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Review

Robust and Distributionally Robust Shortest Path problems: A survey

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ABSTRACT

The availability of frequently updated and reliable data on traversal times of arcs in a network makes the study of non-deterministic Shortest Path problems of high importance nowadays. A large body of literature on robust and distributionally robust models is emerging, allowing reliable decisions to be taken that consider the worst-case condition. The literature differs in the assumptions made on the uncertainty of the traversal times, on the information available, and on the objective function that guides the optimization.

In this paper, we review this literature with the goal of identifying open and relevant research directions. We present robust Shortest Path and Distributionally Robust Shortest Path problems including: static, with recourse, and dynamic robust problems; absolute and relative robust problems. For each area, a description of the models and solution approaches is given, with concise excerpts of the related works. Trends and possible research directions are outlined. We review 29 papers on this subject, classifying them in terms of problem description, model characteristics and proposed solution methods.

Contents

1. Introduction

Shortest Path problems are fundamental Combinatorial Optimization problems. They are important to both academics and practitioners for several reasons. Shortest path problems arise very often in applications (see, *e.g.*, Fu et al. (2006) and Schrijver (2012)). Any satellite navigation software needs to continuously compute shortest paths from

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origins of trips to their destinations (see Schulz et al. (1994)). Also, shipments (of merchandise, of data packets, of phone calls) need to be done from one point to another of a network as efficiently as possible, which implies solving Shortest Path problems (see Chen and Nahrstedt (1998)). These problems encapsulate fundamental aspects of network flow, making them a good starting point for analyzing more complex Network Optimization models, such as routing, flow problems and line planning (see Şahin et al. (2023), Masing et al. (2022)). Finally, the basic variants of Shortest Path problems require non-trivial algorithms but can be solved efficiently (see Ahuja et al. (1993)).

Among the various application areas, the most impactful one is probably related to the Shortest Path for vehicles, cars or trucks, traveling on a road network. Nowadays, traffic-related detailed information is available and users expect a satellite navigation software to find the best path taking into account the uncertainty on the traveling time, while only the observed current traveling time is generally used. This observation implies the need of time-dependent non-deterministic models. In fact, the uncertainty in Shortest Path problems is a relevant factor in most applications.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a directed graph, where $\mathcal{N}$ is the set of *nodes* and $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of *arcs* with $a = (i, j) \in \mathcal{A}$ and $i, j \in \mathcal{N}$. A (directed) *path* in $\mathcal{G}$ is a sequence $i_0, a_1, i_1, a_2, i_2, \dots, a_L, i_L$, whose elements alternate between nodes and arcs, such that $a_\ell = (i_{\ell-1}, i_\ell)$ for all $\ell = 1, 2, \dots, L$, and all nodes are distinct, where L is the number of arcs in the path. If $\mathcal{G}$ does not contain parallel arcs, then a path is identified by the sequence $i_0, i_1, i_2, \dots, i_L$ of visited nodes. Given two distinct vertices $s, t \in \mathcal{N}$, an (s,t)*-path* is a path from s to t; the path is *simple* if all its nodes are distinct (Bondy and Murty (1976)). Let a non-negative cost $c_{ij} \geq 0$ be assigned to each arc $(i, j) \in \mathcal{A}$. Notice that c_{ij} may be either a monetary cost, or a metric distance, or a traveling time. We refer to these values as *costs* or *lengths* or *times*. The Deterministic Shortest (s,t)-Path problem seeks to find a path of minimum total cost from the source node s to the terminal node t. The problem can be formally described as follows (cf. Garey and Johnson (1979)).

Deterministic Shortest (s,t)-Path (DSP)

Input: Graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, cost $c_{ij} \geq 0$ for each $(i, j) \in \mathcal{A}$, specified nodes $s, t \in \mathcal{N}$.

Task: Find a simple path from s to t having minimum total cost.

The problem can be formulated as a binary program:

$$\min_{x \in \mathcal{X}} \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}, \tag{DSP}$$

where $\mathcal{X}$ indicates the set of (s,t)-paths as follows:

$$\mathcal{X} := \left\{ x \in \{0,1\}^{|\mathcal{A}|} \mid \sum_{\{j:(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j:(j,i)\in\mathcal{A}\}} x_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{if } i \in \mathcal{N} \setminus \{s,t\} \end{cases} \right\}.$$

The DSP problem is one of the simplest and most studied Combinatorial Optimization (CO) problems, and is a special case of the class of network flow problems with a single source and a single terminal node. An efficient time labeling algorithm for the DSP problem in general networks was given by Dijkstra (1959). In layered networks, an even simpler algorithm can be easily devised based on a standard dynamic programming procedure (see Papadimitriou and Steiglitz (1998)). We remind the reader that a graph is layered when its vertices can be partitioned into a chain of disjoint subsets, in such a way that the cardinality of each subset is limited by a given constant, called width, and arcs exist only from each subset to the following one in the chain.

Several variants of the DSP are proposed in the literature (see, *e.g.*, Turner (2011)). These problems have the same feasible set as the classic DSP, with the objective replaced by a different criterion like minimizing the largest arc cost (*bottleneck DSP*) or the difference between the largest and the smallest arc cost (*balanced DSP*). Other variants differ from each other in the type of origin–destination setting. We can distinguish:

- the *Single-Pair Shortest Path problem*, corresponding to the DSP problem (*one-to-one*);
- the *Single-Source Shortest Path problem*, where one aims to find the paths of minimum length connecting a fixed node (single source) to the remaining nodes (*one-to-all*);
- the *All-Pairs Shortest Path problem*, where one aims to find the paths of minimum length connecting every pair of nodes (*all-to-all*).

The DSP problem assumes that arc costs are known and fixed. This assumption may be reasonable in some contexts, while often the costs depend on a number of factors that are difficult to predict, such as traffic conditions, weather conditions, traffic jams, connection failures. In such cases, the DSP problem may be inadequate and must be modified to handle data uncertainty and variability over time.

The use of a deterministic model, where an "average" or a "most likely" value is used to represent an uncertain quantity, is justified in applications by the lack of data (see *e.g.* Orda and Rom (1990)). In fact, optimization models incorporating uncertainty require much more information than deterministic optimization models, especially if stochastic models are adopted and dynamic factors are explicitly considered. Moreover, such optimization models are often harder to solve than the deterministic ones. Data and computational requirements have, thus, restrained the application to real contexts of stochastic and robust shortest path models. However, both these issues are scaled down by technological advances, approximations, and decomposition techniques.

To handle data uncertainty, a classic approach is to assume a known probability distribution for each stochastic parameter. We refer to the problem based on this kind of assumption as the *Stochastic Shortest Path (SSP)* problem. This type of problem is out of the scope of this survey and we direct the interested reader to Filippi et al. (2025) for more information.

Another approach does not assume that probability distributions are known and adopts a minimax approach minimizing the cost for a worst-case (maximum cost) realization across multiple scenarios or across a predetermined uncertainty set for the random parameters. We refer to the problem based on this kind of assumption as the *Robust Shortest Path (RSP)* problem. Robust Shortest Path problems are essentially characterized by an objective function that can be of minimax cost or minimax regret type. The minimax cost objective function minimizes the maximum cost path over all realizations of arc costs while the minimax regret objective function minimizes the maximum deviation from the optimal path over all realizations of arc costs. For an introduction to general robust optimization approaches we refer the reader to Ben-Tal et al. (2009).

The RSP problem frequently acts as a foundational component in various robust Network Optimization models, showcasing its significance in addressing real-world challenges. For example, it plays a key role in the Robust Traveling Salesman problem, where the objective is to determine the most cost-effective tour that visits all nodes in the network and returns to the starting point under uncertain conditions (see, *e.g.*, Montemanni et al. (2007)). Similarly, it is a crucial element in the Robust Vehicle Routing problem and its variants, which aim to optimize delivery routes while accounting for uncertainties such as fluctuating demand, traffic conditions, and service times (see, *e.g.*, Ordóñez (2010)). Additional examples include the Robust Time Window Assignment Vehicle Routing problem (see, *e.g.*, Hoogeboom et al. (2021)), the Robust Team Orienteering problem (see, *e.g.*, Yu et al. (2022)), and the Robust Arc Routing problem (see, *e.g.*, De Maio et al. (2021)). In these contexts, the RSP problem is used to determine the paths that effectively balance reliability, risk, and efficiency.

An alternative approach is to apply Distributionally Robust Optimization (DRO) techniques to the Shortest Path Problem, which attempt to balance the lack of distributional information of robust optimization and the complete knowledge of the underlying distribution of stochastic optimization. DRO replaces the assumed-known distribution by an ambiguity set of distributions over which an optimal path is to be found in some worst-case sense. For an introduction to general DRO approaches, we refer the reader to the review of Rahimian and Mehrotra (2019) and references therein.

The purpose of this survey is to classify and organize the contributions that appear in the literature on models and algorithms for Robust and Distributionally Robust Shortest Path problems, in order to highlight the opportunities that such contributions may give to practical applications and renew the interest on the rich set of approaches that are proposed. We review results on Robust and Distributionally Robust Shortest Path that are considered standards by experts in the community, but have not been documented in textbooks yet. For each area, a description of the models and solution approaches is given, with concise excerpts of the related works. Trends and possible research directions are outlined. We review 29 papers on this subject, classifying them in terms of problem description, model characteristics and proposed solution methods. The selection of articles was based on works published in scientific journals recognized in the field of Operations Research. To identify relevant works, we used specific search strings across major scientific databases (*e.g.*, Scopus, Web of Science, and Google Scholar). The main keywords employed were: Robust Shortest Path, Relative Robust Shortest Path, Absolute Robust Shortest Path, and Distributionally Robust Shortest Path.

This review aims to serve two main audiences: academic researchers and practitioners. For academic researchers, it provides a comprehensive overview of the Robust Shortest Path problem, emphasizing open issues in the current literature. This perspective is intended to inspire new research directions and foster innovative approaches to address these gaps. For practitioners, the review focuses on presenting the existing solutions and methodologies developed in the literature. The goal is to equip them with the necessary knowledge to apply these approaches effectively to real-world problems, ensuring a smooth transition from theory to practice.

The paper is organized as follows. In Section 2, the Robust Shortest Path problem is introduced and classified in absolute or relative models, depending on how the quality of a path is measured. Differences in the approaches adopted to describe the uncertainty, via uncertainty sets or scenarios are also described through a simple example. Section 3 analyzes the literature on the static Robust Shortest Path problem, while Section 4 is focused on the dynamic version. In Section 5, the Distributionally Robust Shortest Path problem is introduced. In Section 6, we discuss the opportunities for real applications and we point out some further directions for future research.

2. Robust Shortest Path problem

Robust optimization provides a framework for addressing uncertainty without making any specific assumption on the probability distribution of the underlying uncertain parameter (which can be difficult to calibrate), based on minimizing the worst-case objective value, across multiple scenarios or over a prespecified uncertainty set, representing the set of values for the random parameters that are taken into account in the problem. Robust optimization then adopts a minimax approach that handles the uncertainty by guaranteeing the feasibility and optimality of the solution against all instances of the parameters. Robust optimization is popular because of its computational tractability for many classes of uncertainty sets and problem types. For a detailed overview of the robust optimization framework, we refer to Bertsimas and Sim (2004), Ben-Tal et al. (2009); for a survey on recent advances, we refer to Gabrel et al. (2014), and for a survey on nonlinear robust optimization to Leyffer et al. (2020). For a comparison with the stochastic optimization paradigm on a transportation problem, we refer to Maggioni et al. (2017). Formal definitions of the different problems mentioned here will be given in the following dedicated sections.

The formulation of the Robust Shortest Path (RSP) problem with random arc costs depends on whether the underlying network is static or dynamic:

- in case of *static* networks, arc costs (including uncertainty description) are fixed in time;
- in case of *dynamic* networks, the arc costs are time-dependent. One way of dealing with dynamic networks is by considering discrete time intervals with fixed costs. Thus, understanding Shortest Path algorithms in static networks becomes fundamental also to work with dynamic networks for any possible variant of the problem (earliest arrival, minimum cost with fixed starting point, etc.).

The two main optimization problems defined in the literature on robust approaches to the Shortest Path problem are:

- the *Absolute Robust Shortest Path* (ARSP) problem, defined as the problem of finding the path that minimizes the maximum path length from s to t over all realizations of arc costs;
- the *Relative Robust Shortest Path* (RRSP) problem (also called *Robust Deviation Shortest Path* problem or *Maximum Regret Shortest Path problem*), defined as the problem of finding the path that minimizes the maximum deviation from the optimal path length from s to t over all realizations of arc costs.

Other types of criteria are also discussed.

In the literature on robust optimization for Shortest Path problems, two main approaches are adopted to model uncertainty:

- The *uncertain set model* assumes a prespecified set of possible values for the uncertain quantities. Different types of uncertainty sets are proposed, such as box, polyhedral or ellipsoidal uncertainty models allowing to reformulate the problem in a tractable way. A relevant special case is the *interval data model* (or *box uncertainty model*), where an interval of possible cost values is associated with each arc. More specifically, in the interval data model, each arc $(i,j) \in \mathcal{A}$ is associated with an interval with a cost $\tilde{c}_{ij}$ taking values in $[c_{ij}, c_{ij} + d_{ij}]$, where $c_{ij} \geq 0$ is the *nominal value* for $(i,j) \in \mathcal{A}$ and $d_{ij} \geq 0$ the *deviation* from the nominal value c_{ij}. No probability distribution is assumed to be known for $\tilde{c}_{ij}$.
- The *scenario model* assumes a predetermined finite set of instances of the network condition. The scenarios may be realized with positive (but perhaps unknown) probability. More specifically, in the scenario model a finite set of scenarios $\mathcal{K}$ is considered, where each scenario corresponds to a specific cost vector: $c^k = [c^k_{ij}]_{(i,j)\in\mathcal{A}},\ k \in \mathcal{K}$. For a traveler navigating a traffic network, the travel time on each road (or arc) is influenced by various traffic scenarios, including accidents, peak-hour congestion, road conditions, and construction activities. These factors introduce uncertainties in travel times, which are inputs to the optimization model. The various scenarios may not accurately reflect the conditions of a specific day; however, due to the lack of real-time information for all roads, the traveler must rely on a suitable subset of scenarios to make a robust route choice.

The hierarchy of problems is visualized in Fig. 1. The opportunity of a recourse action, after uncertainty is revealed, is also highlighted. Dashed lines and light gray terms show the missing approaches in the current literature that we discuss later in this work.

Example

To illustrate the differences between ARSP and RRSP under the interval data and scenario models, consider the graphs in Fig. 2. For the sake of illustration we assume that the arc costs are not changing in time. We refer the reader to Section 4 for an illustration of the dynamic RRSP under interval data.

In Fig. 2(a), each arc (i,j) is valued by the corresponding interval $[c_{ij}, c_{ij}+d_{ij}]$. Three paths from s to t are possible: $\mu_0 : \{s,t\}$; $\mu_1 : \{s,1,t\}$; $\mu_2 : \{s,1,2,t\}$. Looking at the nominal values c_{ij}, the shortest path is μ_2, which has a nominal total cost equal to 3. However, in the worst

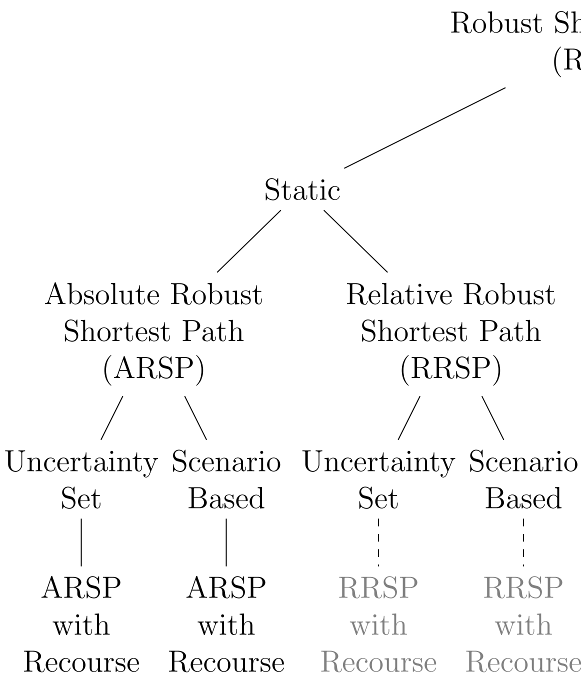

Fig. 1. A hierarchy of Robust Shortest Path (RSP) problems.

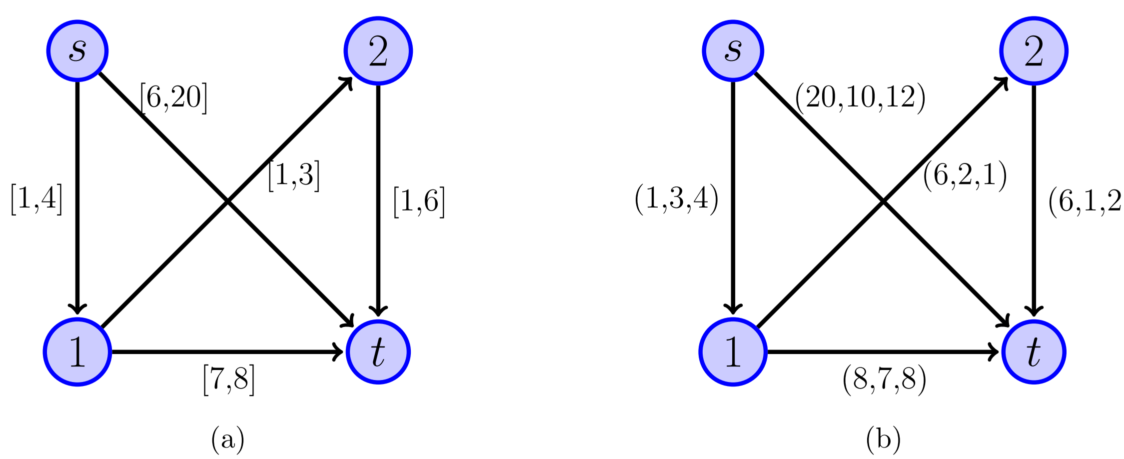

Fig. 2. Example of a RSP problem under the interval data model (a) and under the scenario model (b). In (a) intervals represents $[c_{ij}, c_{ij} + d_{ij}]$, $(i,j) \in \mathcal{A}$ while in (b) triplets represent three possible scenarios $(c^1_{ij}, c^2_{ij}, c^3_{ij})$, $(i,j) \in \mathcal{A}$.

case, *i.e.*, if all arcs assume a cost equal to the nominal value plus the maximum deviation, the total cost of μ_2 is 13. In the worst case, path μ_1 is optimal, because it guarantees a cost not greater than 12.

The ARSP with no additional restrictions can be solved as a DSP by setting all arc costs at their worst value (Karaşan et al. (2001)). However, this approach may lead to solutions that are excessively pessimistic. For this reason, the uncertainty set in ARSP problems is often constrained. For example, for the graph in Fig. 2(a), we may assume that at most two arcs can reach the maximum cost deviation, while all the other arcs remain at their nominal cost values. Under this assumption, path μ_1 still has a worst case cost of 12, while path μ_2 can cost at most 11, and becomes optimal.

The RRSP problem looks for a path that minimizes the maximum difference between its worst cost and the best cost of an alternative path. Consider again the paths on the graph in Fig. 2(a). As we know, path μ_1 has a worst-case cost of 12. If all arcs not included in μ_1 are fixed at their nominal cost, then it is possible to go from s to t at cost 6 using either μ_0 or μ_2. Thus, the regret in choosing path μ_1 is at most $12 - 6 = 6$. Applying the same argument to path μ_2 (respectively μ_0), we discover that its maximum regret is 7 (respectively 17). Hence, μ_1 is optimal for the RRSP problem.

To illustrate ARSP and RRSP problems under the scenario model, consider the graph in Fig. 2(b), where $|\mathcal{K}| = 3$. The three cost values associated with each arc (i,j) are given in their order. We observe that, as opposed to interval models, not all combinations of c_{ij} values constitute a relevant scenario. Again, three paths from s to t are possible: $\mu_0 : \{s,t\}$; $\mu_1 : \{s,1,t\}$; $\mu_2 : \{s,1,2,t\}$. Under scenario 1, the cost of path μ_2 is 1+6+6 = 13, while under scenario 2, the same path has cost 3+2+1 = 6. Considering the costs of each path under each scenario, we discover that in the worst case μ_0 costs 20, μ_1 costs 12, μ_2 costs 13. Thus, μ_1 is optimal for the ARSP problem. On the other hand, μ_1 is the minimum cost path only under scenario 1. Under scenario 2, μ_1 costs 10, whereas μ_2 costs 6. Hence, if we choose μ_1 and scenario 2 happens, then we experience a regret $10 - 6 = 4$. Similarly, under scenario 3, μ_1 costs 12, whereas μ_2 costs 7. So, if we choose μ_1 and scenario 3 happens, then we experience a regret $12 - 7 = 5$. If we choose μ_2 and scenario 1 happens, then μ_2 costs 13, whereas μ_1 costs 9. In this case, we experience a regret $13 - 9 = 4$. Since path μ_0 is never convenient,

Table 1
Classification of references for the Robust Shortest Path problem.

Reference	Objective function[a]	Uncertainty description	Time[b]	Complexity results	Solution methods			
					Exact	Type[c]	Heuristic	Type[d]
Murthy and Her, 1992	ARSP	Scenario	S	–	√	LC	–	
Yu and Yang, 1998	ARSP, RRSP	Scenario	S	√	√	DP	√	US
Karaşan et al., 2001	RRSP	Interval	S	√	√	MILP	–	
Bertsimas and Sim, 2003	ARSP	Interval Budget	S	√	√	P	–	
Averbakh and Lebedev, 2004	RRSP	Interval	S	√	–		–	
Montemanni and Gambardella, 2004	RRSP	Interval	S	–	√	B&B	–	
Montemanni et al., 2004	RRSP	Interval	S	–	√	B&B	–	
Zieliński, 2004	RRSP	Interval	S	√	–		–	
Montemanni and Gambardella, 2005	RRSP	Interval	S	–	√	B	–	
Kasperski and Zieliński, 2006	RRSP	Interval	S	√	√	PP	–	
Bruni and Guerriero, 2010	ARSP	Scenario	S	–	–		√	EF
Catanzaro et al., 2011	RRSP	Interval	S	–	√	PT	–	
Kwon et al., 2013	ARSP	Interval Budget	S	√	√	DVE, PE	–	
Xing and Zhou, 2013	ARSP, α-RSP	Scenario	S	–	–		√	LR
Shahabi et al., 2015	ARSP	Ellipsoidal	S	–	√	OA	–	
Pessoa et al., 2015	ARSP	Interval Budget	S	√	√	DP	–	
Golovin et al., 2015	ARSP	Scenario	R	–	–		√	GP
Li et al., 2015	ARSP	Interval	R	–	–		√	TP
Raith et al., 2018	MOO	Scenario	S	–	√	LC	–	
Bertsekas, 2019	ARSP	Scenario	S	√	√	DP	–	
Chassein et al., 2019	ARSP	Scenario, several approaches	S	–	√	B&B	–	
Duque and Medaglia, 2019	bw-SC	Scenario	S	–	√	PA	–	
Di Puglia Pugliese et al., 2019	ARSP	Interval Budget	S	√	√	DP	–	
Xu and Zhou, 2020	RRSP	Scenario	D	√	√	D	√	ND

[a] ARSP: Absolute Robust Shortest Path, RRSP: Relative Robust Shortest Path, α-RSP: α-Robust Shortest Path, MOO: Multi-objective optimization, bw-SC: bw-robustness criterion.
[b] S: Static, R: with recourse, D: Dynamic.
[c] LC: Label-Correcting algorithm, DP: Dynamic Programming, MILP: Mixed-Integer Linear Program, P: Polynomial algorithm, B&B: Branch-and-Bound, Benders, PP: Pseudo-Polynomial algorithm, PT: reduction procedures (Pegging Tests), DVE: Dual Variable Enumeration algorithm, PE: Path Enumeration algorithm, OA: Outer Approximation algorithm, PA: Pulse Algorithm, D: Dijkstra.
[d] US: Uniform Surrogation, EF: Evaluation Function, LR: Lagrangian Relaxation, GP: Guess and Prune, TP: Two-Phase framework, ND: Nested Dijkstra.

we conclude that μ_2 is the path that minimizes the maximum possible regret, and thus it is optimal for the RRSP problem.

The main references on the RSP problem are summarized in Table 1 and described in details in Sections 3 and 4. The table shows the main characteristics of each contribution. In the column "Objective function" the absolute (ARSP) or relative (RRSP) objective function is indicated. In one case, the α-Robust Shortest Path (α-RSP) function is used that is the α-percentile path cost among all feasible paths, to be minimized; in another case, a Multi-Objective robust Optimization (MOO) approach is considered; finally, in a case bw-robustness is used, where the number of scenarios under which the chosen path does not exceed a threshold cost b is maximized. More details are provided later. The "Uncertainty description" indicates whether an uncertainty set model (interval or ellipsoidal), a scenario model, a budget model or another approach is adopted. The column "Time" says whether the model is static (S), with recourse (R), or dynamic (D). In some papers, "Complexity results" are presented. In the last four columns it is shown whether a solution method, exact and/or heuristic, and which type is proposed.

3. Static Robust Shortest Path problem

Most robust models proposed in the literature assume that the underlying network is static, that is, not dependent on time. Two main criteria are adopted in the Absolute Robust Shortest Path (ARSP) and the Relative Robust Shortest Path (RRSP) models, as detailed below in Sections 3.1 and 3.2, respectively. Other criteria are discussed in Section 3.3.

3.1. Static absolute Robust Shortest Path problem

Let $c = [c_{ij}]_{(i,j)\in\mathcal{A}}$ denote a vector of arc costs, and let $\mathcal{C}$ be a set of possible realizations of arc cost vectors. The Absolute Robust Shortest Path (ARSP) problem may be defined as follows.

Absolute Robust Shortest Path (ARSP)

Input: Graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, set $\mathcal{C}$ of possible arc cost vector realizations, specified nodes $s, t \in \mathcal{N}$.

Task: Find a simple path from s to t minimizing the maximum total cost over all $c \in \mathcal{C}$.

The ARSP problem is tackled by defining set $\mathcal{C}$ both as an uncertainty set (see Section 3.1.1) and as a discrete set of scenarios (see Section 3.1.2). A few works consider a recourse mechanism (see Section 3.1.3).

3.1.1. Static ARSP with uncertainty sets

An interval data model is used in Bertsimas and Sim (2003) to address data uncertainty in general Combinatorial Optimization problems. The authors apply their approach to Shortest Path problems, proposing the ARSP-INT problem defined below which seeks a path that minimizes the maximum cost, given that at most Γ arcs reach the maximum cost $c_{ij} + d_{ij}$ while the others retain the minimum cost c_{ij}.

More formally, let $c = [c_{ij}]_{(i,j)\in\mathcal{A}}$ be a nonnegative vector of nominal arc cost, let $\mathcal{S}^\Gamma = \{S \subseteq \mathcal{A} : |S| \le \Gamma\}$ for some integer $0 \le \Gamma \le |\mathcal{A}|$, and let $d = [d_{ij}]_{(i,j)\in\mathcal{A}}$ be a nonnegative vector of arc cost deviations. For all $S \in \mathcal{S}^\Gamma$, let $d^S = [d^S_{ij}]_{(i,j)\in\mathcal{A}}$ be defined as

$$d^S_{ij} = \begin{cases} d_{ij} & \text{if } (i,j) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the ARSP-INT problem is the ARSP problem where $\mathcal{C} = \{\tilde{c} : \tilde{c} = c + d^S,\ S \in \mathcal{S}^\Gamma\}$. The ARSP-INT problem can be formulated as:

$$\min_{x\in\mathcal{X}} \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + \max_{S\in\mathcal{S}^\Gamma} \sum_{(i,j)\in S} d_{ij}x_{ij}. \tag{ARSP-INT}$$

If $\Gamma = 0$ then the "max" in the objective function is void. If $\Gamma = |\mathcal{A}|$, then all arcs can deviate from their nominal value and the "max" in the objective function boils down to $\sum_{(i,j)\in\mathcal{A}} d_{ij}x_{ij}$. In both cases, we are led to a DSP problem. Hence, the interesting cases are those with $0 < \Gamma < |\mathcal{A}|$.

The ARSP-INT problem is extended in Pessoa et al. (2015) and Di Puglia Pugliese et al. (2019) by introducing resource constraints, which can be capacity constraints or time windows. More precisely, Pessoa et al. (2015) consider two problem variants: in the first one, denoted by $\mathcal{U}$-CSP, the authors assume that a resource consumption is associated with traveling along an arc, and that the total amount of resource consumption along a path is constrained; in the second one, denoted by $\mathcal{U}$-TWSP, the authors assume that a time window is associated with each node. Pessoa et al. (2015) show that $\mathcal{U}$-CSP can be solved in pseudo-polynomial time, while $\mathcal{U}$-TWSP is strongly NP-hard. In their notation, $\mathcal{U}$ is the uncertainty set that in the ARSP-INT formulation is implicitly defined by the collection of arc subsets S^{Γ}.

A generalization of ARSP-INT is considered in Kwon et al. (2013), where the authors consider an uncertainty set $\mathcal{C}$ where, for all $(i,j) \in \mathcal{A}$, the cost coefficient $\tilde{c}_{ij}$ is given by the product of two uncertain factors $\tilde{p}_{ij}$ and $\tilde{q}_{ij}$ (*i.e.*, $\tilde{c}_{ij} = \tilde{p}_{ij}\tilde{q}_{ij}$), both defined in budgeted box-constrained uncertainty sets as follows:

$$\tilde{p}_{ij} = p_{ij} + r_{ij}u_{ij},$$
$$\tilde{q}_{ij} = q_{ij} + t_{ij}v_{ij},$$

where p_{ij}, q_{ij}, r_{ij}, and t_{ij} are all nonnegative,

$$u_{ij} \in U = \left\{ u \in \mathbb{R}^{|\mathcal{A}|} : 0 \le u_{ij} \le 1 \quad \forall (i,j), \sum_{(i,j)} u_{ij} \le \Gamma_u \right\},$$

$$v_{ij} \in V = \left\{ v \in \mathbb{R}^{|\mathcal{A}|} : 0 \le v_{ij} \le 1 \quad \forall (i,j), \sum_{(i,j)} v_{ij} \le \Gamma_v \right\},$$

and Γ_u and Γ_v are positive integers. The uncertainty set $\mathcal{C}$ studied by Kwon et al. (2013) leads to a Disjoint Bilinear Program (DBP), but the authors show that the problem can be reformulated as a Mixed Integer Linear Program (MILP).

A different description of the uncertainty set $\mathcal{C}$ is proposed in Shahabi et al. (2015), where the uncertain cost on arc $(i,j) \in \mathcal{A}$ is modeled as

$$\tilde{c}_{ij} = \bar{c}_{ij} + \sum_{k=1}^{K} b_{ijk} y_k,$$

where $\bar{c}_{ij}$ is the average of $\tilde{c}_{ij}$ and the uncertainty term is defined as the combination of a number of independent variables y_k with an associated arc-dependent weight b_{ijk}. Each variable represents one source of uncertainty affecting the arc cost. The random variables y_k are bounded by an ellipsoidal uncertainty set $U = \{y = (y_1, \dots, y_K) : \|y\| \le \Omega\}$ (see Bertsimas and Sim (2004)), where Ω is referred to as the *budget of uncertainty* and controls the degree of conservatism of the solution. Note that if $\Omega = 0$, the problem is reduced to the classic DSP. In Shahabi et al. (2015), an all-pairs Robust Shortest Path problem is formulated, where a path must be identified between every pair of distinct vertices. Specialized to the case of a distinguished (s,t)-path, the model can be written as follows:

$$\min_{x \in \mathcal{X}} \quad \sum_{(i,j)\in\mathcal{A}} \bar{c}_{ij} x_{ij} + \max_{\{y : \|y\| \le \Omega\}} \sum_{(i,j)\in\mathcal{A}} \left(\sum_{k=1}^{K} b_{ijk} y_k \right) x_{ij}. \qquad \text{(ARSP-INT*)}$$

The above model is a Binary Nonlinear Integer Program (BNIP), which can be reformulated as a Mixed Integer Conic Quadratic Program (MICQP).

A distinct approach is used by Chassein et al. (2019), who follow a data-driven approach. They use real-world traffic measurements provided by the City of Chicago to identify uncertainty sets that describe at best the detected uncertainty. The authors define the uncertainty set $\mathcal{C}$ by considering convex hull uncertainty, interval uncertainty (as in ARSP-INT), ellipsoidal uncertainty, budgeted uncertainty and permutohull uncertainty containing all relevant scenarios for arc costs. They find that permutohull and ellipsoidal uncertainty tends to produce solutions with the best trade-off, while being computationally more challenging than most of the other approaches.

Solution methods

All the aforementioned papers addressing the ARSP-INT problem proposed exact solution methods. Specifically, Bertsimas and Sim (2003) show that the robust counterpart of the Shortest Path problem under interval uncertainty is polynomially solvable. As a specialized version of an algorithmic approach valid for general robust Combinatorial Optimization problems, the authors propose an exact algorithm that solves the ARSP-INT problem by solving a sequence of $|\mathcal{A}|+1$ instances of the DSP problem with a modified objective function. Assuming that the arcs $(i,j) \in \mathcal{A}$ are ordered by nonincreasing value of the deviation d_{ij}, the ℓ-th instance in the sequence corresponds to a DSP problem where only the first ℓ arcs can deviate from their nominal cost. The solution of the problem in the sequence with minimum objective function value is an optimal path for the ARSP-INT problem.

To solve the MILP formulation of their problem, Kwon et al. (2013) propose two exact approaches. The first approach is a dual variable enumeration algorithm that requires the solution of a polynomial number of DSP problems. More specifically, the method enumerates the extreme points in a polyhedral partition of the bidimensional space of the dual variables associated with parameters Γ_u and Γ_v. For each extreme point, a properly defined DSP problem is solved. The total number of DSP problems to be solved turns out to be $|\mathcal{A}|^2+3|\mathcal{A}|+1$. The second approach is a path enumeration algorithm that chooses among a set of candidates selected using any K-shortest path algorithm (see, *e.g.*, Yen (1971)). Since the number K of paths to be selected can be exponential, the second approach is not polynomial. However, for low levels of uncertainty, it can be faster than the first approach.

Pessoa et al. (2015) extend a label setting algorithm for the DSP problem with an elementary resource constraint (see, *e.g.*, Boland et al. (2006)) to a robust setting. More specifically, they observe that the $\mathcal{U}$-TWSP problem can be formulated as a deterministic problem with $|\mathcal{U}|$ time windows constraints. By exploiting a compact description of the labels and dominance rules, the authors are able to reduce the computational complexity of the algorithm derived from the initial observation. The same type of approach can be developed for the $\mathcal{U}$-CSP problem. The running times of the obtained algorithms are pseudopolynomial when the budget parameter Γ is fixed. Considering time windows and capacity constraints as limited resources, Di Puglia Pugliese et al. (2019) solve the resource constrained Shortest Path problem with uncertain data to optimality through a three-phase approach dealing with bounds computation, network reduction, and gap closing. In particular, they compute robust bounds on the resource consumption and cost by solving the robust Shortest Path problem and the dual robust Lagrangian relaxation, respectively. Dynamic programming is used to close the duality gap. Upper and lower bounds are used to reduce the dimension of the network and incorporated in the dynamic programming in order to fathom unpromising states.

Shahabi et al. (2015) solve their reformulation of ARSP-INT* as a MICQP by adapting the Outer Approximation (OA) algorithm proposed by Duran and Grossmann (1986) for mixed-integer nonlinear programs where the nonlinear functions involving continuous variables are convex. As the general OA algorithm, the method proposed by Shahabi et al. (2015) converges to an optimal solution in a finite number of iterations. Numerical experiments on instances derived from publicly available road networks reveal the practical efficiency of the proposed method.

Since the experiments conducted on real data by Chassein et al. (2019) showed that ellipsoidal uncertainty sets are a reasonable choice, the authors propose a Branch-and-Bound method for this case. The method is based on the observation that a Mixed Integer Second Order Cone Programming (MISOCP) formulation of the problem can be linearized and then handled as a biobjective problem. Their method outperforms by several orders of magnitude a commercial solver applied to the original MISOCP formulation.

3.1.2. Static ARSP problem with scenarios

Some contributions have also appeared that adopt the scenario model to tackle the ARSP problem. In this case, the problem aims to find the path that minimizes the maximum path length from s to t over all scenarios. A set of known scenarios $\mathcal{K}$ is introduced. The cost of arc (i,j) under scenario $k \in \mathcal{K}$ is denoted by c_{ij}^k. The ARSP-SC is then an ARSP problem where $C = \{c^k = [c_{ij}^k]_{(i,j)\in\mathcal{A}} \,:\, k \in \mathcal{K}\}$. It can be formulated as

$$\min_{x\in\mathcal{X}} \max_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} c_{ij}^k x_{ij}, \tag{ARSP-SC}$$

and is studied in Murthy and Her (1992), Yu and Yang (1998), Bruni and Guerriero (2010), and Xing and Zhou (2013).

In Yu and Yang (1998), it is shown that the ARSP-SC problem is NP-hard even for the much more restricted layered networks of width two, and with only two scenarios. In the case of unlimited number of scenarios, the problem is strongly NP-hard and the worst-case performance is analyzed, providing a finite tight bound depending on the number of scenarios. In Xing and Zhou (2013), the focus is on the efficient generation of approximate solutions when a finite number of link-based travel-time samples (*e.g.*, from different days) are used to describe the travel-time distribution.

Notice that the ARSP approach tends to emphasize the extreme tail of the travel-time distribution, which may be unlikely to occur. For this reason, in Xing and Zhou (2013) an α-Robust Shortest Path (α-RSP) problem is also considered. This problem aims to minimize the α-percentile path travel time among all feasible paths. The α-RSP problem can meet the needs for travelers with different degrees of risk-avoidance preference.

Solution methods

Exact methods for the ARSP-SC problem are proposed in Murthy and Her (1992) and Yu and Yang (1998). In Murthy and Her (1992), a label-correcting procedure is developed to obtain exact solutions for the ARSP-SC problem. In this context, a label associated with a node i is a vector giving, for each scenario, the cost of a path leading to i from s. Moreover, a number of labels are usually associated with the same node i, corresponding to different paths from s to i. Hence, rules for pruning dominated labels and selecting the next label to correct are crucial for the efficiency of the method. In particular, Murthy and Her (1992) propose two pruning techniques, based on two different relaxations of the ARSP-SC problem where the passage through a given node i is forced. Computational tests on grid networks prove that, in hard instances, the proposed pruning techniques are able to speed up the label-correcting procedure by several orders of magnitude.

Bruni and Guerriero (2010) enhanced the exact approach proposed by Murthy and Her (1992) by using heuristic evaluation functions to better guide the solution search procedure on the most promising labels to correct. Essentially, Bruni and Guerriero (2010) maintain for every label the corresponding Lorenz vector, *i.e.*, the vector obtained by first sorting the entries of the label in nonincreasing order and then considering, as ℓ-th entry, the sum of the first ℓ entries of the sorted label (see, *e.g.*, Marshall and Olkin (1979)). At every iteration, Bruni and Guerriero (2010) choose the label with the lexicographically smallest Lorenz vector. In most cases within a large set of randomly generated instances, the proposed selection rule significantly speeds up the method.

In Yu and Yang (1998), a pseudo-polynomial algorithm based on a dynamic programming approach is devised to solve the ARSP-SC problem under a bounded number of scenarios (recall that the problem is strongly NP-hard for an unbounded number of scenarios). The complexity of the algorithm is improved for the case of layered networks.

Heuristic solution methods have also been proposed for the ARSP-SC problem by Yu and Yang (1998) and Xing and Zhou (2013). Yu and Yang (1998) provide a simple heuristic for finding a good robust shortest path based on uniform surrogation for the ARSP-SC problem. The algorithm solves a DSP problem where the cost of each arc is simply the average cost under all scenarios and returns the corresponding path. The authors provide a bound for the performance ratio of the algorithm.

Xing and Zhou (2013) combine a Lagrangian relaxation approach with a scenario-based representation scheme to reformulate the ARSP-SC and the α-RSP. In both cases, a subgradient algorithm is proposed where the original problem is decomposed into several subproblems, corresponding to standard DSP problems or univariate Linear Programming (LP) problems to be solved in sequence. Extensive numerical experiments using real-world data demonstrate the efficiency of the proposed algorithms.

3.1.3. Static ARSP problem with recourse

A two-phase framework for the ARSP problem is considered in Golovin et al. (2015) and Li et al. (2015), where the goal is to build a two-stage solution to minimize the worst-case cost, where in the first stage the cost is lower but the constraints are uncertain, and in the second stage the constraints are known but the cost is higher. The problem is classified as static since the arc costs are fixed in time. However, unlike the previous cases, the route is enhanced as it is adjusted to the actual network conditions through a recourse action. The problem is tackled by assuming both interval data models and scenario models.

The problem studied in Golovin et al. (2015) can be summarized as follows: we are given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with edge costs $c_f : \mathcal{E} \to \mathbb{R}_+$, a root vertex s, and a list $\mathcal{K}$ of scenarios, only one of which is realized in the second-stage. Each scenario $k \in \mathcal{K}$ specifies the terminal t_k that must be connected to s if that scenario is realized, and an inflation factor β_k for edge costs. The goal is to build a two-stage solution, *i.e.*, to select a set of edges $\mathcal{E}_f$ in the first stage and, for each scenario, a set of recourse edges $\mathcal{E}_s^k$ in the second-stage such that $\mathcal{E}_f \cup \mathcal{E}_s^k$ connects s and t_k for all scenarios $k \in \mathcal{K}$ as follows:

$$\begin{aligned} \min_{\mathcal{E}_f, \mathcal{E}_s^k \subseteq \mathcal{E},\ k\in\mathcal{K}} \quad & c_f(\mathcal{E}_f) + \max_{k\in\mathcal{K}} \beta_k \cdot c_f(\mathcal{E}_s^k) \\ \text{subject to} \quad & \mathcal{E}_f \cup \mathcal{E}_s^k \text{ connects } s \text{ and } t_k\,, \end{aligned} \tag{ARSP-2-S}$$

where for all $\mathcal{E}' \subseteq \mathcal{E}$, $c_f(\mathcal{E}') = \sum_{e\in\mathcal{E}'} c_f(e)$.

Model ARSP-2-S proposed in Golovin et al. (2015) guesses the worst-case second stage cost and, based on the guess, selects a subset of "costly" scenarios for the first stage solution to address. The second stage solution for any scenario is the DSP problem in the residual graph. The ARSP-2-S is NP-hard, as it contains the Steiner tree problem when all inflation factors are infinite. The authors show that there is a near-optimal first stage solution that completely satisfies the subset of costly scenarios selected by the procedure.

Considering a transit itinerary planning problem for public transport, Li et al. (2015) incorporate in their model the randomness that arises in transit vehicle arrival/departure and passenger transfer. The authors adopt a minimax robust approach inside a two-phase framework with uncertainty, described through an interval data approach.

Solution methods

In Golovin et al. (2015) a 1.39-approximation algorithm for the two-stage robust Shortest Path is presented based on a guess-and-prune strategy that exploits the nature of the robust objective. More specifically, the algorithm guesses the worst-case second-stage cost and selects a subset of terminals to connect to the root s in an approximate first-stage solution. This guess produces an approximated Steiner tree $\mathcal{E}_f$. The second-stage solution for any scenario k is simply a shortest path $\mathcal{E}_s^k$ from terminal t_k to the first-stage tree.

The contribution of Li et al. (2015) is much more application-oriented. In their approach, the second stage involves querying one or more itineraries in a transit network to connect a specific origin to a specific destination. This query is addressed by solving a MILP model using a commercial solver for a particular formulation of the ARSP problem. To ensure the second stage problem is manageable in real time, a first stage is implemented where a set of relevant itineraries is selected for each possible origin–destination pair. This extensive set of itineraries is stored in a database, and the appropriate pre-computed itineraries are retrieved whenever the second stage problem needs to be solved.

3.2. Static relative Robust Shortest Path problem

The *Relative Robust Shortest Path* problem (also called *Robust Deviation Shortest Path* problem or *Maximum Regret Shortest Path problem*) is defined as the problem of finding the path that minimizes the maximum deviation from the optimal path length from s to t over all the realizations of arc costs.

The *regret* of a path μ from s to t under a cost realization $\tilde{c}$ is the difference between the cost of μ with respect to $\tilde{c}$ and the cost of a minimum path from s to t with respect to $\tilde{c}$. Given this notion, the Relative Robust Shortest Path (RRSP) problem can be formally defined as follows.

Relative Robust Shortest Path (RRSP)

Input: Graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, set C of possible arc cost vector realizations, specified nodes $s, t \in \mathcal{N}$.

Task: Find a simple path from s to t minimizing the maximum regret over all $c \in C$.

The RRSP problem is tackled by assuming both interval data models and scenario models.

3.2.1. Static RRSP problem with interval data

The RRSP problem with interval data is considered in Karaşan et al. (2001), Averbakh and Lebedev (2004), Montemanni and Gambardella (2004), Montemanni et al. (2004), Zieliński (2004), Montemanni and Gambardella (2005), Kasperski and Zieliński (2006), Catanzaro et al. (2011).

Let $c = [c_{ij}]_{(i,j)\in\mathcal{A}}$ be a nonnegative vector of nominal arc cost and let $d = [d_{ij}]_{(i,j)\in\mathcal{A}}$ be a nonnegative vector of arc cost deviations. The RRSP with Interval data (RRSP-INT) problem is the RRSP problem where $C = \{\tilde{c} :\ c \le \tilde{c} \le c + d\}$. In this context, Karaşan et al. (2001) observe that given a path μ from s to t, a scenario which makes the robust deviation maximum is the one where each arc (i,j) on μ has a cost $c_{ij} + d_{ij}$ and each arc (i,j) not in μ has a cost c_{ij}.

Applying this observation, Karaşan et al. (2001), Montemanni and Gambardella (2004), Montemanni et al. (2004), Montemanni and Gambardella (2005) describe the RRSP problem through a MILP formulation with interval data as follows:

$$\min \sum_{(i,j)\in\mathcal{A}} (c_{ij} + d_{ij})x_{ij} - y_t \qquad \text{(RRSP-INT)}$$

$$\begin{aligned} \text{subject to}\quad & x \in \mathcal{X} \\ & y_j \le y_i + c_{ij} + d_{ij}x_{ij},\ \forall (i,j) \in \mathcal{A} \\ & y_s = 0 \\ & y_j \ge 0,\ \forall j \in \mathcal{N} \cup \{s,t\}. \end{aligned}$$

In the above model, x_{ij} are indicator variables encoding if an arc (i,j) is in a robust shortest path ($x_{ij} = 1$) or not ($x_{ij} = 0$). Variable y_j contains the cost of a shortest path from node s to node j under the assumption that the cost of an arc (i,j) is equal to $c_{ij} + d_{ij}$ if $x_{ij} = 1$, equal to c_{ij} if $x_{ij} = 0$. The first constraint ensures that the resulting x vector defines a path in the graph. The second set of constraints specify shortest distances between nodes based on whether arcs (x variables) are on the path or not while the third constraint sets to 0 the distance to the source node s and prevents an unbounded solution. The last constraint defines the domain of the y variables.

The objective is to find a path μ for which the difference between the length of path μ and the length of the shortest path in the graph is the smallest when the lengths of all arcs on path μ are at their maximum value and the lengths of all other arcs are at their minimum (nominal) value. Notice that the problem defined by the y variables is the dual of a classic Shortest Path problem formulation, where distances on arcs are defined by the x variables as explained before.

Catanzaro et al. (2011) extend the problem considered by Karaşan et al. (2001), assuming that cycles may occur and that arc lengths may be known. They provide sufficient conditions for a node and an arc to be always or never in an optimal solution.

The computational complexity of problem RRSP with interval data is studied in Zieliński (2004), Averbakh and Lebedev (2004), Kasperski and Zieliński (2006). In particular, Zieliński (2004) prove that the problem is NP-hard even when a graph is restricted to be directed, acyclic, and planar with vertex degrees at most three. This result is also valid if the graph is undirected. At the same time, Averbakh and Lebedev (2004) prove that the problem is strongly NP-hard for a general directed (also for undirected) graph even if all the intervals of uncertainty are equal to $[0,1]$. Kasperski and Zieliński (2006) study the computational complexity of the robust Shortest Path Problem in edge series-parallel multidigraphs with interval costs and they show that it is still NP-hard.

Solution methods

Exact solution methods for the RRSP problem are addressed in Karaşan et al. (2001), Montemanni and Gambardella (2004), Montemanni et al. (2004), Montemanni and Gambardella (2005), Kasperski and Zieliński (2006) and Catanzaro et al. (2011).

In particular, Karaşan et al. (2001) define and identify paths which perform satisfactorily under any likely input data and give a MILP model to find them. They classify arcs based on the realization of the input data and identify those arcs which are never on shortest paths. Finally, they propose to solve the RRSP-INT problem by a standard MILP software after a preprocessing phase where arcs that are never part of an optimal path are removed. Computational results show that this approach can significantly reduce the solution time. Unfortunately, as pointed out by Montemanni and Gambardella (2004), this approach works only for acyclic and layered networks. Following the same line of thought, Catanzaro et al. (2011) provide sufficient conditions for a node and an arc to be always or never in an optimal solution of the RRSP-INT. Such results are valid for general graphs, and the authors exploit them to develop "pegging tests" to fix the value of variables to either 0 or 1. Similarly to Karaşan et al. (2001), they use their tests to reduce the size of the MILP formulation of the problem. As a result, on a large set of benchmark instances, the overall running time necessary to a commercial solver to solve the problem is vastly reduced.

Montemanni et al. (2004) provide a branch-and-bound algorithm for the RRSP-INT problem embedding a special lower bound and some reduction rules. The algorithm is a refinement of the method presented by Montemanni and Gambardella (2004), based on the conjecture that, on a given scenario, a path ranked in terms of cost is also well ranked in terms of robust deviation. Exploiting this approach, it is possible to notice that if we examine the paths from s to t by non-decreasing values of cost, we are able to provide at each iteration a lower bound for the robustness cost of the paths from s to t not yet examined.

Montemanni and Gambardella (2005) apply a Benders decomposition to a reformulation of the RRSP-INT based on the dualization of the LP subproblem obtained when the value of the x variables is fixed. As the size of the master problem tends to become too large, the authors adapt an acceleration technique by McDaniel and Devine (1977) for general MILP models. On a large set of benchmark instances, the suggested method outperforms the exact methods previously proposed.

For a series-parallel multidigraph $\mathcal{G}$, Kasperski and Zieliński (2006) propose a pseudopolynomial algorithm for solving the RRSP-INT problem, based on the exploration of the binary decomposition tree of $\mathcal{G}$ (see Valdes et al. (1979)). No computational tests are available.

3.2.2. Static RRSP problem with scenarios

Some contributions have also appeared that adopt the scenario model to tackle the RRSP problem. As before, a set of known scenarios $\mathcal{K}$ is introduced. For each arc $(i,j) \in \mathcal{A}$, the cost of (i,j) under scenario $k \in \mathcal{K}$ is denoted by c_{ij}^k. The RRSP problem with Scenarios (RRSP-SC) is then the RSPP where $C = \{c^k = [c_{ij}^k]_{(i,j)\in\mathcal{A}} : k \in \mathcal{K}\}$. The RRSP-SC problem can be formulated as:

$$\min_{x\in\mathcal{X}} \max_{k\in\mathcal{K}} \left\{ \sum_{(i,j)\in\mathcal{A}} c_{ij}^k x_{ij} - z^k \right\}, \tag{RRSP-SC}$$

where $z^k = \min_{x'\in\mathcal{X}} \sum_{(i,j)\in\mathcal{A}} c_{ij}^k x'_{ij}$. The RRSP-SC problem is considered in Yu and Yang (1998), where it is shown that the RRSP-SC problem is NP-hard even for layered networks of width two, and with only two scenarios. It is also shown that in the case of unbounded number of scenarios, the problem is strongly NP-hard even for layered networks.

The authors provide exact solution methods for the RRSP-SC problem, where a pseudo-polynomial algorithm is devised for general networks and for layered networks under a bounded number of scenarios. Besides, they provide a simple heuristic for finding a good robust shortest path, useful when the number of scenarios is large. They also show a worst-case bound on the heuristic performance.

3.3. Other approaches

While the primary focus of this section has been on reviewing the static Absolute Robust Shortest Path (ARSP) problem and the Relative Robust Shortest Path (RRSP) problem, other researchers have explored a variety of approaches to address similar challenges in other Robust Shortest Path problems. Below, we review some contributions that highlight diverse strategies, ranging from algorithmic extensions to game-theoretic formulations, applied to static Robust Shortest Path problems.

Raith et al. (2018) consider multi-objective Shortest Path problems in which the edge lengths are uncertain. Five different concepts of robust efficiency are considered such as *multi-scenario efficient*, if it cannot be improved in one scenario without worsening it in another scenario, *flimsily* (respectively *highly*) *robust efficient*, if it is efficient for at least one scenario (respectively for all scenarios), *point-based* and *set-based robust efficient*. These concepts generalize the single-objective concept of minimax robustness. The authors investigate whether and how a generic label correcting algorithm for the multi-objective Shortest Path problem can be extended to find robust efficient solutions for the multi-objective uncertain Shortest Path problem. In particular, an extended and repeated labeling algorithm is presented for each one of the robust efficiency concepts mentioned above.

Duque and Medaglia (2019) adopt the *bw-robustness* (bw-SC) criterion proposed by Gabrel et al. (2013) for robust Shortest Path problems and given as follows:

$$\begin{aligned} \max \quad & \sum_{k\in\mathcal{K}} y^k && \text{(bw-SC)} \\ \text{subject to} \quad & x \in \mathcal{X} \\ & \sum_{(i,j)\in\mathcal{A}} c_{ij}^k x_{ij} \le w(1-y^k) + b y^k \quad \forall k \in \mathcal{K} \\ & y^k \in \{0,1\} \quad \forall k \in \mathcal{K}, \end{aligned}$$

where parameter b represents a desirable upper bound on the cost that the decision maker wants for most of the scenarios, while parameter w (with $w > b$) strictly bounds the cost and represents a value that the decision maker is not willing to exceed in any scenario; y^k is a binary variable that takes value 1 if the cost of the path is less than or equal to b in scenario $k \in \mathcal{K}$, and takes value 0 otherwise. The objective function aims to maximize the number of scenarios in which the cost of the robust path is below the target threshold b. To solve this problem, the authors extend the *pulse framework*, a solution strategy proposed for Shortest Path problems with side constraints (see Lozano and Medaglia (2013)). The underlying algorithm follows a depth-first search recursive exploration of the graph. To avoid a complete enumeration of all paths in the graph, the algorithm uses pruning strategies to discard, at early stages of the execution, partial paths that would not lead to a feasible or an improved solution.

Bertsekas (2019) analyzes a Robust Shortest Path problem formulated as a sequential game where, at a given node, we may choose one subset out of a given collection of subsets of nodes, and the successor node on the path is chosen from this subset by an antagonistic opponent. As our goal is to reach the destination t at minimum cost, while the goal of the opponent is to maximize our cost, the problem consists in determining an optimal policy under a minimax criterion. Bertsekas (2019) analyzes the problem using the theory of abstract semicontractive dynamic programming models introduced in Bertsekas (2013). The author proves existence and uniqueness of solution of Bellman's equation for the considered problem, and obtains conditions for optimality of a policy. Moreover, he proposes a finitely terminating Dijkstra-like algorithm for problems with nonnegative arc lengths.

4. Dynamic Robust Shortest Path problem

In the context of urban transportation networks, travel times and travel costs can be both dynamic and uncertain. Dynamic refers to the systematic variability in mean travel times as a function of time-of-day, while uncertain refers to the inherent random fluctuations due to unpredictable factors (*e.g.*, incidents, car accidents, weather conditions). The uncertain and time-dependent (dynamic) Shortest Path problem, where link travel times are time-dependent random variables, provides a natural way to capture both the dynamic and uncertain factors. In this class of problems, a further distinction concerns the information available on the realization of travel time. In problems with online information, travelers have knowledge of the realizations of all link travel times up to the current time period. In problems without online information, travelers have no knowledge of any of the realizations and the only knowledge they have about the current state is the current node and current time. Routing in an uncertain time-dependent network can take one of two forms: a path prescription or a routing policy. A path is a pre-specified set of concatenated links. Travelers following a path make decisions a priori and travel on a fixed set of links, regardless of the network conditions revealed during the trip. In contrast, travelers following a routing policy make decisions en route, depending on actual network conditions. The corresponding strategies can also be called *a priori path based* and *adaptive strategy based* formulations, respectively. Adaptive strategies obviously lead to better solutions. Most of the literature in the dynamic Shortest Path problem under uncertainty assumes to know probability distributions of the random parameters. We refer to Filippi et al. (2025) for a detailed analysis of the dynamic stochastic Shortest Path problem. However, if improper distribution functions are used, a significant loss may occur from a misleading decision, especially in emergency situations. In this case, modeling the arc costs by uncertainty sets or scenarios, and making decision in the worst-case scenario, can be considered a more reliable strategy.

4.1. Dynamic relative Robust Shortest Path problem

The *Dynamic Relative Robust Shortest Path* problem dynamically searches among all scenarios the path with minimized maximum deviation from the optimal shortest path. A visitor starts at node s, that is, $i_0 = s$. At the first stage, it optimizes the path from i_0 to t, and visits the node i_1 in the optimized path directly connected to i_0. At the second stage, it optimizes the path from i_1 to t, and visits the node i_2 in the optimized path directly connected to i_1. In general, at the ℓ-th stage, $\ell = 1, 2, \dots$, the visitor optimizes the path from $i_{\ell-1}$ to t, and visits the node i_ℓ in the optimized path directly connected to $i_{\ell-1}$. Finally, suppose at the Lth stage, the visitor reaches the destination node t, then $i_L = t$. So the whole path from s to t for this visitor is $i_0, i_1, \dots, i_\ell, \dots, i_L$.

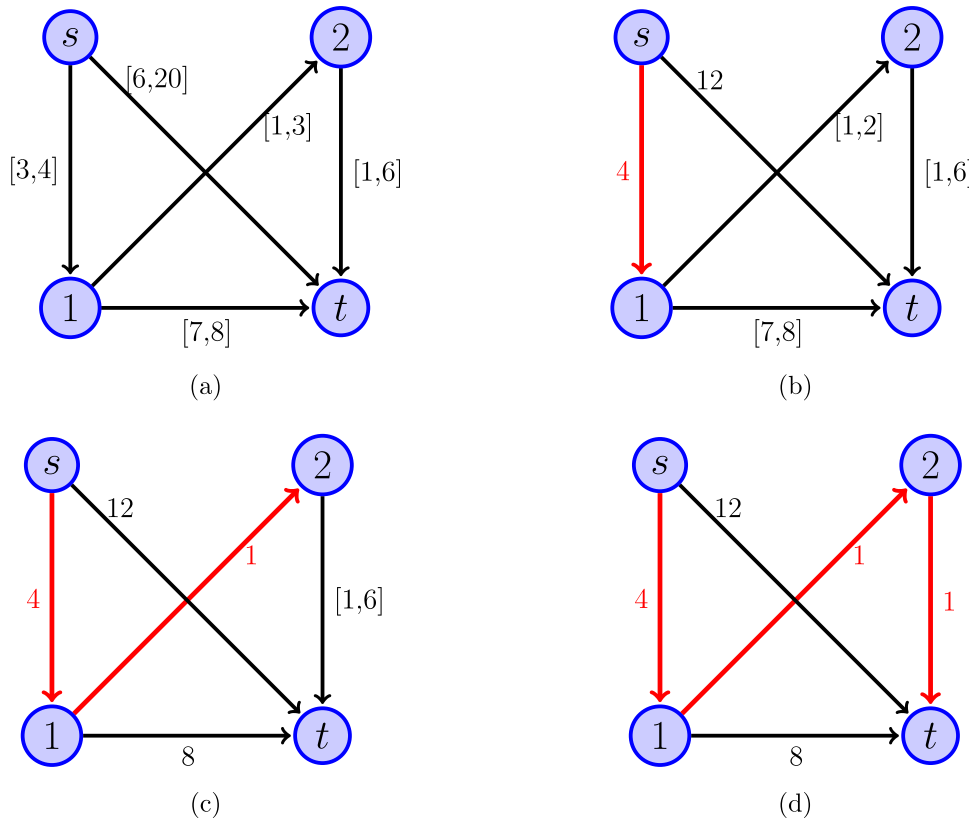

Fig. 3. Example of a graph with interval data to show the advantage of the Dynamic Relative Robust Shortest Path (DRRSP) approach with respect to the Static Relative Robust Shortest Path (RRSP-INT) one.

Table 2
Maximum regrets, associated with paths and visitor location on the graph represented in Fig. 3 and final cost of each path. Bold characters highlight the smallest value per column.

Path	Maximum Regret at:				Final Cost
	Node s	Node 1	Node 2	Node t	
$\mu_0 : \{s,t\}$	15	6	6	6	12
$\mu_1 : \{s,1,t\}$	**6**	6	6	6	12
$\mu_2 : \{s,1,2,t\}$	7	**0**	**0**	**0**	**6**

Example

An example illustrating the advantages of the dynamic approach for the case of interval data is presented in Fig. 3 (cf. Xu and Zhou (2020)). The maximum regret for each path is reported in Table 2. When the visitor is in the source node s (Fig. 3(a)), path $\mu_1 : \{s,1,t\}$ has the minimized maximum regret of $6 = 4+8-6$. When the visitor reaches node 1 (Fig. 3(b)), the cost intervals are updated and path $\mu_2 : \{s,1,2,t\}$ has the minimized maximum regret of 0. When the visitor reaches node 2 (Fig. 3(c)), cost intervals are further updated and path $\mu_2 : \{s,1,2,t\}$ still has the minimized maximum regret of 0. Finally, node t is reached (Fig. 3(d)), path $\mu_2 : \{s,1,2,t\}$ still has the minimized maximum regret of 0.

Hence, if we use the static RRSP-INT model and follow the fixed path $\mu_1 : \{s,1,t\}$ as in the example of Fig. 2, the final realization of regret is 6. If instead we use a dynamic approach, which changes the path in node 1 from $\mu_1 : \{s,1,t\}$ to $\mu_2 : \{s,1,2,t\}$, the final realization of regret is 0. More importantly, path $\mu_2 : \{s,1,2,t\}$ also has the minimum final cost realization of 6, while the other paths have costs of 12.

In general, let $[0,T]$ be a time horizon, and let $C(\tau)$ be the set of possible cost vector realizations in time $\tau \in [0,T]$. Given two nodes $i,j \in \mathcal{N}$, let $\mu(i,j)$ denote a simple path from i to j in $\mathcal{G}$. Given two paths $\mu(i,j)$ and $\mu(j,k)$, with $i \neq k$, let $\mu(i,j) \oplus \mu(j,k)$ be the concatenation of the two paths. The Dynamic Relative Robust Shortest Path (DRRSP) problem can be formally described as follows.

Dynamic Relative Robust Shortest Path (DRRSP)

Input: Graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, collection $C(\tau)$ of possible arc cost vector realizations for all $\tau \in [0,T]$, specified nodes $s,t \in \mathcal{N}$.

Task: Find a strategy $\pi(\tau, i, \mu(s,i))$ that, given a node $i \in \mathcal{N}$ reached from s in time τ through a path $\mu(s,i)$, decides the next node to visit in a path $\mu(i,t)$ such that $\mu(s,i) \oplus \mu(i,t)$ minimizes the regret with respect to $C(\tau)$.

The DRRSP problem is proposed for the first time in Xu and Zhou (2020), adopting the interval data model to describe uncertainty. They assume that in time τ the uncertainty on the cost of arc (i,j) is described by the interval $[c^\tau_{ij}, c^\tau_{ij} + d^\tau_{ij}]$. Given that $i_{\tau-1}$ is the node reached in time $\tau-1$, the next node to be visited is determined by solving the following model:

$$\min \quad \sum_{(i,j)\in\mathcal{A}} (c^\tau_{ij} + d^\tau_{ij}) x_{ij} - y_t \tag{RRSP-INT-DYN}$$

$$\text{subject to} \quad \sum_{\{j:(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j:(j,i)\in\mathcal{A}\}} x_{ji} = \begin{cases} 1 & \text{if } i = i_{\tau-1} \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in \mathcal{N}$$

$$y_j \leq y_i + c^\tau_{ij} + d^\tau_{ij} x_{ij}, \ \forall (i,j) \in \mathcal{A}$$

$$y_{i_{\tau-1}} = 0$$

$$x_{ij} \in \{0,1\}, \ \forall (i,j) \in \mathcal{A}$$

$$y_j \geq 0, \ \forall j \in \mathcal{N} \cup \{s,t\}.$$

Model (RRSP-INT-DYN) is a reformulation of model (RRSP-INT) where, at each stage, the origin of the sought path and the cost intervals are updated. A possible strategy is then to iterate the solution of model (RRSP-INT-DYN) from stage to stage, until the destination node t is reached.

Solution methods

To get a computationally feasible approach, Xu and Zhou (2020) design a heuristic algorithm, named Nested Dijkstra (ND), including a traditional Dijkstra algorithm (Dijkstra, 1959) nested inside another Dijkstra algorithm. The inner Dijkstra algorithm calculates out the maximum regret value of a path, while the external Dijkstra algorithm finds a heuristic solution for the DRRSP problem. Xu and Zhou (2020) prove that ND finds a global optimal solution provided that two conditions related to the uniqueness of optimal paths are satisfied. Otherwise, their method returns a heuristic solution.

5. Distributionally Robust Shortest Path problem

The *Distributionally Robust Shortest Path* (DRSP) problem consists in finding a path that minimizes the expected path cost from s to t, where the arc costs are subject to distributional uncertainty. Basically, the decision-maker attempts to minimize the worst-case expected loss over an *ambiguity set* (or a family) of candidate distributions that are consistent with the decision-maker initial information (see, *e.g.*, related studies in Wiesemann et al. (2014), Delage and Ye (2010), Goh and Sim (2010), Bertsimas et al. (2019) and a review on distributionally robust optimization in Rahimian and Mehrotra (2019)).

This approach attempts to balance the lack of distributional information typical of robust optimization and the complete knowledge of the underlying distribution of stochastic optimization approaches. Some attempts in this direction are proposed in Cheng et al. (2013, 2016), Wang et al. (2020), Ketkov et al. (2021) for the static case and in Ketkov (2023) for the dynamic one.

The general problem in this class can be formalized as follows.

Distributionally Robust Shortest Path (DRSP)

Input: Graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, a family of probability distributions $\mathbb{Q}$ over the set of arc cost vectors $\tilde{c}$, and specified nodes $s, t \in \mathcal{N}$.

Task: Find a simple path μ^* from s to t such that

$$\min_{x\in\mathcal{X}} \sup_{Q\in\mathbb{Q}} \mathbb{E}_Q \left[\sum_{(i,j)\in\mathcal{A}} \tilde{c}_{ij} x_{ij} \right],$$

where $\mathbb{E}_Q$ denotes the mathematical expectation with respect to the worst probability distribution in the family $\mathbb{Q}$.

Specifically, for an ambiguity set $\mathbb{Q}$ that satisfies linear expectation constraints and quantile constraints, Ketkov et al. (2021) express the static DRSP as follows:

$$\min_{x\in\mathcal{X}} \max_{Q\in\mathbb{Q}} \mathbb{E}_Q \left[\sum_{(i,j)\in\mathcal{A}} \tilde{c}_{ij} x_{ij} \right]. \tag{DRSP}$$

The problem considered in Cheng et al. (2013, 2016) consists in finding a directed path between two given vertices s and t such that the sum of the cost and the worst-case delay within a set of probability distributions is minimal. The delay cost is based on a penalty per time unit $\delta > 0$ that must be paid whenever the total delay exceeds a given threshold $D > 0$ that may represent the preferred arrival time. Denoting with $\tilde{d}_{ij}$ the random delay associated to arc (i, j), the *Distributionally Robust Shortest Path with Delay* (DRSP-D) problem is formulated as follows:

$$\min_{x\in\mathcal{X}} \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij} + \delta \max_{Q\in\mathbb{Q}} \mathbb{E}_Q \left[\sum_{(i,j)\in\mathcal{A}} [\tilde{d}_{ij} x_{ij} - D]^+ \right], \tag{DRSP-D}$$

where $[x]^+ = \max\{x, 0\}$.

Wang et al. (2020) proposes the *worst case reliable Mean-Excess Travel Time* (w-METT), formulated as follows:

$$\min_{x\in\mathcal{X},\ t\in\mathbb{R}} \left\{ t + \frac{1}{\alpha} \max_{Q\in\mathbb{Q}} \mathbb{E}_Q \left[\sum_{(i,j)\in\mathcal{A}} [\tilde{c}_{ij} x_{ij} - t]^+ \right] \right\}, \tag{w-METT}$$

where α is the confidence level and t is (at optimality) the Value-at-Risk (VAR). Notice that when the probability distribution Q is known, (w-METT) coincides with the Conditional Value-at-Risk measure (CVaR) proposed by Rockafellar et al. (2000).

The only contribution which considers a multistage version of the DRSP problem, is in Ketkov (2023). Similar to Bertsekas (2019) in a deterministic setting, Ketkov (2023) interprets the problem from the game theoretical perspective, as a dynamic zero-sum game between two decision-makers, which are referred to as a user and an attacker. The user attempts to minimize its expected loss by traveling between two fixed nodes in a given network. On the other hand, the attacker aims at maximizing the user's expected loss by selecting a distribution of arc costs within a given family of probability distributions. The outlined game is dynamic in the sense that both the user and the attacker are able to adjust their decisions at particular nodes of the user's path.

In the literature of DRSP, the approaches typically adopted to define the ambiguity set are using some moments information or the Wasserstein metric. In particular, Ketkov et al. (2021), Ketkov (2023) follow the approach proposed by Wiesemann et al. (2014), where the ambiguity set is formed by all distributions that satisfy prescribed linear first-order moment constraints with respect to subsets of arcs and individual probability constraints with respect to particular arcs. In Ketkov (2023), some additional auxiliary distributional constraints that can be verified by the user dynamically while traveling through the network are introduced. In Cheng et al. (2013, 2016), the ambiguity set $\mathbb{Q}$ is described via first and second moments as:

$$\mathbb{Q}_M = \left\{ Q \in \mathcal{M}(C) : \quad \mathbb{E}_Q(\tilde{c}) = \mu, \quad \mathbb{E}_Q[(\tilde{c} - \mu)(\tilde{c} - \mu)^\top] \le \Sigma \right\},$$

where $\mathcal{M}(C)$ is the set of all probability distributions supported on C. The support C, as well as the mean vector μ and the second-moment matrix Σ of the uncertain costs $\tilde{c}$, are assumed to be known, but the exact information of the distribution is unknown.

In Wang et al. (2020), the ambiguity set $\mathbb{Q}$ is described via a Wasserstein ball as:

$$\mathbb{Q}_W = \left\{ Q \in \mathcal{M}(C) : d_W(Q, \mathcal{P}) \le \epsilon_W \right\},$$

where $\epsilon_W \ge 0$ reflects the confidence in the empirical probability distribution $\mathcal{P}$ and $\mathcal{M}(C)$ is the set of some probability distributions supported on C, *i.e.*, $\mathbb{Q}_W$ contains all distributions within the ϵ_W-Wasserstein distance from $\mathcal{P}$, using the Wasserstein metric d_W (see Definition 2 in Wang et al. (2020)). Note that when the sample dataset is reasonably large, the Wasserstein ball $\mathbb{Q}_W$ includes the true distribution with a high probability (Mohajerin Esfahani and Kuhn (2018)) and thus the DRSP model is expected to exhibit good out-of-sample performance.

The main references on the DRSP problem and its variants are summarized in Table 3. The table shows the main characteristics of each contribution. In the column "Objective function" the Distributionally Robust Shortest Path (DRSP), the Distributionally Robust Shortest Path with Delay (DRSP-D) or Mean-Excess Travel Time (METT) objective function is indicated. The "Ambiguity set description" indicates which type (Moment based or Wasserstein ball) of ambiguity set model is adopted. The column "Time" says whether the model is static (S) or dynamic (D). In some papers, "Complexity results" are presented. In the last two columns it is shown whether a solution method, exact and/or heuristic, is proposed.

Solution methods

Exact solution methods for the DRSP problem are addressed in Cheng et al. (2013, 2016), Wang et al. (2020), Ketkov et al. (2021), Ketkov (2023).

As the problem considered in Cheng et al. (2013, 2016) is NP-hard, they propose new reformulations and approximations using a sequence of Semi-Definite Programming (SDP) problems, which provide tight lower bounds that can be solved in polynomial time. Wang

Table 3
Classification of references for the Distributionally Robust Shortest Path problem.

Reference	Objective function[a]	Ambiguity set description	Time[b]	Complexity results	Solution methods Exact	Type[c]	Heuristic	Type[d]
Cheng et al., 2013	DRSP-D	Moment based	S	–	√	SDP	–	
Cheng et al., 2016	DRSP-D	Moment based	S	√	√	SDP	√	AD
Wang et al., 2020	METT	Wasserstein ball	S	√	√	MILP-BSOCP	–	
Ketkov et al., 2021	DRSP	Moment based	S	√	√	MILP	–	
Ketkov, 2023	DRSP	Moment based	D	–	√	MILP	–	

[a] DRSP: Distributionally Robust Shortest Path, DRSP-D: Distributionally Robust Shortest Path with Delay, METT: Mean-Excess Travel Time.
[b] S: Static, D: dynamic.
[c] SDP: semidefinite programming, MILP: mixed integer linear program, BSOCP: Binary Second Order Cone Program.
[d] AD: alternating direction method.

et al. (2020) reformulate the w-METT problem as a mixed 0–1 Convex Program and explicitly derive the worst-case distribution to achieve the worst-case METT. They solve the model through existing optimization techniques. Experimental results based on a real road network validate that the proposed model provides good out-of-sample performance. Ketkov et al. (2021) propose equivalent robust and MILP reformulations of the DRSP. In particular, the problem without linear expectation constraints has been shown to be polynomially solvable. They show that the proposed MILP formulations can be solved effectively using state-of-the-art solvers. Finally, in Ketkov (2023), by using some properties of the related one-stage formulation (see Ketkov et al. (2021)) and LP duality, the multistage problem is reformulated as one potentially large MILP problem and solved via commercial solvers.

The only heuristic approach is provided in Cheng et al. (2016). By leveraging an SDP relaxation and a further SDP formulation obtained by variable fixing, the authors propose an Alternative Direction method to solve an approximation of the DRSP-D problem.

6. Conclusions and future research

Computing a shortest path from an origin to a destination in a transportation or communication network is a fundamental problem in many applications, ranging from freight distribution, to private transport, to data exchange. In virtually any real world problem, the state of the network is subject to uncertainty. Sometimes the role of uncertainty is negligible, for example if we are considering paths of unmanned automatic vehicles through the aisles of a warehouse. Very often uncertainty is a key factor, for example if we are considering paths in an urban road network. Optimization models incorporating uncertainty require much more information than deterministic optimization models, especially if models of uncertainty are assumed and dynamic factors are explicitly considered. Moreover, such optimization models are often harder to solve than deterministic ones.

Data and computational requirements have restrained the application of Robust Shortest Path models to real contexts. However, such requirements are scaled down by technological advances. After the review of models and methods for Robust and Distributionally Robust Shortest Path problems, we could argue that another factor could have restrained their application, that is, their conceptual difficulty with respect to a plain, classic "Shortest Path problem".

We now summarize the conclusions we draw from this review:

- For the static Robust Shortest Path (RSP) problem, two main formulations have emerged in the literature: the *Absolute Robust Shortest Path* (ARSP) and the *Relative Robust Shortest Path* (RRSP). While ARSP focuses on minimizing the absolute worst-case cost, RRSP emphasizes minimizing the maximum deviation (or regret) from the optimal path, providing a better framework for handling regret-related decisions. Both problems have been tackled using interval data models and discrete scenario frameworks.
  Exact methods (*e.g.*, dynamic programming, MILP, Branch-and-Bound, Benders decomposition) dominate the literature for interval-based models, providing robust guarantees at the cost of computational complexity. For this reason, preprocessing techniques, such as pegging tests and reduction rules, have been applied to enhance the computational efficiency of these methods, by simplifying the problem before optimization. Scenario-based methods face scalability issues with large scenario sets. For this reason, heuristics and approximation algorithms such as uniform surrogation, function-based evaluation, and Lagrangian relaxation are revealed to be more practical in the literature of ARSP and RRSP. However, we believe there is still a lack of fast heuristic methods. Finding good solutions with respect to robustness criteria in a short time would increase the range of applicability of RSP problems, and encourage the inclusion of RSP methods in software platforms.
- For the static Distributionally Robust Shortest Path (DRSP) problem, the main formulations emerged in the literature model the uncertainty through moment-based ambiguity sets and Wasserstein balls. Exact solution methods, including MILP, SDP, and SOCP, have proven computationally efficient, enabling the application of risk-adjusted metrics like CVaR and METT. Future work could explore richer ambiguity sets, data-driven approaches, and scalable algorithms, while building a foundation for dynamic extensions to address evolving uncertainties.
- Most of the contributions in the literature of the Robust and Distributionally Robust Shortest Path Problem refers to the static case, that is, they assume that the arc costs are fixed in time. This approach implies that the route cannot be modified once selected. However, the path-operational constraints inevitably change during time. Making use of the updated data is important, reducing the uncertainty and improving the efficiency. Our analysis has shown that only two contributions allow a recourse action after the realization of the uncertainty, both considering the ARSP objective (see Section 3.1.3). It would be interesting to consider two-stage robust Shortest Path problems where the objective is finding the RRSP or the α-Robust Shortest Path and the randomness of the problem is described via uncertainty sets, budgeted uncertainty or scenarios. This approach would lead to the definition of all the new approaches highlighted in light gray in the left sub-tree of Fig. 1. Adjustable Robust Optimization (ARO) techniques, where some of the decision variables can be modified at a later moment in time according to a decision rule, could be considered (see *e.g.* Ben-Tal et al. (2009), Yanıkoğlu et al. (2019)), as they balance the need for proactive planning with the flexibility to react to unfolding scenarios.
- On the same stream of the previous point, an interesting future research direction is also the study of dynamic extensions for real-time decision making via multi-stage robust optimization and multi-stage distributionally robust optimization. The dynamic RRSP problem is already proposed in Xu and Zhou (2020) using scenarios defined in intervals, and it is addressed using a Nested Dijkstra (ND) heuristic.
  It would be interesting to analyze the properties and the complexity of a dynamic robust Shortest Path problem where the objective

is finding an Absolute Robust Shortest Path or an α-Robust Shortest Path, and uncertainty is described via interval, ellipsoids, or budgeted uncertainty. These new approaches would lead to the definition of the new problems highlighted in light gray in the right sub-tree of Fig. 1. Once again ARO techniques could provide a computationally tractable methodology for many parameterized adjustable decision variables via affine or non-linear decision rules and uncertainty sets.

In the context of Distributionally Robust Shortest Path problems, the only contribution which considers a multistage version, is in Ketkov (2023), where the problem is formulated as a dynamic zero-sum game between a user and an attacker. The use of moment-based ambiguity sets in this context allows for adaptive decision-making as the user traverses the network, while the attacker seeks to maximize the expected cost by dynamically adjusting distributions. A promising direction for future research would be the use of alternative methods, such as Phi-Divergences and Wasserstein ball descriptions, to define conditional ambiguity sets in the dynamic case. Several new models could be devised, opening the question for efficient optimization techniques.

- The models and algorithms analyzed in this paper are very different from each other, and implemented on different platforms over a period of many years, when the evolution of hardware and optimization software has been enormous. At present, it is therefore not possible to directly compare the effectiveness and efficiency of the different proposed solutions. We believe that a research of computational flavor – where different approaches are implemented and tested on a common updated software and hardware platform using a standardized set of benchmark instances – would be very useful to draw some conclusions on the real applicability of the different proposed methods.
- Finally, an interesting avenue of exploration involves coupling multistage RO/DRO approaches for the Shortest Path Problem with real-time data updates, leveraging machine learning techniques to dynamically refine uncertainty/ambiguity sets as new data becomes available. By utilizing historical and spatio-temporal data (such as weather conditions, time-of-day patterns, or congestion levels) Machine Learning (ML) models can dynamically refine uncertainty description based on current contextual information (see Kallus and Mao (2023)). These data-driven methods could estimate conditional distributions of uncertain parameters, allowing decision-makers to make more informed and adaptive choices. Such advancements would enhance the robustness and adaptability of DRSP models, bridging the gap between theoretical optimization and practical real-world applications (see Chenreddy and Delage (2024)).

CRediT authorship contribution statement

Carlo Filippi: Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Francesca Maggioni:** Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **M. Grazia Speranza:** Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

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Data availability

No data was used for the research described in the article.

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