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A short note on the Painlevé XXV–Ermakov equation

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ABSTRACT

Solutions of the second member of the Riccati chain and of the corresponding third order linear differential equation are related to solutions of the so-called Painlevé XXV–Ermakov equation via the Schwarzian derivative. The reduction to the generalised Ermakov equation is shown to arise naturally from the Painlevé XXV–Ermakov equation. Specifically, the first order system of ordinary differential equations, equivalent to the Painlevé XXV–Ermakov equation, is analysed by resolving points of indeterminacy of the vector field over $\mathbb{P}^1 \times \mathbb{P}^1$.

1. Introduction

In [1] the second member of the so-called Riccati chain [2] is considered. It is given by

$$\frac{d^2v}{dz^2} + 3v\frac{dv}{dz} + v^3 + p(z)(v' + v^2) + q(z)v + r(z) = 0, \tag{1}$$

where all the coefficients are assumed to be analytic functions. This equation, as the usual Riccati equation, is linearisable by the transformation $v = y'/y$; the corresponding linear equation is

$$y''' + p(z)y'' + q(z)y' + r(z)y = 0. \tag{2}$$

When considering the ratio of two independent solutions of the corresponding linear equation

$$y_1(z) = w(z)y_2(z), \tag{3}$$

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where both y_1 and y_2 satisfy (2), we define the function $\xi(z)$ to be the Schwarzian derivative of $w(z)$, i.e.

$$\xi(z) \doteq \{w(z), z\} = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2. \tag{4}$$

In [1] it is shown that $\xi(z)$ satisfies the following non-linear second order differential equation

$$(12\xi(z) + b(z))\xi'' = 15\xi'^2 - h_0\xi' - 8\xi^3 - h_1\xi^2 - h_2\xi - h_3, \tag{5}$$

where the functions b, h_i, $i = 0, \ldots, 3$, are determined in terms of the functions p, q, r and their derivatives as

$$\begin{aligned}
&b = 2(p^2 - 3q + 3p'),\ h_1 = 4b,\\
&h_0 = 2(4pp' - 3p'' - 9pq + 2p^3 - 6q' + 27r),\\
&h_2 = 2(-4pp'' - 12qp' + 2p^2p' + 5p'^2 + 6qp' - 6p^2q + p^4 + 6q'' + 9q^2 - 18r'),\\
&h_3 = -6p''q' - 2pqp'' + 18rp'' + 6p'q'' + 2pp'q' - 2p^2qp' - 2qp'^2 + 6q^2p' - 18p'r' +\\
&+ 2p^2q'' + 2p^3q' - 6pqq' + 18pqr + p^2q^2 - 6p^2r' - 4p^3r - 6qq'' + 3q'^2 +\\
&+ 18qr' - 4q^3 - 27r^2.
\end{aligned}$$

It is shown in [3] that the following change of the dependent variable

$$12\xi(z) + b(z) = 12y(z), \tag{6}$$

combined with the definition of the functions $A(z)$ and $B(z)$:

$$\begin{aligned}
A(z) &= \frac{1}{4}p'' + \frac{1}{2}pp' - \frac{3}{4}q' + \frac{1}{9}p^3 - \frac{1}{2}pq + \frac{3}{2}r,\\
4B(z) &= p' + \frac{1}{3}p^2 - q,
\end{aligned} \tag{7}$$

allows to recast Eq. (5) for $\xi(z)$ to a more concise form as

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 3Ay' + 4By^2 - 2A'y - A^2 = 0. \tag{8}$$

This equation has many interesting properties [1,3]. It can be reduced to both the Painlevé XXV equation in the Ince list [4] and to the Ermakov equation, which is widely applied in mathematics and mathematical physics [5]. Moreover, the Painlevé XXV–Ermakov equation admits two families of Bäcklund transformations where an important role is played by the Schwarzian derivative. The following statement holds.

Theorem 1 (*[3]*). *Setting*

$$y(z) = \frac{g(z)}{u(z)^4}, \quad \text{with } g' = 2A(z)u(z)^4 \tag{9}$$

in Eq. (8), the following generalised Ermakov equation for $u(z)$ is obtained:

$$u'' = B(z)u(z) + \frac{g(z)}{6u(z)^3}. \tag{10}$$

In the case $A = 0$, from (9) one gets that $g(z)$ is constant and in this case it follows that (10) is a proper Ermakov equation.

In this paper we would like to present an alternative proof of Theorem 1 by using the method of blowing up points of indeterminacy of the vector field of a system of first order differential equations equivalent to (8). This method turned out to be very useful in studying second order differential equations [6].

2. Main results

For completeness, let us present some details on how to blow up points of indeterminacy of the vector field. Further details and references can be found in [6]. We shall use the same letters p and q but this time without any reference to the coefficients of the linear equation (2). Let p and q be functions of z. For a system of first order differential equations

$$q' = \frac{P_1(z,p,q)}{Q_1(z,p,q)}, \quad p' = \frac{P_2(z,p,q)}{Q_2(z,p,q)}, \tag{11}$$

where P_i and Q_i, $i = 1, 2$, are polynomials in p and q with coefficients rational in z such that P_i and Q_i have no common factors, the points where $P_i(z,p,q) = Q_i(z,p,q) \equiv 0$ for $i = 1$ or $i = 2$ are called the points of indeterminacy of the system. Like for algebraic curves in algebraic geometry [7], the aim is to remove these indeterminacies by a suitable bi-rational transformation. The blow-up at a point $(p,q) = (a,b)$, where $a = a(z)$ and $b = b(z)$, is defined as follows. One introduces new coordinate charts, $p = a + u = a + UV$ and $q = b + uv = b + V$ and re-writes the system in new coordinates (u,v) and (U,V). The exceptional line is then given by $u = 0$ or $V = 0$. The sequence of blowups starting from a given point of indeterminacy can be infinite for a differential system even if no rigorous proof is available. Indeed, via subsequent iterations of blowups, one can see that the form of the numerators and denominators in the right-hand sides of equations in the system does not change much but the expressions for subsequent points of indeterminacy become more and more cumbersome. If the cascade is infinite then we cannot extract any further properties for the system or its solutions. However, whenever the cascade is finite (or all the cascades are finite), it is possible to extract some useful information concerning the system or a singularity structure of its solutions [6] depending on the behaviour of the system on the exceptional line after the final blowup. When the system regularises on the last exceptional line, this gives rise to either holomorphic or polar expansions of solutions of the original system [6]. When the system does not regularise, the question on the existence of algebraic singularities should be examined further [8]. The original system should be considered over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In this paper we choose the compactification $\mathbb{P}^1 \times \mathbb{P}^1$ to take into account infinite values of q and p.

The easiest way to re-write Eq. (8) in the form (11) is to set $y = q$, $q' = p$ which yields

$$q' = p, \; p' = \frac{12A^2 - 36Ap + 15p^2 - 8q(6Bq + q^2 - 3A')}{12q}. \tag{12}$$

In case $A \neq 0$ we immediately see two points of indeterminacy $p_1 = (q = 0, p = 2A/5)$ and $p_2 = (q = 0, p = 2A)$. These points coalesce if $A = 0$. After the compactification $\mathbb{P}^1 \times \mathbb{P}^1$ one more point appears $p_3 = (Q = 1/q = 0, P = 1/p = 0)$. The first cascade starting from p_1 is possibly infinite, the expressions for points of indeterminacy become more and more cumbersome at each step. The second point p_2 gives rise to the finite cascade which regularises and, remarkably, we can easily deduce the connection to the generalised Ermakov equation. In particular, by setting $q = u_2$ and $p = 2A + u_2v_2$ the system (12) becomes

$$u_2' = 2A + u_2v_2, \; v_2' = \frac{3v_2^2 - 8u_2 - 48B}{12}. \tag{13}$$

The third point also gives rise to the finite cascade which finally terminates and regularises, though the number of required blowups is higher than in the previous case.

When $A = 0$ in the system (12), if we choose $v_2 = -12w'/(3w)$ (considering the second equation as a Riccati equation for v_2), we obtain the linear equation

$$w'' = \frac{(6B + u_2)w}{6} \tag{14}$$

and equation $u_2'w = -4u_2w'$, which can be easily solved giving $u_2 = C/w^4$, where C is an arbitrary constant. Thus, we obtain the Ermakov equation

$$w'' = Bw + \frac{C}{6w^3}.$$

For $A \neq 0$ we still obtain the linear equation (14) but the equation linking u_2 and w is more complicated,

$$u_2 = 2A - 4u_2\frac{w'}{w}.$$

Nevertheless, it can also be solved by quadratures giving

$$u_2 = \frac{C}{w^4} + \frac{2\int A(t)w(t)^4 dt}{w^4}.$$

This gives

$$w'' = Bw + \frac{C + 2\int A(t)w(t)^4 dt}{6w^3}$$

and an alternative proof of Theorem 1.

Finally, we would like to recall connection of Eq. (8) to a particular case of the Painlevé XXV equation, which is linearisable [4]. The transformation, leading to the Painlevé XXV equation, is given in the following Proposition.

Theorem 2 (*[3]*)**.** *By setting*

$$y(z) = \frac{2A(z)}{u(z)} \tag{15}$$

in Eq. (8), the Painlevé XXV equation is obtained for $u(z)$:

$$u'' = \frac{3u'^2}{4u} + \left(\frac{A'}{2A} - \frac{3u}{2}\right)u' - \frac{1}{4}u^3 + \frac{A'}{2A}u^2 + \left(4B - \frac{5A'^2}{4A^2} + \frac{A''}{A}\right)u + \frac{4}{3}A. \tag{16}$$

Here clearly $A \neq 0$. By similarly studying Eq. (16) and rewriting in a similar way as the previous equation in the form of an equivalent system of two first order equations (i.e., setting $u = q,\ q' = p$), we see that the there are two cascades. The first one arises from the point $p_1 = (q = 0,\ p = 0)$ and the system regularises after two more blowups. The second cascade from the point $p_2 = (Q = 1/q = 0,\ P = 1/p = 0)$ is splitting after one more blowup into the infinite part and the regularisable finite one. If we study the general Painlevé XXV equation as in [4], we can spot some difference, in particular we notice that one cascade does not regularise but is finite.

3. Conclusions

To sum up, as already mentioned in [6], the method of resolving indeterminacies of the vector field for a system of two differential equations (or, equivalently, for a second order ordinary scalar differential equation) is very useful and can give or explain some nontrivial results. Therefore, it should be used often along with other methods. It is an open question to understand what kind of spaces of initial conditions (see [6,8,9] for more information) appear, if any, for linearisable equations and what infinite cascades of points of indeterminacy for such systems are associated with.

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