous selection $p_{C,n} : X \to C$.

The set $\widehat{C} = \operatorname{cl}\operatorname{co} K(C)$ is compact since X is a Banach space [8, Theorem 5.35]; moreover, since K is closed and $K(C)$ is relatively compact, it follows that K is upper semicontinuous, that is, K is continuous. Then, we introduce the set-valued map $K_n : \widehat{C} \rightrightarrows \widehat{C}$ defined as $K_n(y) = K(p_{C,n}(y))$: as composition of two continuous maps, it is continuous with nonempty, closed, and convex values [8, Theorem 17.23]. So, from the closed graph theorem the map K_n has closed graph; hence, $\operatorname{fix} K_n$ is closed and, since subset of the compact set $\widehat{C}$, it is compact itself.

Let us consider the set-valued map $F : \operatorname{fix} K_n \rightrightarrows X$ defined as

$$F(y) = \bigcap_{y^* \in \Phi(y)} \{z \in X : \langle y^*, z - y\rangle < 0\} = \left\{ z \in X : \max_{y^* \in \Phi(y)} \langle y^*, z - y\rangle < 0 \right\}.$$

Clearly, F is with convex values. To prove that F has open graph, it is sufficient to show that the function $m : \operatorname{fix} K_n \times X \to \mathbb{R}$ defined as

$$m(y, z) = \max_{y^* \in \Phi(y)} \langle y^*, z - y\rangle$$

is upper semicontinuous. From [8, Lemma 17.8], the subset $\Phi(\operatorname{fix} K_n)$ is weak*-compact; hence, it is norm bounded. Thanks to [8, Corollary 6.40] the duality pairing $\langle\cdot,\cdot\rangle$ restricted to $\Phi(\operatorname{fix} K_n) \times X$ is jointly continuous; hence, [8, Lemma 17.30] guarantees the upper semicontinuity of m.

By contradiction, assume that $F(y) \cap K_n(y) \neq \emptyset$ for all $y \in \operatorname{fix} K_n$. From [13, Corollary 1.11.1] $F \cap K_n$ admits a continuous selection $f : \operatorname{fix} K_n \to \widehat{C}$. Therefore, the set-valued map $\Upsilon : \widehat{C} \rightrightarrows \widehat{C}$ defined as

$$\Upsilon(y) = \begin{cases} K_n(y) & \text{if } y \notin \operatorname{fix} K_n \\ \{f(y)\} & \text{if } y \in \operatorname{fix} K_n \end{cases}$$

is lower semicontinuous [14, Lemma 2.3] with closed convex values. Hence [12, Theorem 3.2″] guarantees that f can be extended to a continuous selection φ for Υ. The Brouwer–Schauder–Tychonoff fixed point theorem guarantees that φ has a fixed point, that is, there exists $y \in \widehat{C}$ such that $y = \varphi(y) \in \Upsilon(y)$. Clearly $y \in \operatorname{fix} K_n$ and this implies $y = f(y) \in F(y)$ which is absurd.

Therefore, for every $n \in \mathbb{N}$, there exists $y_n \in \operatorname{fix} K_n$ such that $F(y_n) \cap K_n(y_n) = \emptyset$, that is,

$$\max_{y^* \in \Phi(y_n)} \langle y^*, z - y_n\rangle \geq 0, \qquad \forall z \in K_n(y_n). \tag{1}$$

Since $\widehat{C}$ is compact, without loss of generality, we may assume that there exists $\bar{y} \in \widehat{C}$ such that $y_n \to \bar{y}$. Define $x_n = p_{C,n}(y_n)$; then, the following chain of inequalities holds

$$\begin{aligned} \operatorname{dist}(\bar{y}, C) &\leq \|x_n - \bar{y}\| \\ &\leq \|x_n - y_n\| + \|y_n - \bar{y}\| \\ &\leq \operatorname{dist}(y_n, C) + \|y_n - \bar{y}\| + \frac{1}{n}. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have that $\|x_n - \bar{y}\| \to \operatorname{dist}(\bar{y}, C)$. The approximative compactness of C ensures the existence of a limit point $\bar{x} \in P_C(\bar{y})$ of the minimizing sequence $\{x_n\}$. Again, for the sake of simplicity, assume that $x_n \to \bar{x}$. Since K is upper semicontinuous with compact values and $y_n \in K_n(y_n) = K(x_n)$, then $\bar{y} \in K(\bar{x})$. Now, fix $z \in K(\bar{x})$ arbitrarily. Since K is lower semicontinuous, then there exist a subsequence $\{x_{n_k}\}$ and elements $z_k \in K(x_{n_k})$ such that $z_k \to z$. Therefore, from (1) we have

$$m(y_{n_k}, z_k) = \max_{y^* \in \Phi(y_{n_k})} \langle y^*, z_k - y_{n_k}\rangle \geq 0.$$

The upper semicontinuity of m and the arbitrariness of z guarantee that

$$\max_{y^*\in\Phi(\bar{y})}\langle y^*, z-\bar{y}\rangle \geq 0, \qquad \forall z\in K(\bar{x})$$

which is equivalent to affirm that

$$\min_{z\in K(\bar{x})}\max_{y^*\in\Phi(\bar{y})}\langle y^*, z-\bar{y}\rangle \geq 0.$$

Invoking the Sion's minimax theorem we deduce that

$$\max_{\bar{y}^*\in\Phi(\bar{y})}\min_{z\in K(\bar{y})}\langle \bar{y}^*, z-\bar{y}\rangle \geq 0$$

which means that $\bar{x}=p_C(\bar{y})$ is a projected solution. □

Generalized quasivariational inequality problems over product sets is of great interest in game theory and economics. This particular format is when

$$X=\prod_{i\in\mathcal{I}}X_i, \qquad C=\prod_{i\in\mathcal{I}}C_i, \qquad K=\prod_{i\in\mathcal{I}}K_i,$$

where $\mathcal{I}=\{1,\dots,I\}$ is a finite index set and, for each $i\in\mathcal{I}$, X_i is a normed space with X_i^* its topological dual, $C_i\subseteq X_i$ is a nonempty set, and $K_i : C\rightrightarrows X_i$ is a set-valued map. Denote by x_i the i-component of an element $x\in X$, and by $\langle\cdot,\cdot\rangle_i$ the duality pairing of (X_i^*, X_i). Here, the product map $K : C\rightrightarrows X$ is defined as $K(x)=\prod_{i\in\mathcal{I}}K_i(x)$.

The problem consists in finding a fixed point $\bar{x}\in K(\bar{x})$ such that for each $i\in\mathcal{I}$ there exists $x_i^*\in\Phi_i(\bar{x})$ with

$$\sum_{i\in\mathcal{I}}\langle x_i^*, x_i-\bar{x}_i\rangle_i \geq 0, \quad \forall x\in K(\bar{x}),$$

where $\Phi_i : X\rightrightarrows X_i^*$. If we denote by $\Phi : X\rightrightarrows\prod_{i\in\mathcal{I}}X_i^*$ the product map $\Phi=\prod_{i\in\mathcal{I}}\Phi_i$, then the designation of this problem as $GQVI(\Phi,K)$ is a certain abuse of notation due to the fact that the range space of Φ is the product of the duals instead of the dual of the product $X^*=(\prod_{i\in\mathcal{I}}X_i)^*$. However, we stress the fact that these two vector spaces are isomorphic taking the bijection

$$x^*\mapsto\sum_{i\in\mathcal{I}}x_i^*$$

and that this map is an homeomorphism when considering the product of the weak* topologies on $\prod_{i\in\mathcal{I}}X_i^*$ and the weak* topology on X^*.

The study of the existence of solutions to $GQVI(\Phi,K)$ by requiring the regularity of the component set-valued maps Φ_i and K_i only is not always possible if certain generalized monotonicity and continuity assumptions are needed. For a comprehensive analysis of the problem, the interested reader can refer to [15] and the references therein. Working without monotonicity conditions, we obtain an existence result for projected solutions of product-type generalized quasivariational inequalities as a natural consequence of Theorem 3.1.

Corollary 3.1. *Let X_i be Banach spaces, $C_i\subseteq X_i$ approximatively compact convex, and $K_i(C_i)$ relatively compact. Then, $GQVI(\Phi,K)$ admits a projected solution if the following properties hold for each $i\in\mathcal{I}$:*

(i) *K_i is lower semicontinuous and closed with nonempty convex values;*

(ii) *Φ_i is norm-to-weak* upper semicontinuous on $K_i(C_i)$ with nonempty weak*-compact convex values.*

Proof. Thanks to the Tychonoff's theorem, $K(C)$ is relatively compact. Moreover, according to [9, Theorems VI.2.4 and VI.2.4′], lower and upper semicontinuity are preserved by the product of set-valued maps. Hence, assumptions (i) and (ii) of Theorem 3.1 are verified. Finally, the approximative compactness of C descends from Lemma 2.1 and the conclusion follows from Theorem 3.1. □

In Theorem 3.1, it would be nice if approximatively compact could be replaced by closed, but this is impossible since there exists an example of a closed convex body C in the Banach space of sequences converging to zero c_0 such that $P_C(y) = \emptyset$, for every $y \in c_0 \setminus C$ [11, Remark 4.7]. The reflexivity of the space X is necessary but not sufficient to guarantee that every nonempty closed convex subset $C \subseteq X$ is approximatively compact. Indeed, in a Banach space X, every nonempty closed convex subset is approximatively compact if and only if X is reflexive and it has the Kadec–Klee property, that is, strong convergence and weak convergence are equivalent on the unit sphere of the space [11, Theorem 9.1]. Thanks to this, we can state the following.

Corollary 3.2. *Let X be a reflexive Banach space with the Kadec–Klee property and $C \subseteq X$ closed and convex. Assume that $K(C)$ is relatively compact. Then, $GQVI(\Phi, K)$ admits a projected solution if the following properties hold:*

(i) *K is lower semicontinuous, closed with nonempty convex values;*

(ii) *Φ is norm-to-weak* upper semicontinuous on $K(C)$ with nonempty weak*-compact convex values.*

The assumptions of Corollary 3.2 do not ensure the uniqueness of the projection. Instead, if in addition the norm is strictly convex, then for each C nonempty closed and convex and for each $y \in X$ there exists a unique projection that will be denoted $p_C(y)$. Clearly, p_C is continuous.

The class of uniformly convex Banach spaces constitutes a well-known subclass of reflexive Banach spaces with strictly convex norm and satisfying the Kadec–Klee property: in this setting, as we will prove in Theorem 3.3, the closedness of the map K in Corollary 3.2 can be weakened.

Before proceeding with the statement of our next result we review a notion of relative interior for convex sets which was introduced by Michael [12]. Let $A \subseteq X$ be convex; a convex set $S \subseteq A$ is a face of A if $x_1, x_2 \in A$, $t \in (0, 1)$ and $tx_1 + (1 - t)x_2 \in S$ imply $x_1, x_2 \in S$. The collection of all proper closed faces of $\mathrm{cl}\, A$ is denoted by $\mathcal{F}_A$. A point $x \in A$ is an inside point of A if it is not in any proper closed face of $\mathrm{cl}\, A$: the set of the inside points of A is denoted by $I(A)$. Finally we consider the family of convex sets $\mathcal{D}(X)$ defined as

$$\mathcal{D}(X) = \{A \subseteq X : A \text{ is convex and } I(\mathrm{cl}\, A) \subseteq A\}.$$

This class contains all the convex sets which are either closed, or with nonempty relative interior. In particular, when X is finite dimensional the class $\mathcal{D}(X)$ coincides with the family of all convex sets.

We recall the following selection result, recently proved in [14], which will be crucial for our purposes.

Theorem 3.2. *Let X be a metric space, Y be a normed space, and $\Gamma : X \rightrightarrows Y$ be a lower semicontinuous set-valued map with nonempty values in the class $\mathcal{D}(Y)$ such that $\Gamma(X)$ is relatively compact. Then, Γ admits a continuous selection.*

In particular, Theorem 3.2, with respect to the classical Michael selection result [12, Theorem 3.2′′] used in the proof of Theorem 3.1, does not require the closedness of the values of the set-valued map Γ.

Theorem 3.3. *Let X be a uniformly convex Banach space and $C \subseteq X$ closed and convex. Assume that $K(C)$ is relatively compact. Then, $GQVI(\Phi, K)$ admits a projected solution if the following properties hold:*

(i) *K is lower semicontinuous with nonempty convex values in the class $\mathcal{D}(X)$ and* $\operatorname{fix}(K \circ p_C)$ *is closed;*

(ii) *Φ is norm-to-weak* upper semicontinuous on $K(C)$ with nonempty weak*-compact convex values.*

Proof. In outline, the proof follows the same line of reasoning as in Theorem 3.1. Now, the projection function $p_C : X \to C$ is well defined and continuous and there is no need to approximate it with $p_{C,n}$. We introduce the set-valued map $\widehat{K} : \widehat{C} \rightrightarrows \widehat{C}$ defined as $\widehat{K}(y) = K(p_C(y))$: it is lower semicontinuous with nonempty values in the class $\mathcal{D}(X)$. Thanks to the closedness of $\operatorname{fix} \widehat{K}$, we get that $F : \operatorname{fix} \widehat{K} \rightrightarrows X$ is convex valued with open graph. Assuming by contradiction that $F(y) \cap \widehat{K}(y) \neq \emptyset$ for all $y \in \operatorname{fix} \widehat{K}$, as deduced in Theorem 3.1, there exists a continuous selection $f : \operatorname{fix} \widehat{K} \to \widehat{C}$ of $F \cap \widehat{K}$ and the set-valued map $\Upsilon : \widehat{C} \rightrightarrows \widehat{C}$ defined as

$$\Upsilon(y) = \begin{cases} \widehat{K}(y) & \text{if } y \notin \operatorname{fix} \widehat{K} \\ \{f(y)\} & \text{if } y \in \operatorname{fix} \widehat{K} \end{cases}$$

is lower semicontinuous with convex values in the class $\mathcal{D}(X)$. Hence, according to Theorem 3.2, there exists a continuous selection of Υ and, arguing as before, we obtain a contradiction. Therefore, there exists $\bar{y} \in \operatorname{fix} \widehat{K}$ such that

$$\max_{y^* \in \Phi(\bar{y})} \langle y^*, z - \bar{y} \rangle \geq 0, \qquad \forall z \in K(\bar{x})$$

where $\bar{x} = p_C(\bar{y})$. The proof ends with the application of the Sion's minimax theorem. □

When $K(C)$ is relatively compact, the closedness of the set-valued map K is sufficient (but not necessary) to guarantee that $\operatorname{fix}(K \circ p_C)$ is closed. Moreover, if X is a Hilbert space, the Kolmogorov's criterion for the best approximation provides a way to characterize the closedness of $\operatorname{fix}(K \circ p_C)$. We recall that, given a nonempty closed convex set $C \subseteq X$ and $y \in X$,

$$x = p_C(y) \quad \Leftrightarrow \quad x \in C \text{ and } \langle y - x, z - x \rangle \leq 0 \quad \forall z \in C;$$

this is equivalent to affirms that $y \in x + N_C(x)$ where $N_C(x)$ is the normal cone of C at x defined as

$$N_C(x) = \{x^* \in X^* : \langle x^*, z - x \rangle \leq 0, \ \forall z \in C\}.$$

Hence, defining $T : C \rightrightarrows \mathbb{R}^n$ as

$$T(x) = K(x) \cap (x + N_C(x)), \tag{2}$$

it follows that $T(C) = \operatorname{fix}(K \circ p_C)$. This characterization is the core of the next result.

Corollary 3.3. *Let X be a Hilbert space and $C \subseteq X$ closed and convex. Assume that $K(C)$ is relatively compact. Then, $GQVI(\Phi, K)$ admits a projected solution if the following properties hold:*

(i) *K is lower semicontinuous with nonempty convex values in the class $\mathcal{D}(X)$ and T is closed;*

(ii) *Φ is norm-to-weak* upper semicontinuous on $K(C)$ with nonempty weak*-compact convex values.*

Proof. Thanks to Theorem 3.3 and the fact that $\operatorname{fix}(K \circ p_C) = T(C)$, it is sufficient to verify the closedness of $T(C)$. Take $\{y_n\} \subseteq T(C)$ with $y_n \to y \in X$. Hence, the sequence $\{(p_C(y_n), y_n)\} \subseteq \operatorname{gph} T$ and it converges to $(p_C(y), y)$ since p_C is continuous. According to the closedness of T, it results that $(p_C(y), y) \in \operatorname{gph} T$, that is, $y \in T(p_C(y)) \subseteq T(C)$. □

Remark 3.1. In the proof of Corollary 3.3, we have seen that the closedness of the set-valued map T is sufficient to guarantee the closedness of $T(C)$. Also the converse implication holds. Indeed, fix $\{(x_n, y_n)\} \subseteq \operatorname{gph} T$ with $(x_n, y_n) \to (x, y) \in C \times X$; hence, $x_n = p_C(y_n)$ and $y_n \in T(x_n) \subseteq T(C)$. Thanks to the continuity of p_C and the closedness of $T(C)$ we have $x = p_C(y)$ and $y \in T(C)$; from the uniqueness of the projection we have $(x, y) \in \operatorname{gph} T$. In other words, the equivalence

$$\operatorname{fix}(K \circ p_C) \text{ is closed} \quad \Leftrightarrow \quad T(C) \text{ is closed} \quad \Leftrightarrow \quad T \text{ is closed}$$

holds true.

Now, we provide a comparison with related results in the literature. The first existence result for projected solution of a generalized quasivariational inequality was established in [3, Theorem 3.2] for finite dimensional spaces and under a pseudomonotonicity assumption. The authors underline that their result cannot be proved in an arbitrary infinite dimensional topological vector space due to the assumptions $\operatorname{int} K(x) \neq \emptyset$ and $K(x)$ compact, for all $x \in C$. Notice that, when X is finite dimensional, our result require the convexity of the values of K only. Subsequently, without any monotonicity assumption, the existence of a projected solution was proved in [5] requiring the compactness of $C \subseteq \mathbb{R}^n$. Recently, an analogous result has been established in [4] avoiding the compactness of C: Corollary 3.3 extends this result to Hilbert spaces.

Existence results in the infinite dimensional case have been proved in [6,7]. Our results hold under weaker assumptions: adapting opportunely the notations to our setting, we point out the following facts.

- ▷ Theorem 4.2 in [6] requires Φ norm-to-norm upper semicontinuous with norm-compact values and $\Phi(X)$ relatively compact. Instead, we require that Φ is norm-to-weak* upper semicontinuous with weak*-compact values. Moreover, we do not require the compactness of C but only its approximative compactness in Theorem 3.1 and its closedness in Corollary 3.2, Theorem 3.3, and Corollary 3.3.
- ▷ Theorem 4 in [7] requires the pseudomonotonicity of Φ that we do not need, the norm-to-norm upper semicontinuity and the norm-compact valuedness of Φ. As previously pointed out, we only assume that Φ is norm-to-weak* upper semicontinuous with weak*-compact values.

In addition, with respect to the quoted results, Theorem 3.3 and Corollary 3.3 provide further improvements in terms of requirements on K as observed above. Anyway, in order to support these statements, we propose the following counter-examples.

Example 3.1. Let X be a Hilbert space, with $X^* = X$, and $C = B \subseteq X$. Fixed $u \in X$ with $\|u\| = 1$, let $K : C \rightrightarrows X$ and $\Phi : X \rightrightarrows X$ be defined as

$$K(x) = \begin{cases} (2u, (3 + \|x\|)u] & \text{if } x \neq u \\ [2u, 4u] & \text{if } x = u \end{cases}$$

and

$$\Phi(x) = -2(x - 2u) + (x - 2u)B,$$

where $K(C) = [2u, 4u] = \operatorname{fix}(K \circ p_C)$. Clearly, $K(C) \cap C = \emptyset$ and $GQVI(\Phi, K)$ has no classic solution. Instead, $\bar{x} = u$ is a projected solution associated to $\bar{y} = 2u$: all the requirements of Theorem 3.3 are satisfied but not the ones of Theorem 4.2 in [6] and Theorem 4 in [7]. Indeed, Theorem 4.2 in [6] cannot be applied as C is not compact, K is not closed, and $\operatorname{cl} \Phi(X)$ is not compact; the same holds for Theorem 4 in [7] as K is not closed and Φ is neither with norm-compact values nor pseudomonotone on $K(C)$, for instance in the points $2u$ and $4u$ where $\Phi(2u) = \{0\}$ but $-2u \in \Phi(4u)$ and $\langle -2u, 2u \rangle < 0$.

Example 3.2. Let $X = l_2$ be the Hilbert space of the 2-summable sequences of real numbers, with $(l_2)^* = l_2$, and $C = B \subseteq l_2$. Fixed $u = \left(\frac{1}{n}\right) \in l_2$ with $\|u\| = \frac{\pi}{\sqrt{6}}$, let $K : C \rightrightarrows X$ and $\Phi : X \rightrightarrows X$ be defined as

$$K(x) = \left\{ z \in l_2 : 0 \leq z_n \leq \frac{4 - \|x\|}{n} \right\}$$

and

$$\Phi(x) = \begin{cases} 10B & \text{if } x \in \prod_{n\in\mathbb{N}} \left[\frac{3}{n}, \frac{4}{n}\right] \\ -x+u & \text{otherwise} \end{cases}$$

where $K(C) = \prod_{n\in\mathbb{N}} \left[0, \frac{4}{n}\right]$ is a Hilbert cube. Clearly, $K(C) \cap C \neq \emptyset$ but $K(C) \nsubseteq C$. Notice that $\bar{x} = \frac{u}{\|u\|}$ is a projected solution associated to $\bar{y} = u$: all the requirements of Theorem 3.1 are satisfied but not the ones of Theorem 4.2 in [6] and Theorem 4 in [7]. Indeed, Theorem 4.2 in [6] cannot be applied as C is not compact; the same holds for Theorem 4 in [7] as Φ is neither with norm-compact values nor pseudomonotone on $K(C)$, for instance in the points u and $2u$ where $\Phi(u) = \{0\}$ but $-u \in \Phi(2u)$ and $\langle -u, u\rangle < 0$.

4. Projected solution for quasioptimization problems

The requirement that Φ is norm-to-weak* upper semicontinuous with nonempty weak*-compact convex values allows us to study applications involving optimization problems in the formulation under no too restrictive assumptions. In this section, we focus on the study of a quasioptimization problem: it is an optimization problem in which the constraint set is subject to modifications depending on the considered point. Given $C \subseteq X$ nonempty, $K : C \rightrightarrows X$ and $f : X \to \mathbb{R}$, a quasioptimization problem $QOP(f, K)$ consists in finding

$$\bar{x} \in K(\bar{x}) \text{ such that } \min\{f(y) : y \in K(\bar{x})\} = f(\bar{x}).$$

Clearly, if $K(x) = C$ for all $x \in C$, $QOP(f, K)$ reduces to a classical optimization problem.

In [3], the authors introduced the concept of projected solution when K is a non-self map. A vector $\bar{x} \in C$ is called projected solution of $QOP(f, K)$ if there exists $\bar{y} \in X$ such that

- ▷ $\bar{x}$ is a metric projection of $\bar{y}$ on C,
- ▷ $\bar{y}$ solves the problem $\min\{f(y) : y \in K(\bar{x})\}$.

As in [3], we study $QOP(f, K)$ by using a normal cone approach under the assumption of quasiconvexity of f. We need some concepts. We denote by $S_\alpha = \{y \in X : f(y) \le \alpha\}$ and $S_\alpha^< = \{y \in X : f(y) < \alpha\}$ the sublevel and the strict sublevel set at $\alpha \in \mathbb{R}$, respectively. The function f is quasiconvex if and only if S_α is convex for any $\alpha \in \mathbb{R}$. Following [16], the adjusted sublevel set of f can be associated at any $x \in X$:

$$S_f^a(x) = \begin{cases} S_{f(x)} & \text{if } x \in \operatorname{argmin} f \\ S_{f(x)} \cap B(S_{f(x)}^<) & \text{if } x \notin \operatorname{argmin} f \end{cases}$$

where

$$B(S_{f(x)}^<) = \{y \in X : \operatorname{dist}(y, S_{f(x)}^<) \le \operatorname{dist}(x, S_{f(x)}^<)\}.$$

The quasiconvexity of f is characterized by the convexity of the adjusted sublevel sets $S_f^a(x)$. In this way, the set-valued map $N^a : X \rightrightarrows X^*$ is the normal cone operator to the adjusted sublevel set defined as

$$N^a(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le 0, \quad \forall y \in S_f^a(x)\}.$$

In [16], continuity properties of N^a in relation with continuity of f were investigated. We recall that a convex subset A of a convex cone H in X^* is called base of H if 0 does not belong to the weak*-closure of A and $H = \bigcup_{t\ge 0} tA$. The authors proved in [16, Proposition 3.5] that for each $x \notin \operatorname{argmin} f$ there exists a norm-to-weak* upper semicontinuous base-valued submap of the normal operator N^a defined on a neighborhood of x. When the space X is finite-dimensional, a globally defined upper semicontinuous base-valued submap is obtained taking the intersection of the unit sphere, which is compact, with the normal operator N^a, which is closed [17]. Unfortunately, this technique does not work in the infinite dimensional case since the sphere is not compact. Nevertheless, a partition of unity technique has been recently used in establishing global existence from local existence in Banach spaces [18]. The quoted result is the following.

Theorem 4.1. *Let X be a Banach space and $f : X \to \mathbb{R}$ be a quasiconvex continuous function. Then, there exists a norm-to-weak* upper semicontinuous set-valued map $A : X \setminus \operatorname{argmin} f \rightrightarrows B^*$ such that $A(x)$ is a weak*-compact base of $N^a(x)$ for all x.*

Having a base-valued operator defined globally finds an interesting use in variational inequalities, and we are in a position to prove the following result.

Theorem 4.2. *Let X be a Banach space and $C \subseteq X$ approximatively compact convex. Assume that $K(C)$ is relatively compact. Then, $QOP(f, K)$ admits a projected solution if the following properties hold:*

(i) *K is lower semicontinuous and closed with nonempty convex values;*

(ii) *f is continuous and quasiconvex.*

Proof. Let $\Phi : X \rightrightarrows X^*$ be defined as

$$\Phi(x) = \begin{cases} B^* & \text{if } x \in \operatorname{argmin} f \\ A(x) & \text{if } x \notin \operatorname{argmin} f \end{cases}$$

where A is the norm-to-weak* upper semicontinuous set-valued map obtained in Theorem 4.1. Since $\operatorname{argmin} f$ is closed and $A(x) \subseteq B^*$, then Φ is upper semicontinuous. In this way, thanks to Theorem 3.1, it follows that $GQVI(\Phi, K)$ admits a projected solution $\bar{x} \in C$, that is, there exists $\bar{y} \in X$ so that $\bar{x} \in P_C(\bar{y})$ and

$$\bar{y} \in K(\bar{x}) \text{ such that } \exists y^* \in \Phi(\bar{y}) \text{ with } \langle y^*, y - \bar{y}\rangle \geq 0, \quad \forall y \in K(\bar{x}).$$

Clearly, if $\bar{y} \in \operatorname{argmin} f$, then $f(\bar{y}) \leq f(y)$ for all $y \in K(\bar{x})$. Instead, if $\bar{y} \notin \operatorname{argmin} f$, then it results that

$$y^* \in \Phi(\bar{y}) = A(\bar{y}) \subseteq N^a(\bar{y}) \setminus \{0\}.$$

Hence, $\bar{y}$ is a solution to $GVI(N^a \setminus \{0\}, K(\bar{x}))$ and, thanks to [19, Proposition 3.2], the thesis follows. □

To the best of our knowledge, all the existence results of projected solutions for quasioptimization problems work in a finite dimensional setting [3–5].

5. Conclusions

In this paper, we make use of suitable approximating techniques and continuous selection theorems to get the existence of projected solutions for $GQVI(\Phi, K)$ with non-self map. In particular, the proposed existence results work under different assumptions in suitable infinite dimensional spaces. In doing this, our main motivation is to use assumptions that allow us to capture the study of as broader as possible range of applications arising from game theory, economics, finance, etc. With this spirit, we analyze a quasioptimization problem under minimal assumptions: the considered problem is central in the study of several real-world phenomena.

On this basis, our future developments go in two directions: from a theoretical viewpoint, we aim to weaken the relative compactness of $K(C)$; at the same time, to support our investigation, we propose to provide further applications that take advantage of the refinements studied in this paper.

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