Nonlinear and nonlocal models of heat conduction in continuum thermodynamics

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Abstract

The aim of this paper is to develop a general constitutive scheme within continuum thermodynamics to describe the behavior of heat flow in deformable media. Starting from a classical thermodynamic approach, the rate-type constitutive equations are defined in the material (Lagrangian) description where the standard time derivative satisfies the principle of objectivity. All constitutive functions are required to depend on a common set of independent variables and to be consistent with thermodynamics. The statement of the Second Law is formulated in a general nonlocal form, where the entropy production rate is prescribed by a non-negative constitutive function and the extra entropy flux obeys a no-flow boundary condition. The thermodynamic response is then developed based on a variant of the Coleman-Noll procedure. In the local formulation, the free energy potential and the rate of entropy production function are assumed to depend on temperature, temperature gradient and heat-flux vector along with their time derivatives. This approach results in rate-type constitutive equations for the heat-flux vector that are intrinsically consistent with the Second Law and easily amenable to analysis. Many linear and nonlinear models of the rate type are recovered (e.g., Cattaneo-Maxwell's, Jeffreys-like, Green-Naghdi's, Quintanilla's and Burgers-like). Owing to the (weakly) nonlocal formulation of the second law, weakly nonlocal models based on the heat-flux vector and its gradients are obtained within this (classical) thermodynamic framework. In particular, the nonlocal Guyer-Krumhansl model and some nonlinear generalizations devised by Cimmelli and Sellitto are obtained. Finally, we propose a new model where the heat flux dependence on temperature gradients is allowed up to second-order.

Keywords: Heat conduction; Nonlocal Second Law; Nonlinear models; Nonlocal rate type equations; Continuum Thermodynamics; Higher-order temperature gradients

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1 Introduction

In the last years, numerous heat conduction models beyond Fourier have been developed to account for relaxational and nonlocal effects, fast phenomena or wave propagation, such as being typical for biological systems, nanomaterials or nanosystems. Non-Fourier models mainly differ for their various thermodynamic backgrounds (thermodynamics of irreversible processes, extended irreversible thermodynamics, etc.). A challenging question is their possible compatibility with the Second Law. In the book of Straughan [1] many of them are presented and discussed in connection with wave propagation properties. A recent review by Kovács [2] discusses properties concerning their possible practical applications in light of experiments. This article aims to discuss their deduction in the context of classical Continuum Thermodynamics and possible compatibility with the Second Law stated therein.

Fourier's law gives a macroscopic description of the microscopic phenomena associated with heat diffusion and is an excellent approximation at length scales much greater than the mean free path and at time scales much greater than the thermal relaxation time. Nevertheless, one of the predicted results of the Fourier law is that temperature disturbances propagate at infinite speeds. This violates the law of special relativity, and also, since in metals the heat conduction is typically attributed to the migration of free electrons and in semiconductors it is explained by the migration of phonons, which are collective vibrations of the atoms in the crystal structure, it would be a reasonable requirement that the propagation speed proves to be finite. This view has spurred much academic interest in the last half century towards seeking a model compatible with a finite speed of propagation (see, e.g., [1, 3–7] and refs therein).

The first non-classical heat conduction model capable of predicting the propagation of heat waves is the Maxwell-Cattaneo-Vernotte law (see [4, 8]). The MCV theory is based on a *rate-type* constitutive equation for the heat flux that predicts heat-wave propagation. Nevertheless, in the recent past some controversy raised about the non-objective character of the constitutive equation which limits its application to rigid bodies at rest only. Several efforts have been devoted to circumvent such a difficulty (cf. [9, 10] and references therein). Yet the MCV model suffer from some other drawbacks. In particular, the non-negativity of the absolute temperature at any time is not preserved (see, e.g., [11, 12]).

A new class of models for heat conduction in a rigid body has been developed in the nineties by Green and Naghdi [13]. In the framework of their general theory, the propagation of thermal waves at finite speed is allowed. Unlike the MCV model, their constitutive equations are completely immune from criticism of objectivity, since do not contain the material derivative of any vector field. Green and Naghdi proposed three types of models, named *type I, type II* and *type III*, respectively, the latter being the most general, which (formally) includes the others as particular instances. An empirical temperature scale is used, not necessarily the absolute one.

In the past decades, Green-Naghdi's theories have attracted growing interest. In particular, Green-Naghdi type III (GN III) theory owes its success to the capability of describing heat propagation by means of thermal waves in addition to diffusive propagation and for this reason it has been applied in a number of disparate physical circumstances, where propagation of heat is coupled with elastic deformations of solids, flow of viscoelastic fluids, etc. (see [1, 14, 15] and references therein). On the other hand, some criticisms have been raised about the GN III heat conduction theory [16–18]. Mainly, it has not been demonstrated that the internal dissipation of entropy (which is assumed to be the subject of a constitutive prescription) is non-negative, as required by the Second Law. Moreover, it has been observed that the GN III model fits well to propagation processes of finite duration (transient regime), but leads to unrealistic effects, at least when either asymptotic or stationary phenomena are involved [19].

Recently, Quintanilla [20] proposed a new thermoelastic heat conduction model which emerged from the development of the GN III by adding a relaxation factor. This theory gives rise to a third-order differential equation for the temperature that looks like the linearized Moore-Gibson-Thompson (MGT) equation in high intensity ultrasound [21]. A similar equation had already been obtained by Joseph and Preziosi starting from the linearization of heat conduction with memory according to the Gurtin-Pipkin model (see [7, eqn.(5.7)]). Despite the advantages and wide diffusion of this new theory [22–24], a convincing proof of its thermodynamic consistency is currently lacking.

Nonlocal effects in generalized equations for heat conduction are assuming particular relevance, also by considering that it is possible to design and manifacture metamaterials and metasurfaces with given nonlocal properties. From another point of view, a local descriptions of solids behavior can be seen as a good approximation of their response to the thermodynamic stresses, the corrections to these descriptions being given by the nonlocal effects [25].

Guyer and Krumhansl [26] studied the heat wave propagation in dielectric crystals at low temperature. They observed that in the regime of low temperature the heat flux $\mathbf{q}$ is proportional to the momentum flux of the phonon gas. On the basis of kinetic theory they found a macroscopic equation governing its evolution. When the relaxation time is negligible, the Guyer-Krumhansl (GK) equation reduces to a nonlocal perturbation of the Fourier law (see [27]). The possibility of utilizing a continuum approach to derive the Guyer-Krumhansl equation results in a model with a wide range of validity since it allows to fit the parameters to the observed phenomenon. Considering an undeformable medium, the GK equation has been derived within the framework of classical irreversible thermodynamics with internal variables [28, 29], GENERIC [30] and extended irreversible thermodynamics [31]. A simple nonlinear extension of the GK model was proposed and scrutinized in [32, 33]. Such a model illustrates relaxational and nonlocal effects of the heat flow in nanosystems. Consistency of the GK model with the Second Law in the context of classical continuum thermodynamics is still an open question. Moreover, it is not entirely clear which boundary conditions should be assigned for the heat flux and its gradient.

1.1 Aims and plan of the paper

In this work we propose a new approach to heat conduction theories that is inherently thermodynamic. Local constitutive equations derive directly from the Clausius-Duhem (C-D) inequality by applying a representation formula and the appropriate choice of the specific Helmholtz free energy ψ and the non-negative entropy production σ. The idea that σ be given by a constitutive equation traces back to Green and Naghdi [13]. However, unlike the Green-Naghdi theories, here σ is independent of the constitutive prescription of the free energy ψ and is non-negative along whatever process.

The set of independent variables includes only the macroscopically observable fields and their temporal and spatial derivatives, without resorting to internal variables or ambiguous state variables (such as thermal displacement). No constitutive prescription on the energy influx $\mathbf{q}$ is made, rather it is treated as an independent variable.

This strategy, previously devised in [34], is compatible with both rigid and deformable bodies. We mention that a similar approach was developed in [35] for deformable ferroelectrics, in [36] for elastic-plastic materials, in [37] for viscoelastic and viscoplastic materials and in [40] for heat conduction in crystals.

We consider both local and (weakly) nonlocal models of heat conduction. Constitutive local theories have to satisfy the classical local formulation of the Second Law where the specific entropy production σ is prescribed as a constitutive function in terms of the independent variables, as well as ψ, and is required to be non-negative in all processes. In Sects 4, 5 and 6 we consider different classes of rate-type models by choosing appropriate sets of variables and expressions for free energy and entropy production. Since time derivatives are involved, we adopt a material description in order to avoid the problem of their objectivity. The temperature evolution equations corresponding to each model of Sects 5 and 6 are discussed for isotropic rigid conductors. Some differential forms of GNIII and Quintanilla models are obtained; their thermodynamic consistency is shown by exhibiting (non-unique) explicit expressions of ψ and σ. A more general heat flow model inspired by the Burgers fluid, also known as dual-phase-lag model [39], is considered and its thermodynamic consistency is proved. Some results from Section 6 appeared in a previous paper [41] without proof. Here, the proofs have been given in detail in the Appendices. All these models can be applied also to fluid heat conductors by rewriting their constitutive rate-type equations into the spatial description (see Sect. 6.3).

As for non-local theories, we refer to Green and Laws [42] who first investigated the restrictions imposed on constitutive equations by a global entropy inequality. According to [43] we reformulated the Green-Laws pioneering idea by arriving at an entropy equation in which the entropy production density $\rho\sigma$ is the sum of two terms; a local entropy supply, $\rho\zeta$, which is required to be non-negative along whatever process, and the divergence of a vector field $\mathbf{k}$, called *extra entropy flux*, whose flow across the boundary of the body is zero. Both ζ and $\mathbf{k}$ are the object of a constitutive prescriptions in terms of the same independent variables as ψ. Among others, linear and nonlinear Guyer-Krumhansl-like heat conduction models are derived in Section 7 and their thermodynamic consistency is shown by exhibiting an explicit expressions of ψ, ζ and $\mathbf{k}$. A connection with the evolution equation of the heat flux in the framework of extended thermodynamics is also established (see, for instance, [31, 44]). Finally, restrictions imposed on $\mathbf{q}$ and its gradient by the no-flow boundary condition for the extra entropy flux $\mathbf{k}$ are discussed. In our opinion these conditions provide an important suggestion for the choice of the most appropriate phonon-boundary conditions in applications to nanosystem.

2 Balance laws and the thermodynamic principles

We consider a body occupying the time dependent region $\Omega_t \subset \mathcal{E}^3$. The motion is described by function $\boldsymbol{\chi}(\mathbf{X}, t)$ providing the position vector $\mathbf{x} \in \Omega_t$ in terms of the position vector $\mathbf{X}$, in a reference configuration $\mathcal{R}$, and the time t, so that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad \Omega_t = \boldsymbol{\chi}(\mathcal{R}, t).$$

The deformation is described by means of the deformation gradient

$$\mathbf{F}(\mathbf{X}, t) = \nabla_{\!R}\, \boldsymbol{\chi}(\mathbf{X}, t), \qquad \text{(in suffix notation } F_{iK} = \partial_{X_K} \chi_i),$$

satisfying the constraint $J := \det \mathbf{F} > 0$. Here, $\nabla_{\!R} := \partial_{\mathbf{X}}$ denotes the gradient in the reference configuration $\mathcal{R}$, whereas the symbol $\nabla := \partial_{\mathbf{x}}$ denotes the gradient in the current configuration Ω. For any regular vector field $\mathbf{w}(\mathbf{x}, t)$, they are related as follows

$$\nabla_{\!R}\, \hat{\mathbf{w}} = \nabla \mathbf{w} \mathbf{F}, \qquad \nabla \mathbf{w} = \nabla_{\!R}\, \hat{\mathbf{w}} \mathbf{F}^{-1},$$

where $\hat{\mathbf{w}} = \mathbf{w}\big(\boldsymbol{\chi}(\mathbf{X}, t), t\big)$. In addition, using the Nanson's formula, we have

$$\nabla_{\!R} \cdot \hat{\mathbf{w}} = J \nabla \cdot \left(J^{-1} \mathbf{F} \mathbf{w}\right), \qquad \nabla \cdot \mathbf{w} = J^{-1} \nabla_{\!R} \cdot (J \mathbf{F}^{-1} \hat{\mathbf{w}}). \tag{1}$$

Hereafter, a superposed dot denotes the standard derivative with respect to time. In particular,

$$\dot{f}(\mathbf{X}, t) := \partial_t f(\mathbf{X}, t), \qquad \dot{g}(\mathbf{x}, t) := \frac{d}{dt} g(\boldsymbol{\chi}(\mathbf{X}, t), t) = \partial_t g(\mathbf{x}, t) + (\mathbf{v} \cdot \nabla) g(\mathbf{x}, t),$$

where f and g are scalar-, vector- or tensor-valued differentiable functions of the reference and current position, respectively. The velocity field $\mathbf{v}(\mathbf{x}, t)$ is such that $\mathbf{v}(\boldsymbol{\chi}(\mathbf{X}, t), t) = \partial_t \boldsymbol{\chi}(\mathbf{X}, t)$ and the velocity gradient $\mathbf{L} := \nabla \mathbf{v}$ is related to $\dot{\mathbf{F}}$ as follows

$$\mathbf{L} = \dot{\mathbf{F}} \mathbf{F}^{-1}. \tag{2}$$

Let ε be the internal energy density (per unit mass), $\mathbf{T}$ the Cauchy stress, $\mathbf{q}$ the heat flux vector, ρ the mass density, r the (external) heat supply and $\mathbf{b}$ the mechanical body force per unit mass. The local form of the linear momentum and internal energy balance equations can be written as

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{T} + \rho\mathbf{b}, \tag{3}$$

$$\rho\dot{\varepsilon} = \mathbf{T}\cdot\mathbf{L} - \nabla\cdot\mathbf{q} + \rho r. \tag{4}$$

The total energy balance, also named First Law of continuum thermodynamics, follows from taking the dot product of (3) by $\mathbf{v}$ and then adding the result to (4),

$$\rho\big[\tfrac{1}{2}|\mathbf{v}|^2 + \varepsilon\big]^{\cdot} = -\nabla\cdot\big[\mathbf{q} - \mathbf{T}\mathbf{v}\big] + \rho[\mathbf{b}\cdot\mathbf{v} + r].$$

Balance of entropy. *Let $\mathcal{P}_t$ be any convecting subregion of Ω_t. All processes which are compatible with the balance equations (3)-(4) must satisfy the following integral equation,*

$$\frac{d}{dt}\int_{\mathcal{P}_t}\rho\,\eta\,dv = -\int_{\partial\mathcal{P}_t}\mathbf{h}\cdot\mathbf{n}\,da + \int_{\mathcal{P}_t}\rho s\,dv,$$

where η is the entropy density and s the entropy supply (per unit mass), $\mathbf{h}$ the entropy-flux vector, $\mathbf{n}$ the outward normal to the boundary.

This general statement has been adopted by Müller [45] by allowing the entropy flux $\mathbf{h}$ to be an unknown function. The classical form of the Second Law is obtained by letting

$$\mathbf{h} = \frac{\mathbf{q}}{\theta}, \qquad s = \frac{r}{\theta} + \sigma,$$

where θ denotes the (positive) absolute temperature and the quantity σ is referred to as specific entropy production rate [50, 51] or simply *entropy production*. Exploiting the arbitrariness of the convecting domain $\mathcal{P}_t$, we obtain the local form of the entropy balance,

$$\rho\dot{\eta} + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\rho r}{\theta} = \rho\sigma. \tag{5}$$

Within continuum thermodynamics two versions of the Second Law can be established depending on the assumptions about σ.

Second Law of Thermodynamics (local form). *Along all compatible processes the specific entropy production rate σ is nonnegative,*

$$\sigma(\mathbf{x}, t) \geq 0. \tag{6}$$

Owing to (5) and (6), this statement leads to the classical form of the C-D inequality

$$\rho\dot{\eta} + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\rho r}{\theta} \geq 0.$$

A more general formulation was introduced and discussed in [42] and can be summarized as follows.

Second Law of Thermodynamics (nonlocal weak form). *The specific entropy production σ is assumed to satisfy the integral condition*

$$\int_{\Omega_t} \rho\, \sigma(\mathbf{x}, t)\, dv \geq 0. \tag{7}$$

This statement is weaker than (6) because the inequality is assumed to be valid only over the entire domain Ω_t and not point-wise. Thus (7) allows for local violations of inequality (6), according to Prigogine's Second Law for open systems [46].

Now, let ζ be a nonnegative scalar field and $\mathbf{k}$ be a regular vector field, usually called *entropy extra flux*. Then, condition (7) can be satisfied by letting

$$\rho\sigma = \rho\zeta - \nabla \cdot \mathbf{k} \tag{8}$$

and assuming

$$\int_{\Omega_t} \rho\, \zeta\, dv - \int_{\partial\Omega_t} \mathbf{k} \cdot \mathbf{n}\, da \geq 0. \tag{9}$$

First we note that equation (8) could be interpreted as a microscopic local balance law at any point $\mathbf{x} \in \Omega_t$, where the entropy production rate (per unit volume), $\rho\sigma$, is due to a *local entropy supply*, $\rho\zeta$, and a *local entropy production flux*, $\mathbf{k}$, that is exchanged on a microscopic level with the surrounding points. The idea to introduce the extra entropy flux traces back to [45] and was taken up and discussed in [43]. In connection with phase transition, a balance law of microscopic forces with the same form as (8) was first introduced by Fried and Gurtin in [47].

A sufficient condition for inequality (9) to be verified is given by

$$\int_{\Omega_t} \rho\, \zeta\, dv \geq 0, \qquad \int_{\partial\Omega_t} \mathbf{k} \cdot \mathbf{n}\, da = 0. \tag{10}$$

The first condition is required when $\mathbf{k}$ vanishes identically. The last condition implies that entropy production cannot be exchanged with the external environment, so that the entropy production rate is due only to the local supply within the body. This assumption is consistent with the definition of a 'closed' system according to Prigogine [46]. Conditions (10) are also suggested by applications to phase-field theories [43] and ferroelectricity [48] where no evidence emerges for a net flow of $\mathbf{k}$ across the external boundary of the body.

A point-wise formulation of (10) leads to a stronger formulation of the Second Law.

Second Law of Thermodynamics (nonlocal strong form). *During all compatible processes the entropy extra flow at any point of the body boundary is zero and the local entropy supply at any point of the body interior is nonnegative, namely*

$$\mathbf{k} \cdot \mathbf{n}\big|_{\partial\Omega_t} = 0, \quad \zeta \geq 0 \quad \text{in } \Omega_t. \tag{11}$$

After replacing $(8)_1$ into (5) it follows the local inequality

$$\rho\dot{\eta} + \nabla \cdot \left(\frac{\mathbf{q}}{\theta} + \mathbf{k} \right) - \frac{\rho r}{\theta} = \rho\zeta \geq 0, \tag{12}$$

that resembles the classical entropy inequality except that the flux vector is redefined by adding the extra contribution $\mathbf{k}$. This suggests that $\mathbf{k}$ is a characteristic nonlocal contribute and it must disappear when only uniform processes are involved (see, e.g., [49]).

Assumption 2.1. *We assume that $\mathbf{k}$ and ζ, as well as the internal energy ε and the entropy η, are given independently as constitutive functions of the assigned set of variables.*

Upon substitution of $\nabla \cdot \mathbf{q} - \rho r$ from the energy equation (4) into (12) and multiplication by θ we obtain the basic thermodynamic relation

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T} \cdot \mathbf{L} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta + \theta\nabla \cdot \mathbf{k} = \rho\theta\zeta, \tag{13}$$

where $\psi = \varepsilon - \theta\eta$ denotes the *Helmholtz free energy density*. Due to $(11)_2$, equation (13) becomes an inequality that must be satisfied along whatever process.

Finally, multiplying (13) by J and using the identities (1), (2) and $\rho_R = J\rho$, we obtain the basic thermodynamic inequality in the material description

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\,\theta + \theta\nabla_R \cdot \mathbf{k}_R = \rho_R\theta\zeta \geq 0, \tag{14}$$

where $\nabla_R := \mathbf{F}^T\nabla$, $\mathbf{E}$ is the Green-St Venant strain tensor and

$$\mathbf{T}_{RR} := J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}, \qquad \mathbf{q}_R := J\mathbf{F}^{-1}\mathbf{q}, \qquad \mathbf{k}_R := J\mathbf{F}^{-1}\mathbf{k}, \quad \mathbf{k}_R \cdot \mathbf{n}_R\big|_{\partial\mathcal{R}} = 0.$$

3 Local models of heat conduction

In the first part of this paper we restrict our attention to *local* models, that is we assume $\mathbf{k} = 0$ and then $\sigma = \zeta \geq 0$. Consequently (14) reduces to

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta}\,\mathbf{q}_R \cdot \nabla_R\,\theta = \rho_R\theta\sigma \geq 0. \tag{15}$$

Within the isothermal setting, the term $\rho_R\theta\sigma$ is usually referred to as *rate of (mechanical) dissipation*.

In recent papers [36, 37] several local models of viscoelastic, viscoplastic and elastic-plastic materials (solids and fluids) are developed on the basis of (15) by specifying three elements;

- the set Ξ_R of admissible variables,
- the free energy density function $\psi = \psi(\Xi_R)$,
- the entropy production function $\sigma = \sigma(\Xi_R)$.

The idea that the entropy production σ be given by a constitutive equation traces back to Green and Naghdi [13]. However, they do not require $\sigma \geq 0$ along whatever process and implicitly assume that constitutive equations of entropy production and free energy are related. Their approach was extended in [34] assuming from the outset that all constitutive functions, including that of entropy production, depend on a common set of physical variables and imposing $\sigma \geq 0$. Conceptually, our contribution follows a very similar scheme by exhibiting explicit expression for the entropy production.

Likewise, thermodynamically consistent constitutive equations of the rate type are devised here for heat-conduction in solids. The key idea is to exploit the formal similarity of the scalar products

$$\mathbf{T}_{RR} \cdot \dot{\mathbf{E}}, \qquad -\mathbf{q}_R \cdot \nabla_R (\ln \theta)$$

that represent the mechanical and thermal powers of internal forces, respectively. In [37] the exploitation of the thermodynamic inequality involving the mechanical power leads to some well-known viscoelastic models. By mimicking such a procedure, the correspondence between $\mathbf{T}_{RR}$ and $\mathbf{q}_R$, as well as that between $\dot{\mathbf{E}}$ and $-\nabla_R (\ln \theta)$, gives rise to as many models of thermal conductivity.

3.1 Basic local models

The simplest local models can be obtained by choosing a small set of independent variables, namely the temperature field and its gradient along with a strain measure and its rate. Since invariance requirements demand that the dependence on time and space derivatives occurs in an objective way, we let

$$\Xi_R := (\theta, \mathbf{E}, \nabla_R \theta, \dot{\mathbf{E}})$$

be the basic set of invariant variables. Accordingly, Euclidean invariance of ψ, η and σ implies that their dependence be a function of Ξ_R. Here and below, we assume η is continuous while ψ is continuously differentiable.

Upon evaluation of $\dot{\psi}$ and substitution in (15) we obtain

$$\rho_R(\partial_\theta \psi + \eta)\dot{\theta} + (\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\nabla_R \theta} \psi \cdot \nabla_R \dot{\theta} + \rho_R \partial_{\dot{\mathbf{E}}} \psi \cdot \ddot{\mathbf{E}} + \frac{1}{\theta} \mathbf{q}_R \cdot \nabla_R \theta = -\rho_R \theta \sigma.$$

Due to the constitutive assumption on ψ, η and σ, this expression depends linearly on $\dot{\theta}$, $\ddot{\mathbf{E}}$ and $\nabla_R \dot{\theta}$. Hence, their arbitrariness implies that

$$\psi = \psi(\theta, \mathbf{E}), \qquad \eta = -\partial_\theta \psi, \qquad (\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \frac{1}{\theta} \mathbf{q}_R \cdot \nabla_R \theta = -\rho_R \theta \sigma \leq 0$$

If σ is independent of $\dot{\mathbf{E}}$ then the linearity and arbitrariness of $\dot{\mathbf{E}}$ give

$$\mathbf{T}_{RR} = \rho_R \partial_{\mathbf{E}} \psi, \qquad \mathbf{q}_R \cdot \nabla_R \theta = -\rho_R \theta^2 \sigma \leq 0.$$

As is well known, this occurs in heat conducting hyperelastic materials.

Otherwise, we split the last inequality as follows[1] [38]

$$(\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi) \cdot \dot{\mathbf{E}} = \rho_R \theta \sigma^{ET} \geq 0, \quad \mathbf{q}_R \cdot \nabla_R \theta = -\rho_R \theta^2 \sigma^q \leq 0. \tag{16}$$

For example, it is easy to check that the former inequality $(16)_1$ is satisfied by the Kelvin-Voigt viscoelastic model,

$$\mathbf{T}_{RR} = \mathbb{C}\mathbf{E} + \mathbb{K}\dot{\mathbf{E}},$$

and thermodynamic consistency is ensured by letting

$$\rho_R \psi = \frac{1}{2} \mathbf{E} \cdot \mathbb{C}\mathbf{E}, \qquad \rho_R \theta \sigma^{ET} = \dot{\mathbf{E}} \cdot \mathbb{K}\dot{\mathbf{E}} \geq 0, \tag{17}$$

where $\mathbb{K}$ is a positive semidefinite fourth-order tensor. On the other hand, the latter inequality $(16)_2$ is satisfied by assuming Fourier's law for the heat flux vector,

$$\mathbf{q}_R = -\boldsymbol{\kappa} \nabla_R \theta.$$

In this case ψ is not involved and thermodynamic consistency requires

$$\rho_R \sigma^q = \nabla_R (\ln \theta) \cdot \boldsymbol{\kappa} \nabla_R (\ln \theta) \geq 0, \tag{18}$$

so that $\boldsymbol{\kappa}$ must be a positive semidefinite second-order tensor.

A comparison of the expressions of σ^{ET} and σ^q as given by (17) and (18) respectively, provides immediate evidence of our key idea.

In order to derive more general constitutive equations for $\mathbf{T}_{RR}$ and $\mathbf{q}_R$ we take advantage of the following property (see, *e.g.*, [38, Proposition 1]).

Representation Lemma 3.1. *Given a vector (or a tensor)* $\mathbf{A}$*, let* $\mathbf{N} = \mathbf{A}/|\mathbf{A}|$*. If* $\mathbf{Z}$ *is a vector (or a tensor) such that only its projection on* $\mathbf{N}$ *is known, namely*

$$\mathbf{Z} \cdot \mathbf{N} = g,$$

then for any vector (or tensor) $\mathbf{G}$ *we can write*

$$\mathbf{Z} = g\mathbf{N} + (\mathbf{1} - \mathbf{N} \otimes \mathbf{N})\mathbf{G} \tag{19}$$

where $\mathbf{1}$ *is the second-order (or fourth-order) identity tensor.*

Let $\mathbb{M}$ be a symmetric non-singular fourth-order tensor. After applying the Representation Lemma to $(16)_1$ with

$$\mathbf{N} = \mathbb{M}\dot{\mathbf{E}}/|\mathbb{M}\dot{\mathbf{E}}|, \qquad \mathbf{Z} = \mathbb{M}^{-1}[\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi], \qquad \mathbf{G} = \mathbf{0},$$

[1]Assuming that $\mathbf{T}_{RR}$ is independent of $\nabla_R \theta$ and $\mathbf{q}_R$ is independent of $\dot{\mathbf{E}}$, as usually happens, then $\sigma^{ET} = \sigma|_{\nabla_R \theta = \mathbf{0}}$ and $\sigma^q = \sigma|_{\dot{\mathbf{E}} = \mathbf{0}}$.

we obtain the nonlinear constitutive equation

$$\mathbf{T}_{RR} = \rho_R \partial_{\mathbf{E}} \psi + \rho_R \theta \sigma^{ET} \frac{\mathbb{M}^2 \dot{\mathbf{E}}}{|\mathbb{M}\dot{\mathbf{E}}|^2}.$$

In particular, the linear Kelvin-Voigt relation follows from (17) with the special choice $\mathbb{K} = \mathbb{M}^2$.

Let $\mathbf{A}$ be a any symmetric, non-singular, second-order tensor, possibly parametrized by the temperature θ. Applying (19) with $\mathbf{N} = \mathbf{A}\nabla_R \theta / |\mathbf{A}\nabla_R \theta|$, $\mathbf{Z} = \mathbf{A}^{-1}\mathbf{q}_R$, and $\mathbf{G} = \mathbf{0}$, from $(16)_2$ it follows that

$$\mathbf{q}_R = -\frac{\rho_R \theta^2 \sigma^q}{|\mathbf{A}\nabla_R \theta|^2} \mathbf{A}^2 \nabla_R \theta.$$

In particular, assuming $\boldsymbol{\kappa} = \mathbf{A}^2$ and σ^q as in (18), we recover Fourier's law. Note however that in this case $\boldsymbol{\kappa}$ turns out to be strictly positive definite because $\mathbf{A}$ is non-singular.

3.2 Rate-type local models

To describe rate-type models we expand the basic set of Euclidean invariant variables by adding some quantities that are usually not assumed to be independent, but considered as constitutive functions; specifically, $\mathbf{T}_{RR}$ and $\mathbf{q}_R$. Hence we let

$$\Xi_R := (\theta, \mathbf{E}, \mathbf{T}_{RR}, \mathbf{q}_R, \nabla_R \theta, \dot{\mathbf{E}})$$

be the set of independent variables and assume that ψ, η, σ are scalar-valued functions of Ξ_R. In view of the introduction of rate-type models, we look for a scheme where $\dot{\mathbf{q}}_R$ and $\dot{\mathbf{T}}_{RR}$, as well as ψ, η and σ, are considered as constitutive functions depending on Ξ_R. In particular, this approach leads to differential relations in which $\dot{\mathbf{q}}_R$ and $\nabla_R \theta$, as well as $\dot{\mathbf{T}}_{RR}$ and $\dot{\mathbf{E}}$, appear to be mutually dependent variables[2].

Upon evaluation of $\dot{\psi}$ and substitution in (15) we obtain

$$\rho_R(\partial_\theta \psi + \eta)\dot{\theta} + (\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} + \rho_R \partial_{\mathbf{q}_R} \psi \cdot \dot{\mathbf{q}}_R$$
$$+ \rho_R \partial_{\nabla_R \theta} \psi \cdot \nabla_R \dot{\theta} + \rho_R \partial_{\dot{\mathbf{E}}} \psi \cdot \ddot{\mathbf{E}} + \frac{1}{\theta} \mathbf{q}_R \cdot \nabla_R \theta = -\rho_R \theta \sigma.$$

The linearity and arbitrariness of $\dot{\theta}$, $\ddot{\mathbf{E}}$, $\nabla_R \dot{\theta}$, imply that

$$\psi = \psi(\theta, \mathbf{E}, \mathbf{T}_{RR}, \mathbf{q}_R), \qquad \eta = -\partial_\theta \psi. \tag{20}$$

and the thermodynamic inequality reduces to

$$(\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} + \rho_R \partial_{\mathbf{q}_R} \psi \cdot \dot{\mathbf{q}}_R + \frac{\mathbf{q}_R}{\theta} \cdot \nabla_R \theta = -\rho_R \theta \sigma \le 0, \tag{21}$$

[2] A similar approach occurs in anholonomic system described by a set of independent parameters subject to differential constraints that make their rates mutually dependent.

where $\dot{\mathbf{T}}_{RR}$ and $\dot{\mathbf{q}}_R$ are functions of Ξ_R. The independent variables appearing as arguments of ψ in (20) are referred to as *state variables*,

$$\Sigma_R = (\theta, \mathbf{E}, \mathbf{T}_{RR}, \mathbf{q}_R).$$

It is reasonable to assume that $\dot{\mathbf{T}}_{RR}$ and $\dot{\mathbf{q}}_R$ are functionally independent. Moreover, we let $\dot{\mathbf{T}}_{RR}$ be independent of $\nabla_R \theta$, as usually happens. Since ψ is also independent of $\nabla_R \theta$, then letting $\dot{\mathbf{q}}_R = \nabla_R \theta = \mathbf{0}$ we can write (21) in the form

$$(\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} = -\rho_R \theta \sigma^{ET} \leq 0, \tag{22}$$

where σ^{ET} is the entropy production density σ when $\dot{\mathbf{q}}_R = \nabla_R \theta = \mathbf{0}$. Likewise, assuming that $\dot{\mathbf{q}}_R$ is independent of $\dot{\mathbf{E}}$ then

$$\rho_R \partial_{\mathbf{q}_R} \psi \cdot \dot{\mathbf{q}}_R + \frac{\mathbf{q}_R}{\theta} \cdot \nabla_R \theta = -\rho_R \theta \sigma^q \leq 0, \tag{23}$$

where σ^q is the entropy production density when $\dot{\mathbf{T}}_{RR} = \dot{\mathbf{E}} = \mathbf{0}$. Apparently, $\sigma = \sigma^{ET} + \sigma^q$. The entropy productions σ^{ET} and σ^q, as well as σ, are nonnegative constitutive functions of Ξ_R to be determined according to the constitutive model.

By exploiting inequality (22), memory properties of viscoelasticity, elastoplasticity and viscoplasticity were modeled with suitable nonlinear rate-type stress-strain relations [36, 37]. However, since the aim of this paper is to establish nonlinear rate-type models of heat conduction, in the following we limit our attention to (23) and disregard (22).

4 Rate-type local models of heat conduction

Although inequality (23) is common to many approaches where both $\mathbf{q}_R$ and $\nabla_R \theta$ are independent variables [52, 53], our scheme has the advantage that the material time derivative is objective and makes consistency with thermodynamics much easier than it happens when heat conduction involves histories [54], summed histories [6] or internal variables [55, 56].

In this Section, we restrict our attention to the C-D inequality in the form (23) and all constitutive functions, including $\dot{\mathbf{q}}_R$, are assumed to depend on the independent variables

$$\Xi_R := (\theta, \mathbf{q}_R, \nabla_R \theta),$$

whereas the free energy (20) reduces to $\psi = \psi(\theta, \mathbf{q}_R)$.

In the following we present some examples by choosing special expressions for the constitutive functions ψ and σ. Many of them match with well-established thermal conduction models. Despite this, it is instructional to see how they fit within our approach.

4.1 Green-Naghdi type II heat conductors

A (linear) type II heat conductor according to Green and Naghdi [13] is characterized by

$$\mathbf{q}_R = -\boldsymbol{\xi}\nabla_R\,\alpha, \tag{24}$$

where α is an internal variable, named thermal displacement, that represents, by definition, a time primitive of the temperature, namely $\dot{\alpha} = \theta$ and $\boldsymbol{\xi}$ is a constant tensor depending on the material properties. Since this relation enters the heat equation for the temperature θ only after a differentiation with respect to time, namely

$$\dot{\mathbf{q}}_R = -\boldsymbol{\xi}\nabla_R\,\theta, \tag{25}$$

it is reasonable to consider (25) instead of (24) as constitutive equation, especially since such a rate-type model can be easily obtained from (23) by appealing to the Representation Lemma 3.1.

Assuming that $\sigma = 0$ eqn. (23) becomes

$$\rho_R\partial_{\mathbf{q}_R}\psi \cdot \dot{\mathbf{q}}_R = -\frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\,\theta. \tag{26}$$

This condition characterizes heat conductors without entropy intrinsic production, that means without any dissipation. An explicit relation for $\dot{\mathbf{q}}_R$ then follows by applying (19) with

$$\mathbf{N} = \partial_{\mathbf{q}_R}\psi/|\partial_{\mathbf{q}_R}\psi|, \qquad \mathbf{Z} = \dot{\mathbf{q}}_R, \qquad \mathbf{G} = \mathbf{J}_R\nabla_R\,\theta,$$

where $\mathbf{J}_R = \mathbf{J}_R(\theta, \mathbf{q}_R)$ is an arbitrary second-order tensor-valued function. Using (26) and assuming $\partial_{\mathbf{q}_R}\psi \neq \mathbf{0}$ we have

$$\mathbf{Z} \cdot \mathbf{N} = -\frac{\mathbf{q}_R \cdot \nabla_R\,\theta}{\rho_R\theta|\partial_{\mathbf{q}_R}\psi|}$$

and then

$$\dot{\mathbf{q}}_R = -\left[\frac{\mathbf{q}_R \cdot \nabla_R\,\theta}{\rho_R\theta|\partial_{\mathbf{q}_R}\psi|^2}\right]\partial_{\mathbf{q}_R}\psi + (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{J}_R\nabla_R\,\theta. \tag{27}$$

Finally, letting

$$\begin{aligned}\mathbf{K}_R(\theta, \mathbf{q}_R) &= \frac{1}{\rho_R\theta|\partial_{\mathbf{q}_R}\psi|}\mathbf{N}\otimes\mathbf{q}_R - (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{J}_R \\ &= -\mathbf{J}_R + \frac{1}{\rho_R|\partial_{\mathbf{q}_R}\psi|^2}\partial_{\mathbf{q}_R}\psi\otimes(\mathbf{q}_R/\theta + \rho_R\mathbf{J}_R^T\partial_{\mathbf{q}_R}\psi),\end{aligned} \tag{28}$$

we can write (27) in the more compact form,

$$\dot{\mathbf{q}}_R = -\mathbf{K}_R\nabla_R\,\theta. \tag{29}$$

and we can identify $\boldsymbol{\xi}$ in 24 with $\mathbf{K}_R$. Owing to the arbitrariness of $\mathbf{J}_R$ there are infinitely many tensor-valued functions $\mathbf{K}_R$ compatible with a given free energy ψ. Moreover, $\mathbf{K}_R$ need not be positive-definite or symmetrical and this is a notable difference from the conductivity tensor in Fourier-like models.

Thanks to the aforementioned correspondence between the pairs $\mathbf{T}_{RR}, \dot{\mathbf{E}}$ and $\mathbf{q}_R, \nabla_R(\ln\theta)$, the heat conduction law (25) finds its mechanical counterpart in hypoelasticity. A nonlinear extension can be obtained as in [36]. Let $\psi(\theta, \mathbf{q}_R)$ depend on $\mathbf{q}_R$ via $\xi = |\mathbf{q}_R|^n$, $n \geq 2$. Hence

$$\partial_{\mathbf{q}_R}\psi = n\,\partial_\xi\psi\,|\mathbf{q}_R|^{n-2}\,\mathbf{q}_R, \quad \partial_\xi\psi \neq 0,$$

and (28) becomes

$$\mathbf{K}_R = -\mathbf{J}_R + \frac{1}{\rho_R\partial_\xi\psi\,n|\mathbf{q}_R|^n}\,\mathbf{q}_R \otimes \Big(\frac{\mathbf{q}_R}{\theta} + n\,\rho_R\partial_\xi\psi\,|\mathbf{q}_R|^{n-2}\,\mathbf{J}_R^T\mathbf{q}_R\Big).$$

Finally, choosing

$$\mathbf{J}_R = -\frac{1}{n\,\theta\rho_R\partial_\xi\psi\,|\mathbf{q}_R|^{n-2}}\mathbf{1}$$

we obtain

$$\mathbf{K}_R = -\mathbf{J}_R = \frac{1}{n\,\theta\rho_R\partial_\xi\psi\,|\mathbf{q}_R|^{n-2}}\mathbf{1}$$

and then

$$\dot{\mathbf{q}}_R = -\frac{\nabla_R\,\theta}{n\,\theta\rho_R\partial_\xi\psi\,|\mathbf{q}_R|^{n-2}}.$$

Note that $\mathbf{K}_R$ is positive definite if and only if $\partial_\xi\psi > 0$, i.e. if ψ is a convex function of $\mathbf{q}_R$. On the contrary, if the constitutive relation is given in advance, for instance

$$\dot{\mathbf{q}}_R = -\hat{\mathbf{K}}_R\nabla_R\,\theta, \tag{30}$$

$\hat{\mathbf{K}}_R$ being a fixed second-order tensor, one can ask whether this model is thermodynamically consistent. Upon substitution into (26) we obtain

$$\rho_R\partial_{\mathbf{q}_R}\psi \cdot \hat{\mathbf{K}}_R\nabla_R\,\theta = \frac{\mathbf{q}_R}{\theta} \cdot \nabla_R\,\theta$$

and the arbitrariness of $\nabla_R\,\theta$ implies

$$\mathbf{q}_R = \rho_R\theta\hat{\mathbf{K}}_R^T\partial_{\mathbf{q}_R}\psi. \tag{31}$$

If we replace (31) into (28) we obtain

$$\mathbf{K}_R = -\mathbf{J}_R - \frac{1}{|\partial_{\mathbf{q}_R}\psi|^2}\partial_{\mathbf{q}_R}\psi \otimes [(\hat{\mathbf{K}}_R^T + \mathbf{J}_R^T)\partial_{\mathbf{q}_R}\psi].$$

Finally, by letting $\mathbf{J}_R = -\hat{\mathbf{K}}_R$, the identity $\mathbf{K}_R = \hat{\mathbf{K}}_R$ follows. Since $\sigma = 0$ we infer that (30) is consistent with the Second Law if and only if there exists a free energy ψ

satisfying (31). This is the case provided that $\hat{\mathbf{K}}_R$ is a non-singular symmetric constant tensor and the free energy is given by

$$\rho_R\psi = \rho_R\psi_0(\theta) + \frac{1}{2\theta}\mathbf{q}_R \cdot \hat{\mathbf{K}}_R^{-1}\mathbf{q}_R.$$

Assuming, for instance, $\hat{\mathbf{K}}_R = f(\theta, |\mathbf{q}_R|)\mathbf{1}$, thermodynamic consistency of the model is ensured by letting

$$\rho_R\psi = \rho_R\psi_0(\theta) + \frac{1}{\theta}\, g(\theta,\xi), \quad \xi = |\mathbf{q}_R|^n,\ n \geq 2,$$

provided that

$$\partial_\xi g(\theta,\xi) = \frac{1}{f(\theta, |\mathbf{q}_R|)\, |\mathbf{q}_R|^{n-2}}.$$

4.2 Maxwell-Cattaneo-Vernotte-like models

Assume $\sigma > 0$ and $\partial_{\mathbf{q}_R}\psi \neq \mathbf{0}$. Equation (23) can then be written in the form

$$\frac{\partial_{\mathbf{q}_R}\psi}{|\partial_{\mathbf{q}_R}\psi|} \cdot \dot{\mathbf{q}}_R = -\frac{1}{\rho_R|\partial_{\mathbf{q}_R}\psi|}\Big(\frac{\mathbf{q}_R}{\theta} \cdot \nabla_R\,\theta + \rho_R\theta\sigma\Big).$$

Eexploiting this relation and applying (19) with $\mathbf{N} = \partial_{\mathbf{q}_R}\psi/|\partial_{\mathbf{q}_R}\psi|$, $\mathbf{Z} = \dot{\mathbf{q}}_R$ we obtain

$$\dot{\mathbf{q}}_R = -\Big(\frac{\mathbf{q}_R}{\theta} \cdot \nabla_R\,\theta + \rho_R\theta\sigma\Big)\frac{\mathbf{N}}{\rho_R|\partial_{\mathbf{q}_R}\psi|} + (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{G}, \tag{32}$$

where $\mathbf{G}$ is an arbitrary vector-valued function dependent on $(\theta, \mathbf{q}_R, \nabla_R\,\theta)$.

We now show that a class of nonlinear isotropic models of the Maxwell-Cattaneo-Vernotte (MCV) type [1], can be derived from (32) by properly choosing ψ and σ (see, for instance, [37]). Let $\psi(\theta, \mathbf{q}_R)$ depend on $\mathbf{q}_R$ via $\xi = |\mathbf{q}_R|^n$, $n \geq 2$. Hence

$$\partial_{\mathbf{q}_R}\psi = n\,\partial_\xi\psi\,|\mathbf{q}_R|^{n-2}\,\mathbf{q}_R,$$

and inequality (23) becomes

$$\Big(n\rho_R\partial_\xi\psi|\mathbf{q}_R|^{n-2}\dot{\mathbf{q}}_R + \frac{1}{\theta}\nabla_R\,\theta\Big) \cdot \mathbf{q}_R = -\rho_R\theta\sigma \leq 0.$$

Let

$$\sigma = \frac{|\mathbf{q}_R|^2}{\rho_R\theta^2\kappa}, \tag{33}$$

where κ is a positive-valued scalar function. In view of (32), if $\partial_\xi\psi \neq 0$ it follows

$$n\rho_R\partial_\xi\psi|\mathbf{q}_R|^{n-2}\dot{\mathbf{q}}_R = -\frac{1}{\theta}\nabla_R\,\theta - \frac{1}{\kappa\theta}\mathbf{q}_R.$$

Consequently,

$$\tau \dot{\mathbf{q}}_R + \mathbf{q}_R = -\kappa \nabla_R \theta, \qquad \tau = \kappa n \rho_R \theta \partial_\xi \psi |\mathbf{q}_R|^{n-2},$$

can be viewed as a MCV equation with τ playing the role of relaxation time and κ representing the heat conductivity. If $n = 2$ then

$$\tau = 2\kappa \rho_R \theta \partial_\xi \psi.$$

In this case, τ reduces to a function of the temperature alone provided that $\kappa = \kappa(\theta)$ and ψ is a linear function of $\xi = |\mathbf{q}_R|^2$ (see [57]). Hence τ can be viewed as a *relaxation time* and the Fourier law is recovered when $\tau \to 0$.

In addition, a more general class of anisotropic models of the MCV type can be derived from (32). To this end let

$$\rho_R \psi = \rho_R \psi_0(\theta) + \frac{\tau}{2\theta} \mathbf{q}_R \cdot \boldsymbol{\kappa}^{-1} \mathbf{q}_R, \qquad \sigma = \frac{1}{\rho_R \theta^2} \mathbf{q}_R \cdot \boldsymbol{\kappa}^{-1} \mathbf{q}_R.$$

where $\boldsymbol{\kappa}$ must be a positive-definite second-order tensor in order to have $\sigma \geq 0$. In view of (32) it follows

$$\dot{\mathbf{q}}_R = -\frac{1}{\tau}(\boldsymbol{\kappa}\mathbf{N} \cdot \nabla_R \theta + \mathbf{q}_R \cdot \mathbf{N})\mathbf{N} + (\mathbf{1} - \mathbf{N} \otimes \mathbf{N})\mathbf{G} = \mathbf{G} - \frac{1}{\tau}\mathbf{N} \otimes \mathbf{N}(\boldsymbol{\kappa}\nabla_R \theta + \mathbf{q}_R + \tau \mathbf{G}),$$

where $\mathbf{N} = \boldsymbol{\kappa}^{-1}\mathbf{q}_R / |\boldsymbol{\kappa}^{-1}\mathbf{q}_R|$, and letting $\tau \mathbf{G} = -\boldsymbol{\kappa}\nabla_R \theta - \mathbf{q}_R$ we obtain

$$\tau \dot{\mathbf{q}}_R + \mathbf{q}_R = -\boldsymbol{\kappa} \nabla_R \theta. \tag{34}$$

The sign of τ is not prescribed by thermodynamic arguments. However, the common assumption $\tau > 0$ implies that ψ has a strict minimum at $\mathbf{q}_R = \mathbf{0}$. This in turn implies that ψ is a strictly convex function of $\mathbf{q}_R$. On a more general, fully anisotropic approach, it could be possible to introduce a matrix of relaxation times, like one has for extended irreversible thermodynamics [31].

5 An enlarged set of independent variables

For simplicity, in the following we restrict our attention to homogeneous rigid heat conductors. If this is the case, material and spatial description coincide so that all subscripts $_R$ can be neglected. Accordingly, we consider the C-D inequality in the reduced form

$$-\rho(\dot{\psi} + \eta \dot{\theta}) - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta = \rho \theta \sigma \geq 0, \tag{35}$$

where $\rho = \rho_R$ is constant and $\dot{} = \partial_t$.

In this Section, we expand the set Ξ_R considered in the previous Section by adding first-order time derivatives of the temperature and its gradient. Hence all constitutive functions, including $\dot{\mathbf{q}}$, are assumed to depend on the independent variables

$$\Xi := (\theta, \mathbf{q}, \nabla \theta, \dot{\theta}, \nabla \dot{\theta}).$$

Upon evaluation of $\dot{\psi}$ and substitution in (35) we obtain

$$\rho(\partial_\theta \psi + \eta)\dot{\theta} + \rho \partial_{\dot{\theta}} \psi \cdot \ddot{\theta} + \rho \partial_{\mathbf{q}} \psi \cdot \dot{\mathbf{q}} + \rho \partial_{\nabla\theta} \psi \cdot \nabla\dot{\theta} + \rho \partial_{\nabla\dot{\theta}} \psi \cdot \nabla\ddot{\theta} + \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta = -\rho\theta\sigma.$$

The linearity and arbitrariness of $\ddot{\theta}$ and $\nabla\ddot{\theta}$ imply that ψ is independent of $\dot{\theta}$ and $\nabla\dot{\theta}$. Assuming, for simplicity, that σ and $\dot{\mathbf{q}}$ are also independent of $\dot{\theta}$, the linearity and arbitrariness of $\dot{\theta}$ imply

$$\psi = \psi(\theta, \mathbf{q}, \nabla\theta), \qquad \eta = -\partial_\theta \psi,$$

so that $\Sigma = (\theta, \mathbf{q}, \nabla\theta)$ and the entropy inequality reduces to

$$\rho \partial_{\mathbf{q}} \psi \cdot \dot{\mathbf{q}} + \rho \partial_{\nabla\theta} \psi \cdot \nabla\dot{\theta} + \frac{\mathbf{q}}{\theta} \cdot \nabla\theta = -\rho\theta\sigma \leq 0, \tag{36}$$

where σ and $\dot{\mathbf{q}}$ depend on $(\theta, \mathbf{q}, \nabla\theta, \nabla\dot{\theta})$. A similar procedure has been adopted in [37] to evaluate the thermodynamic consistency of some non-linear viscoelastic models of the rate-type, such as the Oldroyd-B fluids.

5.1 Heat conductors of the Jeffreys type

The constitutive equation of a heat conductor of the Jeffreys type is given by

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\boldsymbol{\xi}\nabla\theta - \tau\boldsymbol{\kappa}\nabla\dot{\theta}, \qquad \boldsymbol{\xi}, \boldsymbol{\kappa} \in \mathrm{Sym}^+, \tag{37}$$

where $\tau > 0$ is referred to as *relaxation time* and Sym^+ denotes the set of symmetric and positive-definite tensors of the second order. This model can also be written as a system

$$\mathbf{q} = -\boldsymbol{\kappa}\nabla\theta + \mathbf{y}, \qquad \tau\dot{\mathbf{y}} + \mathbf{y} = (\boldsymbol{\kappa} - \boldsymbol{\xi})\nabla\theta.$$

In this way we can interpret the Jeffreys model as the constitutive law of the heat flux vector $\mathbf{q}$ in a mixture of two different conductors, $\mathbf{q} = \mathbf{q}^{(1)} + \mathbf{q}^{(2)}$, one of the Fourier type, $\mathbf{q}^{(1)} = -\boldsymbol{\kappa}^{(1)}\nabla\theta$, $\boldsymbol{\kappa}^{(1)} = \boldsymbol{\kappa}$, the other of the MCV type, $\mathbf{q}^{(2)} = \mathbf{y}$, $\boldsymbol{\kappa}^{(2)} = \boldsymbol{\xi} - \boldsymbol{\kappa}$. A similar result has been obtained in [28] within the framework of classical irreversible thermodynamics.

In the limits as $\tau \to 0^+$ and $\tau \to +\infty$ we have, respectively,

$$\mathbf{q} \simeq -\boldsymbol{\xi}\nabla\theta, \qquad \dot{\mathbf{q}} \simeq -\boldsymbol{\kappa}\nabla\dot{\theta}.$$

Furthermore, when $\boldsymbol{\kappa} = \mathbf{0}$ the anisotropic MCV model (34) is recovered.

We prove that the rate equation (37) can be derived from (36) by applying the representation formula. The selection $\mathbf{Z} = \dot{\mathbf{q}}$ and $\mathbf{N} = \partial_{\mathbf{q}}\psi / |\partial_{\mathbf{q}}\psi|$ into (19) together with the exploitation of (36) provide

$$\dot{\mathbf{q}} = -\Big[\frac{1}{\theta}\,\mathbf{q} \cdot \nabla\theta + \rho \partial_{\nabla\theta}\,\psi \cdot \nabla\dot{\theta} + \rho\theta\sigma\Big] \frac{\partial_{\mathbf{q}}\psi}{\rho|\partial_{\mathbf{q}}\psi|^2} + (\mathbf{1} - \mathbf{N} \otimes \mathbf{N})\mathbf{G}, \tag{38}$$

where $\mathbf{G}$ is an arbitrary second-order tensor-valued function of Ξ.

Let Sym* be the set of symmetric and non singular second-order tensors and let

$$\rho\psi = \rho\psi_0(\theta) + \frac{\tau}{2\theta}[\mathbf{q} + \boldsymbol{\kappa}\nabla\theta] \cdot (\boldsymbol{\xi} + \boldsymbol{\kappa})^{-1}[\mathbf{q} + \boldsymbol{\kappa}\nabla\theta],$$

with $\boldsymbol{\xi} + \boldsymbol{\kappa} \in \mathrm{Sym}^*$. After denoting $\mathbf{Q} := (\boldsymbol{\xi} + \boldsymbol{\kappa})^{-1}\big(\mathbf{q} + \boldsymbol{\kappa}\nabla\theta\big)$, we obtain

$$\rho\partial_{\mathbf{q}}\psi = \frac{\tau}{\theta}\mathbf{Q}, \qquad \rho\partial_{\nabla\theta}\,\psi = \frac{\tau}{\theta}\boldsymbol{\kappa}\mathbf{Q}, \qquad \mathbf{N} = \partial_{\mathbf{q}}\psi/|\partial_{\mathbf{q}}\psi| = \mathbf{Q}/|\mathbf{Q}|.$$

Now we define the entropy production as

$$\rho\sigma = \frac{1}{\theta^2}\mathbf{q} \cdot (\boldsymbol{\xi} + \boldsymbol{\kappa})^{-1}\mathbf{q} + \frac{1}{\theta^2}\nabla\theta \cdot \boldsymbol{\kappa}(\boldsymbol{\xi} + \boldsymbol{\kappa})^{-1}\boldsymbol{\xi}\nabla\theta. \tag{39}$$

In order to ensure $\sigma \geq 0$ we are forced to require

i) $\boldsymbol{\xi} \in \mathrm{Sym}^+$;
ii) there exists $\beta \geq 0$ such that $\boldsymbol{\kappa} = \beta\boldsymbol{\xi}$.

A straightforward but boring calculation yields

$$\frac{1}{\theta}\,\mathbf{q} \cdot \nabla\theta + \rho\partial_{\nabla\theta}\,\psi \cdot \nabla\dot{\theta} + \rho\theta\sigma = \frac{1}{\theta}\,[\mathbf{q} + \boldsymbol{\xi}\nabla\theta + \tau\boldsymbol{\kappa}\nabla\dot{\theta}] \cdot \mathbf{Q}.$$

Inserting this result into the representation formula (38) we obtain

$$\dot{\mathbf{q}} = -\frac{1}{\tau}\Big([\mathbf{q} + \boldsymbol{\xi}\nabla\theta + \tau\boldsymbol{\kappa}\nabla\dot{\theta}] \cdot \mathbf{Q}\Big)\frac{\mathbf{N}}{|\mathbf{Q}|} + (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{G}.$$

Equation (37) follows by taking into account that $(\mathbf{N}\otimes\mathbf{N})\mathbf{G} = (\mathbf{G}\cdot\mathbf{N})\mathbf{N}$ and choosing

$$\tau\mathbf{G} = -\mathbf{q} - \boldsymbol{\xi}\nabla\theta - \tau\boldsymbol{\kappa}\nabla\dot{\theta}.$$

This completes the proof of the following Proposition.

Proposition 5.1. *The thermodynamic consistency of the Jeffreys model is ensured if and only if $\boldsymbol{\xi} \in \mathrm{Sym}^+$ and $\boldsymbol{\kappa} = \beta\boldsymbol{\xi}$, $\beta \geq 0$. In the isotropic case*[3] $\boldsymbol{\xi} = \xi\mathbf{1}$ *and* $\boldsymbol{\kappa} = \kappa\mathbf{1}$, *so that the consistency conditions reduce to* $\xi > 0$, $\kappa \geq 0$.

It is worth noting that it is possible to make an alternative choice of free energy and entropy production. To be precise, we can define

$$\rho\psi^* = \rho\psi_0(\theta) + \frac{\tau}{2\theta}[\mathbf{q} + \boldsymbol{\kappa}\nabla\theta] \cdot (\boldsymbol{\xi} - \boldsymbol{\kappa})^{-1}[\mathbf{q} + \boldsymbol{\kappa}\nabla\theta],$$

[3]A material is said isotropic if it has symmetry properties that are invariant with respect to all rotations and inversions of the frame of axes.

with $\boldsymbol{\xi} - \boldsymbol{\kappa} \in \mathrm{Sym}^*$, so that

$$\rho \partial_{\mathbf{q}} \psi^* = \frac{\tau}{\theta} \mathbf{Q}^*, \qquad \rho \partial_{\nabla\theta} \psi^* = \frac{\tau}{\theta} \boldsymbol{\kappa} \, \mathbf{Q}^*, \qquad \mathbf{N} = \frac{\mathbf{Q}^*}{|\mathbf{Q}^*|},$$

where $\mathbf{Q}^* := (\boldsymbol{\xi} - \boldsymbol{\kappa})^{-1}\big(\mathbf{q} + \boldsymbol{\kappa}\nabla\theta\big)$. After defining the entropy production as

$$\rho\sigma^* = \frac{1}{\theta^2}\big(\mathbf{q} + \boldsymbol{\kappa}\nabla\theta\big) \cdot (\boldsymbol{\xi} - \boldsymbol{\kappa})^{-1}\big(\mathbf{q} + \boldsymbol{\kappa}\nabla\theta\big) + \frac{1}{\theta^2}\nabla\theta \cdot \boldsymbol{\kappa}\nabla\theta. \tag{40}$$

we get (37) following the same procedure as above. In this case, however, $\sigma \geq 0$ is ensured by imposing a condition stronger than before, namely $\boldsymbol{\xi} - \boldsymbol{\kappa} \in \mathrm{Sym}^+$ and $\boldsymbol{\kappa}$ positive semidefinite. In the isotropic case, $\boldsymbol{\xi} = \xi\mathbf{1}$ and $\boldsymbol{\kappa} = \kappa\mathbf{1}$, this condition reduces to $\xi > \kappa \geq 0$.

Note that under these conditions both the free energies ψ and ψ^* are convex functions of $\mathbf{q}$ and $\nabla\theta$. However, they are not strictly convex in that $\psi = \psi^* = \psi_0(\theta)$ when $\mathbf{q} = -\boldsymbol{\kappa}\nabla\theta$.

Remark 5.1. *Each function obtained by the convex combination of ψ and ψ^* satisfies*

$$\rho\partial_{\mathbf{q}}\psi_\lambda \cdot \dot{\mathbf{q}} + \rho\partial_{\nabla\theta}\psi_\lambda \cdot \nabla\dot{\theta} + \frac{\mathbf{q}}{\theta} \cdot \nabla\theta = -\rho\theta\sigma_\lambda.$$

where $\psi_\lambda = \lambda\psi + (1-\lambda)\psi^*$ and $\sigma_\lambda = \lambda\sigma + (1-\lambda)\sigma^*$, $\lambda \in [0,1]$. All of them verify the C-D inequality provided that $\sigma, \sigma^* \geq 0$. Accordingly, there is an infinite number of thermodynamically-consistent and convex free energy functions for the Jeffreys model.

Jeffreys' temperature equation

To describe the evolution of temperature we take into account the energy balance equation for a rigid body,

$$\rho c_v \dot{\theta} = -\nabla \cdot \mathbf{q} + \rho r. \tag{41}$$

After combining (37) with (41) we get

$$\rho c_v(\tau\ddot{\theta} + \dot{\theta}) = \nabla \cdot (\boldsymbol{\xi}\nabla\theta + \tau\boldsymbol{\kappa}\nabla\dot{\theta}) + \rho(r + \tau\dot{r}).$$

For the sake of simplicity, we assume the isotropic case, i.e. $\boldsymbol{\xi} = \xi\mathbf{1}$ and $\boldsymbol{\kappa} = \kappa\mathbf{1}$ where ξ, κ are constant, and let $r = 0$ (no external heat source). So we get

$$\rho c_v(\tau\ddot{\theta} + \dot{\theta}) = \xi\nabla^2\theta + \tau\kappa\nabla^2\dot{\theta}. \tag{42}$$

In [29] a similar equation, namely eqn.(45), is derived in the framework od classical irreversible thermodynamics with internal variables.

Equation (42) is linear and can be solved by the standard techniques. By setting $\theta(\mathbf{X}, t) = T(t)Y(\mathbf{X})$ we get

$$\rho c_v \left(\tau\ddot{T} + \dot{T}\right) Y = \left(\xi T + \tau\kappa\dot{T}\right) \nabla^2 Y,$$

Then applying the separation of variables we obtain

$$-\nabla^2 Y = \Lambda Y, \tag{43}$$

where Λ denote a set of constants, and

$$\tau\ddot{T} + \dot{T} = -\tilde{\Lambda}(\xi T + \tau\kappa\dot{T}), \qquad \tilde{\Lambda} = \Lambda/(\rho c_v). \tag{44}$$

For a given bounded domain, Λ denotes an eigenvalue of the Laplacian operator (as defined by (43) and the boundary conditions imposed to Y). Notice that for the most common boundary conditions (Dirichlet, Neumann, Robin, etc.) the eigenvalues are countable infinite, let say Λ_n, $n \in \mathbb{N}$, non-negative and not bounded by any constant value. Usually, the eigenvalues are ordered to form an ascending sequence, $\Lambda_n < \Lambda_{n+1}$.

For any fixed value of Λ_n, the corresponding solution T_n of (44) is given by $T_n(t) = C_1 e^{w_+ t} + C_2 e^{w_- t}$, where

$$w^+ + w^- = -\left(\frac{1}{\tau} + \kappa\tilde{\Lambda}_n\right), \quad w^+ w^- = \frac{\tilde{\Lambda}_n \xi}{\tau} \tag{45}$$

The temperature evolution depends on the relative size of the material parameters. In order to have a solution described by bounded functions eventually approaching the equilibrium steady state, we require that both the real parts of $w^{\pm}$ are negative. Owing to (45), if $\tau > 0$ this requirement implies $\kappa \geq 0$, since $\kappa > -1/(\tau\tilde{\Lambda}_n)$ $\forall n$ and $\tilde{\Lambda}_n$ is an unbounded sequence, and $\xi > 0$, since $\tilde{\Lambda}_n \geq 0$. Even if the case $\tau < 0$ is acceptable from a purely algebraic point of view, it implies some unphysical conditions. Indeed, in this case one should have $\kappa > 1/(|\tau|\tilde{\Lambda}_1)$, i.e. there is a threshold for the values of κ that depends on Λ_1. But Λ_1 is not a material parameter as it depends on the domain of the Laplacian operator and on the boundary conditions. To avoid such anomalous conditions we limit our analysis to $\tau > 0$. When $\xi = 0$ equation (42) reduces to the weakly damped wave equation, the same one obtained from the linear MCV model (34).

Finally, we note that in general for any given small value of κ there are a certain number of eigenvalues Λ_n that give rise to complex values of $w_{\pm}$: the number of these eigenvalues increases by decreasing κ. Depending on the existence of complex values of $w_{\pm}$, the solution T_n can be oscillatory but modulated by a an exponential decay in time. For a numerical characterization of these properties the reader can see e.g. [58].

5.2 Green-Naghdi type III heat conductors

A type III heat conductor according to Green and Naghdi [13] is characterized by

$$\mathbf{q} = -\boldsymbol{\kappa}\nabla\theta - \boldsymbol{\xi}\nabla\alpha.$$

where α is the thermal displacement such that $\dot{\alpha} = \theta$. As remarked in [17] this relation enters the heat equation for the temperature θ only after a differentiation with respect to time. Therefore it is reasonable to consider here the rate-type model equation

$$\dot{\mathbf{q}} = -\boldsymbol{\xi}\nabla\theta - \boldsymbol{\kappa}\nabla\dot{\theta}, \tag{46}$$

which is obtained from Green-Naghdi type III model by derivation with respect to time when $\boldsymbol{\xi}, \boldsymbol{\kappa}$ are constant. When $\boldsymbol{\kappa} = \mathbf{0}$ the type III model collapses to type II (25).

Following the procedure of the previous subsection, we prove the thermodynamic consistency of this model with the C-D inequality (36) by providing explicit expressions for ψ and σ. We start by letting

$$\rho\psi = \rho\psi_0(\theta) + \frac{1}{2\theta}[\mathbf{q} + \boldsymbol{\kappa}\nabla\theta] \cdot \boldsymbol{\xi}^{-1}[\mathbf{q} + \boldsymbol{\kappa}\nabla\theta], \qquad \boldsymbol{\xi} \in \mathrm{Sym}^*,$$

from which we obtain

$$\rho\partial_{\mathbf{q}}\psi = \frac{1}{\theta}\mathbf{Q}_0, \quad \rho\partial_{\nabla\theta}\,\psi = \frac{1}{\theta}\boldsymbol{\kappa}\,\mathbf{Q}_0, \quad \mathbf{N} = \frac{\mathbf{Q}_0}{|\mathbf{Q}_0|},$$

where $\mathbf{Q}_0 = \boldsymbol{\xi}^{-1}\big(\mathbf{q} + \boldsymbol{\kappa}\nabla\theta\big)$. Now, if we define the entropy production as

$$\rho\sigma = \frac{1}{\theta^2}\nabla\theta \cdot \boldsymbol{\kappa}\nabla\theta, \tag{47}$$

a straightforward calculation yields

$$\frac{1}{\theta}\,\mathbf{q}\cdot\nabla\theta + \rho\partial_{\nabla\theta}\,\psi\cdot\nabla\dot{\theta} + \rho\theta\sigma = \frac{1}{\theta}\,[\boldsymbol{\xi}\nabla\theta + \boldsymbol{\kappa}\nabla\dot{\theta}]\cdot\boldsymbol{\xi}^{-1}\big(\mathbf{q} + \boldsymbol{\kappa}\nabla\theta\big).$$

After replacing this result within the representation formula (38) we obtain

$$\dot{\mathbf{q}} = -\big([\boldsymbol{\xi}\nabla\theta + \boldsymbol{\kappa}\nabla\dot{\theta}]\cdot\mathbf{N}\big)\mathbf{N} + (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{G}.$$

Then choosing the arbitrary vector $\mathbf{G} = -\boldsymbol{\xi}\nabla\theta - \boldsymbol{\kappa}\nabla\dot{\theta}$ we recover (46).
Proposition 5.2. *The thermodynamic consistency of the GN III model in its differential form* (46) *is guaranteed if and only if* $\boldsymbol{\xi} \in \mathrm{Sym}^*$ *and* $\boldsymbol{\kappa} \in \mathrm{Sym}^+$*. If in addition* $\boldsymbol{\xi} \in \mathrm{Sym}^+$ *then the corresponding free energy is a convex function of* $\mathbf{q}$ *and* $\nabla\theta$*.*

It is worth noting that the GN III model does not proper reduces to the Fourier model, since the solutions of the GN III temperature equation do not converge to the solutions of the Fourier heat equation when $|\boldsymbol{\xi}| \to 0$ (see [19] for a detailed discussion).

GN III temperature equation

The properties of the temperature evolution in a rigid isotropic GN III heat conductor are described by combining (41) and (46) to obtain

$$\rho c_v\ddot{\theta} = \xi\nabla^2\theta + \kappa\nabla^2\dot{\theta} + \rho\dot{r}. \tag{48}$$

This is a linear version of the GN III temperature equation (see, e.g., [17, eqn.(37)]. In absence of heat sources and applying the separation of variables we obtain (43) and

$$\ddot{T}_n = -\tilde{\Lambda}_n(\xi T_n + \kappa \dot{T}_n), \qquad \tilde{\Lambda}_n = \frac{\Lambda_n}{\rho c_v}. \tag{49}$$

As $n \in \mathbb{N}$, the values of the eigenvalues Λ_n (and therefore also $\tilde{\Lambda}_n$) are countable infinite, non-negative, sorted in ascending order and unbounded above. Due to linearity, we let $T_n(t) = e^{w_n t}$ so that each w_n solves the equation

$$w_n^2 + \tilde{\Lambda}_n \kappa w_n + \tilde{\Lambda}_n \xi = 0. \tag{50}$$

For all solutions to (50) have negative real part, we must impose that all Λ_n are positive and $\kappa > 0$, $\xi > 0$. For these choices of the parameters the model is thermodynamically consistent and the solutions approaches the steady state exponentially in time. Note that if $\kappa = 0$ and $\xi > 0$ equation (50) admits purely imaginary roots so that solutions to (49) oscillates over time without decaying. In this case, equation (48) reduces to the wave equation.

6 Higher-order rate-type local models

To describe local effects of higher order in time we expand the previously considered set Ξ by adding the first-order time derivative of the heat flux. Hence we let

$$\Xi := (\theta, \dot{\theta}, \mathbf{q}, \dot{\mathbf{q}}, \nabla\theta, \nabla\dot{\theta})$$

and moreover, we assume ψ, η, σ and $\ddot{\mathbf{q}}$ be constitutive functions dependent on Ξ. Upon evaluation of $\dot{\psi}$ and substitution in (35) we obtain

$$\rho(\partial_\theta \psi + \eta)\dot{\theta} + \rho \partial_{\dot{\theta}} \psi \cdot \ddot{\theta} + \rho \partial_{\mathbf{q}} \psi \cdot \dot{\mathbf{q}} + \rho \partial_{\dot{\mathbf{q}}} \psi \cdot \ddot{\mathbf{q}}$$
$$+ \rho \partial_{\nabla\theta} \psi \cdot \nabla\dot{\theta} + \rho \partial_{\nabla\dot{\theta}} \psi \cdot \nabla\ddot{\theta} + \frac{1}{\theta} \mathbf{q} \cdot \nabla\theta = -\rho\theta\sigma.$$

The linearity and arbitrariness of $\ddot{\theta}$ and $\nabla\ddot{\theta}$ imply that ψ is independent of $\dot{\theta}$ and $\nabla\dot{\theta}$, Assuming, for simplicity, that σ and $\ddot{\mathbf{q}}$ are also independent of $\dot{\theta}$, the linearity and arbitrariness of $\dot{\theta}$ imply

$$\psi = \psi(\theta, \mathbf{q}, \dot{\mathbf{q}}, \nabla\theta), \qquad \eta = -\partial_\theta \psi,$$

so that $\Sigma = (\theta, \mathbf{q}, \dot{\mathbf{q}}, \nabla\theta)$ and the entropy inequality reduces to

$$\rho \partial_{\mathbf{q}} \psi \cdot \dot{\mathbf{q}} + \rho \partial_{\dot{\mathbf{q}}} \psi \cdot \ddot{\mathbf{q}} + \rho \partial_{\nabla\theta} \psi \cdot \nabla\dot{\theta} + \frac{\mathbf{q}}{\theta} \cdot \nabla\theta = -\rho\theta\sigma \le 0. \tag{51}$$

We provide higher-order rate-type local models of heat conduction by applying the representation formula to (51). The selection $\mathbf{Z} = \ddot{\mathbf{q}}$ and $\mathbf{N} = \partial_{\dot{\mathbf{q}}}\psi / |\partial_{\dot{\mathbf{q}}}\psi|$ into (19) together with the exploitation of (51) provide

$$\ddot{\mathbf{q}} = -\Big[\frac{1}{\theta}\,\mathbf{q}\cdot\nabla\theta + \rho\partial_{\mathbf{q}}\psi\cdot\dot{\mathbf{q}} + \rho\partial_{\nabla\theta}\,\psi\cdot\nabla\dot{\theta} + \rho\theta\sigma\Big]\frac{\partial_{\dot{\mathbf{q}}}\psi}{\rho|\partial_{\dot{\mathbf{q}}}\psi|^2} + (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{G}, \tag{52}$$

where $\mathbf{G}$ is an arbitrary second-order tensor-valued function of Ξ. Suitable choices of ψ, σ and $\mathbf{G}$ give rise to thermodynamically consistent models.

Alternatively, we start with a given constitutive equation,

$$\ddot{\mathbf{q}} = \boldsymbol{\mathcal{Q}}(\theta, \mathbf{q}, \dot{\mathbf{q}}, \nabla\theta, \nabla\dot{\theta})$$

and then, upon substitution for $\ddot{\mathbf{q}}$ in (51), we obtain σ as a function of Ξ and partial derivatives of $\psi(\Sigma)$. At least to design linear models, we can assume a quadratic form (with respect to $\mathbf{q}$, $\dot{\mathbf{q}}$, $\nabla\theta$) of the free energy ψ and then discuss the nonnegative definiteness of the resulting expression of σ. A similar procedure has been adopted in [37] to evaluate the thermodynamic consistency of some viscoelastic models of the rate-type, such as Burgers' fluids.

6.1 Quintanilla's heat conduction model

A new theory of thermoelasticity phenomena have been proposed by Quintanilla in [20] by modifying the Green-Naghdi's type III theory that make use of the thermal displacement α as an independent variable. We can obtain the constitutive equation proposed by Quintanilla by adding to the MCV model (34) a term involving the gradient of thermal displacement α,

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\boldsymbol{\kappa}\nabla\theta - \boldsymbol{\xi}\nabla\alpha.$$

Exploiting the term by term correspondences $\mathbf{T}_{RR} \leftrightarrow \mathbf{q}$, $\dot{\mathbf{E}} \leftrightarrow -\nabla\theta$ (and $\mathbf{E} \leftrightarrow -\nabla\alpha$) we notice that this equation has a mechanical counterpart in the rate-type model of linear viscoelasticity.

Assuming that $\tau, \boldsymbol{\xi}, \boldsymbol{\kappa}$ take constant values, deriving with respect to time and taking into account that $\dot{\alpha} = \theta$ (the absolute temperature), we obtain the rate-type Quintanilla model

$$\tau\ddot{\mathbf{q}} + \dot{\mathbf{q}} = -\boldsymbol{\xi}\nabla\theta - \boldsymbol{\kappa}\nabla\dot{\theta}. \tag{53}$$

Equation (46) for GN III conductors is recovered when $\tau \to 0^+$.

Proposition 6.1. *The rate-type Quintanilla model* (53) *with* $\tau \neq 0$ *is thermodynamically consistent with the reduced thermodynamic inequality (51) if and only if* $\boldsymbol{\xi} \in \mathrm{Sym}^*$ *and* $\boldsymbol{\kappa} - \tau\boldsymbol{\xi} \in \mathrm{Sym}^+$*. In the isotropic version*

$$\tau\ddot{\mathbf{q}} + \dot{\mathbf{q}} = -\xi\nabla\theta - \kappa\nabla\dot{\theta}, \tag{54}$$

that follows from (53) *by letting* $\boldsymbol{\xi} = \xi\mathbf{1}$ *and* $\boldsymbol{\kappa} = \kappa\mathbf{1}$*, the consistency condition becomes* $\xi \neq 0$ *and* $\kappa > \tau\xi$*.*

The proof of this result is given in detail in Appendix A.

MGT temperature equation

The properties of temperature evolution can be understood by looking at the equation obtained from (41) and (54), namely

$$\tau \dddot{\theta} + \ddot{\theta} = \frac{1}{\rho c_v}\left(\xi \nabla^2 \theta + \kappa \nabla^2 \dot{\theta}\right).$$

This looks like a linear version of the well-known Moore-Gibson-Thompson equation which arises in the modeling of wave propagation in viscous thermally relaxing fluids [21]. Since the unknown function of the MGT model is mass density and not temperature, the two equations show a purely formal analogy. By setting $\theta(\mathbf{X}, t) = T(t)Y(\mathbf{X})$ it follows

$$\left(\tau \dddot{T} + \ddot{T}\right) Y = \frac{1}{\rho c_v}\left(\xi T + \kappa \dot{T}\right) \nabla^2 Y.$$

Then applying the separation of variables we obtain

$$\tau \dddot{T} + \ddot{T} = -\tilde{\Lambda}(\xi T + \kappa \dot{T}), \qquad \tilde{\Lambda} = \frac{\Lambda}{\rho c_v}, \tag{55}$$

where $\Lambda \in \{\Lambda_n\}_{n=\mathbb{N}}$, the sequence of eigenvalues. Letting $T(t) = e^{wt}$, w solves the cubic equation

$$\tau w^3 + w^2 + \tilde{\Lambda}\kappa w + \tilde{\Lambda}\xi = 0. \tag{56}$$

To avoid a blow up of the solutions at infinity, we look at decaying or oscillating solutions of the equation (55). It is known that a polynomial $a_3 w^3 + a_2 w^2 + a_1 w + a_0$ has all three roots with negative real part if and only if all the coefficients are of the same sign and $a_2 a_1 > a_0 a_3$ (the so-called Routh-Hurwitz criterion). This provides the following conditions on our parameters,

$$\tau > 0, \ \xi > 0, \ \kappa > 0, \ \kappa > \tau\xi, \tag{57}$$

which are consistent with conditions given in Proposition 6.1. To get a more precise understanding of the behavior of the solutions, we look at the discriminant of (56). After defining

$$\mathrm{Discr}_w = \tau^4 (w_1 - w_2)^2 (w_1 - w_3)^2 (w_2 - w_3)^2,$$

where w_i, $i = 1, 2, 3$ are the roots of equation (56), we get[4]

$$\mathrm{Discr}_w = -\tilde{\Lambda}\left(4\kappa^3 \tau \tilde{\Lambda}^2 + \left(9\tau\xi(3\tau\xi - 2\kappa)\right)\tilde{\Lambda} + 4\xi\right).$$

Accordingly, for any suitable positive value of the constants τ, κ, ξ and for sufficiently large values of Λ the discriminant is negative, and then a couple of roots, say w_2 and w_3, takes conjugate complex values (with negative real part if the conditions (57) are fulfilled). Since, in general, the sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ is unbounded, this implies that solutions to (55) oscillates with an exponential damping. This type of behavior is close to that of the MCV model [59]. Remarkably, conditions (57) also ensures well-posedness and stability of solutions in three dimensions [20].

[4] Given a cubic equation $ax^3 + bx^2 + cx + d = 0$ its discriminant is $18abcd - 4b^3 d + b^2 c^2 - 4ac^3 - 27a^2 d^2$.

6.2 Heat conductors of the Burgers type (dual-phase-lag model)

Heat conductors of the Burgers type are characterized by the rate-type equation

$$\lambda\ddot{\mathbf{q}} + \tau\dot{\mathbf{q}} + \mathbf{q} = -\boldsymbol{\mu}\nabla\theta - \tau\boldsymbol{\nu}\nabla\dot{\theta}. \tag{58}$$

This model can be obtained, as for the Burgers' fluid rheological model, by considering a mixture of two substances characterized by a conduction mechanism described by the Maxwell-Cattaneo equation (34), namely

$$\tau_1\dot{\mathbf{q}}^{(1)} + \mathbf{q}^{(1)} = -\boldsymbol{\kappa}^{(1)}\nabla\theta, \qquad \tau_2\dot{\mathbf{q}}^{(2)} + \mathbf{q}^{(2)} = -\boldsymbol{\kappa}^{(2)}\nabla\theta.$$

Equation (58) can be obtained by combining the previous two equations and setting

$$\tau = \tau_1 + \tau_2, \qquad \lambda = \tau_1\tau_2, \qquad \boldsymbol{\mu} = \boldsymbol{\kappa}^{(1)} + \boldsymbol{\kappa}^{(2)}, \qquad \tau\boldsymbol{\nu} = \tau_1\boldsymbol{\kappa}^{(2)} + \tau_2\boldsymbol{\kappa}^{(1)}. \tag{59}$$

For simplicity we restrict our attention to the isotropic equation

$$\lambda\ddot{\mathbf{q}} + \tau\dot{\mathbf{q}} + \mathbf{q} = -\mu\nabla\theta - \tau\nu\nabla\dot{\theta}, \tag{60}$$

that follows from (58) by letting $\boldsymbol{\mu} = \mu\mathbf{1}$ and $\boldsymbol{\nu} = \nu\mathbf{1}$, and we investigate the consistency of this evolution equation with the reduced thermodynamic inequality (51). A constitutive equation similar to (60) was proposed by Tzou [39] assuming

$$\lambda = \tau_q^2/2, \qquad \tau = \tau_q, \qquad \mu = k, \qquad \tau\nu = \tau_\theta k. \tag{61}$$

This approach introduces two lag-parameters; the delay time τ_θ which is related to microstructural interactions (such as phonon scattering or phonon-electron interactions) and the delay τ_q which represents the relaxation time due to fast-transient effects of thermal inertia.

For definiteness, we let $\lambda \neq 0$. Otherwise, the Burgers model degenerates into simpler models. The Jeffreys model is recovered if $\lambda = 0$. When λ and $\boldsymbol{\nu}$ vanish, then (60) reduces to the MCV model and if in addition $\tau = 0$ the Fourier law is recovered. Consequently, the Burgers' model, unlike the GN III and Quintanilla's models, contains the Fourier theory in a proper sense.

Owing to $\lambda \neq 0$, we consider (60) as a constitutive equation for $\ddot{\mathbf{q}}$. Upon substitution of $\ddot{\mathbf{q}}$ from (60) into (51), we have

$$\rho\Big(\partial_{\mathbf{q}}\psi - \frac{\tau}{\lambda}\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\dot{\mathbf{q}} - \rho\frac{1}{\lambda}\partial_{\dot{\mathbf{q}}}\psi\cdot\mathbf{q} + \Big(\frac{\mathbf{q}}{\theta} - \frac{\mu}{\lambda}\rho\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\nabla\theta$$
$$+\rho\Big(\partial_{\nabla\theta}\psi - \frac{\tau\nu}{\lambda}\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\nabla\dot{\theta} = -\rho\theta\sigma \leq 0.$$

Since we cannot invoke the linearity and arbitrariness of $\nabla\dot{\theta}$, we impose the relation

$$\partial_{\nabla\theta}\psi = \frac{\tau\nu}{\lambda}\partial_{\dot{\mathbf{q}}}\psi, \tag{62}$$

which implies

$$\rho\Big(\partial_{\mathbf{q}}\psi - \frac{\tau}{\lambda}\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\dot{\mathbf{q}} - \rho\frac{1}{\lambda}\partial_{\dot{\mathbf{q}}}\psi\cdot\mathbf{q} + \Big(\frac{\mathbf{q}}{\theta} - \frac{\mu}{\lambda}\rho\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\nabla\theta = -\rho\theta\sigma \le 0. \tag{63}$$

Now we select the free energy ψ as in (A3), so that upon substitution into (62) and (63) we obtain

$$\gamma_2 = \frac{\tau\nu}{\lambda}\gamma_1, \qquad \gamma_3 = \frac{\tau\nu}{\lambda}\alpha_2, \qquad \alpha_3 = \frac{\tau\nu}{\lambda}\gamma_3, \tag{64}$$

$$A_{11}|\mathbf{q}|^2 + A_{22}|\dot{\mathbf{q}}|^2 + A_{33}|\nabla\theta|^2 + 2A_{12}\dot{\mathbf{q}}\cdot\mathbf{q} + 2A_{13}\mathbf{q}\cdot\nabla\theta + 2A_{23}\dot{\mathbf{q}}\cdot\nabla\theta = \rho\theta\sigma \ge 0, \tag{65}$$

where

$$A_{11} = \frac{1}{\lambda}\gamma_1, \quad A_{22} = \frac{\tau}{\lambda}\alpha_2 - \gamma_1, \quad A_{33} = \frac{\mu}{\lambda}\gamma_3,$$
$$2A_{12} = \frac{\tau}{\lambda}\gamma_1 + \frac{1}{\lambda}\alpha_2 - \alpha_1, \quad 2A_{13} = -\frac{1}{\theta} + \frac{1}{\lambda}\gamma_3 + \frac{\mu}{\lambda}\gamma_1, \quad 2A_{23} = \frac{\tau}{\lambda}\gamma_3 - \gamma_2 + \frac{\mu}{\lambda}\alpha_2.$$

Proposition 6.2. *The Burgers-like model (60) with $\lambda \neq 0$ is thermodynamically consistent if and only if one of the following hypotheses occurs*

i) $\tau\nu = 0$ *and* $\mu > 0$, $\lambda < 0$;
ii) $\nu\tau \neq 0$ *and* $\mu = 0$, $\nu > 0$;
iii) $\nu\tau \neq 0$ *and* $\mu > 0$, $\nu\tau^2 \ge \lambda\mu$.

The proof of this proposition is too cumbersome to be included in the body of the paper, so we postpone it to Appendix B.

Joseph-Preziosi temperature equation

Let us briefly discuss the temperature equation corresponding to Burgers-like model. For simplicity we restrict the discussion to the isotropic case with all the parameters constant. Also, we assume there is no external heat supply. By combining the energy equation (41) with (60) we get

$$\lambda\dddot{\theta} + \tau\ddot{\theta} + \dot{\theta} = \frac{1}{\rho c_v}\left(\mu\nabla^2\theta + \tau\nu\nabla^2\dot{\theta}\right). \tag{66}$$

This third order equation looks like a generalization of the MGT temperature equation. An equation of this form was obtained by Joseph and Preziosi starting from the linearization of heat conduction with memory according to the Gurtin-Pipkin model (see [7, eqn.(5.7)]). The same equation with parameters defined by (61) was later proposed in [39] as an approximation in dual-phase-lag heat conduction.

Letting $\theta(\mathbf{X},t) = T(t)Y(\mathbf{X})$, we obtain the temperature equation

$$\left(\lambda\dddot{T} + \tau\ddot{T} + \dot{T}\right)Y = \frac{1}{\rho c_v}\left(\mu T + \tau\nu\dot{T}\right)\nabla^2 Y,$$

and then the equation for $T(t)$ reads

$$\lambda \dddot{T} + \tau \ddot{T} + \dot{T} = -\tilde{\Lambda}(\mu T + \tau \nu \dot{T}), \qquad \tilde{\Lambda} = \frac{\Lambda}{\rho c_v}, \tag{67}$$

where the sequence $\tilde{\Lambda}_n$, $n \in \mathbb{N}$, is non-negative and unbounded. If $T(t) = e^{wt}$, then w solves the cubic equation

$$\lambda w^3 + \tau w^2 + (1 + \tilde{\Lambda}_n \tau \nu) w + \tilde{\Lambda}_n \mu = 0, \qquad n \in \mathbb{N}. \tag{68}$$

To avoid blow up of solutions at large time, we seek only for decaying or oscillating solutions to equation (67). Accordingly, we look at the conditions giving all roots with negative real part. We can assume $\tau \neq 0$, otherwise the sum of the three roots of equation (68) would be zero, contrary to the assumptions that all roots have negative real part. Thus, we rearrange equation (68) as

$$\frac{\lambda}{\tau} w_n^3 + w_n^2 + \left(\frac{1}{\tau} + \nu \tilde{\Lambda}_n \right) w_n + \frac{\mu \tilde{\Lambda}_n}{\tau} = 0. \tag{69}$$

By applying the Routh-Hurwitz criterion,

$$\frac{\lambda}{\tau} > 0, \quad \frac{1}{\tau} + \nu \tilde{\Lambda}_n > 0, \quad \frac{\mu \tilde{\Lambda}_n}{\tau} > 0, \quad \frac{1}{\tau} + \nu \tilde{\Lambda}_n > \frac{\mu \lambda \tilde{\Lambda}_n}{\tau^2} \tag{70}$$

We let $\tau > 0$. The second inequality implies $\nu \geq 0$, since $\nu > -1/(\tau \tilde{\Lambda}_n)$ and $\tilde{\Lambda}_n$ is an unbounded sequence. Also, the first and third inequalities imply $\lambda > 0$ and $\mu > 0$. Finally, the last inequality gives $(\mu\lambda - \nu\tau^2) < \tau/\tilde{\Lambda}_n$, i.e. $\mu\lambda \leq \nu\tau^2$. We remark that these conditions are compatible with the conditions iii) given in Proposition (6.2). From the above conditions we must exclude the case $\nu = 0$, since, according to the condition i) of Proposition (6.2) we should have $\lambda < 0$ to get the thermodynamic consistency, but in this case we lost the dynamic consistency. Furthermore, we exclude $\tau < 0$. Indeed, in this case from $(70)_2$ one should have $\nu > 1/(|\tau|\tilde{\Lambda}_1)$, i.e. there is a threshold for the values of the material parameter ν that depends on Λ_1.

Ultimately, we let one of the roots of equation (68) to be zero. This is the case if $\mu = 0$, corresponding to condition ii) in Proposition (6.2). The conditions on the other parameters to get the dynamic consistency do not change, i.e. one has $\lambda/\tau > 0$ and $1/\tau + \nu\tilde{\Lambda}_n > 0$, again entailing $\tau > 0$, $\lambda > 0$ and $\nu > 0$.

Summarizing: the Burgers-like model are thermodinamically and dinamically consistent (i.e. the entropy production is nonnegative and all solutions of the temperature equation are bounded and eventually approach equilibrium values) if and only if the following conditions are satisfied:

$$\mu \geq 0, \ \nu > 0, \ \lambda > 0, \ \tau > 0, \ \nu\tau^2 \geq \lambda\mu.$$

According to (61), these conditions can be rewritten as

$$k, \tau_q, \tau_\theta > 0, \qquad \tau_\theta \geq \tau_q/2$$

and represent sufficient conditions to ensure the stability of the solutions (see [60–62]).

6.3 Local heat conduction models of rate-type in spatial description

The models described in the previous sections can be applied also to fluid heat conductors by rewriting their constitutive rate-type equations into the spatial description. We illustrate the procedure by considering the Burgers-like model (58),

$$\lambda \ddot{\mathbf{q}}_R + \tau \dot{\mathbf{q}}_R + \mathbf{q}_R = -\boldsymbol{\mu} \nabla_R \theta - \tau \boldsymbol{\nu} \nabla_R \dot{\theta}.$$

To this end we remember that $\nabla_R \dot{\theta} = (\nabla_R \theta)^{\cdot}$ and

$$\mathbf{q}_R = J\mathbf{F}^{-1}\mathbf{q}, \qquad \nabla_R = \mathbf{F}^T \nabla.$$

Therefore it follows

$$\lambda (J\mathbf{F}^{-1}\mathbf{q})^{\cdot\cdot} + \tau (J\mathbf{F}^{-1}\mathbf{q})^{\cdot} + J\mathbf{F}^{-1}\mathbf{q} = -\boldsymbol{\mu}\mathbf{F}^T \nabla\theta - \tau\boldsymbol{\nu}(\mathbf{F}^T\nabla\theta)^{\cdot}.$$

After introducing the convective time derivative (also referred to as *Truesdell vector rate* [34, p.59]),

$$\overset{\square}{\mathbf{q}} = \partial_t \mathbf{q} + \nabla \times (\mathbf{q} \times \mathbf{v}) + (\nabla \cdot \mathbf{q})\mathbf{v} = \dot{\mathbf{q}} - \mathbf{L}\mathbf{q} + (\nabla \cdot \mathbf{v})\mathbf{q},$$

and the Cotter-Rivlin vector rate,

$$(\nabla\theta)^{\triangle} = (\nabla\theta)^{\cdot} + \mathbf{L}^T \nabla\theta,$$

a straightforward calculation provides

$$(J\mathbf{F}^{-1}\mathbf{q})^{\cdot} = J\mathbf{F}^{-1}\overset{\square}{\mathbf{q}}, \qquad (J\mathbf{F}^{-1}\mathbf{q})^{\cdot\cdot} = J\mathbf{F}^{-1}\overset{\square\square}{\mathbf{q}}, \qquad (\mathbf{F}^T\nabla\theta)^{\cdot} = \mathbf{F}^T(\nabla\theta)^{\triangle}.$$

Accordingly, we can write

$$J\mathbf{F}^{-1}[\lambda \overset{\square\square}{\mathbf{q}} + \tau \overset{\square}{\mathbf{q}} + \mathbf{q}] = -\boldsymbol{\mu}\mathbf{F}^T\nabla\theta - \tau\boldsymbol{\nu}\mathbf{F}^T(\nabla\theta)^{\triangle},$$

from which it follows

$$\lambda \overset{\square\square}{\mathbf{q}} + \tau \overset{\square}{\mathbf{q}} + \mathbf{q} = -\boldsymbol{\mu}_s \nabla\theta - \tau\boldsymbol{\nu}_s(\nabla\theta)^{\triangle}, \tag{71}$$

where

$$\boldsymbol{\mu}_s := J^{-1}\mathbf{F}\boldsymbol{\mu}\mathbf{F}^T, \qquad \boldsymbol{\nu}_s := J^{-1}\mathbf{F}\boldsymbol{\nu}\mathbf{F}^T$$

are the viscosity tensors in the spatial description. Note that all time derivatives involved in (71) are objective. In particular, when $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are isotropic, namely $\boldsymbol{\mu} = \mu \mathbf{1}$ and $\boldsymbol{\nu} = \nu \mathbf{1}$, then

$$\boldsymbol{\mu}_s = J^{-1}\mu\mathbf{B}, \qquad \boldsymbol{\nu}_s = J^{-1}\nu\mathbf{B}, \qquad \mathbf{B} := \mathbf{F}\mathbf{F}^T.$$

Neglecting that the viscosity tensors depend on $\mathbf{F}$, (71) represents a linear rate-type constitutive equation for the heat flux vector in the spatial description.

7 Weakly nonlocal theories of heat conduction

Nonlocal effects in generalized equations for heat conduction are of particular interest in systems where the mean free path is comparable to (or bigger than) the characteristic volume size, i.e., where the Knudsen number is equal or greater than one. This may occur when considering nanosystems (the size of the system is comparable to the mean free path) or studying phonon propagation at low temperatures or in rarefied systems (the mean free path is comparable to the size of the system). In our approach weakly non-local means that the set of independent variables Ξ contains temperature and heat-flux gradients up to some order $n \geq 2$.

We focus here on the entropy production in the form $\sigma = \zeta - \nabla \cdot \mathbf{k}$ subject to (11), in agreement with the nonlocal statement of the Second Law. As in the previous sections, we neglect the dependence on $\mathbf{E}$ and $\mathbf{T}_{RR}$. Hence (14) reduces to

$$-\rho(\dot{\psi} + \eta\dot{\theta}) - \frac{1}{\theta}\,\mathbf{q}\cdot\nabla\theta + \theta\nabla\cdot\mathbf{k} = \rho\theta\zeta \geq 0. \tag{72}$$

Weakly nonlocal models are developed here on the basis of four elements,

- the set Ξ of admissible variables;
- the free energy density function $\psi = \psi(\Xi)$;
- the local entropy supply function $\zeta = \zeta(\Xi)$
- the extra entropy flux function $\mathbf{k} = \mathbf{k}(\Xi)$

In the following we consider some basic nonlocal models.

7.1 A weakly nonlocal model for rigid heat conductors

To describe weakly nonlocal effects in Fourier-like heat conduction, we consider a rigid body where higher-order gradients of the temperature are involved. Let $\psi, \eta, \mathbf{k}$ and $\gamma \geq 0$ be functions of the following set of independent variables

$$\Xi = (\theta, \dot{\theta}, \nabla\theta, \nabla\nabla\theta, \nabla\nabla\nabla\theta).$$

Upon evaluation of $\dot{\psi}$ and substitution into (72) we obtain

$$-\rho(\partial_\theta\psi + \eta)\dot{\theta} - \rho\partial_{\dot{\theta}}\psi\,\ddot{\theta} - \rho\partial_{\nabla\theta}\psi\cdot\nabla\dot{\theta} - \rho\partial_{\nabla\nabla\theta}\psi\cdot\nabla\nabla\dot{\theta}$$
$$-\rho\partial_{\nabla\nabla\nabla\theta}\psi\cdot\nabla\nabla\nabla\dot{\theta} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta + \theta\nabla\cdot\mathbf{k} = \rho\theta\zeta.$$

If ζ and $\mathbf{k}$ are independent of $\dot{\theta}$, we follow the Coleman-Noll (C-N) procedure. Indeed, the arbitrariness and linearity of $\dot{\theta}, \ddot{\theta}, \nabla\dot{\theta}, \nabla\nabla\dot{\theta}, \nabla\nabla\nabla\dot{\theta}$ imply

$$\eta = \partial_\theta \psi. \quad \partial_{\dot{\theta}} \psi = 0, \quad \partial_{\nabla\theta}\psi = \mathbf{0}, \quad \partial_{\nabla\nabla\theta}\psi = \mathbf{0}, \quad \partial_{\nabla\nabla\nabla\theta}\psi = \mathbf{0},$$

and then $\psi = \psi(\theta)$, $\eta = \eta(\theta)$ and

$$\frac{1}{\theta^2}\mathbf{q} \cdot \nabla\theta = \nabla \cdot \mathbf{k} - \rho\zeta. \tag{73}$$

Note that these thermodynamic restrictions differ from the classical conclusions of the C-N procedure by the the presence of $\nabla \cdot \mathbf{k}$ in the C-D relation (73), which imply the independence of $\mathbf{k}$ from the third spatial gradient of θ. By virtue of this peculiarity we are able to prove the thermodynamic consistency of the following constitutive equation

$$\mathbf{q}(\Xi) = -k(\theta)\nabla\theta + h^2(\theta)\nabla \cdot [\nabla\nabla\theta + 2(\nabla^2\theta)\mathbf{1}]. \tag{74}$$

where $\nabla^2 = \nabla \cdot \nabla$, $h^2(\theta)$ is a (possibly small) positive function and $\kappa(\theta) > 0$ is the usual bulk thermal conductivity. This equation provides a nonlocal perturbation of the Fourier law, which is recovered when $h = 0$. Taking into account the identity

$$\Big(\nabla \cdot [\nabla\nabla\theta + 2(\nabla^2\theta)\mathbf{1}]\Big) \cdot \nabla\theta = \nabla \cdot [(\nabla\theta \cdot \nabla)\nabla\theta + 2(\nabla^2\theta)\nabla\theta] - |\nabla\nabla\theta|^2 - 2|\nabla^2\theta|^2,$$

after multiplying (74) by $\nabla\theta/\theta^2$ and letting $h^2(\theta) = h_0^2\,\theta^2$ it follows

$$\frac{1}{\theta^2}\mathbf{q} \cdot \nabla\theta = h_0^2 \nabla \cdot [(\nabla\theta \cdot \nabla)\nabla\theta + 2(\nabla^2\theta)\nabla\theta] - \frac{k}{\theta^2}|\nabla\theta|^2 - h_0^2|\nabla\nabla\theta|^2 - 2h_0^2|\nabla^2\theta|^2$$

Hence, letting

$$\rho\zeta = \frac{k(\theta)}{\theta^2}|\nabla\theta|^2 + h_0^2|\nabla\nabla\theta|^2 + 2h_0^2|\nabla^2\theta|^2 > 0, \quad \mathbf{k} = h_0^2(\nabla\theta \cdot \nabla)\nabla\theta + 2h_0^2(\nabla^2\theta)\nabla\theta, \tag{75}$$

the C-D relation (73) is recovered.

7.2 A weakly nonlocal model based on the heat-flux vector and its gradients

We consider a rigid heat conductor where the heat-flux vector and their gradients up to the second order are involved as independent variables. Hence, ψ, η, $\mathbf{k}$ and $\zeta \geq 0$ are assumed to depend on

$$\Xi := (\theta, \nabla\theta, \mathbf{q}, \nabla\mathbf{q}, \nabla\nabla\mathbf{q}).$$

In addition, we assume η is continuous while ψ is continuously differentiable. Upon evaluation of $\dot{\psi}$ and substitution in (72) we obtain

$$\rho(\partial_\theta\psi + \eta)\dot{\theta} + \rho\partial_{\nabla\theta}\psi\cdot\nabla\dot{\theta} + \rho\partial_{\mathbf{q}}\psi\cdot\dot{\mathbf{q}} + \rho\partial_{\nabla\mathbf{q}}\psi\cdot\nabla\dot{\mathbf{q}}$$
$$+\rho\partial_{\nabla\nabla\mathbf{q}}\psi\cdot\nabla\nabla\dot{\mathbf{q}} + \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta - \theta\nabla\cdot\mathbf{k} = -\rho\theta\zeta.$$

Following the usual Coleman-Noll procedure, the arbitrariness and linearity of $\dot{\theta}, \nabla\dot{\theta}$, $\dot{\mathbf{q}}$, $\nabla\dot{\mathbf{q}}$ and $\nabla\nabla\dot{\mathbf{q}}$ imply

$$\eta = \partial_\theta\psi, \quad \partial_{\nabla\theta}\psi = \mathbf{0}, \quad \partial_{\mathbf{q}}\psi = \mathbf{0}, \quad \partial_{\nabla\mathbf{q}}\psi = \mathbf{0}, \quad \partial_{\nabla\nabla\mathbf{q}}\psi = \mathbf{0},$$

so that $\psi = \psi(\theta)$ and (72) reduces to

$$\frac{\mathbf{q}}{\theta^2}\cdot\nabla\theta = \nabla\cdot\mathbf{k} - \rho\zeta \leq 0. \tag{76}$$

Note that $\mathbf{k}$ must be independent of the second gradient of $\mathbf{q}$.

According to [27], a nonlocal perturbation of the Fourier law is given by the following macroscopic equation,

$$\mathbf{q} = -\kappa\nabla\theta + \lambda^2[\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})], \tag{77}$$

where $\lambda^2(\theta)$ is a (possibly small) positive function related to the mean-free path of phonons and $\kappa(\theta) > 0$ is the usual bulk thermal conductivity (the Fourier model is recovered when $\lambda = 0$).

Now, we prove that (77) is thermodynamically consistent in that it satisfies (76) by properly choosing $\mathbf{k}$ and $\zeta > 0$. After multiplying (77) by $\mathbf{q}/\lambda^2(\theta)$, owing to the identity

$$\Big[\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})\Big]\cdot\mathbf{q} = \nabla\cdot[(\mathbf{q}\cdot\nabla)\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{q}] - |\nabla\mathbf{q}|^2 - 2|\nabla\cdot\mathbf{q}|^2 \tag{78}$$

it follows

$$\frac{\kappa(\theta)}{\lambda^2(\theta)}\mathbf{q}\cdot\nabla\theta = \nabla\cdot[(\mathbf{q}\cdot\nabla)\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{q}] - \frac{1}{\lambda^2(\theta)}|\mathbf{q}|^2 - |\nabla\mathbf{q}|^2 - 2|\nabla\cdot\mathbf{q}|^2.$$

Finally, letting $\lambda^2(\theta) = \kappa(\theta)\theta^2$ and

$$\rho\zeta = \frac{1}{\kappa(\theta)\theta^2}|\mathbf{q}|^2 + |\nabla\mathbf{q}|^2 + 2|\nabla\cdot\mathbf{q}|^2 > 0, \qquad \mathbf{k} = (\mathbf{q}\cdot\nabla)\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{q}, \tag{79}$$

the reduced thermodynamic inequality (76) is recovered.

Since all the quantities involved in equation (77) are also present in the Ξ set, this equation does not represent a constitutive relation, but rather a constraint between the variables of Ξ resulting from the nonlocal statement of the Second Law.

7.3 Some rate-type nonlocal and nonlinear models

In this case the set of independent variables Ξ is the same as before

$$\Xi := (\theta, \nabla\theta, \mathbf{q}, \nabla\mathbf{q}, \nabla\nabla\mathbf{q}),$$

but, in addition to $\psi, \eta, \mathbf{k}$ and ζ, also the time derivative of the heat flux vector, $\dot{\mathbf{q}}$, is considered as a constitutive function depending on Ξ. Upon evaluation of $\dot{\psi}$ and substitution in (72) we obtain

$$\begin{aligned} &\rho(\partial_\theta\psi + \eta)\dot{\theta} + \rho\partial_{\nabla\theta}\psi \cdot \nabla\dot{\theta} + \rho\partial_{\mathbf{q}}\psi \cdot \dot{\mathbf{q}} \\ &+\rho\partial_{\nabla\mathbf{q}}\psi \cdot \nabla\dot{\mathbf{q}} + \rho\partial_{\nabla\nabla\mathbf{q}}\psi \cdot \nabla\nabla\dot{\mathbf{q}} + \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta - \theta\nabla\cdot\mathbf{k} = -\rho\theta\zeta. \end{aligned}$$

The linearity and arbitrariness of $\dot{\theta}$, $\nabla\dot{\theta}$, $\nabla\dot{\mathbf{q}}$ and $\nabla\nabla\dot{\mathbf{q}}$ imply that[5]

$$\psi = \psi(\theta, \mathbf{q}), \qquad \eta = -\partial_\theta\psi,$$

so that the C-D inequality reduces to

$$\rho\partial_{\mathbf{q}}\psi \cdot \dot{\mathbf{q}} + \frac{\mathbf{q}}{\theta}\cdot\nabla\theta - \theta\nabla\cdot\mathbf{k} = -\rho\theta\zeta \le 0. \tag{80}$$

7.3.1 Guyer-Krumhansl conductors

Guyer and Krumhansl in [26] studied the heat wave propagation in dielectric crystals at low temperature. They observed that in the regime of low temperature the heat flux $\mathbf{q}$ is proportional to the momentum flux of the phonon gas and then found the following macroscopic equation governing its evolution

$$\tau\dot{\mathbf{q}} + \mathbf{q} + \kappa\nabla\theta = \lambda^2[\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})], \tag{81}$$

where τ is the relaxation time for resistive phonon scattering, λ^2 is the mean-free path of phonons and κ is the usual bulk thermal conductivity [63]. Both λ and κ possibly depend on θ. When τ is negligible the Guyer-Krumhansl model reduces to (77).

Hereafter, we prove that (81) is a thermodynamically consistent constitutive equation for $\dot{\mathbf{q}}$ in that it can be directly derived from the C-D inequality by properly choosing ψ, $\mathbf{k}$ and $\zeta > 0$ (see also [64]). A similar approach can be found in [65]. Assuming $\partial_{\mathbf{q}}\psi \neq \mathbf{0}$ equation (80) can be written in the form

$$\frac{\partial_{\mathbf{q}}\psi}{|\partial_{\mathbf{q}}\psi|}\cdot\dot{\mathbf{q}} = -\frac{\theta}{\rho_R|\partial_{\mathbf{q}}\psi|}\Big(\frac{\mathbf{q}}{\theta^2}\cdot\nabla\theta + \rho\zeta - \nabla\cdot\mathbf{k}\Big),$$

[5] Since $\dot{\mathbf{q}}$ is assumed to depend on the variables of the set Ξ then $\nabla\dot{\mathbf{q}}$ depends on the gradients of these variables, in particular on $\nabla\nabla\theta$ and $\nabla\nabla\nabla\mathbf{q}$ which can be arbitrarily chosen. Hence, if $\dot{\mathbf{q}}$ actually depends on $\nabla\theta$ and $\nabla\nabla\mathbf{q}$ then also $\nabla\dot{\mathbf{q}}$ can be arbitrarily chosen. The same argument is applied to $\nabla\nabla\dot{\mathbf{q}}$. However, if the linearity and arbitrariness of $\nabla\dot{\mathbf{q}}$ and $\nabla\nabla\dot{\mathbf{q}}$ cannot be applied, we can still assume that $\partial_{\nabla\mathbf{q}}\psi = \partial_{\nabla\nabla\mathbf{q}}\psi = \mathbf{0}$.

and applying (19) with $\mathbf{N} = \partial_{\mathbf{q}}\psi/|\partial_{\mathbf{q}}\psi|$, $\mathbf{Z} = \dot{\mathbf{q}}$ we obtain

$$\dot{\mathbf{q}} = \Big(\mathbf{q}\cdot\nabla\frac{1}{\theta} - \rho\zeta + \nabla\cdot\mathbf{k}\Big)\frac{\theta\mathbf{N}}{\rho_R|\partial_{\mathbf{q}}\psi|} + (\mathbf{1} - \mathbf{N}\otimes\mathbf{N})\mathbf{G}, \tag{82}$$

where $\mathbf{G}$ is an arbitrary vector-valued function dependent on $(\theta, \mathbf{q}, \nabla\theta)$. Now we let

$$\rho\psi = \rho\psi_0(\theta) + \frac{\tau\theta}{2\varkappa^2(\theta)}|\mathbf{q}|^2, \quad \rho\zeta = \frac{1}{\varkappa^2(\theta)}|\mathbf{q}|^2 + \ell^2|\nabla\mathbf{q}|^2 + 2\ell^2|\nabla\cdot\mathbf{q}|^2, \tag{83}$$

$$\mathbf{k} = \ell^2[(\mathbf{q}\cdot\nabla)\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{q}].$$

where ℓ is a constant parameter and $\varkappa$ a function of θ. Exploiting (78) the representation formula (82) gives

$$\dot{\mathbf{q}} = \mathbf{G} + \frac{1}{\tau}\mathbf{N}\otimes\mathbf{N}\Big(\varkappa^2(\theta)\nabla\frac{1}{\theta} - \frac{1}{\varkappa^2(\theta)}\mathbf{q} + \ell^2\varkappa^2(\theta)[\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})] - \tau\mathbf{G}\Big),$$

where $\mathbf{N} = \mathbf{q}/|\mathbf{q}|$. Finally, letting

$$\tau\mathbf{G} = \varkappa^2(\theta)\Big(\nabla\frac{1}{\theta} - \frac{1}{\varkappa^2(\theta)}\mathbf{q} + \ell^2[\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})]\Big)$$

we obtain (81) provided that

$$\lambda(\theta) = \ell\varkappa(\theta), \qquad \kappa(\theta) = \varkappa^2(\theta)/\theta^2. \tag{84}$$

These conditions reveal that temperature-dependent parameters of the Guyer-Krumhansl model are functionally connected. Notably, this can also be viewed as a non-trivial consequence of the *Onsagerian relation* (see, e.g., [66, eqn.7]).

Remark 7.1. *Let $\varkappa(\theta) = \varkappa_0$. Hence $\lambda(\theta) = \ell\varkappa_0$. Due to the identity*

$$\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q}) = \nabla\cdot[\nabla\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{1}],$$

the constitutive equation (81) can be rewritten as

$$\tau\dot{\mathbf{q}} + \mathbf{q} + \kappa\nabla\theta = \nabla\cdot\mathfrak{Q}, \tag{85}$$

where

$$\mathfrak{Q} = \ell^2\varkappa_0^2\big[\nabla\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{1}\big] = \ell^2\varkappa_0^2\,\partial_{\mathbf{q}}\mathbf{k}\,.$$

Hence (85) can also be viewed as the evolution equation of the heat flux in the framework of extended irreversible thermodynamics [44] and the tensor $\mathfrak{Q}$ looks like the flux of the heat flux.

In the special case $\varkappa(\theta) = \varkappa_0$ the corresponding internal energy depends only on θ,

$$\varepsilon := \psi + \theta\eta = \psi - \theta\partial_\theta\psi = \psi_0(\theta) - \partial_\theta\psi_0(\theta),$$

so that we are allowed to assume the usual form $\varepsilon = c_v(\theta)\theta$. In [66] well-posedness is proved for the one-dimensional version of the GK equation (81) under adiabatic (perfect thermal insulation) boundary conditions and letting the specific heat c_v be constant.

Local boundary conditions imposed on the basic field $\mathbf{q}$ must satisfy condition $(11)_1$ which is involved in the nonlocal strong form of the Second Law, namely

$$\mathbf{k}\cdot\mathbf{n}\Big|_{\partial\Omega} = \mathbf{q}\cdot\frac{\partial\mathbf{q}}{\partial n}\Big|_{\partial\Omega} + 2(\nabla\cdot\mathbf{q})\mathbf{q}\cdot\mathbf{n}\Big|_{\partial\Omega} = 0. \tag{86}$$

In the seminal paper [26] steady one-dimensional solutions are derived in a cylinder assuming $\mathbf{q} = \mathbf{0}$ on its boundary, a condition that certainly complies (86). As for nanowires, the boundary conditions applied in [67] to the stationary solutions are perfectly consistent with (86). Furthermore, if the heat flow across the boundary, q_n, is given then assuming

$$\mathbf{q}\cdot\mathbf{n}\Big|_{\partial\Omega} = q_n, \qquad \mathbf{q}\cdot\frac{\partial\mathbf{q}}{\partial n}\Big|_{\partial\Omega} = -2q_n(\nabla\cdot\mathbf{q})\Big|_{\partial\Omega}.$$

thermodynamic condition (86) is satisfied.

7.3.2 Nonlinear Guyer-Krumhansl conductors

A simple nonlinear extension of the Guyer-Krumhansl model can be written as[6]

$$\tau\dot{\mathbf{q}} + \mathbf{q} + \kappa\nabla\theta = \lambda^2[\nabla^2\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})] + \mu^2(\mathbf{q}\cdot\nabla)\mathbf{q} + \nu^2(\nabla\cdot\mathbf{q})\mathbf{q}. \tag{87}$$

This model illustrates relaxational and nonlocal effects of the heat flow in nanosystems [33]. Borrowing the procedure of the previous subsection, we choose ψ and ζ as in (83) and we let

$$\mathbf{k} = \ell^2[\mathbf{q}\nabla\mathbf{q} + 2(\nabla\cdot\mathbf{q})\mathbf{q}] + \nu^2|\mathbf{q}|^2\mathbf{q}.$$

First we compute $\nabla\cdot\mathbf{k}$ by exploiting the identity

$$\nabla\cdot\left[|\mathbf{q}|^2\mathbf{q}\right] = 2(\mathbf{q}\cdot\nabla)\mathbf{q}\cdot\mathbf{q} + |\mathbf{q}|^2\nabla\cdot\mathbf{q},$$

then we apply the representation formula (82) as in the previous section. As a result we recover the constitutive equation (87) provided that

$$\lambda(\theta) = \ell\varkappa(\theta), \qquad \kappa(\theta) = \varkappa^2(\theta)/\theta^2, \qquad \mu(\theta) = \sqrt{2}\delta\varkappa(\theta), \qquad \nu(\theta) = \delta\varkappa(\theta). \tag{88}$$

Note that thermodynamics does not impose any restrictions on $\varkappa(\theta)$, ℓ and δ.

Proposition 7.1. *Summarizing, (87) is thermodynamically consistent with the nonlocal statement of the Second Law (11) provided that*

i) $\kappa(\theta)$ *is positive and proportional to* $\lambda^2(\theta)/\theta^2$*;*
ii) $\mu^2(\theta) = 2\nu^2(\theta)$ *is proportional to* $\lambda^2(\theta)$*;*

[6] When $\nu = 0$, see for instance [32].

iii) boundary conditions for $\mathbf{q}$ *must satisfy the nonlinear condition*

$$\mathbf{q}\cdot\frac{\partial\mathbf{q}}{\partial n}\Big|_{\partial\Omega}+\left[2\nabla\cdot\mathbf{q}+|\mathbf{q}|^2\right]\mathbf{q}\cdot\mathbf{n}\Big|_{\partial\Omega}=0. \tag{89}$$

8 Conclusions

This paper is devoted to develop a general constitutive scheme within continuum thermodynamics to describe the behavior of heat flow in deformable media. Starting from a classical thermodynamic approach in the material (Lagrangian) description, a wide class of rate-type constitutive equations for the heat flux vector $\mathbf{q}_R$ are obtained. Their thermodynamic consistency with the Second Law in the local form (15) is established by exhibiting functional expressions of the specific free energy ψ and entropy production rate σ. Since deformable conductors are considered, mechanical and thermal entropic productions are treated separately via the inequalities (22) and (23).

It is important to underline that non-Fourier heat conduction is relevant for modern applications such e.g. as microelectronics, nanomaterials and bioheat transfer (see e.g. [68–70]), where classical Fourier law fails. Further, the rigorous thermodynamic basis could inform future numerical models and multiscale simulations that require both finite-speed and nonlocal behavior.

Some nonlinear anisotropic models of the rate type are generated by means of a suitable Representation Lemma 3.1. For instance, heat conduction models with zero dissipation (29) including the Green-Naghdi type II equation, nonlinear Maxwell-Cattaneo-Vernotte-like equations (32) and anisotropic equations of the Jeffreys type (37). For each of these models, restrictions on the material parameters are derived to ensure non-negativity of entropy production.

Linear models involving the notion of "thermal displacement" (such as Green-Naghdi and Quintanilla constitutive equations) are reformulated here as equations of the rate-type, namely (46) and (53), and their free energy and entropy production are explicitly determined as quadratic functions. Mainly, this approach allows us to overcome two "severe limitations" of the GN theories reported in [17]. When Quintanilla's model is considered, the necessary and sufficient condition on for entropy production σ to be non-negative (see Proposition 6.1) is equivalent to the estimate *iii*) of [20] which guarantees the stability of the thermoelastic problem. Solutions to the resulting third order temperature equation, also referred as Moore-Gibson-Thompson (linearized) equation, are outlined.

By considering a mixture of two different substances both characterized by a Maxwell-Cattaneo type heat exchange mechanism, a linear model is proposed. This results in the rate equation (58) involving the second time derivative of the heat flux; it looks like the dual-phase-lag model and its thermodynamic consistency is proved in Proposition 6.2 as restrictions on the material parameters that ensure non-negativity of the entropy production. We are not aware of any such result having been obtained in the context of classical continuum thermodynamics.Our procedure is inspired by the constitutive equation of a Burgers fluid . For isotropic rigid bodies the corresponding third order temperature equation (66) is scrutinized. It was first proposed by Joseph and Preziosi in [7] on the basis of a linearized Gurtin-Pipkin model with exponential kernel and represents an approximation in dual-phase-lag heat-conduction theory.

In the second part of the paper, linear and nonlinear Guyer-Krumhansl heat conduction models are derived using the Representation Lemma 3.1. Their thermodynamic consistency with the Second Law in the nonlocal form (14) is established by exhibiting explicit expressions of the specific free energy ψ, local entropy supply ζ and extra entropy flux $\mathbf{k}$. Notably, this is a rigorous proof that the Guyer-Krumhansl equation can be derived in the framework of classical thermodynamics, offering a predictive alternative to empirical modeling. Temperature-dependent material parameters of the model are required to be functionally connected (in the linear and nonlinear case we have (84) and (88), respectively). Notably, this condition can also be obtained within different frameworks and can be viewed as a non-trivial consequence of the Onsagerian relations (see, e.g., [66]). Restrictions (86) imposed by the no-flow boundary condition (11) for $\mathbf{k}$ are discussed. The resulting analysis carries important suggestions for the choice of the most appropriate boundary conditions on the heat flux vector in applications to nanosystem. In Remark 7.1, a connection is established with the evolution equation of the heat flux in the framework of extended irreversible thermodynamics (see [31, 55]).

As pointed out in the Introduction, nonlocal heat conduction models are assuming a particular relevance in the literature also due to the possible applications. Numerical simulations or exact solutions of the Guyer-Krumhansl model can be found for example in [71], with a comparison with available experimental data for silicon thin layers, in [72], where boundary conditions from laser flash experiment are considered and in [73], where an ad-hoc finite element method for the Guyer–Krumhansl heat conductivity model is given.

Some final remarks are in order.

First we note that different choices of the thermodynamic functions are possible in connection with the same heat conduction model (local or non-local). As explicitly shown in connection with the Jeffreys-type equation (37), different free energies ψ and entropy productions σ are allowed for the same given rate equation (see Remark 5.1).

Second, in this paper free energies are allowed to depend on the temperature gradient. Apparently this dependence contrasts with some classical results by Coleman et al. [74, 75] where the C-D inequality is used to show that the internal energy and the entropy are independent of the temperature gradients. It should be noted, however, that their results are limited to constitutive relation in functional form, whereas local constitutive relations considered here (i.e. Jeffreys, GNIII, Quintanilla and Burgers models) are of the rate type.

In the third remark we point out that the entropy production (rate) σ in all models derived in the first part of the paper is explicitly represented as a positive-definite quadratic function of non-equilibrium quantities[7], in general $\dot{\mathbf{E}}, \nabla\theta, \mathbf{q}, \dot{\mathbf{q}}$ (see eqns (17), (18), (33), (39), (40), (65) and (A7)). Also for nonlocal models, in the second part of the paper, the local entropy supply ζ is a quadratic function of $\nabla\theta, \nabla\nabla\theta, \mathbf{q}, \nabla\mathbf{q}$, (see eqns (75), (79) and (83)). Hence, σ and ζ decrease when the system approaches the

[7] "To extend thermodynamics to non-equilibrium processes we need an explicit expressions for the entropy production" in I. Prigogine, Time, structure and fluctuations, Nobel lecture, 1977.

steady-state (constant deformation, uniform temperature distribution and zero heat flux) and vanish at the steady-state, according to the Prigogine's theorem [46].

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Appendix A Proof of Proposition 6.1

For simplicity we give the proof in the isotropic case only. Let $\tau \neq 0$ and consider (54) as the constitutive equation of $\ddot{\mathbf{q}}$. Upon substitution of $\ddot{\mathbf{q}}$ from (54) into (51), we have

$$\rho\Big(\partial_{\mathbf{q}}\psi - \frac{1}{\tau}\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\dot{\mathbf{q}} + \Big(\frac{\mathbf{q}}{\theta} - \frac{\xi}{\tau}\rho\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\nabla\theta + \rho\Big(\partial_{\nabla\theta}\psi - \frac{\kappa}{\tau}\partial_{\dot{\mathbf{q}}}\psi\Big)\cdot\nabla\dot{\theta} = -\rho\theta\sigma \leq 0.$$

Although ψ is independent of $\nabla\dot{\theta}$, σ is not necessarily independent, so we cannot invoke the linearity and arbitrariness of $\nabla\dot{\theta}$ in the C-D inequality. However we can impose the relation

$$\partial_{\nabla\theta}\psi = \frac{\kappa}{\tau}\partial_{\dot{\mathbf{q}}}\psi \tag{A1}$$

which implies

$$\rho\Big(\frac{1}{\tau}\partial_{\dot{\mathbf{q}}}\psi - \partial_{\mathbf{q}}\psi\Big)\cdot\dot{\mathbf{q}} + \Big(\frac{\xi}{\tau}\rho\partial_{\dot{\mathbf{q}}}\psi - \frac{\mathbf{q}}{\theta}\Big)\cdot\nabla\theta = \rho\theta\sigma \geq 0. \tag{A2}$$

Now we select the dependence on $\mathbf{q}, \dot{\mathbf{q}}, \nabla\theta$ of the free energy ψ in a quadratic form,

$$\rho\psi = \rho\psi_0 + \frac{\alpha_1}{2}|\mathbf{q}|^2 + \frac{\alpha_2}{2}|\dot{\mathbf{q}}|^2 + \frac{\alpha_3}{2}|\nabla\theta|^2 + \gamma_1\dot{\mathbf{q}}\cdot\mathbf{q} + \gamma_2\mathbf{q}\cdot\nabla\theta + \gamma_3\dot{\mathbf{q}}\cdot\nabla\theta, \tag{A3}$$

whence

$$\rho\partial_{\mathbf{q}}\psi = \alpha_1\mathbf{q} + \gamma_1\dot{\mathbf{q}} + \gamma_2\nabla\theta, \qquad \rho\partial_{\dot{\mathbf{q}}}\psi = \gamma_1\mathbf{q} + \alpha_2\dot{\mathbf{q}} + \gamma_3\nabla\theta,$$
$$\rho\partial_{\nabla\theta}\psi = \gamma_2\mathbf{q} + \gamma_3\dot{\mathbf{q}} + \alpha_3\nabla\theta.$$

All parameters and ψ_0 possibly depend on θ. Upon substitution into (A1) we obtain

$$\gamma_2 = \frac{\kappa}{\tau}\gamma_1, \qquad \gamma_3 = \frac{\kappa}{\tau}\alpha_2, \qquad \alpha_3 = \frac{\kappa}{\tau}\gamma_3, \tag{A4}$$

while (A2) becomes

$$A_{22}|\dot{\mathbf{q}}|^2 + A_{33}|\nabla\theta|^2 + 2A_{12}\dot{\mathbf{q}}\cdot\mathbf{q} + 2A_{13}\mathbf{q}\cdot\nabla\theta + 2A_{23}\dot{\mathbf{q}}\cdot\nabla\theta = \rho\theta\sigma \geq 0, \tag{A5}$$

where $A_{11} = 0$ and

$$A_{22} = \frac{1}{\tau}\alpha_2 - \gamma_1, \quad A_{33} = \frac{\xi}{\tau}\gamma_3, \quad 2A_{12} = \frac{1}{\tau}\gamma_1 - \alpha_1,$$
$$2A_{13} = -\frac{1}{\theta} + \frac{\xi}{\tau}\gamma_1, \quad 2A_{23} = \frac{1}{\tau}\gamma_3 - \gamma_2 + \frac{\xi}{\tau}\alpha_2.$$

Inequality (A5) is satisfied for all values of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\nabla\theta$ provided that the symmetric matrix $A = \{A_{ij}\}_{i,j=1,2,3}$ is positive-semidefinite, that is if and only if all principal minors of A are nonnegative (see, for instance [76, § 7.6]). Since $A_{11} = 0$, we consider the 2-by-2 principal minor

$$d_3 := \det\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22}\end{pmatrix} = A_{11}A_{22} - A_{12}^2 = -\frac{1}{4\tau^2}(\gamma_1 - \tau\alpha_1)^2.$$

If $\gamma_1 \neq \tau\alpha_1$ it takes a negative value and the thesis could not be proved. Hence we let $\alpha_1 = \gamma_1/\tau$ so that $A_{12} = 0$. Then we consider

$$d_2 := \det\begin{pmatrix} A_{11} & A_{13} \\ A_{13} & A_{33}\end{pmatrix} = A_{11}A_{33} - A_{13}^2 = -\frac{1}{4\theta^2\tau^2}(\xi\theta\gamma_1 - \tau)^2.$$

If $\xi = 0$ or $\gamma_1 \neq \tau/\xi\theta$ this minor takes a negative value; therefore, we let

$$\xi \neq 0, \qquad \gamma_1 = \tau/\xi\theta,$$

so that $A_{13} = 0$. Summarizing and applying $(\text{A4})_1$

$$\gamma_1 = \frac{\tau}{\xi\theta}, \quad \alpha_1 = \frac{1}{\xi\theta}, \quad \gamma_2 = \frac{\kappa}{\xi\theta}, \quad \xi \neq 0. \tag{A6}$$

Finally we consider

$$d_1 := \det\begin{pmatrix} A_{22} & A_{23} \\ A_{23} & A_{33}\end{pmatrix} = A_{33}A_{22} - A_{23}^2 = \left(\frac{\kappa}{\tau^2}\alpha_2 - \frac{\kappa}{\xi\theta}\right)\frac{\xi}{\tau}\alpha_2 - \frac{1}{4}\left(\frac{\kappa}{\tau^2}\alpha_2 - \frac{\kappa}{\xi\theta} + \frac{\xi}{\tau}\alpha_2\right)^2$$
$$= -\frac{1}{4}\left(\frac{\kappa}{\tau^2}\alpha_2 - \frac{\kappa}{\xi\theta} - \frac{\xi}{\tau}\alpha_2\right)^2$$

where we applied (A6) and $(A4)_2$. If the quantity in brackets does not vanish this minor takes a negative value; therefore, we assume it vanishes by letting

i) either $\kappa = 0$ and $\alpha_2 = 0$,
ii) or $\kappa \neq 0$, $\kappa \neq \tau\xi$ and

$$\alpha_2 = \frac{\kappa\tau^2}{\xi\theta(\kappa - \tau\xi)}.$$

In the former case (i) we have $\alpha_2 = \alpha_3 = \gamma_2 = \gamma_3 = 0$ and then

$$A_{11} = A_{33} = A_{12} = A_{13} = A_{23} = 0, \quad A_{22} = -\frac{\tau}{\xi\theta}.$$

Accordingly, A is positive-semidefinite if and only if $\kappa = 0$, $\tau, \xi \neq 0$ and $\tau\xi < 0$. Unfortunately, under these conditions, the free energy (A3) takes the form

$$\rho\psi_0 + \frac{1}{2\theta}\mathbf{q} \cdot \xi^{-1}\mathbf{q} + \frac{\tau}{\theta}\dot{\mathbf{q}} \cdot \xi^{-1}\mathbf{q},$$

which is not a convex function of $\mathbf{q}, \dot{\mathbf{q}}$.

Under the assumptions (ii), exploiting $(A4)_{2,3}$ we obtain

$$\gamma_3 = \frac{\kappa^2\tau}{\xi\theta(\kappa - \tau\xi)}, \qquad \alpha_3 = \frac{\kappa^3}{\xi\theta(\kappa - \tau\xi)},$$

and then we are able to represent A as a function of θ and the material parameters τ, ξ, κ,

$$A_{11} = A_{12} = A_{13} = 0, \quad A_{22} = \frac{\tau^2}{\theta(\kappa - \tau\xi)}, \quad A_{33} = \frac{\kappa^2}{\theta(\kappa - \tau\xi)}, \quad A_{23} = \frac{\kappa\tau}{\theta(\kappa - \tau\xi)}.$$

Accordingly, we infer that A is positive-semidefinite if and only if $\tau, \kappa, \xi \neq 0$ and $\kappa > \tau\xi$. It can easily be verified by direct substitution into (A5) that the entropy production takes the form

$$\rho\sigma = \frac{1}{\theta^2}(\tau\dot{\mathbf{q}} + \kappa\nabla\theta) \cdot (\kappa - \tau\xi)^{-1}(\tau\dot{\mathbf{q}} + \kappa\nabla\theta) \tag{A7}$$

and the free energy (A3) can be written as

$$\rho\psi = \rho\psi_0(\theta) + \frac{1}{2\theta}\big[\mathbf{q} + \tau\dot{\mathbf{q}} + \kappa\nabla\theta\big] \cdot \xi^{-1}\big[\mathbf{q} + \tau\dot{\mathbf{q}} + \kappa\nabla\theta\big]$$
$$+ \frac{1}{2\theta}\big[\tau\dot{\mathbf{q}} + \kappa\nabla\theta\big] \cdot \tau(\kappa - \tau\xi)^{-1}\big[\tau\dot{\mathbf{q}} + \kappa\nabla\theta\big].$$

Therefore, if $\tau, \xi > 0$ in addition to $\kappa > \tau\xi$, this free energy is a convex function of $\mathbf{q}, \dot{\mathbf{q}}, \nabla\theta$. Strict convexity holds only with respect to $\mathbf{q}$.

Note that letting $\tau \to 0$ the entropy production and free energy of the GN III model are recovered. Hence, the Quintanilla model represents a proper generalization of the GN III linear theory.

Appendix B Proof of Proposition 6.2

To satisfy inequality (65) for all values of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\nabla\theta$ we have to prove that the symmetric matrix $A = \{A_{ij}\}_{i,j=1,2,3}$ is positive semidefinite. First we consider the 2-by-2 principal minor

$$d_1 := \det \begin{pmatrix} A_{22} & A_{23} \\ A_{23} & A_{33} \end{pmatrix} = A_{22}A_{33} - A_{23}^2 = \Big(\frac{\tau}{\lambda}\alpha_2 - \gamma_1\Big)\frac{\mu}{\lambda}\gamma_3 - \frac{1}{4}\Big(\frac{\tau}{\lambda}\gamma_3 - \gamma_2 + \frac{\mu}{\lambda}\alpha_2\Big)^2$$
$$= -\frac{1}{4\lambda^2}\Big(\frac{\nu\tau^2 - \mu\lambda}{\lambda}\alpha_2 - \nu\tau\gamma_1\Big)^2$$

where we applied $(64)_{1,2}$. If the quantity in brackets does not vanish this minor takes a negative value; therefore, we are forced to impose $d_1 = 0$. This can be achieved in different ways depending on the parameter values:

1. when $\tau\nu = 0$ and $\mu = 0$, whatever the values of γ_1 and α_2 are.
2. when $\tau\nu = 0$ and $\mu \neq 0$, by setting $\alpha_2 = 0$.
3. when $\nu\tau \neq 0$ and $\mu = 0$ by setting

$$\gamma_1 = \frac{\tau}{\lambda}\alpha_2. \tag{B8}$$

4. when $\nu\tau \neq 0$ and $\mu \neq 0$ by setting

$$\gamma_1 = \frac{\nu\tau^2 - \mu\lambda}{\nu\tau\lambda}\alpha_2. \tag{B9}$$

Case (1). If $\mu = 0$ and either $\tau = 0$ or $\nu = 0$ then from (64) it follows $\gamma_2 = \gamma_3 = \alpha_3 = 0$ and

$$A_{11} = \frac{1}{\lambda}\gamma_1, \quad A_{33} = 0, \quad A_{13} = -\frac{1}{2\theta}, \quad A_{23} = 0.$$

Hence, in any case $d_2 := A_{11}A_{33} - A_{13}^2 = -1/4\theta^2 < 0$ and A cannot be positive semidefinite.

Case (2). If $\mu \neq 0$ and either $\tau = 0$ or $\nu = 0$ then from (64) it follows $\gamma_2 = \gamma_3 = \alpha_3 = 0$. By setting $\alpha_2 = 0$ we get $d_1 = 0$ and

$$A_{11} = \frac{1}{\lambda}\gamma_1, \quad A_{22} = -\gamma_1, \quad A_{33} = 0,$$
$$2A_{12} = \frac{\tau}{\lambda}\gamma_1 - \alpha_1, \quad 2A_{13} = -\frac{1}{\theta} + \frac{\mu}{\lambda}\gamma_1, \quad 2A_{23} = 0.$$

Since

$$d_2 = -\frac{1}{4}\Big(-\frac{1}{\theta} + \frac{\mu}{\lambda}\gamma_1\Big)^2$$

we are forced to set $\gamma_1 = \frac{\lambda}{\mu\theta}$. Accordingly, $d_2 = 0$ and

$$A_{11} = \frac{1}{\mu\theta}, \qquad A_{22} = -\frac{\lambda}{\mu\theta}, \qquad 2A_{12} = \frac{\tau}{\mu\theta} - \alpha_1,$$

so that the diagonal elements of A are non negative only if $\mu > 0$ and $\lambda < 0$. If this is the case,

$$d_3 := A_{11}A_{22} - A_{12}^2 = \frac{|\lambda|}{\mu^2\theta^2} - \frac{1}{4}\Big(\frac{\tau}{\mu\theta} - \alpha_1\Big)^2 \geq 0$$

is ensured to be positive by choosing, for instance, $\alpha_1 = \tau/\mu\theta$ so that $A_{12} = 0$. Finally,

$$\det A = d_1 A_{11} - d_2 A_{22} + d_3 A_{33} = 0$$

and we conclude that A is positive semidefinite provided that i) occurs.

Case (3). Let $\nu\tau \neq 0$ and $\mu = 0$. After replacing (B8) into (64) we obtain

$$\gamma_2 = \frac{\nu\tau^2}{\lambda^2}\alpha_2, \qquad \gamma_3 = \frac{\tau\nu}{\lambda}\alpha_2, \qquad \alpha_3 = \frac{\tau^2\nu^2}{\lambda^2}\alpha_2,$$

and then $A_{22} = A_{33} = A_{23} = 0$

$$A_{11} = \frac{\tau}{\lambda^2}\alpha_2, \qquad 2A_{12} = \frac{\tau^2 + \lambda}{\lambda^2}\alpha_2 - \alpha_1, \qquad 2A_{13} = -\frac{1}{\theta} + \frac{\nu\tau}{\lambda^2}\alpha_2.$$

By letting

$$\alpha_1 = \frac{\tau^2 + \lambda}{\lambda^2}\alpha_2, \qquad \alpha_2 = \frac{\lambda^2}{\nu\tau\theta}$$

we get $d_2 = d_3 = 0$ and A turns out to be positive semidefinite if and only if $A_{11} = 1/\nu\theta \geq 0$, that is when ii) occurs. Accordingly

$$\alpha_1 = \frac{\tau^2 + \lambda}{\nu\tau\theta}, \quad \alpha_2 = \frac{\lambda^2}{\nu\tau\theta}, \quad \alpha_3 = \frac{\nu\tau}{\theta}, \quad \gamma_1 = \frac{\lambda}{\nu\theta}, \quad \gamma_2 = \frac{\tau}{\theta}, \quad \gamma_3 = \frac{\lambda}{\theta}.$$

$$A_{11} = \frac{1}{\nu\theta}, \qquad A_{22} = A_{33} = A_{12} = A_{13} = A_{23} = 0.$$

Under the same assumptions, the quadratic form in the free energy (A3) is positive-semidefinite provided that in addition $\lambda > 0$.

Case (4). Let $\nu\tau \neq 0$, $\mu \neq 0$. After replacing (B9) into (64) we obtain

$$\gamma_2 = \frac{\nu\tau^2 - \mu\lambda}{\lambda^2}\alpha_2, \qquad \gamma_3 = \frac{\tau\nu}{\lambda}\alpha_2, \qquad \alpha_3 = \frac{\tau^2\nu^2}{\lambda^2}\alpha_2,$$

and then

$$A_{11} = \frac{\nu\tau^2 - \mu\lambda}{\nu\tau\lambda^2}\alpha_2, \quad A_{22} = \frac{\mu}{\nu\tau}\alpha_2, \quad A_{33} = \frac{\mu\nu\tau}{\lambda^2}\alpha_2, \quad 2A_{12} = \frac{\nu\tau^2 + (\nu - \mu)\lambda}{\nu\lambda^2}\alpha_2 - \alpha_1,$$
$$2A_{13} = -\frac{1}{\theta} + \frac{\nu^2\tau^2 + \mu(\nu\tau^2 - \mu\lambda)}{\nu\tau\lambda^2}\alpha_2, \quad 2A_{23} = \frac{2\mu}{\lambda}\alpha_2.$$

If we assume $A_{12} = 0$ by letting

$$\alpha_1 = \frac{\nu\tau^2 + (\nu - \mu)\lambda}{\nu\lambda^2}\alpha_2$$

then $d_3 = A_{11}A_{22}$ and since $d_1 = 0$, $d_2 = A_{11}A_{33} - A_{13}^2$ it follows

$$\det A = d_3 A_{33} - d_2 A_{22} = (A_{11}A_{33} - d_2)A_{22} = A_{13}^2 A_{22}.$$

For simplicity, let $\mathcal{D} = \nu^2\tau^2 + \mu(\nu\tau^2 - \mu\lambda)$. If we choose

$$\alpha_2 = \frac{\nu\tau\lambda^2}{\theta\mathcal{D}}$$

then $A_{13} = 0$ and A is positive semidefinite if and only if its diagonal terms

$$A_{11} = \frac{\nu\tau^2 - \mu\lambda}{\theta\mathcal{D}},\ A_{22} = \frac{\mu\lambda^2}{\theta\mathcal{D}},\ A_{33} = \frac{\mu\nu^2\tau^2}{\theta\mathcal{D}}$$

take nonnegative values. This is achieved by assuming conditions iii) on the material parameters. Accordingly

$$\alpha_1 = \frac{\tau(\nu\tau^2 + (\nu - \mu)\lambda)}{\theta\mathcal{D}}, \qquad \alpha_2 = \frac{\nu\tau\lambda^2}{\theta\mathcal{D}}, \qquad \alpha_3 = \frac{1}{\nu\tau\theta\mathcal{D}},$$

$$\gamma_1 = \frac{\lambda(\nu\tau^2 - \mu\lambda)}{\theta\mathcal{D}}, \qquad \gamma_2 = \frac{\nu\tau(\nu\tau^2 - \mu\lambda)}{\theta\mathcal{D}}, \qquad \gamma_3 = \frac{\nu^3\tau^3}{\theta\mathcal{D}}.$$

$$A_{11} = \frac{\nu\tau^2 - \mu\lambda}{\theta\mathcal{D}}, \quad A_{22} = \frac{\mu\lambda^2}{\theta\mathcal{D}}, \quad A_{33} = \frac{\mu\nu^2\tau^2}{\theta\mathcal{D}}, \quad A_{12} = A_{13} = 0, \quad A_{23} = \frac{\mu\nu\tau\lambda}{\theta\mathcal{D}}.$$

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