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A reduction of the spectrum problem for odd sun systems and the prime case

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Abstract

A k-cycle with a pendant edge attached to each vertex is called a k-sun. The existence problem for k-sun decompositions of K_v, with k odd, has been solved only when $k = 3$ or 5. By adapting a method used by Hoffmann, Lindner, and Rodger to reduce the spectrum problem for odd cycle systems of the complete graph, we show that if there is a k-sun system of K_v (k odd) whenever v lies in the range $2k < v < 6k$ and satisfies the obvious necessary conditions, then such a system exists for every admissible $v \geq 6k$. Furthermore, we give a complete solution whenever k is an odd prime.

KEYWORDS

crown graph, cycle systems, graph decompositions, partial mixed differences, sun systems

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1 | INTRODUCTION

We denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and the list of edges of a graph Γ, respectively. Also, we denote by $\Gamma + w$ the graph obtained by adding to Γ an independent set $W = \{\infty_i \mid 1 \leq i \leq w\}$ of $w \geq 0$ vertices each adjacent to every vertex of Γ, namely,

$$\Gamma + w := \Gamma \cup K_{V(\Gamma), W},$$

where $K_{V(\Gamma),W}$ is the complete bipartite graph with parts $V(\Gamma)$ and W. Denoting by K_v the *complete graph* of order v, it is clear that $K_v + 1$ is isomorphic to K_{v+1}.

We denote by $x_1 \sim x_2 \sim \cdots \sim x_k$ the *path* with edges $\{x_{i-1}, x_i\}$ for $2 \le i \le k$. By adding the edge $\{x_1, x_k\}$ when $k \ge 3$, we obtain a *cycle of length* k (briefly, a k-*cycle*) denoted by $(x_1, x_2, ..., x_k)$. A k-cycle with further $v - k \ge 0$ isolated vertices will be referred to as a k-cycle of order v. By adding to $(x_1, x_2, ..., x_k)$ an independent set of edges $\{\{x_i, x_i'\} | 1 \le i \le k\}$, we obtain the k-*sun* on $2k$ vertices (sometimes referred to as k-*crown* graph) denoted by

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k \\ x_1' & x_2' & \cdots & x_{k-1}' & x_k' \end{pmatrix},$$

whose edge-set is therefore $\{\{x_i, x_{i+1}\}, \{x_i, x_i'\} | 1 \le i \le k\}$, where $x_{k+1} = x_1$.

A *decomposition* of a graph K is a set $\{\Gamma_1, \Gamma_2, ..., \Gamma_t\}$ of subgraphs of K whose edge-sets between them partition the edge-set of K; in this case, we briefly write $K = \oplus_{i=1}^t \Gamma_i$. If each Γ_i is isomorphic to Γ, we speak of a Γ-*decomposition of* K. If Γ is a k-cycle (resp., k-sun), we also speak of a k-*cycle system* (resp., k-*sun system*) of K.

In this paper we study the existence problem for k-sun systems of K_v ($v > 1$). Clearly, for such a system to exist we must have

$$v \ge 2k \quad \text{and} \quad v(v-1) \equiv 0 \pmod{4k}. \qquad (*)$$

As far as we know, this problem has been completely settled only when $k = 3, 5$ [8,10], $k = 4, 6, 8$ [12], and when $k = 10, 14$ or $2^t \ge 4$ [9]. It is important to notice that, as a consequence of a general result proved in [14], condition (*) is sufficient whenever v is large enough with respect to k. These results seem to suggest the following.

Conjecture 1. *Let $k \ge 3$ and $v > 1$. There exists a k-sun system of K_v if and only if* (*) *holds.*

Our constructions rely on the existence of k-cycle systems of K_v, a problem that has been completely settled in [1,4,5,11,13]. More precisely, [4] and [11] reduce the problem to the orders v in the range $k \le v < 3k$, with v odd. These cases are then solved in [1,13]. For odd k, an alternative proof based on 1-rotational constructions is given in [5]. Further results on k-cycle systems of K_v with an automorphism group acting sharply transitively on all but at most one vertex can be found in [2,6,7,15].

The main results of this paper focus on the case where k is odd. By adapting a method used in [11] to reduce the spectrum problem for odd cycle systems of the complete graph, we show that if there is a k-sun system of K_v (k odd) whenever v lies in the range $2k < v < 6k$ and satisfies the obvious necessary conditions, then such a system exists for every admissible $v \ge 6k$. In other words, we show the following.

Theorem 1.1. *Let $k \ge 3$ be an odd integer and $v > 1$. Conjecture 1 is true if and only if there exists a k-sun system of K_v for all v satisfying the necessary conditions in* (*) *with $2k < v < 6k$.*

We would like to point out that we strongly believe the reduction methods used in [4,11] could be further developed to reduce the spectrum problem of other types of graph decompositions of K_v.

In Section 6, we construct k-sun systems of K_v for every odd prime k whenever $2k < v < 6k$ and (*) holds. Therefore, as a consequence of Theorem 1.1, we solve the existence problem for k-sun systems of K_v whenever k is an odd prime.

Theorem 1.2. *For every odd prime p there exists a p-sun system of K_v with $v > 1$ if and only if $v \geq 2p$ and $v(v-1) \equiv 0 \pmod{4p}$.*

Both results rely on the difference methods described in Section 2. These methods are used in Section 3 to construct specific k-cycle decompositions of some subgraphs of $K_{2k} + w$, which we then use in Section 4 to build k-sun systems of $K_{4k} + n$. This is the last ingredient we need in Section 5 to prove Theorem 1.1. Difference methods are finally used in Section 6 to construct k-sun systems of K_v for every odd prime k whenever $2k < v < 6k$ and (*) holds.

2 | PRELIMINARIES

Henceforward, $k \geq 3$ is an odd integer, and $\ell = \frac{k-1}{2}$. Also, given two integers $a \leq b$, we denote by $[a, b]$ the interval containing the integers $\{a, a+1, \ldots, b\}$. If $a > b$, then $[a, b]$ is empty.

In our constructions we make extensive use of the method of partial mixed differences which we now recall but limited to the scope of this paper.

Let G be an abelian group of odd order n in additive notation, let $W = \{\infty_u \mid 1 \leq u \leq w\}$, and denote by Γ a graph with vertices in $V = (G \times [0, m-1]) \cup W$. For any permutation f of V, we denote by $f(\Gamma)$ the graph obtained by replacing each vertex of Γ, say x, with $f(x)$. Letting τ_g, with $g \in G$, be the permutation of V fixing each $\infty_u \in W$ and mapping $(x, i) \in G \times [0, m-1]$ to $(x+g, i)$, we call τ_g the *translation by* g and $\tau_g(\Gamma)$ the related translate of Γ.

We denote by $Orb_G(\Gamma) = \{\tau_g(\Gamma) \mid g \in G\}$ the *G-orbit* of Γ, that is, the set of all distinct translates of Γ, and by $Dev_G(\Gamma) = \bigcup_{g \in G} \tau_g(\Gamma)$ the graph union of all translates of Γ. Further, by $Stab_G(\Gamma) = \{g \in G \mid \tau_g(\Gamma) = \Gamma\}$ we denote the *G-stabilizer* of Γ, namely, the set of translations fixing Γ. We recall that $Stab_G(\Gamma)$ is a subgroup of G, hence $s = |Stab_G(\Gamma)|$ is a divisor of $n = |G|$. Henceforward, when $G = \mathbb{Z}_k$, we will simply write $Orb(\Gamma)$, $Dev(\Gamma)$, and $Stab(\Gamma)$.

Suppose now that Γ is either a k-cycle or a k-sun with vertices in V. For every $i, j \in [0, m-1]$, the list of (i, j)-differences of Γ is the multiset $\Delta_{ij}\Gamma$ defined as follows:

1. if $\Gamma = (x_1, x_2, \ldots, x_k)$, then

$$\begin{aligned}\Delta_{ij}\Gamma = &\{a_{h+1} - a_h \mid x_h = (a_h, i), x_{h+1} = (a_{h+1}, j), 1 \leq h \leq k/s\} \\ &\cup \{a_h - a_{h+1} \mid x_h = (a_h, j), x_{h+1} = (a_{h+1}, i), 1 \leq h \leq k/s\};\end{aligned}$$

2. if $\Gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ x_1' & x_2' & \cdots & x_k' \end{pmatrix}$, then

$$\begin{aligned}\Delta_{ij}\Gamma = &\Delta_{ij}(x_1, x_2, \ldots, x_k) \cup \{a_h' - a_h \mid x_h = (a_h, i), x_h' = (a_h', j), 1 \leq h \leq k/s\} \\ &\cup \{a_h - a_h' \mid x_h = (a_h, j), x_h' = (a_h', i), 1 \leq h \leq k/s\}.\end{aligned}$$

We notice that when $s = 1$ we find the classic concept of list of differences. Usually, one speaks of *pure or mixed differences* according to whether $i = j$ or not, and when $m = 1$ we simply write

$\Delta\Gamma$. This concept naturally extends to a family $\mathcal{F}$ of graphs with vertices in V by setting $\Delta_{ij}\mathcal{F} = \bigcup_{\Gamma\in\mathcal{F}} \Delta_{ij}\Gamma$. Clearly, $\Delta_{ij}\Gamma = -\Delta_{ji}\Gamma$, hence $\Delta_{ij}\mathcal{F} = -\Delta_{ji}\mathcal{F}$, for every $i, j \in [0, m-1]$.

We also need to define the *list of neighbors* of ∞_u in $\mathcal{F}$, that is, the multiset $N_{\mathcal{F}}(\infty_u)$ of the vertices in V adjacent to ∞_u in some graph $\Gamma \in \mathcal{F}$.

Finally, we introduce a special class of subgraphs of K_{mn}. To this purpose, we take $V(K_{mn}) = G \times [0, m-1]$. Letting $D_{ii} \subseteq G\setminus\{0\}$ for every $0 \le i \le m-1$, and $D_{ij} \subseteq G$ for every $0 \le i < j \le m-1$, we denote by

$$\langle D_{ij} \mid 0 \le i \le j \le m-1\rangle$$

the spanning subgraph of K_{mn} containing exactly the edges $\{(g, i), (g+d, j)\}$ for every $g \in G$, $d \in D_{ij}$, and $0 \le i \le j \le m-1$. The reader can easily check that this graph remains unchanged if we replace any set D_{ii} with $\pm D_{ii}$.

The following result, standard in the context of difference families, provides us with a method to construct Γ-decompositions for subgraphs of $K_{mn} + w$.

Proposition 2.1. *Let G be an abelian group of odd order n, let m and w be nonnegative integers, and denote by $\mathcal{F}$ a family of k-cycles (resp., k-suns) with vertices in $(G \times [0, m-1]) \cup \{\infty_u \mid u \in \mathbb{Z}_w\}$ satisfying the following conditions:*

1. *$\Delta_{ij}\mathcal{F}$ has no repeated elements, for every $0 \le i \le j < m$;*
2. *$N_{\mathcal{F}}(\infty_u) = \{(g_{u,i}, i) \mid 0 \le i < m, g_{u,i} \in G\}$ for every $1 \le u \le w$.*

Then $\bigcup_{\Gamma\in\mathcal{F}} Orb_G(\Gamma) = \{\tau_g(\Gamma) \mid g \in G, \Gamma \in \mathcal{F}\}$ is a k-cycle (resp., k-sun) system of $\langle \Delta_{ij}\mathcal{F} \mid 0 \le i \le j \le m-1\rangle + w$.

Proof. Let $\mathcal{F}^* = \bigcup_{\Gamma\in\mathcal{F}} Orb_G(\Gamma)$, $K = \langle \Delta_{ij}\mathcal{F} \mid 0 \le i \le j \le m-1\rangle$, and let ϵ be an edge of $K + w$. We are going to show that ϵ belongs to exactly one graph of $\mathcal{F}^*$.

If $\epsilon \in E(K)$, by recalling the definition of K we have that $\epsilon = \{(g, i), (g+d, j)\}$ for some $g \in G$ and $d \in \Delta_{ij}\mathcal{F}$, with $0 \le i \le j < m$. Hence, there is a graph $\Gamma \in \mathcal{F}$ such that $d \in \Delta_{ij}\Gamma$. This means that Γ contains the edge $\epsilon' = \{(g', i), (g'+d, j)\}$ for some $g' \in G$, therefore $\epsilon = \tau_{g-g'}(\epsilon') \in \tau_{g-g'}(\Gamma) \in \mathcal{F}^*$. To prove that ϵ only belongs to $\tau_{g-g'}(\Gamma)$, let Γ' be any graph in $\mathcal{F}$ such that $\epsilon \in \tau_x(\Gamma')$, for some $x \in G$. Since translations preserve differences, we have that $d \in \Delta_{ij}\tau_x(\Gamma') = \Delta_{ij}\Gamma'$. Considering that $d \in \Delta_{ij}\Gamma \cap \Delta_{ij}\Gamma'$ and, by condition (1), $\Delta_{ij}\mathcal{F}$ has no repeated elements, we necessarily have that $\Gamma' = \Gamma$, hence $\tau_{-x}(\epsilon) \in \Gamma$. Again, since $\Delta_{ij}\Gamma$ has no repeated elements (condition 1), and considering that ϵ' and $\tau_{-x}(\epsilon)$ are edges of Γ that yield the same differences, then $\tau_{-x}(\epsilon) = \epsilon' = \tau_{g'-g}(\epsilon)$, that is, $\tau_{g'-g+x}(\epsilon) = \epsilon$. Since G has odd order, it has no element of order 2, hence $g' - g + x = 0$, that is, $x = g - g'$, therefore $\tau_{g-g'}(\Gamma)$ is the only graph of $\mathcal{F}^*$ containing ϵ.

Similarly, we show that every edge of $(K + w)\setminus K$ belongs to exactly one graph of $\mathcal{F}^*$. Let $\epsilon = \{\infty_u, (g, i)\}$ for some $u \in \mathbb{Z}_w$ and $(g, i) \in G \times [0, m-1]$. By assumption, there is a graph $\Gamma \in \mathcal{F}^*$ containing the edge $\epsilon' = \{\infty_u, (g_{u,i}, i)\}$ with $g_{u,i} \in G$. Hence, $\epsilon = \tau_{g-g_{u,i}}(\epsilon') \in \tau_{g-g_{u,i}}(\Gamma)$. Finally, if $\epsilon \in \tau_x(\Gamma')$ for some $x \in G$ and $\Gamma' \in \mathcal{F}$, then $\{\infty_u, (g-x, i)\} = \tau_{-x}(\epsilon) \in \Gamma'$. Since condition (2) implies that $N_{\mathcal{F}}(\infty_u)$ contains exactly one pair from $G \times \{i\}$, we necessarily have that $\Gamma = \Gamma'$ and $x = g - g_{u,i}$; therefore, there is exactly one graph of $\mathcal{F}^*$ containing ϵ. Condition (2) also implies that $N_{\mathcal{F}}(\infty_u)$ is disjoint from $\{\infty_u \mid u \in \mathbb{Z}_w\}$, and this guarantees that no graph in $\mathcal{F}^*$ contains edges joining two infinities. Therefore, $\mathcal{F}^*$ is the desired decomposition of $K + w$. □

Considering that $K_{mn} = \langle D_{ij} \mid 0 \le i \le j \le m-1 \rangle$ if and only if $\pm D_{ii} = G \setminus \{0\}$ for every $i \in [0, m-1]$, and $D_{ij} = G$ for every $0 \le i < j \le m-1$, the proof of the following corollary to Proposition 2.1 is straightforward.

Corollary 2.2. *Let G be an abelian group of odd order n, let m and w be nonnegative integers, and denote by $\mathcal{F}$ a family of k-cycles (resp., k-suns) with vertices in $(G \times [0, m-1]) \cup \{\infty_u \mid u \in \mathbb{Z}_w\}$ satisfying the following conditions:*

1. $\Delta_{ij}\mathcal{F} = \begin{cases} G \setminus \{0\} & \text{if } 0 \le i = j \le m-1, \\ G & \text{if } 0 \le i < j \le m-1, \end{cases}$
2. $N_{\mathcal{F}}(\infty_u) = \{(g_{u,i}, i) \mid 0 \le i < m, g_{u,i} \in G\}$ *for every* $1 \le u \le w$.

Then $\bigcup_{\Gamma \in \mathcal{F}} Orb_G(\Gamma)$ is a k-cycle (resp., k-sun) system of $K_{mn} + w$.

3 | CONSTRUCTING k-CYCLE SYSTEMS OF $\langle D_{00}, D_{01}, D_{11} \rangle + w$

In this section, we recall and generalize some results from [11] to provide conditions on $D_{00}, D_{01}, D_{11} \subseteq \mathbb{Z}_k$ that guarantee the existence of a k-cycle system for the subgraph $\langle D_{00}, D_{01}, D_{11} \rangle + w$ of $K_{2k} + w$, where $V(K_{2k}) = \mathbb{Z}_k \times \{0, 1\}$.

We recall that every connected 4-regular Cayley graph over an abelian group has a Hamilton cycle system [3] and show the following.

Lemma 3.1. *Let $[a, b], [c, d] \subseteq [1, \ell]$. The graph $\langle [a, b], \varnothing, [c, d] \rangle$ has a k-cycle system whenever both $[a, b]$ and $[c, d]$ satisfy the following condition: the interval has even size or contains an integer coprime with k.*

Proof. The graph $\langle [a, b], \varnothing, [c, d] \rangle$ decomposes into $\langle [a, b], \varnothing, \varnothing \rangle$ and $\langle \varnothing, \varnothing, [c, d] \rangle$. The first one is the Cayley graph $\Gamma = Cay(\mathbb{Z}_k, [a, b])$ with further k isolated vertices, while the second one is isomorphic to $\langle [c, d], \varnothing, \varnothing \rangle$. Therefore, it is enough to show that Γ has a k-cycle system.

Note that Γ decomposes into the subgraphs $Cay(\mathbb{Z}_k, D_i)$, for $0 \le i \le t$, whenever the sets D_i between them partition $[a, b]$. By assumption, $[a, b]$ has even size or contains an integer coprime with k. Therefore, we can assume that for every $i > 0$ the set D_i is a pair of integers at distance 1 or 2, and D_0 is either empty or contains exactly one integer coprime with k. Clearly, $Cay(\mathbb{Z}_k, D_0)$ is either the empty graph or a k-cycle, and the remaining $Cay(\mathbb{Z}_k, D_i)$ are 4-regular Cayley graphs. Also, for every $i > 0$ we have that D_i is a generating set of $\mathbb{Z}_k$ (since k is odd and D_i contains integers at distance 1 or 2), hence the graph $Cay(\mathbb{Z}_k, D_i)$ is connected. It follows that each $Cay(\mathbb{Z}_k, D_i)$, with $i > 0$, decomposes into two k-cycles, thus the assertion is proven. □

Lemma 3.2. *Let $S \subseteq \{2i - 1 \mid 1 \le i \le \ell\}$. Then there exist k-cycle systems for the graphs $\langle \{\ell\}, S \cup (S+1), \varnothing \rangle$ and $\langle \{\ell\}, (S+1) \cup (S+2), \varnothing \rangle$.*

Proof. We note that the result is trivial when $S = \varnothing$, since $\langle\{\ell\}, \varnothing, \varnothing\rangle$ is a k-cycle.

The existence of a k-cycle system of $\Gamma = \langle\{\ell\}, S \cup (S+1), \varnothing\rangle$ has been proven in [11, Lemma 3] when $S \subseteq \{2i-1 \mid 1 \le i \le \ell\}$. Consider now the permutation f of $\mathbb{Z}_k \times \{0,1\}$ fixing $\mathbb{Z}_k \times \{0\}$ pointwise, and mapping $(i,1)$ to $(i+1,1)$ for every $i \in \mathbb{Z}_k$. It is not difficult to check that $f(\Gamma) = \langle\{\ell\}, (S+1) \cup (S+2), \varnothing\rangle$ which is therefore isomorphic to Γ, and hence it has a k-cycle system. □

Lemma 3.3. *Let r, s, and s' be integers such that $1 \le s \le s' \le \min\{s+1, \ell\}$, and $0 < r \not\equiv s + s' \pmod 2$. Also, let $D \subseteq [0, k-1]$ be a nonempty interval of size $k - (s + s' + 2r)$. Then there is a cycle $C = (x_1, x_2, \ldots, x_k)$ of $\Gamma = \langle[1+\epsilon, s+\epsilon], D, [1+\epsilon, s'+\epsilon]\rangle + r$, for every $\epsilon \in \{0,1\}$, such that $Orb(C)$ is a k-cycle system of Γ. Furthermore, if $u = 0$ or $u = 1 - \epsilon = 1 \le s - 1$, then*

1. *$Dev(\{x_{2-u}, x_{3-u}\})$ is a k-cycle with vertices in $\mathbb{Z}_k \times \{0\}$;*
2. *$Dev(\{x_{4+u}, x_{5+u}\})$ is a k-cycle with vertices in $\mathbb{Z}_k \times \{1\}$.*

Proof. Set $t = k - (s + s' + 2r)$ and let $\Omega = \langle[1+\epsilon, s+\epsilon], [0, t-1], [1+\epsilon, s'+\epsilon]\rangle + r$. For $i \in [0, s+s'+1]$ and $j \in [0, t+r-1]$, let a_i and b_j be the elements of $\mathbb{Z}_k \times \{0,1\}$ defined as follows:

$$a_i = \begin{cases} \left(-\dfrac{i}{2}, 0\right) & \text{if } i \in [0,s] \text{ is even}, \\ \left(-s-\epsilon+\dfrac{i-1}{2}, 0\right) & \text{if } i \in [1,s] \text{ is odd}, \\ a_{2s+1-i} + (0,1) & \text{if } i \in [s+1, 2s+1], \\ (-s'-\epsilon, 1) & \text{if } i = s+s'+1 > 2s+1, \end{cases}$$

$$b_j = \begin{cases} \left(\dfrac{j}{2}, 0\right) & \text{if } j \in [0, t+r-2] \text{ is even}, \\ \left(t - \dfrac{j+1}{2}, 1\right) & \text{if } j \in [1, t-1] \text{ is odd}, \\ \left(t + \left\lfloor \dfrac{j-t}{2} \right\rfloor, 1\right) & \text{if } j \in [t, t+r-2] \text{ is odd}, \\ a_{s+s'+1} & \text{if } j = t+r-1. \end{cases}$$

Since the elements a_i and b_j are pairwise distinct, except for $a_0 = b_0$ and $a_{s+s'+1} = b_{t+r-1}$, then the union F of the following two paths is a k-cycle:

$$P = a_0 \sim a_1 \sim \cdots \sim a_{s+s'+1},$$
$$Q = b_0 \sim b_1 \sim \cdots \sim b_{t-1} \sim \infty_1 \sim b_t \sim \infty_2 \sim b_{t+1} \sim \cdots \sim \infty_r \sim b_{t+r-1}.$$

Since $\Delta_{ij}F = \Delta_{ij}P \cup \Delta_{ij}Q$, for $i, j \in \{0,1\}$, where

$$\Delta_{00}P = \pm[1+\epsilon, s+\epsilon], \quad \Delta_{01}P = \{0\}, \quad \Delta_{11}P = \pm[1+\epsilon, s'+\epsilon],$$
$$\Delta_{00}Q = \varnothing, \quad \Delta_{01}Q = [1, t-1], \quad \Delta_{11}Q = \varnothing,$$

and considering that $N_F(\infty_h) = N_Q(\infty_h) = \{b_{t+h-2}, b_{t+h-1}\}$ for every $h \in [1, r]$, Proposition 2.1 guarantees that $Orb(F)$ is a k-cycle system of Ω. Furthermore, if $u = 0$ or $u = 1 - \epsilon = 1 \leq s - 1$, then

$$\pm(a_{s-u} - a_{s-u-1}) = \pm(a_{s+u+2} - a_{s+u+1}) = \pm(u + \epsilon + 1, 0).$$

Since k is odd, we have that $Dev(\{a_{s-u-1}, a_{s-u}\})$ and $Dev(\{a_{s+u+2}, a_{s+u+1}\})$ are k-cycles with vertices in $\mathbb{Z}_k \times \{0\}$ and $\mathbb{Z}_k \times \{1\}$, respectively.

If $D = [g, g + t - 1]$ is any interval of $[0, k - 1]$ of size t, and f is the permutation of $\mathbb{Z}_k \times \{0, 1\}$ fixing $\mathbb{Z}_k \times \{0\}$ pointwise, and mapping $(i, 1)$ to $(i + g, 1)$ for every $i \in \mathbb{Z}_k$, one can check that $C = f(F)$ is the desired k-cycle of $\Gamma = f(\Omega)$. □

Lemma 3.4.

1. *Let ℓ be odd. If Γ is a 1-factor of K_{2k}, then $\Gamma + \ell$ decomposes into k cycles of length k, each of which contains exactly one edge of Γ. Furthermore, if $\Gamma = \langle\varnothing, \{d\}, \varnothing\rangle$, then there exists a k-cycle $C = (c_1, c_2, ..., c_k)$ of $\Gamma + \ell$, with $c_1 \in \mathbb{Z}_k \times \{0\}$ and $c_2 \in \mathbb{Z}_k \times \{1\}$, such that*

$$Dev(\{c_1, c_2\}) = \Gamma \quad \textit{and} \quad Orb(C) \textit{ is a } k\textit{-cycle system of } \Gamma + \ell.$$

2. *Let ℓ be even. If Γ is a k-cycle of order $2k$, then $\Gamma + \ell$ decomposes into k cycles of length k, each of which contains exactly one edge of Γ. Furthermore, if $\Gamma = \langle\{d\}, \varnothing, \varnothing\rangle$ and d is coprime with k, then there exists a k-cycle $C = (c_1, c_2, ..., c_k)$ of $\Gamma + \ell$, with $c_1, c_2 \in \mathbb{Z}_k \times \{0\}$, such that*

$$Dev(\{c_1, c_2\}) \textit{ is the } k\textit{-cycle of } \Gamma \quad \textit{and} \quad Orb(C) \textit{ is a } k\textit{-cycle system of } \Gamma + \ell.$$

Proof. Permuting the vertices of K_{2k} if necessary, we can assume that Γ is the 1-factor $\Gamma_0 = \langle\varnothing, \{0\}, \varnothing\rangle$ when ℓ is odd, and the k-cycle $\Gamma_1 = \langle\{\ell\}, \varnothing, \varnothing\rangle$ (of order $2k$) when ℓ is even. For $h \in \{0, 1\}$, let $C_h = (c_{h,1}, c_{h,2}, \infty_1, c_3, \infty_2, c_4, ..., \infty_{\ell-1}, c_{\ell+1}, \infty_\ell)$ be the k-cycle of $\Gamma_h + \ell$, where

$$c_{h,1} = (0, 1 - h), \quad c_{h,2} = (h\ell, 0), \quad \text{and} \quad c_j = \begin{cases} \left(\dfrac{j-1}{2}, 1\right) & \text{if } j \in [3, \ell + 1] \text{ is odd}, \\ \left(\dfrac{j}{2}, 0\right) & \text{if } j \in [4, \ell + 1] \text{ is even}. \end{cases}$$

Note that the sets $\Delta_{ij}C_h$ are empty, except for $\Delta_{01}C_0 = \{0\}$ and $\Delta_{00}C_1 = \{\pm\ell\}$. Also, the two neighbors of ∞_u in C_h belong to $\mathbb{Z}_k \times \{0\}$ and $\mathbb{Z}_k \times \{1\}$, respectively. Hence, Proposition 2.1 guarantees that $Orb(C_h)$ is a k-cycle system of $\Gamma_h + \ell$, for $h \in \{0, 1\}$. We finally notice that $Dev(\{c_{h,1}, c_{h,2}\}) = \Gamma_h$ (up to isolated vertices) and this completes the proof. □

The following result has been proven in [11].

Lemma 3.5. *Let $D \subseteq [1, \ell]$. The subgraph $\langle D, \{0\}, D\rangle$ of K_{2k} has a 1-factorization.*

Remark 3.6. Considering the permutation f of $\mathbb{Z}_k \times \{0, 1\}$ such that $f(i, j) = (i, 1 - j)$, and a graph $\Gamma = \langle D_0, D_1, D_2\rangle$, we have that $f(\Gamma) = \langle D_2, -D_1, D_0\rangle$. Therefore, Lemmas 3.1–3.5 continue to hold when we replace Γ by $f(\Gamma)$.

4 | k-SUN SYSTEMS OF $K_{4k} + n$

In this section we provide sufficient conditions for a k-sun system of $K_{4k} + n$ to exist, when $n \equiv 0, 1 \pmod 4$. More precisely, we show the following.

Theorem 4.1. *Let $k \geq 7$ be an odd integer and let $n \equiv 0, 1 \pmod 4$ with $2k < n < 10k$; then there exists a k-sun system of $K_{4k} + n$, except possibly when*

- *$k = 7$ and $n = 20, 21, 32, 33, 44, 45, 56, 57, 64, 65, 68, 69$,*
- *$k = 11$ and $n = 100, 101, 112, 113$.*

To prove Theorem 4.1, we start by introducing some notions and prove some preliminary results. Let M be a positive integer and take $V(K_{2^iM}) = \mathbb{Z}_M \times [0, 2^i - 1]$ and $V(K_{2^iM} + w) = V(K_{2^iM}) \cup \{\infty_h \mid h \in \mathbb{Z}_w\}$, for $i \in \{1, 2\}$ and $w > 0$.

Now assume that $w = 2u$, and let $x \mapsto \overline{x}$ be the permutation of $V(K_{4M} + 2u)$ defined as follows:

$$\overline{x} = \begin{cases} (a, 2 - j) & \text{if } x = (a, j) \in \mathbb{Z}_M \times \{0, 2\}, \\ (a, 4 - j) & \text{if } x = (a, j) \in \mathbb{Z}_M \times \{1, 3\}, \\ \infty_{h+u} & \text{if } x = \infty_h. \end{cases}$$

For any subgraph Γ of $K_{4M} + 2u$, we denote by $\overline{\Gamma}$ the graph (isomorphic to Γ) obtained by replacing each vertex x of Γ with $\overline{x}$.

Given a subgraph Γ of $K_{2M} + u$, we denote by $\Gamma[2]$ the spanning subgraph of $K_{4M} + 2u$ whose edge-set is

$$E(\Gamma[2]) = \{\{x, y\}, \{x, \overline{y}\}, \{\overline{x}, y\}, \{\overline{x}, \overline{y}\} \mid \{x, y\} \in E(\Gamma)\},$$

and let $\Gamma^*[2] = \Gamma[2] \oplus I$ be the graph obtained by adding to $\Gamma[2]$ the 1-*factor*

$$I = \{\{x, \overline{x}\} \mid x \in \mathbb{Z}_M \times \{0, 1\}\}.$$

Note that, up to isolated vertices, $\Gamma[2]$ is the *lexicographic product* of Γ with the empty graph on two vertices.

The proof of the following elementary lemma is left to the reader.

Lemma 4.2. *Let $\Gamma = \oplus_{i=1}^{n} \Gamma_i$ and let $w = \sum_{i=1}^{n} w_i$ with $w_i \geq 0$. If Γ and the Γ_is have the same vertex-set (possibly with isolated vertices), then*

1. *$\Gamma + w = \oplus_{i=1}^{n} (\Gamma_i + w_i)$;*

2. $\Gamma[2] = \oplus_{i=1}^{n}\Gamma_i[2]$;
3. $(\Gamma + w)[2] = \Gamma[2] + 2w$.

We start showing that if C is a k-cycle, then $C[2]$ decomposes into two k-suns.

Lemma 4.3. *Let $C = (c_1, c_2, ..., c_k)$ be a cycle with vertices in $(\mathbb{Z}_M \times \{0, 1\}) \cup \{\infty_h \mid h \in \mathbb{Z}_u\}$ and let S be the k-sun defined as follows:*

$$S = \begin{pmatrix} s_1 & \dots & s_{k-1} & s_k \\ \overline{s_2} & \dots & \overline{s_k} & \overline{s_1} \end{pmatrix}, \tag{1}$$

where $s_i \in \{c_i, \overline{c_i}\}$ for every $i \in [1, k]$. Then $C[2] = S \oplus \overline{S}$.

Proof. It is enough to notice that S contains the edges $\{s_i, s_{i+1}\}$ and $\{s_i, \overline{s_{i+1}}\}$, while $\overline{S}$ contains $\{\overline{s_i}, \overline{s_{i+1}}\}$ and $\{\overline{s_i}, s_{i+1}\}$, for every $i \in [1, k]$, where $s_{k+1} = s_1$ and $\overline{s_{k+1}} = \overline{s_1}$. □

Example 4.4. In Figure 1 we have the graph $C_7[2]$ which can be decomposed into two 7-suns S and $\overline{S}$. The nondashed edges are those of S, while the dashed edges are those of $\overline{S}$.

For every cycle $C = (c_1, c_2, ..., c_k)$ with vertices in $\mathbb{Z}_M \times \{0, 1\}$, we set

$$\sigma(C) = \begin{pmatrix} c_1 & \dots & c_{k-1} & c_k \\ \overline{c_2} & \dots & \overline{c_k} & \overline{c_1} \end{pmatrix}.$$

Clearly, $C[2] = \sigma(C) \oplus \overline{\sigma(C)}$ by Lemma 4.3.

Lemma 4.5. *If $\mathcal{C} = \{C_1, C_2, ..., C_t\}$ is a k-cycle system of $\Gamma + u$, where Γ is a subgraph of K_{2M}, and S_i is a k-sun obtained from C_i as in Lemma 4.3, then $\mathcal{S} = \{S_i, \overline{S_i} \mid i \in [1, t]\}$ is a k-sun system of $\Gamma[2] + 2u$. In particular, if $\mathcal{C} = Orb(C_1)$, then $Orb(S_1) \cup Orb(\overline{S_1})$ is a k-sun system of $\Gamma[2] + 2u$.*

Proof. By assumption $\Gamma + u = \oplus_{i=1}^{t} C_i$, where each C_i is a k-cycle. Also, by Lemma 4.2, we have that $\Gamma[2] + 2u = (\Gamma + u)[2] = \oplus_{i=1}^{t} C_i[2]$. Since $C_i[2] = S_i \oplus \overline{S_i}$ by Lemma 4.3, then $\mathcal{S}$ is a k-sun system of $\Gamma[2] + 2u$.

The second part easily follows by noticing that if $C_i = \tau_g(C_1)$ for some $g \in \mathbb{Z}_M$, then $C_i[2] = \tau_g(C_1[2]) = \tau_g(S_1) \oplus \tau_g(\overline{S_1})$. □

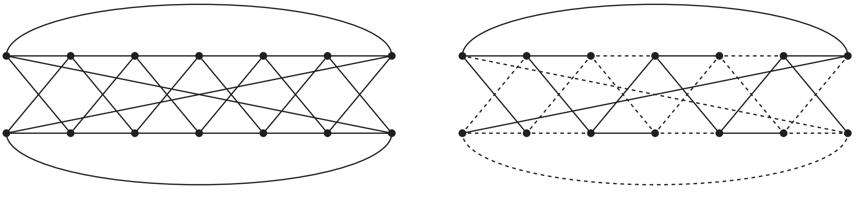

FIGURE 1 $C_7[2] = S \oplus \overline{S}$

The following lemma describes the general method we use to construct k-sun systems of $K_{4k}+n$. We point out that throughout the rest of this section we take $V(K_{2k})=\mathbb{Z}_k\times\{0,1\}$ and $V(K_{4k})=\mathbb{Z}_k\times[0,3]$.

Lemma 4.6. *Let $K_{2k}=\Gamma_1\oplus\Gamma_2$ with $V(\Gamma_1)=V(\Gamma_2)=V(K_{2k})$. If Γ_1+w_1 has a k-cycle system and $\Gamma_2^*[2]+w_2$ has a k-sun system, then $K_{4k}+(2w_1+w_2)$ has a k-sun system.*

Proof. The result follows by Lemma 4.2. In fact, noting that $K_{4k}=K_{2k}[2]\oplus I$, where $I=\{\{z,\bar{z}\}|z\in\mathbb{Z}_k\times\{0,1\}\}$, we have that

$$\begin{aligned}K_{4k}+(2w_1+w_2)&=(\Gamma_1[2]\oplus(\Gamma_2[2]\oplus I))+2w_1+w_2\\&=(\Gamma_1[2]+2w_1)\oplus(\Gamma_2^*[2]+w_2)=(\Gamma_1+w_1)[2]\oplus(\Gamma_2^*[2]+w_2).\end{aligned}$$

The result then follows by Lemma 4.5. □

We are now ready to prove the main result of this section, Theorem 4.1. The case $k\equiv 1\pmod 4$ is proven in Theorem 4.7, while the case $k\equiv 3\pmod 4$ is dealt with in Theorems 4.9–4.12.

Theorem 4.7. *If $k\equiv 1\pmod 4\geq 9$ and $n\equiv 0,1\pmod 4$ with $2k<n<10k$, then there exists a k-sun system of $K_{4k}+n$.*

Proof. Let $n=2(q\ell+r)+\nu$ with $1\leq r\leq\ell$ and $\nu\in\{2,3\}$. Note that $\ell\geq 4$ is even and r is odd, since $n\equiv 0,1\pmod 4\geq 9$ and $k\equiv 1\pmod 4$. Considering also that $2k<n<10k$, we have that $2\leq q\leq 10\leq k+2r-1$. Furthermore, let $V(K_{4k}+n)=(\mathbb{Z}_k\times[0,3])\cup\{\infty_h|h\in\mathbb{Z}_{n-\nu}\}\cup\{\infty_1',\infty_2',\infty_\nu'\}$.

We start decomposing K_{2k} into the following two graphs:

$$\Gamma_1=\langle[2,\ell],[k-2r-2,k-1],[2,\ell-1]\rangle\quad\text{and}\quad\Gamma_2=\langle\{1\},[0,k-2r-3],\{1,\ell\}\rangle.$$

We notice that Γ_1 further decomposes into the following graphs:

$$\langle[2,\ell-1],\varnothing,\varnothing\rangle,\quad\langle\varnothing,\varnothing,[2,\ell-1]\rangle,\quad\langle\{\ell\},[k-2r-2,k-1],\varnothing\rangle,$$

each of which decomposes into k-cycles by Lemmas 3.1 and 3.2; hence Γ_1 has a k-cycle system $\{C_1,C_2,\ldots,C_\gamma\}$, where $\gamma=k+2r-2$. Note that this system is nonempty, since $1\leq q-1\leq\gamma$. Without loss of generality, we can assume that each cycle C_i has order $2k$ and

$$C_1\text{ is a subgraph of }\langle[2,\ell-1],\varnothing,\varnothing\rangle.\tag{2}$$

Now set $\Omega_1=\Gamma_1\backslash C_1$ and $\Omega_2=\Gamma_2\oplus C_1$. Letting $w_1=(q-2)\ell=\sum_{j=2}^{\gamma}w_{1,j}$, where $w_{1,j}=\ell$ when $j<q$, and $w_{1,j}=0$ otherwise, by Lemma 4.2 we have that $\Omega_1+w_1=\oplus_{i=2}^{\gamma}(C_i+w_{1,i})$. Therefore, Ω_1+w_1 has a k-cycle system, since each $C_i+w_{1,i}$ decomposes into k-cycles by Lemma 3.4. Setting $w_2=n-2w_1=2(2\ell+r)+\nu$ and

considering that $K_{2k} = \Gamma_1 \oplus \Gamma_2 = \Omega_1 \oplus \Omega_2$, by Lemma 4.6 it is left to show that $\Omega_2^*[2] + w_2$ has a k-sun system.

Set $\Gamma_3 = C_1$, and recall that $\Omega_2^*[2] = \Omega_2[2] \oplus I = \Gamma_2[2] \oplus \Gamma_3[2] \oplus I$, where I denotes the 1-factor $\{\{z, \overline{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$ of K_{4k}. Hence,

$$\Omega_2^*[2] + w_2 = (\Gamma_2 + (\ell + r))[2] \oplus (\Gamma_3 + \ell)[2] \oplus (I + \nu) \tag{3}$$

by Lemma 4.2. Clearly, $\Gamma_2 = \Gamma_{2,1} \oplus \Gamma_{2,2}$ where $\Gamma_{2,1} = \langle\{1\}, [0, k-2r-3], \{1\}\rangle$ and $\Gamma_{2,2} = \langle\varnothing, \varnothing, \{\ell\}\rangle$, hence $\Gamma_2 + (\ell + r) = (\Gamma_{2,1} + r) \oplus (\Gamma_{2,2} + \ell)$. By Lemmas 3.3 and 3.4, there exists a k-cycle $A = (x_1, x_2, y_3, y_4, a_5, \ldots, a_k)$ of $\Gamma_{2,1} + r$ and a k-cycle $B = (y_1, y_2, b_3, \ldots, b_k)$ of $\Gamma_{2,2} + \ell$ satisfying the following properties:

$$Orb(A) \cup Orb(B) \quad \text{is a } k\text{-cycle system of } \Gamma_2 + (\ell + r), \tag{4}$$

$$Dev(\{x_1, x_2\}) \quad \text{is a } k\text{-cycle with vertices in } \mathbb{Z}_k \times \{0\}, \tag{5}$$

$$Dev(\{y_1, y_2\}) \quad \text{and} \quad Dev(\{y_3, y_4\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{1\}. \tag{6}$$

Furthermore, denoted by $(c_1, c_2, \ldots, c_k)$ the cycle in Γ_3, Lemma 3.4 guarantees that

$$\begin{aligned} &\Gamma_3 + \ell \quad \text{has a } k\text{-cycle system } \{F_1, F_2, \ldots, F_k\} \text{ such that} \\ &F_j = (c_j, c_{j+1}, f_{j,3}, f_{j,4}, \ldots, f_{j,k}) \quad \text{for every } j \in [1, k] \text{ (with } c_{k+1} = c_1). \end{aligned}$$

Let $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$ and $\mathcal{S}' = \{S_{3+2j}, S_{4+2j} | j \in [1, k]\}$, where

$$\begin{aligned} S_1 &= \sigma(x_1, \overline{x_2}, y_3, y_4, a_5, \ldots, a_k), \quad S_3 = \sigma(y_1, \overline{y_2}, b_3, \ldots, b_k), \\ S_{3+2j} &= \sigma(c_j, \overline{c_{j+1}}, f_{j,3}, f_{j,4}, \ldots, f_{j,k}) \quad \text{for } j \in [1, k], \quad \text{and} \\ S_{2i} &= \overline{S_{2i-1}} \quad \text{for } i \in [1, k+2]. \end{aligned}$$

By Lemma 4.5 we have that $\bigcup_{S \in \mathcal{S}} Orb(S)$ is a k-sun system of $(\Gamma_2 + (\ell + r))[2]$, and $\mathcal{S}'$ is a k-sun system of $(\Gamma_3 + \ell)[2]$. It follows by (3) that $\bigcup_{S \in \mathcal{S}} Orb(S) \cup \mathcal{S}'$ decomposes $(\Omega_2^*[2] + w_2) \setminus (I + \nu)$.

To construct a k-sun system of $\Omega_2^*[2] + w_2$, we first modify the k-suns in $\mathcal{S} \cup \mathcal{S}'$ by replacing some of their vertices with ∞_1', ∞_2', and possibly ∞_3' when $\nu = 3$. More precisely, following Table 1, we obtain T_i from S_i by replacing the ordered set V_i of vertices of S_i with V_i'. This yields a set M_i of 'missing' edges no longer covered by T_i after this substitution, but replaced by those in N_i, namely,

$$E(T_i) = (E(S_i) \setminus M_i) \cup N_i.$$

We point out that $T_{3+2j} = S_{3+2j}$, and $T_{4+2j} = S_{4+2j}$ when $\nu = 2$, for every $j \in [1, k]$. The remaining graphs T_i are explicitly given below, where the elements in bold are the replaced vertices.

TABLE 1 From S_i to T_i

i	$V_i \to V_i'$	M_i	N_i	ν
1	$(x_2, y_3) \to (\infty_1', \infty_2')$	$\{x_1, x_2\}, \{\overline{x_2}, y_3\}, \{y_3, y_4\}, \{y_3, \overline{y_4}\}$	$\{\infty_1', x_1\}, \{\infty_2', \overline{x_2}\}, \{\infty_2', y_4\}, \{\infty_2', \overline{y_4}\}$	2, 3
2	$(\overline{x_2}, y_3) \to (\infty_1', \infty_2')$	$\{\overline{x_1}, \overline{x_2}\}, \{x_2, y_3\}$	$\{\infty_1', \overline{x_1}\}, \{\infty_2', x_2\}$	2
2	$(\overline{x_2}, y_3, \overline{y_3}) \to (\infty_1', \infty_2', \infty_3')x$	$\{\overline{x_1}, \overline{x_2}\}, \{x_2, y_3\}, \{x_2, \overline{y_3}\}, \{\overline{y_3}, \overline{y_4}\}, \{\overline{y_3}, y_4\}$	$\{\infty_1', \overline{x_1}\}, \{\infty_2', x_2\}, \{\infty_3', x_2\}, \{\infty_3', \overline{y_4}\}, \{\infty_3', y_4\}$	3
3	$y_2 \to \infty_1'$	$\{y_1, y_2\}$	$\{\infty_1', y_1\}$	2, 3
4	$\overline{y_2} \to \infty_1'$	$\{\overline{y_1}, \overline{y_2}\}$	$\{\infty_1', \overline{y_1}\}$	2, 3
3 + 2j	$\varnothing$	$\varnothing$	$\varnothing$	2, 3
4 + 2j	$\varnothing$	$\varnothing$	$\varnothing$	2
4 + 2j	$\overline{c_{j+1}} \to \infty_3'$	$\{\overline{c_j}, \overline{c_{j+1}}\}$	$\{\infty_3', \overline{c_j}\}$	3

$$T_1 = \begin{pmatrix} x_1 & \overline{x_2} & \infty_2' & y_4 & a_5 & \cdots & a_{k-1} & a_k \\ \infty_1' & \overline{y_3} & \overline{y_4} & \overline{a_5} & \overline{a_6} & \cdots & \overline{a_k} & \overline{x_1} \end{pmatrix},$$

$$T_2 = \begin{cases} \begin{pmatrix} \overline{x_1} & x_2 & \overline{y_3} & \overline{y_4} & \overline{a_5} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \infty_1' & \infty_2' & y_4 & a_5 & a_6 & \cdots & a_k & x_1 \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} \overline{x_1} & x_2 & \infty_3' & \overline{y_4} & \overline{a_5} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \infty_1' & \infty_2' & y_4 & a_5 & a_6 & \cdots & a_k & x_1 \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$T_3 = \begin{pmatrix} y_1 & \overline{y_2} & b_3 & \cdots & b_{k-1} & b_k \\ \infty_1' & \overline{b_3} & \overline{b_4} & \cdots & \overline{b_k} & \overline{y_1} \end{pmatrix}, \quad T_4 = \begin{pmatrix} \overline{y_1} & y_2 & \overline{b_3} & \cdots & \overline{b_{k-1}} & \overline{b_k} \\ \infty_1' & b_3 & b_4 & \cdots & b_k & y_1 \end{pmatrix},$$

$$T_{4+2j} = \begin{pmatrix} \overline{c_j} & c_{j+1} & \overline{f_{j,3}} & \cdots & \overline{f_{j,k-1}} & \overline{f_{j,k}} \\ \infty_3' & f_{j,3} & f_{j,4} & \cdots & f_{j,k} & c_j \end{pmatrix} \quad \text{for every } j \in [1, k].$$

We notice that $\bigcup_{i=1}^{4} Dev(N_i) \cup \bigcup_{i=5}^{2k+4} N_i = \{\{\infty_j', x\} | j \in [1, \nu], x \in \mathbb{Z}_k \times [0, 3]\}$. We finally build the following $2\nu + 1$ graphs:

$$G_1 = \begin{cases} Dev(x_1 \sim x_2 \sim \overline{x_2}) & \text{if } \nu = 2, \\ Dev(x_1 \sim x_2 \sim \overline{y_3}) & \text{if } \nu = 3, \end{cases} \quad G_2 = Dev(\overline{x_1} \sim \overline{x_2} \sim y_3),$$
$$G_3 = Dev(y_4 \sim y_3 \sim x_2), \quad G_4 = Dev(y_1 \sim y_2 \sim \overline{y_2}),$$
$$G_5 = Dev(\{\overline{y_1}, \overline{y_2}\} \oplus \{y_3, \overline{y_4}\}), \quad G_6 = Dev(\overline{y_4} \sim \overline{y_3} \sim y_4),$$
$$G_7 = \begin{pmatrix} \overline{c_1} & \overline{c_2} & \ldots & \overline{c_k} \\ c_1 & c_2 & \ldots & c_k \end{pmatrix}.$$

By recalling (2) and (4)–(6), it is not difficult to check that $G_1, G_2, \ldots, G_{2\nu+1}$ are k-suns. Furthermore,

$$\bigcup_{i=1}^{2\nu+1} E(G_i) = \bigcup_{i=1}^{4} Dev(M_i) \cup \bigcup_{i=5}^{2k+4} M_i \cup E(I),$$

where, we recall, I denotes the 1-factor $\{\{z, \overline{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$ of K_{4k}. Therefore, $\bigcup_{i=1}^{4} Orb(T_i) \cup \{T_5, T_6, \ldots, T_{2k+4}\} \cup \{G_1, G_2, \ldots, G_{2\nu+1}\}$ is a k-sun system of $\Omega_2^*[2] + w_2$, and this concludes the proof. □

Example 4.8. By following the proof of Theorem 4.7, we construct a k-sun system of $K_{4k} + n$ when $(k, n) = (9, 21)$; hence $(\ell, q, r, \nu) = (4, 2, 1, 3)$.

The graphs $\Gamma_1 = \langle [2, 4], [5, 8], [2, 3] \rangle$ and $\Gamma_2 = \langle \{1\}, [0, 4], \{1, 4\} \rangle$ decompose the complete graph K_{18} with vertex-set $\mathbb{Z}_9 \times \{0, 1\}$. Also Γ_1 decomposes into the following 9-cycles of order 18, where $i = 0, 1$:

$$C_{1+i} = ((0, i), (2, i), (8, i), (1, i), (3, i), (5, i), (7, i), (4, i), (6, i)),$$
$$C_{3+i} = ((0, i), (3, i), (6, i), (8, i), (5, i), (2, i), (4, i), (1, i), (7, i)),$$
$$C_{5+i} = ((4i, 0), (8 + 4i, 1), (1 + 4i, 0), (4i, 1), (2 + 4i, 0), (1 + 4i, 1), (3 + 4i, 0), (2 + 4i, 1), (4 + 4i, 0)),$$
$$C_{7+i} = ((8 + 4i, 0), (5 + 4i, 1), (4i, 0), (6 + 4i, 1), (1 + 4i, 0), (7 + 4i, 1), (2 + 4i, 0), (8 + 4i, 1), (3 + 4i, 0)),$$
$$C_9 = ((7, 0), (2, 0), (6, 0), (1, 0), (5, 0), (0, 0), (7, 1), (8, 0), (4, 1)).$$

Clearly, $K_{18} = \Omega_1 \oplus \Omega_2$, where $\Omega_1 = \Gamma_1 \setminus C_1$ and $\Omega_2 = \Gamma_2 \oplus C_1$.

Let $V(K_{36}) = \mathbb{Z}_9 \times [0, 3]$, and denote by I the 1-factor of K_{36} containing all edges of the form $\{(a, i), (a, i + 2)\}$, with $a \in \mathbb{Z}_9$ and $i \in \{0, 1\}$. Then,

$$K_{36} = K_{18}[2] \oplus I = \Omega_1[2] \oplus \Omega_2[2] \oplus I.$$

Considering that $(\Omega_2 + 9)[2] = \Omega_2[2] + 18$, we have

$$K_{36} + 21 = \Omega_1[2] \oplus (\Omega_2[2] + 18) \oplus (I + 3) = \Omega_1[2] \oplus (\Omega_2 + 9)[2] \oplus (I + 3).$$

Since the set $\{\sigma(C_i), \overline{\sigma(C_i)} | i \in [2, 9]\}$ is a 9-sun system of $\Omega_1[2]$, it is left to build a 9-sun system of $\Omega_2^*[2] + 21 = (\Omega_2[2] + 18) \oplus (I + 3)$.

We start by decomposing $\Omega_2 + 9$ into 9-cycles. Since $\Omega_2 = \Gamma_{2,1} \oplus \Gamma_{2,2} \oplus \Gamma_3$ with $\Gamma_{2,1} = \langle \{1\}, [0, 4], \{1\} \rangle$, $\Gamma_{2,2} = \langle \varnothing, \varnothing, \{4\} \rangle$ and $\Gamma_3 = C_1$, then

$$\Omega_2 + 9 = (\Gamma_{2,1} + 1) \oplus (\Gamma_{2,2} + 4) \oplus (\Gamma_3 + 4).$$

Let $A = (x_1, x_2, y_3, y_4, a_5, ..., a_9)$ and $B = (y_1, y_2, b_3, ..., b_9)$ be the 9-cycles defined as follows:

$$(x_1, x_2, y_3, y_4) = ((0, 0), (-1, 0), (-1, 1), (0, 1)),$$
$$(a_5, ..., a_9) = (\infty_1, (2, 0), (3, 1), (1, 0), (4, 1)),$$
$$(y_1, y_2) = ((0, 1), (4, 1)),$$
$$(b_3, ..., b_9) = (\infty_2, (1, 0), \infty_3, (1, 1), \infty_4, (0, 0), \infty_5).$$

One can easily check that $Orb(A)$ (resp., $Orb(B)$) decomposes $\Gamma_{2,1} + 1$ (resp., $\Gamma_{2,2} + 4$). Also, for every edge $\{c_j, c_{j+1}\}$ of C_1, with $j \in [1, 9]$ and $c_{10} = c_1$, we construct the cycle $F_j = (c_j, c_{j+1}, f_{j,3}, f_{j,4}, ..., f_{j,9})$, where

$$(f_{j,3}, f_{j,4}, ..., f_{j,9}) = (\infty_6, (1, 0), \infty_7, (1, 1), \infty_8, (0, 0), \infty_9).$$

One can check that $\{F_1, F_2, ..., F_9\}$ is a 9-cycle system of $\Gamma_3 + 4$. Therefore, $\mathcal{U}_1 = Orb(A) \cup Orb(B) \cup \{F_1, F_2, ..., F_9\}$ provides a 9-cycle system of $\Omega_2 + 9$. Since the set $\{C[2] | C \in \mathcal{U}_1\}$ decomposes $(\Omega_2 + 9)[2]$, and each $C[2]$ decomposes into two 9-suns, we can easily obtain a 9-sun system of $(\Omega_2 + 9)[2]$. Indeed, letting

$$S_1 = \sigma(x_1, \overline{x_2}, y_3, y_4, a_5, ..., a_9), \quad S_3 = \sigma(y_1, \overline{y_2}, b_3, ..., b_9),$$
$$S_{3+2j} = \sigma(c_j, \overline{c_{j+1}}, f_{j,3}, f_{j,4}, ..., f_{j,9}) \quad \text{for } j \in [1, 9], \quad \text{and}$$
$$S_{2i} = \overline{S_{2i-1}} \quad \text{for } i \in [1, 11],$$

we have that $A[2] = S_1 \oplus S_2$, $B[2] = S_3 \oplus S_4$, and $F_j[2] = S_{3+2j} \oplus S_{4+2j}$, for every $j \in [1, 9]$. Therefore $\mathcal{U}_2 = \bigcup_{i=1}^{4} Orb(S_i) \cup \{S_5, S_6, \ldots, S_{22}\}$ is a 9-sun system of $\Omega_2[2] + 18$.

We finally use $\mathcal{U}_2$ to build a 9-sun system of $\Omega_2^*[2] + 21 = (\Omega_2[2] + 18) \oplus (I + 3)$. By replacing the vertices of each S_i, as outlined in Table 1, we obtain the 9-sun T_i. The new 22 graphs, $T_1, T_2, \ldots, T_{22}$, are built in such a way that

(a) $\bigcup_{i=1}^{4} Orb(T_i) \cup \{T_5, T_6, \ldots, T_{22}\}$ decomposes a subgraph K of $\Omega_2^*[2] + 21$;

(b) $(\Omega_2^*[2] + 21) \backslash K$ decomposes into seven 9-suns.

This way we obtain a 9-sun system of $\Omega_2^*[2] + 21$, and hence the desired 9-sun system of $K_{36} + 21$.

Theorem 4.9. *Let $k \equiv 3 \pmod 4 \geq 7$ and $n \equiv 0, 1 \pmod 4$ with $2k < n < 10k$. If $n \not\equiv 2, 3 \pmod{k-1}$ and $\left\lfloor \frac{n-4}{k-1} \right\rfloor$ is even, then there exists a k-sun system of $K_{4k} + n$ except possibly when $(k, n) \in \{(7, 64), (7, 65)\}$.*

Proof. First, $k \equiv 3 \pmod 4 \geq 7$ implies that $\ell \geq 3$ is odd. Now, let $n = 2(q\ell + r) + \nu$ with $1 \leq r \leq \ell$ and $\nu \in \{2, 3\}$. Note that $q = \left\lfloor \frac{n-4}{k-1} \right\rfloor$, hence q is even. Also, since $2k < n < 10k$, we have $2 \leq q \leq 10$. By q even and $n \equiv 0, 1 \pmod 4$ it follows that r is odd, and $n \not\equiv 2, 3 \pmod{k-1}$ implies that $r \neq \ell$. To sum up,

$$q \text{ is even with } 2 \leq q \leq 10, \quad \text{and} \quad r \text{ is odd with } 1 \leq r \leq \ell - 2.$$

As in the previous theorem, let $V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_h \mid h \in \mathbb{Z}_{n-\nu}\} \cup \{\infty'_1, \infty'_2, \infty'_\nu\}$.

We split the proof into two cases.

Case 1. $q \leq 2r + 4$. We start decomposing K_{2k} into the following two graphs:

$$\Gamma_1 = \langle [3, \ell], [k - 2r - 2, k], [3, \ell] \rangle \quad \text{and} \quad \Gamma_2 = \langle \{1, 2\}, [1, k - 2r - 3], \{1, 2\} \rangle.$$

Since $q \leq 2r + 4$, the graph Γ_1 can be further decomposed into the following graphs:

$$\Gamma_{1,1} = \langle \{\ell\}, [k - 2r + q - 3, k], \varnothing \rangle, \quad \Gamma_{1,2} = \langle [3, \ell - 1], \varnothing, [3, \ell] \rangle,$$
$$\Gamma_{1,3} = \langle \varnothing, [k - 2r - 2, k - 2r + q - 4], \varnothing \rangle.$$

The first two graphs have a k-cycle system by Lemmas 3.2 and 3.1, while $\Gamma_{1,3}$ decomposes into $(q-1)$ 1-factors, say $J_1, J_2, \ldots, J_{q-1}$. Setting $w_1 = (q-1)\ell$, by Lemma 4.2 we have that:

$$\Gamma_1 + (q-1)\ell = \oplus_{i=1}^{q-1} (J_i + \ell) \oplus (\Gamma_{1,1} \oplus \Gamma_{1,2}).$$

Hence $\Gamma_1 + (q-1)\ell$ has a k-cycle system since each $J_i + \ell$ decomposes into k-cycles by Lemma 3.4.

Letting $w_2 = n - 2w_1 = 2(\ell + r) + \nu$ and recalling that $K_{2k} = \Gamma_1 \oplus \Gamma_2$, by Lemma 4.6 it remains to construct a k-sun system of $\Gamma_2^*[2] + w_2$. We start decomposing Γ_2 into the following graphs:

$$\Gamma_{2,0} = \langle\{1, 2\}, [1, k - 2r - 4], \{1, 2\}\rangle \quad \text{and} \quad \Gamma_{2,1} = \langle\varnothing, \{k - 2r - 3\}, \varnothing\rangle.$$

Recalling that $\Gamma_2^*[2] = \Gamma_2[2] \oplus I$, where I denotes the 1-factor $\{\{z, \overline{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$ of K_{4k}, by Lemma 4.2 we have that

$$\Gamma_2^*[2] + w_2 = (\Gamma_{2,1} + \ell)[2] \oplus (\Gamma_{2,0} + r)[2] \oplus (I + \nu).$$

By Lemmas 3.3 and 3.4 there exist a k-cycle $A = (x_1, x_2, x_3, y_4, y_5, y_6, a_7, ..., a_k)$ of $\Gamma_{2,0} + r$ and a k-cycle $B = (y, x, b_3, ..., b_k)$ of $\Gamma_{2,1} + \ell$, satisfying the following properties:

$$\begin{aligned}
&Orb(A) \cup Orb(B) \quad \text{is a } k\text{-cycle system of } \Gamma_2 + (\ell + r);\\
&Dev(\{x_1, x_2\}) \quad \text{and} \quad Dev(\{x_2, x_3\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{0\};\\
&Dev(\{y_4, y_5\}) \quad \text{and} \quad Dev(\{y_5, y_6\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{1\};\\
&x \in \mathbb{Z}_k \times \{0\} \quad \text{and} \quad y \in \mathbb{Z}_k \times \{1\}.
\end{aligned}$$

Set $A' = (x_1, \overline{x_2}, x_3, y_4, \overline{y_5}, y_6, a_7, ..., a_k)$ and $B' = (y, \overline{x}, b_3, ..., b_k)$ and let $\mathcal{S} = \{\sigma(A'), \overline{\sigma(A')}, \sigma(B'), \overline{\sigma(B')}\}$. By Lemma 4.5, we have that $\bigcup_{S \in \mathcal{S}} Orb(S)$ is a k-sun system of $(\Gamma_2 + (\ell + r))[2] = \Gamma_2[2] + 2(\ell + r) = (\Gamma_2^*[2] + w_2) \backslash (I + \nu)$.

To construct a k-sun system of $\Gamma_2^*[2] + w_2$ we proceed as in Theorem 4.7. We modify the graphs in $\mathcal{S}$ and obtain four k-suns T_1, T_2, T_3, T_4 whose translates between them cover all edges incident with ∞_1', ∞_2', and possibly ∞_3' when $\nu = 3$. Then we construct further $2\nu + 1$ k-suns $G_1, ..., G_{2\nu+1}$ to cover the missing edges. The reader can check that $\bigcup_{i=1}^4 Orb(T_i) \cup \{G_1, ..., G_{2\nu+1}\}$ is a k-sun system of $\Gamma_2^*[2] + w_2$.

The graphs T_i are the following, where the elements in bold are the replaced vertices:

$$T_1 = \begin{cases}
\begin{pmatrix} x_1 & \overline{x_2} & x_3 & \boldsymbol{\infty_2'} & \overline{y_5} & y_6 & a_7 & \cdots & a_{k-1} & a_k \\ \boldsymbol{\infty}_1' & \overline{x_3} & \overline{y_4} & y_5 & \boldsymbol{y_4} & \overline{a_7} & \overline{a_8} & \cdots & \overline{a_k} & \overline{x_1} \end{pmatrix} & \text{if } \nu = 2,\\[2ex]
\begin{pmatrix} x_1 & \overline{x_2} & x_3 & \boldsymbol{\infty_2'} & \overline{y_5} & y_6 & a_7 & \cdots & a_{k-1} & a_k \\ \boldsymbol{\infty}_1' & \boldsymbol{\infty}_3' & \overline{y_4} & y_5 & \boldsymbol{y_4} & \overline{a_7} & \overline{a_8} & \cdots & \overline{a_k} & \overline{x_1} \end{pmatrix} & \text{if } \nu = 3,
\end{cases}$$

$$T_2 = \begin{cases}
\begin{pmatrix} \overline{x_1} & x_2 & \overline{x_3} & \boldsymbol{\infty_1'} & y_5 & \overline{y_6} & \overline{a_7} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \boldsymbol{\infty}_2' & x_3 & y_4 & \overline{y_5} & y_6 & a_7 & a_8 & \cdots & a_k & x_1 \end{pmatrix} & \text{if } \nu = 2,\\[2ex]
\begin{pmatrix} \overline{x_1} & x_2 & \overline{x_3} & \boldsymbol{\infty_1'} & y_5 & \overline{y_6} & \overline{a_7} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \boldsymbol{\infty}_2' & \boldsymbol{\infty_3'} & y_4 & \overline{y_5} & y_6 & a_7 & a_8 & \cdots & a_k & x_1 \end{pmatrix} & \text{if } \nu = 3,
\end{cases}$$

$$T_3 = \begin{cases}
\sigma(B') & \text{if } \nu = 2,\\
\begin{pmatrix} y & \overline{x} & b_3 & b_4 & \cdots & b_{k-1} & b_k \\ \boldsymbol{\infty_3'} & \overline{b_3} & \overline{b_4} & \overline{b_5} & \cdots & \overline{b_k} & \overline{y} \end{pmatrix} & \text{if } \nu = 3,
\end{cases}$$

$$T_4 = \begin{cases}
\overline{\sigma(B')} & \text{if } \nu = 2,\\
\begin{pmatrix} \overline{y} & x & \overline{b_3} & \overline{b_4} & \cdots & \overline{b_{k-1}} & \overline{b_k} \\ \boldsymbol{\infty_3'} & b_3 & b_4 & b_5 & \cdots & b_k & y \end{pmatrix} & \text{if } \nu = 3.
\end{cases}$$

The graphs G_i, for $i = [1, 2\nu + 1]$, are so defined:

$$G_1 = Dev(x_1 \sim x_2 \sim \overline{x_2}), \qquad G_2 = Dev(y_5 \sim y_4 \sim x_3),$$
$$G_3 = Dev(\{\overline{x_1}, \overline{x_2}\} \oplus \{\overline{x_3}, \overline{y_4}\}), \quad G_4 = Dev(\overline{y_5} \sim \overline{y_4} \sim y_5),$$
$$G_5 = Dev(\overline{y_5} \sim \overline{y_6} \sim y_6), \qquad G_6 = Dev(\{x_2, x_3\} \oplus \{x, y\}),$$
$$G_7 = Dev(\{\overline{x_2}, \overline{x_3}\} \oplus \{\overline{x}, \overline{y}\}).$$

Case 2. $q \geq 2r + 6$. Note that this implies $r = 1$ and $q = 8, 10$. As before $K_{2k} = \Gamma_1 \oplus \Gamma_2$ where

$$\Gamma_1 = \langle [3, \ell], \{0\} \cup [k-5, k-1], [3, \ell] \rangle \quad \text{and} \quad \Gamma_2 = \langle \{1, 2\}, [1, k-6], \{1, 2\} \rangle.$$

Since $(k, n) \neq (7, 64), (7, 65)$ then $(\ell, q) \neq (3, 10)$, hence the graph Γ_1 can be decomposed into the following graphs:

$$\Gamma_{1,1} = \langle \varnothing, [k-5, k-1], \varnothing \rangle, \quad \Gamma_{1,2} = \left\langle \left[3, \frac{q-2}{2}\right], \{0\}, \left[3, \frac{q-2}{2}\right] \right\rangle,$$
$$\Gamma_{1,3} = \left\langle \left[\frac{q}{2}, \ell\right], \varnothing, \left[\frac{q}{2}, \ell\right] \right\rangle.$$

The graph $\Gamma_{1,1}$ decomposes into five 1-factors $J_1, \ldots, J_5$, while by Lemma 3.5 $\Gamma_{1,2}$ decomposes into $(q-5)$ 1-factors $J'_1, \ldots, J'_{q-5}$. Letting $w_1 = q\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = (\Gamma_{1,1} + 5\ell) \oplus (\Gamma_{1,2} + (q-5)\ell) \oplus \Gamma_{1,3} = \oplus_{i=1}^{5}(J_i + \ell) \oplus \left[\oplus_{i=1}^{q-5}(J'_i + \ell)\right] \oplus \Gamma_{1,3}.$$

By Lemmas 3.4 and 3.1, each $J_i + \ell$, each $J'_i + \ell$ and $\Gamma_{1,3}$ decompose into k-cycles. Hence $\Gamma_1 + q\ell$ has a k-cycle system. Let now $w_2 = n - 2w_1 = 2 + \nu$. Note that a k-sun system of $\Gamma_2^*[2] + w_2$ can be obtained as in Case 1, where $\Gamma_{2,1}$ is empty. □

Theorem 4.10. *Let $k \equiv 3 \pmod 4 \geq 11$ and $n \equiv 0, 1 \pmod 4$ with $2k < n < 10k$. If $\left\lfloor \frac{n-4}{k-1} \right\rfloor$ is even, and $n \equiv 2, 3 \pmod{k-1}$, then there is a k-sun system of $K_{4k} + n$, except possibly when $(k, n) \in \{(11, 112), (11, 113)\}$.*

Proof. Let $n = 2(q\ell + r) + \nu$ with $1 \leq r \leq \ell$ and $\nu \in \{2, 3\}$. Clearly, $q = \left\lfloor \frac{n-4}{k-1} \right\rfloor$, hence q is even. Since $k \geq 11$, $2k < n < 10k$, and $n \equiv 2, 3 \pmod{2\ell}$, we have that

$$q \text{ is even with } 2 \leq q \leq 10 \quad \text{and} \quad r = \ell \geq 5 \text{ is odd}.$$

As before, let $V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_h \mid h \in \mathbb{Z}_{n-\nu}\} \cup \{\infty'_1, \infty'_2, \infty'_\nu\}$.

We start decomposing K_{2k} into the following two graphs:

$$\Gamma_1 = \langle [3, \ell], [k-3, k], [4, \ell] \rangle, \quad \Gamma_2 = \langle \{1, 2\}, [1, k-4], \{1, 2, 3\} \rangle.$$

If $q = 2, 4$, Γ_1 can be further decomposed into

$$\Gamma_{1,1} = \langle \varnothing, [k-3, k-4+q], \varnothing \rangle, \quad \Gamma_{1,2} = \langle \varnothing, [k-3+q, k], \{\ell\} \rangle,$$
$$\Gamma_{1,3} = \langle [3, \ell], \varnothing, [4, \ell - 1] \rangle.$$

The graph $\Gamma_{1,1}$ decomposes into q 1-factors, say $J_1, \ldots, J_q$. Letting $w_1 = q\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = (\Gamma_{1,1} + w_1) \oplus \Gamma_{1,2} \oplus \Gamma_{1,3} = \oplus_{i=1}^{q}(J_i + \ell) \oplus \Gamma_{1,2} \oplus \Gamma_{1,3}.$$

Lemmas 3.4, 3.2, and 3.1 guarantee that each $J_i + \ell$, $\Gamma_{1,2}$, and $\Gamma_{1,3}$ decompose into k-cycles, hence $\Gamma_1 + w_1$ has a k-cycle system. Suppose now $q \geq 6$. By $(k, n) \notin \{(11, 112), (11, 113)\}$, we have $(\ell, q) \neq (5, 10)$. In this case Γ_1 can be further decomposed into

$$\Gamma_{1,1} = \langle \varnothing, [k-3, k-1], \varnothing \rangle, \quad \Gamma_{1,2} = \left\langle \left[\ell + 3 - \frac{q}{2}, \ell\right], \{0\}, \left[\ell + 3 - \frac{q}{2}, \ell\right] \right\rangle,$$
$$\Gamma_{1,3} = \left\langle \left[3, \ell + 2 - \frac{q}{2}\right], \varnothing, \left[4, \ell + 2 - \frac{q}{2}\right] \right\rangle.$$

The graph $\Gamma_{1,1}$ can be decomposed into three 1-factors say J_1, J_2, J_3, also by Lemma 3.5 the graph $\Gamma_{1,2}$ can be decomposed into $(q-3)$ 1-factors say $J'_1, \ldots, J'_{q-3}$. Set again $w_1 = q\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = (\Gamma_{1,1} + 3\ell) \oplus (\Gamma_{1,2} + (q-3)\ell) \oplus \Gamma_{1,3} = \oplus_{i=1}^{3}(J_i + \ell) \oplus [\oplus_{j=1}^{q-3}(J'_j + \ell)] \oplus \Gamma_{1,3}.$$

By Lemmas 3.4 and 3.1 we have that each $J_i + \ell$, each $J'_j + \ell$, and $\Gamma_{1,3}$ decompose into k-cycles, hence $\Gamma_1 + w_1$ has a k-cycle system. Therefore for any value of q we have proved that $\Gamma_1 + w_1$ has a k-cycle system.

Now, setting $w_2 = n - 2w_1 = 2\ell + \nu$ and recalling that $K_{2k} = \Gamma_1 \oplus \Gamma_2$, by Lemma 4.6 it is left to show that $\Gamma_2^*[2] + w_2$ has a k-sun system. Let r_1 and $r_2 \geq 2$ be an odd and an even integer, respectively, such that $r_1 + r_2 = r = \ell$. Note that Γ_2 can be further decomposed into

$$\Gamma_{2,1} = \langle \{1\}, [1, k - 2r_1 - 2], \{1\} \rangle, \quad \Gamma_{2,2} = \langle \{2\}, [k - 2r_1 - 1, k - 4], \{2, 3\} \rangle.$$

Recalling that $\Gamma_2^*[2] = \Gamma_2[2] \oplus I$, where I denotes the 1-factor $\{\{z, \overline{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$ of K_{4k}, by Lemma 4.2 we have that

$$\Gamma_2^*[2] + w_2 = \oplus_{i=1}^{2}(\Gamma_{2,i} + r_i)[2] \oplus (I + \nu).$$

By Lemma 3.3 there are a k-cycle $A = (y_1, y_2, x_3, x_4, a_5, \ldots, a_k)$ of $\Gamma_{2,1} + r_1$ and a k-cycle $B = (x_1, x_2, y_3, y_4, b_5, \ldots, b_k)$ of $\Gamma_{2,2} + r_2$ such that

$$\begin{aligned} &Orb(A) \cup Orb(B) \quad \text{is a } k\text{-cycle system of } \Gamma_2 + \ell, \\ &Dev(\{x_1, x_2\}) \quad \text{and} \quad Dev(\{x_3, x_4\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{0\}, \\ &Dev(\{y_1, y_2\}) \quad \text{and} \quad Dev(\{y_3, y_4\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{1\}. \end{aligned} \tag{7}$$

Set $A' = (y_1, \overline{y_2}, x_3, \overline{x_4}, a_5, \ldots, a_k)$ and $B' = (x_1, \overline{x_2}, y_3, \overline{y_4}, b_5, \ldots, b_k)$. Let $\mathcal{S} = \{\sigma(A'), \overline{\sigma(A')}, \sigma(B'), \overline{\sigma(B')}\}$, by Lemma 4.5, we have that $\bigcup_{S \in \mathcal{S}} Orb(S)$ is a k-sun system of $(\Gamma_2 + \ell)[2] = \Gamma_2[2] + 2\ell = (\Gamma_2^*[2] + w_2) \backslash (I + \nu)$. To construct a k-sun system of $\Gamma_2^*[2] + w_2$, we build a family $\mathcal{T} = \{T_1, T_2, T_3, T_4\}$ of k-suns by modifying the graphs in $\mathcal{S}$ so that $\bigcup_{T\in\mathcal{T}} Orb(T)$ covers all the edges incident with ∞_1', ∞_2', and possibly ∞_3' when $\nu = 3$. We then construct further $(2\nu + 1)$ k-suns $G_1, G_2, \ldots, G_{2\nu+1}$ which cover the remaining edges exactly once. Hence, $\bigcup_{T\in\mathcal{T}} Orb(T) \cup \{G_1, G_2, \ldots, G_{2\nu+1}\}$ is a k-sun system of $\Gamma_2^*[2] + w_2$.

The graphs $T_1, \ldots, T_4$ and $G_1, \ldots, G_{2\nu+1}$ are the following, where as before the elements in bold are the replaced vertices.

$$T_1 = \begin{pmatrix} y_1 & \overline{y_2} & x_3 & \overline{x_4} & a_5 & \cdots & a_{k-1} & a_k \\ \boldsymbol{\infty_2'} & \overline{x_3} & x_4 & \overline{a_5} & \overline{a_6} & \cdots & \overline{a_k} & \overline{y_1} \end{pmatrix},$$

$$T_2 = \begin{cases} \begin{pmatrix} \overline{y_1} & \boldsymbol{\infty_1'} & \overline{x_3} & x_4 & \overline{a_5} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \boldsymbol{\infty_2'} & x_3 & \overline{x_4} & a_5 & a_6 & \cdots & a_k & \overline{y_1} \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} \overline{y_1} & \boldsymbol{\infty_1'} & \overline{x_3} & x_4 & \overline{a_5} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \boldsymbol{\infty_2'} & x_3 & \boldsymbol{\infty_3'} & a_5 & a_6 & \cdots & a_k & \overline{y_1} \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$T_3 = \begin{pmatrix} x_1 & \overline{x_2} & y_3 & \overline{y_4} & b_5 & \cdots & b_{k-1} & b_k \\ \boldsymbol{\infty_2'} & \overline{y_3} & \boldsymbol{\infty_1'} & \overline{b_5} & \overline{b_6} & \cdots & \overline{b_k} & \overline{x_1} \end{pmatrix},$$

$$T_4 = \begin{cases} \begin{pmatrix} \overline{x_1} & x_2 & \overline{y_3} & y_4 & \overline{b_5} & \cdots & \overline{b_{k-1}} & \overline{b_k} \\ \boldsymbol{\infty_2'} & y_3 & \overline{y_4} & b_5 & b_6 & \cdots & b_k & x_1 \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} \overline{x_1} & x_2 & \boldsymbol{\infty_3'} & y_4 & \overline{b_5} & \cdots & \overline{b_{k-1}} & \overline{b_k} \\ \boldsymbol{\infty_2'} & y_3 & \overline{y_4} & b_5 & b_6 & \cdots & b_k & x_1 \end{pmatrix} & \text{if } \nu = 3. \end{cases}$$

$$G_1 = Dev(y_1 \sim y_2 \sim x_3), \quad G_2 = Dev(\overline{y_2} \sim \overline{y_1} \sim y_2),$$
$$G_3 = Dev(y_3 \sim y_4 \sim \overline{y_4}), \quad G_4 = Dev(\{\overline{x_1}, \overline{x_2}\} \oplus \{\overline{x_3}, y_2\}),$$
$$G_5 = \begin{cases} Dev(x_1 \sim x_2 \sim \overline{x_2}) & \text{if } \nu = 2, \\ Dev(x_1 \sim x_2 \sim \overline{y_3}) & \text{if } \nu = 3, \end{cases} \quad G_6 = Dev(\overline{x_3} \sim \overline{x_4} \sim x_4\}),$$
$$G_7 = Dev(\overline{y_4} \sim \overline{y_3} \sim y_4).$$

By recalling (7), it is not difficult to check that the graphs G_h are k-suns. □

Theorem 4.11. *Let $k \equiv 3 \pmod 4 \geq 7$ and $n \equiv 0, 1 \pmod 4$ with $2k < n < 10k$. If $\left\lfloor \frac{n-4}{k-1} \right\rfloor$ is odd and $n \not\equiv 0, 1 \pmod{k-1}$, then there is a k-sun system of $K_{4k} + n$.*

Proof. Let $n = 2(q\ell + r) + \nu$ with $1 \leq r \leq \ell$ and $\nu \in \{2, 3\}$. Clearly, $q = \left\lfloor \frac{n-4}{k-1} \right\rfloor$. Also, we have that q and $\ell \geq 3$ are odd, and $n \equiv 0, 1 \pmod 4$; hence r is even. Furthermore, we have that $2 \leq q \leq 10$, since by assumption $2k < n < 10k$. Considering now the hypothesis that $n \not\equiv 0, 1 \pmod{2\ell}$, it follows that $r \neq \ell - 1$. To sum up,

$$q \text{ is odd with } 3 \leq q \leq 9, \quad \text{and} \quad r \text{ is even with } 2 \leq r \leq \ell - 3. \tag{8}$$

As before, let $V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_h \mid h \in \mathbb{Z}_{n-\nu}\} \cup \{\infty_1', \infty_2', \infty_\nu'\}$.

We start decomposing K_{2k} into the following two graphs:

$$\Gamma_1 = \langle [4, \ell], [k - 2r - 1, k], [3, \ell] \rangle \quad \text{and} \quad \Gamma_2 = \langle [1, 3], [1, k - 2r - 2], [1, 2] \rangle.$$

Considering that $3 \le q \le 9 \le 2r + 5$, the graph Γ_1 can be further decomposed into the following graphs:

$$\Gamma_{1,1} = \langle [4, \ell], \varnothing, [3, \ell - 1] \rangle, \quad \Gamma_{1,2} = \langle \varnothing, [k - 2r - 4 + q, k], \{\ell\} \rangle, \quad \text{and}$$
$$\Gamma_{1,3} = \langle \varnothing, [k - 2r - 1, k - 2r - 5 + q], \varnothing \rangle.$$

The first two have a k-cycle system by Lemmas 3.1 and 3.2, while $\Gamma_{1,3}$ decomposes into $(q - 3)$ 1-factors, say $J_1, J_2, \ldots, J_{q-3}$. Letting $w_1 = (q - 3)\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = \oplus_{i=1}^{q-3} (J_i + \ell) \oplus (\Gamma_{1,1} \oplus \Gamma_{1,2}).$$

Therefore, $\Gamma_1 + w_1$ has a k-cycle system, since each $J_i + \ell$ decomposes into k-cycles by Lemma 3.4. Setting $w_2 = n - 2w_1 = 2(3\ell + r) + \nu$ and recalling that $K_{2k} = \Gamma_1 \oplus \Gamma_2$, by Lemma 4.6 it is left to show that $\Gamma_2^*[2] + w_2$ has a k-sun system.

We start decomposing Γ_2 into the following graphs:

$$\Gamma_{2,0} = \langle [1, 3], [1, k - 2r - 5], [1, 2] \rangle \quad \text{and}$$
$$\Gamma_{2,i} = \langle \varnothing, \{k - 2r - 5 + i\}, \varnothing \rangle \quad \text{for } 1 \le i \le 3.$$

Recalling that $\Gamma_2^*[2] = \Gamma_2[2] \oplus I$, where I denotes the 1-factor $\{\{z, \bar{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$ of K_{4k}, by Lemma 4.2 we have that

$$\Gamma_2^*[2] + w_2 = \oplus_{i=1}^{3} (\Gamma_{2,i} + \ell)[2] \oplus (\Gamma_{2,0} + r)[2] \oplus (I + \nu).$$

By Lemmas 3.3 and 3.4 there exist a k-cycle $A = (x_1, x_2, x_3, y_4, y_5, y_6, a_7, \ldots, a_k)$ of $\Gamma_{2,0} + r$, a k-cycle $B_1 = (x_{1,0}, y_{1,1}, b_{1,2}, \ldots, b_{1,k-1})$ of $\Gamma_{2,1} + \ell$, and a k-cycle $B_i = (y_{i,0}, x_{i,1}, b_{i,2}, \ldots, b_{i,k-1})$ of $\Gamma_{2,i} + \ell$, for $2 \le i \le 3$, satisfying the following properties:

$$\begin{aligned} &Dev(\{x_1, x_2\}) \quad \text{and} \quad Dev(\{x_2, x_3\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{0\}, \\ &Dev(\{y_4, y_5\}) \quad \text{and} \quad Dev(\{y_5, y_6\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{1\}, \end{aligned} \tag{9}$$

$$x_{1,0}, x_{2,1}, x_{3,1} \in \mathbb{Z}_k \times \{0\}, \quad y_{1,1}, y_{2,0}, y_{3,0} \in \mathbb{Z}_k \times \{1\}, \tag{10}$$

$$\bigcup_{i=1}^{3} Orb(B_i) \cup Orb(A) \quad \text{is a } k\text{-cycle system of } \Gamma_2 + (3\ell + r). \tag{11}$$

Set $A' = (x_1, \overline{x_2}, x_3, \overline{y_4}, y_5, \overline{y_6}, a_7, a_8, \ldots, a_{k-1}, a_k)$ and let $\mathcal{S} = \{\sigma(A'), \overline{\sigma(A')}\} \cup \{\sigma(B_i), \overline{\sigma(B_i)} | 1 \le i \le 3\}$. By Lemma 4.5, we have that $\bigcup_{S \in \mathcal{S}} Orb(S)$ is a k-sun system of $(\Gamma_2 + (3\ell + r))[2] = \Gamma_2[2] + 2(3\ell + r) = (\Gamma_2^*[2] + w_2) \backslash (I + \nu)$.

To construct a k-sun system of $\Gamma_2^*[2] + w_2$, we build a family $\mathcal{T} = \{T_0, T_1, \ldots, T_7\}$ of k-suns by modifying the graphs in $\mathcal{S}$ so that $\bigcup_{T \in \mathcal{T}} Orb(T)$ covers all the edges incident with ∞_1', ∞_2', and possibly ∞_3' when $\nu = 3$. We then construct further $(2\nu + 1)$ k-suns $G_1, G_2, \ldots, G_{2\nu+1}$

which cover the remaining edges exactly once. Hence, $\bigcup_{T\in\mathcal{T}} Orb(T) \cup \{G_1, G_2, ..., G_{2\nu+1}\}$ is a k-sun system of $\Gamma_2^*[2] + w_2$.

The graphs $T_0, ..., T_7$ and $G_1, ..., G_{2\nu+1}$ are the following, where as before the elements in bold are the replaced vertices.

$$T_0 = \begin{cases} \begin{pmatrix} x_1 & \overline{x_2} & x_3 & \overline{y_4} & y_5 & \overline{y_6} & a_7 & \cdots & a_{k-1} & a_k \\ x_2 & \boldsymbol{\infty}_1' & y_4 & \boldsymbol{\infty}_2' & y_6 & \overline{a_7} & \overline{a_8} & \cdots & \overline{a_k} & \overline{x_1} \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} x_1 & \overline{x_2} & x_3 & \overline{y_4} & y_5 & \overline{y_6} & a_7 & \cdots & a_{k-1} & a_k \\ \boldsymbol{\infty}_3' & \boldsymbol{\infty}_1' & y_4 & \boldsymbol{\infty}_2' & y_6 & \overline{a_7} & \overline{a_8} & \cdots & \overline{a_k} & \overline{x_1} \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$T_1 = \begin{cases} \begin{pmatrix} \overline{x_1} & x_2 & \overline{x_3} & y_4 & \overline{y_5} & y_6 & \overline{a_7} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \overline{x_2} & \boldsymbol{\infty}_1' & \overline{y_4} & \boldsymbol{\infty}_2' & \boldsymbol{y_5} & a_7 & a_8 & \cdots & a_k & x_1 \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} \overline{x_1} & x_2 & \overline{x_3} & y_4 & \overline{y_5} & y_6 & \overline{a_7} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \boldsymbol{\infty}_3' & \boldsymbol{\infty}_1' & \overline{y_4} & \boldsymbol{\infty}_2' & \boldsymbol{y_5} & a_7 & a_8 & \cdots & a_k & x_1 \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$T_2 = \begin{pmatrix} x_{1,0} & y_{1,1} & b_{1,2} & \cdots & b_{1,k-2} & b_{1,k-1} \\ \boldsymbol{\infty}_2' & \overline{b_{1,2}} & \overline{b_{1,3}} & \cdots & \overline{b_{1,k-1}} & \overline{x_{1,0}} \end{pmatrix},$$

$$T_3 = \begin{pmatrix} \overline{x_{1,0}} & \overline{y_{1,1}} & \overline{b_{1,2}} & \cdots & \overline{b_{1,k-2}} & \overline{b_{1,k-1}} \\ \boldsymbol{\infty}_2' & b_{1,2} & b_{1,3} & \cdots & b_{1,k-1} & x_{1,0} \end{pmatrix},$$

$$T_4 = \begin{pmatrix} y_{2,0} & x_{2,1} & b_{2,2} & \cdots & b_{2,k-2} & b_{2,k-1} \\ \boldsymbol{\infty}_1' & \overline{b_{2,2}} & \overline{b_{2,3}} & \cdots & \overline{b_{2,k-1}} & \overline{y_{2,0}} \end{pmatrix},$$

$$T_5 = \begin{pmatrix} \overline{y_{2,0}} & \overline{x_{2,1}} & \overline{b_{2,2}} & \cdots & \overline{b_{2,k-2}} & \overline{b_{2,k-1}} \\ \boldsymbol{\infty}_1' & b_{2,2} & b_{2,3} & \cdots & b_{2,k-1} & y_{2,0} \end{pmatrix},$$

$$T_6 = \begin{cases} \sigma(B_3) & \text{if } \nu = 2, \\ \begin{pmatrix} y_{3,0} & x_{3,1} & b_{3,2} & \cdots & b_{3,k-2} & b_{3,k-1} \\ \boldsymbol{\infty}_3' & \overline{b_{3,2}} & \overline{b_{3,3}} & \cdots & \overline{b_{3,k-1}} & \overline{y_{3,0}} \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$T_7 = \begin{cases} \overline{\sigma(B_3)} & \text{if } \nu = 2, \\ \begin{pmatrix} \overline{y_{3,0}} & \overline{x_{3,1}} & \overline{b_{3,2}} & \cdots & \overline{b_{3,k-2}} & \overline{b_{3,k-1}} \\ \boldsymbol{\infty}_3' & b_{3,2} & b_{3,3} & \cdots & b_{3,k-1} & y_{3,0} \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$\begin{aligned} G_1 &= Dev(x_2 \sim x_3 \sim \overline{x_3}), & G_2 &= Dev(\{\overline{x_2}, \overline{x_3}\} \oplus \{\overline{x_{1,0}}, y_{1,1}\}), \\ G_3 &= Dev(\{y_4, y_5\} \oplus \{y_{2,0}, \overline{x_{2,1}}\}), & G_4 &= Dev(\{\overline{y_4}, \overline{y_5}\} \oplus \{\overline{y_{2,0}}, x_{2,1}\}), \\ G_5 &= Dev(\{\overline{y_5}, \overline{y_6}\} \oplus \{\overline{y_{1,1}}, x_{1,0}\}), & G_6 &= Dev(\{x_1, x_2\} \oplus \{x_{3,1}, \overline{y_{3,0}}\}), \\ G_7 &= Dev(\{\overline{x_1}, \overline{x_2}\} \oplus \{\overline{x_{3,1}}, y_{3,0}\}). \end{aligned}$$

By recalling (9)–(11), it is not difficult to check that the graphs G_h are k-suns. □

Theorem 4.12. *Let* $k \equiv 3 \pmod 4 \geq 7$ *and* $n \equiv 0, 1 \pmod 4$ *with* $2k < n < 10k$. *If* $\left\lfloor \frac{n-4}{k-1} \right\rfloor$ *is odd, and* $n \equiv 0, 1 \pmod{k-1}$, *then there is a* k*-sun system of* $K_{4k} + n$ *except possibly when* $(k, n) \in \{(11, 100), (11, 101)\}$.

Proof. Let $n = 2(q\ell + r) + \nu$ with $1 \leq r \leq \ell$ and $\nu \in \{2, 3\}$. Reasoning as in the proof of Theorem 4.11 and considering that $n \equiv 0, 1 \pmod{2\ell}$ and $(k, n) \notin \{(11, 100), (11, 101)\}$, we have that

$$q \text{ is odd with } 3 \leq q \leq 9, \quad r = \ell - 1 \geq 2, \quad r \text{ is even}, \quad \text{and} \quad (\ell, q) \neq (5, 9). \tag{12}$$

As before, let $V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_h | h \in \mathbb{Z}_{n-\nu}\} \cup \{\infty'_1, \infty'_2, \infty'_\nu\}$.

We start decomposing K_{2k} into the following two graphs

$$\Gamma_1 = \langle [3, \ell], \{0\}, [3, \ell] \rangle \quad \text{and} \quad \Gamma_2 = \langle \{1, 2\}, [1, k-1], \{1, 2\} \rangle.$$

Considering (12), we can further decompose Γ_1 into the following two graphs:

$$\Gamma_{1,1} = \left\langle \left[3, \frac{q+3}{2}\right], \{0\}, \left[3, \frac{q+3}{2}\right] \right\rangle, \quad \Gamma_{1,2} = \left\langle \left[\frac{q+5}{2}, \ell\right], \varnothing, \left[\frac{q+5}{2}, \ell\right] \right\rangle.$$

By Lemma 3.5, the graph $\Gamma_{1,1}$ decomposes into q 1-factors, say $J_1, J_2, \ldots, J_q$. Letting $w_1 = q\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = (\Gamma_{1,1} + w_1) \oplus \Gamma_{1,2} = \oplus_{i=1}^{q} (J_i + \ell) \oplus \Gamma_{1,2}.$$

Lemmas 3.4 and 3.1 guarantee that each $J_i + \ell$ and $\Gamma_{1,2}$ decompose into k-cycles, hence $\Gamma_1 + w_1$ has a k-cycle system. Let r_1 and r_2 be odd positive integers such that $r = \ell - 1 = r_1 + r_2$. Then, setting $w_2 = n - 2w_1 = 2(r_1 + r_2) + \nu$ and recalling that $K_{2k} = \Gamma_1 \oplus \Gamma_2$, by Lemma 4.6 it is left to show that $\Gamma_2^*[2] + w_2$ has a k-sun system.

We start decomposing Γ_2 into the following graphs:

$$\Gamma_{2,1} = \langle \{1\}, [1, k - 2r_1 - 2], \{1\} \rangle \quad \text{and} \quad \Gamma_{2,2} = \langle \{2\}, [k - 2r_1 - 1, k - 1], \{2\} \rangle.$$

Recalling that $\Gamma_2^*[2] = \Gamma_2[2] \oplus I$, where I denotes the 1-factor $\{\{z, \bar{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$ of K_{4k}, by Lemma 4.2 we have that

$$\Gamma_2^*[2] + w_2 = (\Gamma_{2,1} + r_1)[2] \oplus (\Gamma_{2,2} + r_2)[2] \oplus (I + \nu). \tag{13}$$

By Lemma 3.3 there are a k-cycle $A = (y_1, y_2, x_3, x_4, a_5, \ldots, a_k)$ of $\Gamma_{2,1} + r_1$ and a k-cycle $B = (x_1, x_2, y_3, y_4, b_5, \ldots, b_k)$ of $\Gamma_{2,2} + r_2$ such that

$$\begin{aligned} &Orb(A) \cup Orb(B) \quad \text{is a } k\text{-cycle system of } \Gamma_2 + r, \\ &Dev(\{x_3, x_4\}) \quad \text{and} \quad Dev(\{x_1, x_2\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{0\}, \\ &Dev(\{y_1, y_2\}) \quad \text{and} \quad Dev(\{y_3, y_4\}) \quad \text{are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{1\}. \end{aligned}$$

Set $A' = (y_1, \overline{y_2}, x_3, \overline{x_4}, a_5, \dots, a_k)$, $B' = (x_1, \overline{x_2}, \overline{y_3}, y_4, b_5, \dots, b_k)$ and let $\mathcal{S} = \{\sigma(A'), \overline{\sigma(A')}, \sigma(B'), \overline{\sigma(B')}\}$. By Lemma 4.5, we have that $\bigcup_{S\in\mathcal{S}} Orb(S)$ is a k-sun system of $(\Gamma_2^*[2] + w_2)\backslash(I + \nu)$.

To construct a k-sun system of $\Gamma_2^*[2] + w_2$, we build a family $\mathcal{T} = \{T_1, T_2, T_3, T_4\}$ of four k-suns, each of which is obtained from a graph in $\mathcal{S}$ by replacing some of their vertices with ∞_1', ∞_2', and possibly ∞_3' when $\nu = 3$. Then we construct further $(2\nu + 1)$ k-suns $G_1, G_2, \dots, G_{2\nu+1}$ so that $\bigcup_{T\in\mathcal{T}} Orb(T) \cup \{G_1, G_2, \dots, G_{2\nu+1}\}$ is a k-sun system of $\Gamma_2^*[2] + w_2$.

$$T_1 = \begin{cases} \begin{pmatrix} y_1 & \overline{y_2} & x_3 & \overline{x_4} & a_5 & \cdots & a_{k-1} & a_k \\ \infty_1' & \infty_2' & x_4 & \overline{a_5} & \overline{a_6} & \cdots & \overline{a_k} & \overline{y_1} \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} y_1 & \overline{y_2} & \infty_3' & \overline{x_4} & a_5 & \cdots & a_{k-1} & a_k \\ \infty_1' & \infty_2' & x_4 & \overline{a_5} & \overline{a_6} & \cdots & \overline{a_k} & \overline{y_1} \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$T_2 = \begin{pmatrix} \overline{y_1} & y_2 & \overline{x_3} & x_4 & \overline{a_5} & \cdots & \overline{a_{k-1}} & \overline{a_k} \\ \infty_1' & \infty_2' & \overline{x_4} & a_5 & a_6 & \cdots & a_k & y_1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} x_1 & \overline{x_2} & \overline{y_3} & y_4 & b_5 & \cdots & b_{k-1} & b_k \\ \infty_1' & \infty_2' & y_3 & \overline{b_5} & \overline{b_6} & \cdots & \overline{b_k} & \overline{x_1} \end{pmatrix},$$

$$T_4 = \begin{cases} \begin{pmatrix} \overline{x_1} & x_2 & y_3 & \overline{y_4} & \overline{b_5} & \cdots & \overline{b_{k-1}} & \overline{b_k} \\ \infty_1' & \infty_2' & y_4 & b_5 & b_6 & \cdots & b_k & x_1 \end{pmatrix} & \text{if } \nu = 2, \\ \begin{pmatrix} \overline{x_1} & x_2 & y_3 & \overline{y_4} & \overline{b_5} & \cdots & \overline{b_{k-1}} & \overline{b_k} \\ \infty_1' & \infty_2' & \infty_3' & b_5 & b_6 & \cdots & b_k & x_1 \end{pmatrix} & \text{if } \nu = 3, \end{cases}$$

$$G_1 = Dev(y_1 \sim y_2 \sim x_3), \qquad G_2 = Dev(\overline{y_1} \sim \overline{y_2} \sim \overline{x_3}),$$

$$G_3 = Dev(\overline{y_4} \sim \overline{y_3} \sim x_2), \qquad G_4 = Dev(x_1 \sim x_2 \sim \overline{x_2}),$$

$$G_5 = \begin{cases} Dev(\overline{x_1} \sim \overline{x_2} \sim y_3) & \text{if } \nu = 2, \\ Dev(\{\overline{x_1}, \overline{x_2}\} \oplus \{\overline{x_4}, x_3\}) & \text{if } \nu = 3, \end{cases} \qquad G_6 = Dev(y_4 \sim y_3 \sim \overline{x_2}),$$

$$G_7 = Dev(x_4 \sim x_3 \sim \overline{y_2}).$$

By (13), it is not difficult to check that the graphs G_h are k-suns. □

5 | IT IS SUFFICIENT TO SOLVE $2k < \nu < 6k$

In this section we show that if the necessary conditions in (*), for the existence of a k-sun system of K_ν, are sufficient for all ν satisfying $2k < \nu < 6k$, then they are sufficient for all ν. In other words, we prove Theorem 1.1.

We start by showing how to construct k-sun systems of $K_{g\times h}$ (i.e., the complete multipartite graph with g parts each of size h) when $h = 4k$.

Theorem 5.1. *For any odd integer $k \geq 3$ and any integer $g \geq 3$, there exists a k-sun system of $K_{g\times 4k}$.*

Proof. Set $V(K_{g\times 2k}) = \mathbb{Z}_{gk} \times [0, 1]$ and let $K_{g\times 4k} = K_{g\times 2k}[2]$. In [11, Theorem 2] the authors proved the existence of a k-cycle system of $K_{g\times 2k}$. By applying Lemma 4.5 (with $\Gamma = K_{g\times 2k}$ and $u = 0$) we obtain the existence of a k-sun system of $K_{g\times 4k}$. □

The following result exploits Theorem 5.1 and shows how to construct k-sun systems of K_{4kg+n}, for $g \neq 2$, starting from a k-sun system of $K_{4k} + n$ and a k-sun system of either K_n or K_{4k+n}.

Theorem 5.2. *Let $k \geq 3$ be an odd integer and assume that both the following conditions hold*:

1. *there exists a k-sun system of either K_n or K_{4k+n}*;
2. *there exists a k-sun system of $K_{4k} + n$.*

Then there is a k-sun system of K_{4kg+n} for all positive $g \neq 2$.

Proof. Suppose there exists a k-sun system $\mathcal{S}_1$ of K_n, also, by (2), there exists a k-sun system $\mathcal{S}_2$ of $K_{4k} + n$. Clearly, $\mathcal{S}_1 \cup \mathcal{S}_2$ is a k-sun system of $K_{n+4k} = K_n \oplus (K_{4k} + n)$. Hence we can suppose $g \geq 3$. Let V, H, and G be sets of size n, $4k$, and g, respectively, such that $V \cap (H \times G) = \varnothing$. Let $\mathcal{S}$ be a k-sun system of K_n (resp., K_{n+4k}) with vertex-set V (resp., $V \cup (H \times \{x_0\})$ for some $x_0 \in G$). By assumption, for each $x \in G$, there is a k-sun system, say $\mathcal{B}_x$, of $K_{4k} + n$ with vertex-set $V \cup (H \times \{x\})$, where $V(K_{4k}) = H \times \{x\}$. Also, by Theorem 5.1 there is a k-sun system $\mathcal{C}$ of $K_{g\times 4k}$ whose parts are $H \times \{x\}$ with $x \in G$. Hence the k-suns of $\mathcal{B}_x$ with $x \in G$ (resp., $x \in G\backslash\{x_0\}$), $\mathcal{S}$ and $\mathcal{C}$ form a k-sun system of K_{n+4kg} with vertex-set $V \cup (H \times G)$. □

We are now ready to prove Theorem 1.1 whose statement is recalled below.

Theorem 1.1. *Let $k \geq 3$ be an odd integer and $v > 1$. Conjecture 1 is true if and only if there exists a k-sun system of K_v for all v satisfying the necessary conditions in* (*) *with $2k < v < 6k$.*

Proof. The existence of 3-sun systems and 5-sun systems has been solved in [10] and in [8], respectively. Hence we can suppose $k \geq 7$ and $2k < v < 6k$.

We first deal with the case where $(k, v) \neq (7, 21)$. By assumption there exists a k-sun system of K_v, which implies $v(v-1) \equiv 0 \pmod 4$, hence Theorem 4.1 guarantees the existence of a k-sun system of $K_{4k} + v$. Therefore, by Theorem 5.2 there is a k-sun decomposition of K_{4kg+v} whenever $g \neq 2$. To decompose K_{8k+v} into k-suns, we first decompose K_{8k+v} into K_{4k+v} and $K_{4k} + (4k + v)$. By Theorem 5.2 (with $g = 1$), there is a k-sun system of K_{4k+v}. Furthermore, Theorem 4.1 guarantees the existence of a k-sun system of $K_{4k} + (4k + v)$, except possibly when $(k, 4k + v) \in \{(7, 56), (7, 57), (7, 64), (11, 100)\}$. Therefore, by Theorem 5.2, there is a k-sun decomposition of K_{8k+v} whenever $(k, 4k + v) \notin \{(7, 56), (7, 57), (7, 64), (11, 100)\}$. For each of these four cases we construct k-sun systems of K_{8k+v} as follows.

If $k = 7$ and $4k + v = 56$, set $V(K_{84}) = \mathbb{Z}_{83} \cup \{\infty\}$. We consider the following 7-suns:

$$T_1 = \begin{pmatrix} 0 & -1 & 3 & -4 & 6 & -7 & 16 \\ 31 & 27 & 37 & 18 & 43 & 12 & 56 \end{pmatrix},$$
$$T_2 = \begin{pmatrix} 0 & -2 & 3 & -5 & 6 & -8 & 17 \\ 32 & 27 & 38 & 19 & 44 & 12 & 58 \end{pmatrix},$$
$$T_3 = \begin{pmatrix} 0 & -3 & 3 & -6 & 6 & -9 & 18 \\ 33 & 27 & 39 & 20 & 45 & 12 & \infty \end{pmatrix}.$$

One can easily check that $\bigcup_{i=1}^{3} Orb_{\mathbb{Z}_{83}}(T_i)$ is a 7-sun system of K_{84}.

If $k = 7$ and $4k + v = 57$, set $V(K_{85}) = \mathbb{Z}_{85}$. Let T_1 and T_2 be defined as above, and let T_3' be the graph obtained from T_3 replacing ∞ with 60. It is immediate that $\bigcup_{i=1}^{2} Orb_{\mathbb{Z}_{85}}(T_i) \cup Orb_{\mathbb{Z}_{85}}(T_3')$ is a 7-sun system of K_{85}.

If $k = 7$ and $4k + v = 64$, set $V(K_{92}) = (\mathbb{Z}_7 \times \mathbb{Z}_{13}) \cup \{\infty\}$. We consider the following 7-suns:

$$T_1 = \begin{pmatrix} (0,0) & (1,1) & -(2,1) & (3,1) & -(4,1) & (5,1) & -(6,1) \\ \infty & (-1,1) & (2,7) & (-3,5) & -(3,5) & -(5,7) & (6,7) \end{pmatrix},$$
$$T_2 = \begin{pmatrix} (0,0) & (1,2) & -(2,2) & (3,2) & -(4,2) & (5,2) & -(6,2) \\ (0,10) & -(1,8) & (2,8) & (-3,7) & -(3,7) & -(5,8) & (6,8) \end{pmatrix},$$
$$T_3 = \begin{pmatrix} (0,0) & (1,3) & -(2,3) & (3,3) & -(4,3) & (5,3) & -(6,3) \\ (0,12) & -(1,9) & (2,9) & (-3,9) & -(3,9) & -(5,9) & (6,9) \end{pmatrix},$$
$$T_4 = Dev_{\mathbb{Z}_7\times\{0\}}((0,0) \sim (4,0) \sim (6,8)), \quad T_5 = Dev_{\mathbb{Z}_7\times\{0\}}((0,0) \sim (6,0) \sim (6,8)).$$

One can easily check that $\bigcup_{i=1}^{3} Orb_{\mathbb{Z}_7\times\mathbb{Z}_{13}}(T_i) \cup \bigcup_{i=4}^{5} Orb_{\{0\}\times\mathbb{Z}_{13}}(T_i)$ is a 7-sun system of K_{92}.

If $k = 11$ and $4k + v = 100$, set $V(K_{144}) = (\mathbb{Z}_{11} \times \mathbb{Z}_{13}) \cup \{\infty\}$. We consider the following 11-suns:

$$T_1 = \begin{pmatrix} (0,0) & (1,1) & -(2,1) & (3,1) & -(4,1) & (5,1) & -(6,1) & (7,1) & -(8,1) & (9,1) & -(10,1) \\ \infty & (-1,1) & (2,7) & -(3,7) & (4,7) & (-5,1) & -(5,5) & -(7,7) & (8,7) & -(9,7) & (10,7) \end{pmatrix},$$
$$T_2 = \begin{pmatrix} (0,0) & (1,2) & -(2,2) & (3,2) & -(4,2) & (5,2) & -(6,2) & (7,2) & -(8,2) & (9,2) & -(10,2) \\ (0,10) & -(1,8) & (2,8) & -(3,8) & (4,8) & (-5,6) & -(5,7) & -(7,8) & (8,8) & -(9,8) & (10,8) \end{pmatrix},$$
$$T_3 = \begin{pmatrix} (0,0) & (1,3) & -(2,3) & (3,3) & -(4,3) & (5,3) & -(6,3) & (7,3) & -(8,3) & (9,3) & -(10,3) \\ (0,12) & -(1,9) & (2,9) & -(3,9) & (4,9) & (-5,9) & -(5,9) & -(7,9) & (8,9) & -(9,9) & (10,9) \end{pmatrix},$$
$$T_4 = Dev_{\mathbb{Z}_{11}\times\{0\}}((0,0) \sim (4,0) \sim (6,8)), \quad T_5 = Dev_{\mathbb{Z}_{11}\times\{0\}}((0,0) \sim (6,0) \sim (5,8)),$$
$$T_6 = Dev_{\mathbb{Z}_{11}\times\{0\}}((0,0) \sim (8,0) \sim (8,8)).$$

One can check that $\bigcup_{i=1}^{3} Orb_{\mathbb{Z}_{11}\times\mathbb{Z}_{13}}(T_i) \cup \bigcup_{i=4}^{6} Orb_{\{0\}\times\mathbb{Z}_{13}}(T_i)$ is an 11-sun system of K_{144}.

It is left to prove the existence of a k-sun system of K_{4kg+v} when $(k, v) = (7, 21)$ and for every $g \geq 1$. If $g = 1$, a 7-sun system of K_{49} can be obtained as a particular case of the following construction. Let p be a prime, $q = p^n \equiv 1 \pmod 4$ and r be a primitive root of $\mathbb{F}_q$. Setting $S = Dev_{\langle r\rangle}(0 \sim r \sim r + 1)$ where $\langle r\rangle = \{jr \mid 1 \leq j \leq p\}$, we have that $\bigcup_{i=0}^{\frac{q-5}{4}} Orb_{\mathbb{F}_q}(r^{2i}S)$ is a p-sun system of K_q.

If $g \geq 2$, we notice that $K_{28g+21} = K_{28(g-1)+49}$. Considering the 7-sun system of K_{49} just built, and recalling that by Theorem 4.1 there is a 7-sun system of $K_{28} + 49$, then Theorem 5.2 guarantees the existence of a 7-sun system of $K_{28(g-1)+49}$ whenever $g \neq 3$.

When $g = 3$, a 7-sun system of K_{105} is constructed as follows. Set $V(K_{105}) = \mathbb{Z}_7 \times \mathbb{Z}_{15}$. Let $S_{i,j}$ and T be the 7-suns defined below, where $(i, j) \in X = ([1, 3] \times [1, 7]) \setminus \{(1, 3), (1, 6)\}$:

$$S_{i,j} = \begin{pmatrix} (0,0) & (i, j/2) & (2i, j) & (3i, 0) & (4i, j) & (5i, 0) & (6i, j) \\ (i, -j/2) & (2i, 0) & (3i, 2j) & (4i, -j) & (5i, 2j) & (6i, -j) & (0, 2j) \end{pmatrix},$$

$$T = \begin{pmatrix} (0,0) & (0,7) & (0,2) & (0,5) & (0,-1) & (0,3) & (0,1) \\ (2,0) & (3,7) & (1,2) & (1,8) & (1,5) & (1,0) & (1,10) \end{pmatrix}.$$

One can check that $\bigcup_{(i,j)\in X} Orb_{\{0\}\times\mathbb{Z}_{15}}(S_{i,j}) \cup Orb_{\mathbb{Z}_7\times\mathbb{Z}_{15}}(T)$ is a 7-sun system of K_{105}. □

6 | CONSTRUCTION OF p-SUN SYSTEMS, p PRIME

In this section we prove Theorem 1.2. Clearly in view of Theorem 1.1 it is sufficient to construct a p-sun system of K_v for any admissible v with $2p < v < 6p$. Hence, we are going to prove the following result.

Theorem 6.1. *Let p be an odd prime and let $v(v-1) \equiv 0 \pmod{4p}$ with $2p < v < 6p$. Then there exists a p-sun system of K_v.*

Since the existence of p-sun systems with $p = 3, 5$ has been proved in [10] and in [8], respectively, here we can assume $p \geq 7$.

It is immediate to see that by the necessary conditions for the existence of a p-sun system of K_v, it follows that v lies in one of the following congruence classes modulo $4p$:

1. $v \equiv 0, 1 \pmod{4p}$;
2. $v \equiv p, 3p + 1 \pmod{4p}$ if $p \equiv 1 \pmod 4$;
3. $v \equiv p + 1, 3p \pmod{4p}$ if $p \equiv 3 \pmod 4$.

If $v \equiv 0, 1 \pmod{4p}$ we present a direct construction which holds more in general for $p = k$, where k is an odd integer and not necessarily a prime.

Theorem 6.2. *For any $k = 2t + 1 \geq 7$ there exists a k-sun system of K_{4k+1} and a k-sun system of K_{4k}.*

Proof. Let C be the k-cycle with vertices in $\mathbb{Z}$ so defined:

$$C = (0, -1, 1, -2, 2, -3, 3, \dots, 1 - t, t - 1, -t, 2t).$$

Note that the list D_1 of the positive differences in $\mathbb{Z}$ of C is $D_1 = [1, 2t] \cup \{3t\}$. Consider now the ordered k-set $D_2 = \{d_1, d_2, \dots, d_k\}$ so defined:

$$D_2 = [2t + 1, 3t - 1] \cup [3t + 1, 4t + 2].$$

Obviously $D_1 \cup D_2 = [1, 2k]$. Let $\{c_1, c_2, \dots, c_k\}$ be the increasing order of the vertices of the cycle C and set $\ell_r = c_r + d_r$ for every $r \in [1, k]$, with $r \neq \frac{t+1}{2}$, and $\ell_{\frac{t+1}{2}} = c_{\frac{t+1}{2}} - d_{\frac{t+1}{2}}$ when t is odd. It is not hard to see that $V = \{c_1, c_2, ..., c_k, \ell_1, \ell_2, ..., \ell_k\}$ is a set. Note also that $V \subseteq \{-3t - 1\} \cup [-t, 5t] \cup \{6t + 2\}$.

Let S be the sun obtainable from C by adding the pendant edges $\{c_i, \ell_i\}$ for $i \in [1, k]$. Clearly, $\Delta S = \pm(D_1 \cup D_2) = \pm[1, 2k]$. So we can conclude that if we consider the vertices of S as elements of $\mathbb{Z}_{4k+1}$, the vertices are still pairwise distinct and $\Delta S = \mathbb{Z}_{4k+1} \setminus \{0\}$. Then, by applying Corollary 2.2 (with $G = \mathbb{Z}_{4k+1}$, $m = 1$, $w = 0$), it follows that $Orb_{\mathbb{Z}_{4k+1}} S$ is a k-sun system of K_{4k+1}.

Now we construct a k-sun system of K_{4k}. Let S be defined as above and note that $d_k = 2k$. Let S^* be the sun obtained by S setting $\ell_k = \infty$. It is immediate that if we consider the vertices of S^* as elements of $\mathbb{Z}_{4k-1} \cup \{\infty\}$, then Corollary 2.2 (with $G = \mathbb{Z}_{4k-1}$, $m = 1$, $w = 1$) guarantees that $Orb_{\mathbb{Z}_{4k-1}} S^*$ is a k-sun system of K_{4k}. □

Example 6.3. Let $k = 2t + 1 = 9$, hence $t = 4$. By following the proof of Theorem 6.2, we construct a 9-sun system of K_{37}. Taking $C = (0, -1, 1, -2, 2, -3, 3, -4, 8)$, we have that

$$\{d_1, d_2, ..., d_9\} = [9, 11] \cup [13, 18],$$
$$\{c_1, c_2, ..., c_9\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 8\}.$$

Hence $\{\ell_1, \ell_2, ..., \ell_9\} = \{5, 7, 9, 12, 14, 16, 18, 20, 26\}$ and we obtain the following 9-sun S with vertices in $\mathbb{Z}_{37}$:

$$S = \begin{pmatrix} 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 8 \\ 14 & 12 & 16 & 9 & 18 & 7 & 20 & 5 & 26 \end{pmatrix},$$

such that $\Delta S = \mathbb{Z}_{37} \setminus \{0\}$. Therefore, $Orb_{\mathbb{Z}_{37}} S$ is a 9-sun system of K_{37}.

From now on, we assume that p is an odd prime number and denote by Σ the following p-sun:

$$\Sigma = \begin{pmatrix} c_0 & c_1 & \cdots & c_{p-2} & c_{p-1} \\ \ell_0 & \ell_1 & \cdots & \ell_{p-2} & \ell_{p-1} \end{pmatrix}.$$

Lemma 6.4. *Let p be an odd prime. For any $x, y \in \mathbb{Z}_p$ with $x \neq 0$ and any $i, j \in \mathbb{Z}_m$ with $i \neq j$ there exists a p-sun S such that $\Delta_{ii} S = \pm x$, $\Delta_{ij} S = y$, $\Delta_{ji} S = -y$, and $\Delta_{hk} S = \varnothing$ for any $(h, k) \in (\mathbb{Z}_m \times \mathbb{Z}_m) \setminus \{(i, i), (i, j), (j, i)\}$.*

Proof. It is easy to see that $S = Dev_{\mathbb{Z}_p \times \{0\}}((0, i) \sim (x, i) \sim (y + x, j))$ is the required p-sun. □

We will call such a p-sun a *sun of type* (i, j). For the following it is important to note that if S is a p-sun of type (i, j), then $|\Delta_{ii} S| = 2$, $|\Delta_{jj} S| = 0$, and $|\Delta_{ij} S| = |\Delta_{ji} S| = 1$.

The following two propositions provide us p-sun systems of K_{mp+1} whenever $m \in \{3, 5\}$ and $p \equiv m - 2 \pmod 4$.

Proposition 6.5. *Let $p \equiv 1 \pmod 4 \geq 13$ be a prime. Then there exists a p-sun system of K_{3p+1}.*

Proof. We have to distinguish two cases according to the congruence of p modulo 12.

Case 1. Let $p \equiv 1 \pmod{12}$.

If $p = 13$, we construct a 13-sun system of K_{40} as follows. Let S be the following 13-sun whose vertices are labeled with elements of $(\mathbb{Z}_{13} \times \mathbb{Z}_3) \cup \{\infty\}$:

$$S = \begin{pmatrix} \infty & (2,1) & (4,2) & (8,0) & (3,1) & (6,2) & (12,0) & (11,1) & (9,2) & (5,0) & (10,1) & (7,2) & (1,0) \\ (0,2) & (4,1) & (8,1) & (3,2) & (6,0) & (12,1) & (11,2) & (9,0) & (5,1) & (10,2) & (7,0) & (1,1) & (2,2) \end{pmatrix}.$$

We have

$$\Delta_{12}S = \Delta_{21}S = \pm\{2,3,4,6\}, \quad \Delta_{02}S = \Delta_{20}S = \pm\{1,4,5,6\},$$
$$\Delta_{01}S = -\Delta_{10}S = \{-1,2,\pm3,\pm5\}, \quad \Delta_{00}S = \Delta_{22}S = \emptyset, \quad \Delta_{11}S = \pm\{2\}.$$

Now it remains to construct a set $\mathcal{T}$ of edge-disjoint 13-suns such that

$$\Delta_{12}\mathcal{T} = \Delta_{21}\mathcal{T} = \{0,\pm1,\pm5\}, \quad \Delta_{02}\mathcal{T} = \Delta_{20}\mathcal{T} = \{0,\pm2,\pm3\},$$
$$\Delta_{01}\mathcal{T} = -\Delta_{10}\mathcal{T} = \{0,1,-2,\pm4,\pm6\}, \quad \Delta_{00}\mathcal{T} = \Delta_{22}\mathcal{T} = \mathbb{Z}_{13}^*, \quad \Delta_{11}\mathcal{T} = \mathbb{Z}_{13}^* \setminus \{\pm2\}.$$

To do this it is sufficient to take, $\mathcal{T} = \{T_{01}^i \mid i \in [1,4]\} \cup \{T_{02}^i \mid i \in [1,2]\} \cup \{T_{10}^i \mid i \in [1,3]\} \cup \{T_{12}^i \mid i \in [1,2]\} \cup \{T_{20}^i \mid i \in [1,3]\} \cup \{T_{21}^i \mid i \in [1,3]\}$, where

$$T_{01}^i = Dev_{\mathbb{Z}_{13}\times\{0\}}((0,0)\sim(x_i,0)\sim(y_i+x_i,1)), \quad \text{where } x_i \in [1,4], y_i \in \pm\{4,6\},$$
$$T_{02}^i = Dev_{\mathbb{Z}_{13}\times\{0\}}((0,0)\sim(x_i,0)\sim(y_i+x_i,2)), \quad \text{where } x_i \in [5,6], y_i \in \pm\{2\},$$
$$T_{10}^i = Dev_{\mathbb{Z}_{13}\times\{0\}}((0,1)\sim(x_i,1)\sim(y_i+x_i,0)), \quad \text{where } x_i \in \{1,3,4\}, y_i \in \{0,-1,2\},$$
$$T_{12}^i = Dev_{\mathbb{Z}_{13}\times\{0\}}((0,1)\sim(x_i,1)\sim(y_i+x_i,2)), \quad \text{where } x_i \in [5,6], y_i \in \pm\{1\},$$
$$T_{20}^i = Dev_{\mathbb{Z}_{13}\times\{0\}}((0,2)\sim(x_i,2)\sim(y_i+x_i,0)), \quad \text{where } x_i \in [1,3], y_i \in \{0,\pm3\},$$
$$T_{21}^i = Dev_{\mathbb{Z}_{13}\times\{0\}}((0,2)\sim(x_i,2)\sim(y_i+x_i,1)), \quad \text{where } x_i \in [4,6], y_i \in \{0,\pm5\}.$$

We have that $\mathcal{T} \cup Orb_{\mathbb{Z}_{13}\times\{0\}}S$ is a 13-sun system of K_{40}.

Suppose now that $p \geq 37$. We proceed in a very similar way to the previous case. Let r be a primitive root of $\mathbb{Z}_p$. Consider the $((\mathbb{Z}_p \times \mathbb{Z}_3) \cup \{\infty\})$-labeling B of Σ so defined:

$$B(c_0) = \infty, \quad B(c_i) = (r^i, i) \quad \text{for } 1 \leq i \leq p-1,$$
$$B(\ell_0) = (0,2), \quad B(\ell_i) = (r^{i+1}, i+2)$$

except for $\frac{p-9}{4}$ values of $i \equiv 1 \pmod 3$ for which we set $B(\ell_i) = (r^{i+1}, i)$. Letting $S = B(\Sigma)$, it is immediate that the labels of the vertices of S are pairwise distinct. Note that

$$|\Delta_{00}S| = |\Delta_{22}S| = 0, \quad |\Delta_{11}S| = \frac{p-9}{2}, \quad |\Delta_{01}S| = |\Delta_{10}S| = \frac{5p+7}{12},$$
$$|\Delta_{ij}S| = \frac{2p-2}{3} \quad \text{for } (i,j) \in \{(0,2),(1,2),(2,0),(2,1)\}.$$

Hence, reasoning as in the previous case, we have to construct a set $\mathcal{T}$ of p-suns such that if $i \neq j$, then $\Delta_{ij}\mathcal{T} = \mathbb{Z}_p \setminus \Delta_{ij}S$ is a set and also $\Delta_{ii}\mathcal{T} = \mathbb{Z}_p^* \setminus \Delta_{ii}S$ is a set. In particular, this implies that for any $T, T' \in \mathcal{T}$ we have $\Delta_{ij}T \cap \Delta_{ij}T' = \varnothing$ and that $|\Delta_{00}\mathcal{T}| = |\Delta_{22}\mathcal{T}| = p-1$, $|\Delta_{11}\mathcal{T}| = \frac{p+7}{2}$, $|\Delta_{ij}\mathcal{T}| = \frac{p+2}{3}$ for $(i,j) \in \{(0,2),(1,2),(2,0),(2,1)\}$, and $|\Delta_{01}\mathcal{T}| = |\Delta_{10}\mathcal{T}| = \frac{7p-7}{12}$. To do this it is sufficient to take $\mathcal{T}$ as a set consisting of $\frac{p-1}{2}$ suns of type $(0,1)$, $\frac{p-1}{12}$ suns of type $(1,0)$, $\frac{p+11}{6}$ suns of type $(1,2)$, $\frac{p+2}{3}$ suns of type $(2,0)$, and $\frac{p-7}{6}$ suns of type $(2,1)$, which exist in view of Lemma 6.4. We have that $Orb_{\mathbb{Z}_p\times\{0\}}S \cup \mathcal{T}$ is a p-sun system of K_{3p+1}.

Case 2. Let $p \equiv 5 \pmod{12}$. Let r be a primitive root of $\mathbb{Z}_p$. Consider the $((\mathbb{Z}_p \times \mathbb{Z}_3) \cup \{\infty\})$-labeling B of Σ so defined:

$$B(c_0) = \infty, \quad B(c_i) = (r^i, i) \quad \text{for } 1 \le i \le p-2, \quad B(c_{p-1}) = (1,0),$$

$$B(\ell_0) = (0,2), \quad B(\ell_1) = (r,2), \quad B(\ell_i) = \begin{cases} (r^{i-1}, i+1) & \text{for } i \in \left[2, \dfrac{p-1}{2}\right], \\ (r^{i+1}, i+2) & \text{for } i \in \left[\dfrac{p+1}{2}, p-3\right], \end{cases}$$

$$B(\ell_{p-2}) = (1,1), \quad B(\ell_{p-1}) = (1,2)$$

except for $\frac{p-17}{6}$ values of $i \equiv 0 \pmod 3$ with $i \in \left[3, \frac{p-1}{2}\right]$ for which we set $B(\ell_i) = (r^{i-1}, i)$ and $\frac{p-5}{12}$ values of $i \equiv 0 \pmod 3$ with $i \in \left[\frac{p+1}{2}, p-5\right]$ for which we set $B(\ell_i) = \left(r^{i+1}, i\right)$. Letting $S = B(\Sigma)$, it is easy to see that the labels of the vertices of S are pairwise distinct. Note that

$$|\Delta_{00}S| = \frac{p-9}{2}, \quad |\Delta_{11}S| = |\Delta_{22}S| = 0, \quad |\Delta_{01}S| = |\Delta_{10}S| = \frac{p+1}{2},$$
$$|\Delta_{02}S| = |\Delta_{20}S| = \frac{7p+1}{12}, \quad |\Delta_{12}S| = |\Delta_{21}S| = \frac{2p-4}{3}.$$

Hence, we have to construct a set $\mathcal{T}$ of p-suns such that

$$|\Delta_{11}\mathcal{T}| = |\Delta_{22}\mathcal{T}| = p-1, \quad |\Delta_{00}\mathcal{T}| = \frac{p+7}{2}, \quad |\Delta_{01}\mathcal{T}| = |\Delta_{10}\mathcal{T}| = \frac{p-1}{2},$$
$$|\Delta_{02}\mathcal{T}| = |\Delta_{20}\mathcal{T}| = \frac{5p-1}{12}, \text{ and } |\Delta_{12}\mathcal{T}| = |\Delta_{21}\mathcal{T}| = \frac{p+4}{3}.$$

To do this it is sufficient to take $\mathcal{T}$ as a set consisting of $\frac{p+7}{4}$ suns of type $(0,1)$, $\frac{p-9}{4}$ suns of type $(1,0)$, $\frac{p+7}{4}$ suns of type $(1,2)$, $\frac{5p-1}{12}$ suns of type $(2,0)$, and $\frac{p-5}{12}$ suns of type $(2,1)$ which exist in view of Lemma 6.4. We have that $Orb_{\mathbb{Z}_p}S \cup \mathcal{T}$ is a p-sun system of K_{3p+1}. □

Proposition 6.6. *For any prime* $p \equiv 3 \pmod 4$ *there exists a* p*-sun system of* K_{5p+1}.

Proof. Set $p = 4n+3$, and let $Y = [1,n]$ and $X = [n+1, 2n+1]$. Consider the following $(\mathbb{Z}_p \times \mathbb{Z}_5) \cup \{\infty\}$-labeling B of Σ defined as follows:

$$B(c_0) = (0, 0), \quad B(c_i) = (-1)^{i+1}(i, 1) \quad \text{for every } i \in [1, p-1];$$
$$B(\ell_0) = \infty, \quad B(\ell_y) = (-1)^y(y, -1) \quad \text{for every } y \in Y;$$
$$B(\ell_{2n+1}) = (-2n-1, 3), \quad B(\ell_{2n+2}) = (-2n-1, -3);$$
$$B(\ell_i) = (-1)^i(i, 3) \quad \text{for every } i \in [1, p-1] \setminus (Y \cup \{2n+1, 2n+2\}).$$

One can directly check that the vertices of $S = B(\Sigma)$ are pairwise distinct. Also, it is not hard to verify that ΔS does not have repetitions and that its complement in $(\mathbb{Z}_p \times \mathbb{Z}_5) \setminus \{(0, 0)\}$ is the set

$$D = \{\pm(2x, 0) | x \in X\} \cup \{\pm(2y, 4) | y \in Y\} \cup \{\pm(0, 1)\}.$$

Clearly, D can be partitioned into $n+1$ quadruples of the form $D_x = \{\pm(2x, 0), \pm(r_x, s_x)\}$ with $x \in X$ and $s_x \neq 0$. Letting

$$S_x = Dev_{\mathbb{Z}_p \times \{0\}}((0, 0) \sim (2x, 0) \sim (r_x + 2x, s_x))$$

for $x \in X$, it is clear that $\Delta S_x = D_x$, hence $\Delta\{S_x | x \in X\} = D$. Therefore, Corollary 2.2 guarantees that $\bigcup_{x \in X} Orb_{\{0\} \times \mathbb{Z}_5}(S_x) \cup Orb_{\mathbb{Z}_p \times \mathbb{Z}_5}(S)$ is a p-sun system of K_{5p+1}. □

Example 6.7. Here, we construct a 7-sun system of K_{36} following the proof of Proposition 6.6. In this case, $Y = \{1\}$ and $X = \{2, 3\}$. Now consider the 7-sun S defined below, whose vertices lie in $(\mathbb{Z}_7 \times \mathbb{Z}_5) \cup \{\infty\}$:

$$S = \begin{pmatrix} (0, 0) & (1, 1) & -(2, 1) & (3, 1) & -(4, 1) & (5, 1) & -(6, 1) \\ \infty & -(1, -1) & (2, 3) & (-3, 3) & -(3, 3) & -(5, 3) & (6, 3) \end{pmatrix}.$$

We have

$$\Delta S = \pm\{(1, 1), (3, 2), (5, 2), (0, 2), (2, 2), (4, 2), (6, 1), (2, 0), (4, 4), (6, -2), (1, -2), (3, 4), (5, 4)\}.$$

Hence ΔS does not have repetitions and its complement in $(\mathbb{Z}_7 \times \mathbb{Z}_5) \setminus \{(0, 0)\}$ is the set

$$D = \pm\{(4, 0), (6, 0), (2, 4), (0, 1)\}.$$

Now it is sufficient to take

$$S_2 = Dev_{\mathbb{Z}_7 \times \{0\}}((0, 0) \sim (4, 0) \sim (6, 4)), \quad S_3 = Dev_{\mathbb{Z}_7 \times \{0\}}((0, 0) \sim (6, 0) \sim (6, 1)).$$

One can check that $\bigcup_{x \in X} Orb_{\{0\} \times \mathbb{Z}_5}(S_x) \cup Orb_{\mathbb{Z}_7 \times \mathbb{Z}_5} S$ is a 7-sun system of K_{36}.

We finally construct p-sun systems of K_{mp} whenever $p \equiv m \pmod 4$.

Proposition 6.8. *Let m and p be odd prime numbers with $m \leq p$ and $m \equiv p \pmod 4$. Then there exists a p-sun system of K_{mp}.*

Proof. For each pair $(r, s) \in \mathbb{Z}_p^* \times \mathbb{Z}_m$, let $B_{r,s} : V(\Sigma) \to \mathbb{Z}_p \times \mathbb{Z}_m$ be the labeling of the vertices of Σ defined as follows:

$$B_{r,s}(c_0) = (0, 0),$$

$$B_{r,s}(c_i) = B_{r,s}(c_{i-1}) + \begin{cases} (r, s) & \text{if } i \in [1, m+1] \cup \{m+3, m+5, \ldots, p-1\}, \\ (r, -s) & \text{if } i \in \{m+2, m+4, \ldots, p-2\}, \end{cases}$$

$$B_{r,s}(\ell_i) = B_{r,s}(c_i) + \begin{cases} (r, -s) & \text{if } i \in [0, m] \cup \{m+2, m+4, \ldots, p-2\}, \\ (r, s) & \text{if } i \in \{m+1, m+3, \ldots, p-1\}. \end{cases}$$

Since $B_{r,s}$ is injective, for every $h \in \mathbb{Z}_m$ the graph $S_{r,s}^h = \tau_{(0,h)}(B_{r,s}(\Sigma))$ is a p-sun. For $i, j \in \mathbb{Z}_m$, we also notice that $\Delta_{ij}\{S_{r,s}^h \mid h \in \mathbb{Z}_m\} = \{\pm r\}$ whenever $i - j = \pm s$, otherwise it is empty.

Letting $\mathcal{S}$ be the union of the following two sets of p-suns:

$$\{S_{r,1}^h \mid h \in \mathbb{Z}_m, r \in [1, (p+m-2)/4]\},$$
$$\{S_{r,s}^h \mid h \in \mathbb{Z}_m, r \in [1, (p-1)/2], s \in [2, (m-1)/2]\},$$

it is not difficult to see that for every $i, j \in \mathbb{Z}_m$

$$\Delta_{ij}\mathcal{S} = \begin{cases} \varnothing & \text{if } i = j, \\ \pm\left[1, \dfrac{p+m-2}{4}\right] & \text{if } i - j = \pm 1, \\ \mathbb{Z}_p^* & \text{otherwise}. \end{cases}$$

It is left to construct a set $\mathcal{T}$ of p-suns such that $\Delta_{ij}\mathcal{T} = \mathbb{Z}_p \setminus \Delta_{ij}\mathcal{S}$ whenever $i \neq j$, and $\Delta_{ii}\mathcal{T} = \mathbb{Z}_p^* \setminus \Delta_{ii}\mathcal{S} = \mathbb{Z}_p^*$. Therefore,

$$|\Delta_{ij}\mathcal{T}| = \begin{cases} p - 1 & \text{if } i = j, \\ \dfrac{p-m}{2} + 1 & \text{if } i - j = \pm 1, \\ 1 & \text{otherwise}. \end{cases}$$

It is enough to take $\mathcal{T}$ as a set consisting of one sun of type $(h, h+x)$ and $\frac{p-m}{2}$ suns of type $(h, h+1)$, for every $h \in \mathbb{Z}_m$ and $x \in \left[1, \frac{m-1}{2}\right]$. These p-suns exist by Lemma 6.4, therefore $\mathcal{S} \cup \mathcal{T}$ is the desired p-sun system of K_{mp}. □

Example 6.9. Let $(m, p) = (3, 11)$. Following the proof of Proposition 6.8, we construct an 11-sun system of K_{33}. For every $h \in \mathbb{Z}_3$ and $r \in [1, 3]$, let $S_{r,1}^h$ be the 11-sun defined below:

$$S_{r,1}^h = \begin{pmatrix} (0,h) & (r,h+1) & (2r,h+2) & (3r,h) & (4r,h+1) & (5r,h) & (6r,h+1) & (7r,h) & (8r,h+1) & (9r,h) & (10r,h+1) \\ (r,h+2) & (2r,h) & (3r,h+1) & (4r,h+2) & (5r,h+2) & (6r,h+2) & (7r,h+2) & (8r,h+2) & (9r,h+2) & (10r,h+2) & (0,h+2) \end{pmatrix}.$$

One can check that $\Delta_{ij}\{S_{r,1}^0, S_{r,1}^1, S_{r,1}^2\} = \{\pm r\}$ if $i \neq j$, otherwise it is empty. Therefore, letting $\mathcal{S} = \{S_{r,1}^h \mid h \in \mathbb{Z}_3, r \in [1, 3]\}$, we have that $\Delta_{ij}\mathcal{S}$ is nonempty only when $i \neq j$, in which case we have $\Delta_{ij}\mathcal{S} = \pm[1, 3]$.

Now let $\mathcal{T} = \{T_{hg} \mid h \in \mathbb{Z}_3, g \in [1, 5]\}$ where T_{hg} is the 11-sun defined as follows:

$$T_{h1} = Dev_{\mathbb{Z}_{11}\times\{0\}}((0, h) \sim (1, h) \sim (1, h + 1)),$$
$$T_{hg} = Dev_{\mathbb{Z}_{11}\times\{0\}}((0, h) \sim (g, h) \sim (9, h + 1)) \quad \text{for every } g \in [2, 5].$$

Note that each T_{hg} is an 11-sun of type $(h, h + 1)$. Therefore we have that

$$\Delta_{ij}\mathcal{T} = \begin{cases} \pm[1, 5] & \text{if } 0 \le i = j \le 2, \\ \{0\} \cup [4, 7] & \text{otherwise.} \end{cases}$$

By Corollary 2.2, it follows that $\mathcal{S} \cup \mathcal{T}$ is an 11-sun system of K_{33}.

We are now ready to show that the necessary conditions for the existence of a p-sun system of K_v are also sufficient whenever p is an odd prime. In other words, we end this section by proving Theorem 6.1.

Proof of Theorem 6.1. If $p = 3, 5$ the result can be found in [10] and in [8], respectively. For $p \ge 7$, the result follows from Propositions 6.5, 6.6, and 6.8. □

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