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On the Basic Properties and the Structure of Power Cells

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Abstract
Given a set $T \subseteq \mathbb{R}^n$ and a nonnegative function r defined on T, we consider the power of $x \in \mathbb{R}^n$ with respect to the sphere with center $t \in T$ and radius $r(t)$, that is, $p_r(x, t) := \|x - t\|^2 - r^2(t)$, with $\|\cdot\|$ denoting the Euclidean distance. The corresponding power cell of $s \in T$ is the set

$$C_T^r(s) := \{x \in \mathbb{R}^n : p_r(x, s) \le p_r(x, t), \text{ for all } t \in T\}.$$

We study the structure of such cells and investigate the assumptions on r that allow for generalizing known results on classical Voronoi cells.

Keywords Power cell · Voronoi diagram · Structure of power cells

Mathematics Subject Classification 52A20 · 52C22 · 52B11 · 51M20

1 Introduction and Preliminaries

This paper presents a systematic study of the fundamental properties of power cells. Power cells have been extensively studied in the literature, the main reference being the seminal paper [2], which contains a detailed theoretical and algorithmic study of power diagrams and reviews some of its practical applications. Some more recent

Dedicated to Professor Boris Sh. Mordukhovich on the occasion of his 75th birthday.

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developments can be found in [3, 7, 8, 12–14]. Power cells and power diagrams are used in several applications. In [4], for example, power diagrams are used to design contracts to incentivize self-minded agents to provide honest information. Applications to harvest planning can be found in [15], while in [1, 5, 18] algorithms for assignment problems are developed based on the concept of power cells.

In this paper we mainly investigate the conditions on the weight function r under which the basic properties of classical Voronoi cells extend to power cells. Among other results, we give conditions for power cells to have a nonempty interior and to be bounded. One of the main results gives sufficient conditions for a closed convex set to be a power cell with respect to a given weight function. Another main result states sufficient conditions on a given set of sites and a given weight function, for the corresponding collection of power cells to make a tesselation of the space.

We will use standard convex analytic terminology and notation, following the classical reference [17]. We will denote $\mathbb{R}_+$, $\mathbb{R}_{++}$ and $\mathbb{R}_{--}$ the sets of nonnegative, strictly positive and strictly negative real numbers, respectively. The zero vector, the open unit ball and the unit sphere in $\mathbb{R}^n$ are denoted by $\mathbf{0}$, $\mathbf{B}$ and $\mathbf{S}$, respectively. The Euclidean inner product of $x, y \in \mathbb{R}^n$ and the Euclidean norm of x will be denoted by $\langle x, y\rangle$ and $\|x\|$, respectively. Given $C \subseteq \mathbb{R}^n$, with $\text{int}C$ and $\text{cl}C$ we denote the interior and the closure of C, respectively. The convex hull of C, denoted $\text{conv}C$, is the smallest convex set containing C. We denote $\text{cone}C$ the convex conical hull of C, that is, $\text{cone}C := \mathbb{R}_+\text{conv}C$. The dimension of C, denoted $\dim C$, is the one of the affine variety generated by C; if $C = \emptyset$, we set $\dim C := 0$. The linearity space of a convex cone $K \subseteq \mathbb{R}^n$ is the largest linear subspace contained in K, that is, $\text{lin}K := K \cap (-K)$. The orthogonal of a linear subspace $L \subseteq \mathbb{R}^n$ is $L^\perp := \{x^* \in \mathbb{R}^n : \langle x, x^*\rangle = 0, \text{ for all } x \in L\}$.

Let $T \subseteq \mathbb{R}^n$, with $n \geq 1$, be a set whose elements are called *sites*. We consider a weight function $r : T \to \mathbb{R}_+$ on the set of sites. One can interpret $r(t)$ as the radius of a sphere centered at t; under such an interpretation, the function $p_r : \mathbb{R}^n \times T \to \mathbb{R}$ defined by $p_r(x, t) := ||x - t||^2 - r^2(t)$ assigns to every pair consisting of a point x and a site t the power of x with respect to the sphere associated to t. We recall that the power of a point with respect to a sphere is the product of the distances from that point to the two intersections with the sphere of an arbitrary line through the point that intersects the sphere; a remarkable property, which is nevertheless easy to prove, says that such a product is independent of the chosen line. In particular, the power equals the product of the shortest and farthest distances from the point to the sphere. Moreover, as a limiting case, considering a line through the point that is tangent to the sphere, the power equals the square of the distance from the given point to the tangency point. The *cell* of $s \in T$, denoted by $C_T^r(s)$, consists of all points for which the power of x with respect to the sphere centered at t with radius $r(t)$ is minimized at $t = s$, i.e.

$$C_T^r(s) := \{x \in \mathbb{R}^n : p_r(x, s) \leq p_r(x, t), \text{ for all } t \in T\} \tag{1}$$

or equivalently,

$$C_T^r(s) = \{x \in \mathbb{R}^n : \langle t - s, x\rangle \leq b_r(t, s), \text{ for all } t \in T\} \tag{2}$$

with

$$b_r(t,s) := \frac{1}{2}\left(||t||^2 - ||s||^2 + r^2(s) - r^2(t)\right). \tag{3}$$

In view of the discussion above, the difference between $C_T^r(s)$ and the ordinary Voronoi cell of s is that, in the latter case, a point x belons to the cell of s when s is the closest site, whereas, in the former case, the proximity criterion from x to a site $t \in T$ corresponds to the geometric mean of the shortest and the largest distances from x to a sphere centered at $t \in T$ with radius $r(t)$.

Clearly, the condition $t \in T$ in (1) and (2) can be equivalently replaced with $t \in T \setminus \{s\}$.

The representation (2) shows that $C_T^r(s)$, being the solution set of a (possibly infinite) linear inequality system, is a closed convex set and, in the case when T is finite, is a convex polyhedron.

The classical Voronoi cell of $s \in T$ with respect to T, defined by

$$V_T(s) := \{x \in \mathbb{R}^n : \|x - s\| \leq \|x - t\|, \text{ for all } t \in T\},$$

corresponds to the particular case of $C_T^r(s)$ when r is constant. In such a case, one has $b_r(t,s) = \frac{1}{2}\left(||t||^2 - ||s||^2\right)$.

The rest of the paper consists of four sections. In Sect. 2, we establish the basic properties of power cells; in particular, we give conditions on T and r that guarantee their nonemptiness. In Sect. 3, we study the interior of power cells and their properties. In Sect. 4, we give conditions for a point to belong to some power cell and, as a consequence, conditions on T and r implying that the collection of all the power cells make a tessellation of the space. Section 5 contains the conclusions.

2 Basic Properties of Power Cells

Unlike in the case of ordinary Voronoi cells, which are always nonempty (since $s \in V_T(s)$), power cells may be empty, even for very simple sets of sites T and a very well behaved function r. Indeed; consider, for instance, the case when $T := \{\mathbf{0}, t, -t\}$, with $t \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and $r := \alpha \|\cdot\|_{|T}$, with $\alpha > 1$. Then $C_T^r(\mathbf{0}) = \emptyset$, since, for $x \in C_T^r(\mathbf{0})$, from (2) one gets $\langle t, x\rangle \leq b_r(t,\mathbf{0})$ and $\langle -t, x\rangle \leq b_r(-t,\mathbf{0})$, which yields the absurd conclusion $0 \leq b_r(t,\mathbf{0}) + b_r(-t,\mathbf{0}) = \left(1-\alpha^2\right)||t||^2 < 0$.

Using Gale Theorem [9, Propositions 1 and 2 in Table 3.1], one immediately obtains the characterization of the nonemptiness of power cells provided by the next proposition, in which we make use of the second-moment cone [9] of the linear inequality system in the representation (2):

$$K_T^r(s) := \operatorname{cone}\{(t - s, b_r(t,s)) : t \in T\}.$$

Proposition 2.1 *The following statements are equivalent:*

i) $C_T^r(s) \neq \emptyset$.
ii) $(\mathbf{0}, -1) \notin \operatorname{cl} K_T^r(s)$.

A sufficient condition for the nonemptiness of power cells, which does not involve second-moment cones, is provided next.

Proposition 2.2 *If* $s \notin \operatorname{clconv}(T \setminus \{s\})$ *and the function* $b_r(\cdot, s)$ *is bounded, then* $C_T^r(s) \neq \emptyset$.

Proof Suppose by contradiction that $C_T^r(s) = \emptyset$. By Proposition 2.1, there exist sequences $\lambda_j^k \geq 0$, $t_j^k \in T \backslash \{s\}$, $j = 1, \ldots, n$, such that

$$\begin{aligned} \mathbf{0} &= \lim_{k \to \infty} \sum_{j=1}^{n} \lambda_j^k (t_j^k - s), \\ -1 &= \lim_{k \to \infty} \sum_{j=1}^{n} \lambda_j^k b_r(t_j^k, s). \end{aligned} \tag{4}$$

Let $M > 0$ be an upper bound of $b_r(\cdot, s)$. Then, by the continuity and subadditivity of $|.|$, we have

$$1 \leq \lim_{k \to \infty} \sum_{j=1}^{n} \lambda_j^k \left| b_r(t_j^k, s) \right| \leq M \liminf_{k \to \infty} \sum_{j=1}^{n} \lambda_j^k,$$

and hence

$$\liminf_{k \to \infty} \sum_{j=1}^{n} \lambda_j^k \geq \frac{1}{M} > 0.$$

Without loss of generality, we can assume that the sequence $\sum_{j=1}^n \lambda_j^k$ converges, so that $\lim_{k\to\infty} \sum_{j=1}^n \lambda_j^k \geq \frac{1}{M}$. Dividing both sides of (4) by $\sum_{j=1}^n \lambda_j^k$ and defining

$$\mu_j^k := \frac{\lambda_j^k}{\sum_{j=1}^n \lambda_j^k},$$

we have $\sum_{j=1}^n \mu_j^k = 1$ and

$$\mathbf{0} = \lim_{k \to \infty} \sum_{j=1}^{n} \mu_j^k (t_j^k - s) = \lim_{k \to \infty} \sum_{j=1}^{n} (\mu_j^k t_j^k) - s \in \operatorname{clconv}(T \setminus \{s\}) - s,$$

which contradicts the assumption $s \notin \operatorname{clconv}(T \backslash \{s\})$. □

The following proposition establishes a necessary condition for a power cell to be nonempty.

Proposition 2.3 *If* $C_T^r(s) \neq \emptyset$, *then* $\inf_{t \in T \backslash \{s\}} \frac{b_r(t,s)}{||t-s||} > -\infty$.

Proof Taking $x \in C_T^r(s)$ and using (2), we obtain

$$\inf_{t\in T\setminus\{s\}} \frac{b_r(t,s)}{||t-s||} \geq \inf_{t\in T\setminus\{s\}} \frac{\langle t-s, x\rangle}{||t-s||} \geq -\|x\| > -\infty.$$

□

As mentioned above, the ordinary Voronoi cell of s always contains s, but this is not the case of power cells. Our next proposition, which follows easily from (2), gives a necessary and sufficient condition for $C_T^r(s)$ to contain s.

Proposition 2.4 *The following statements are equivalent:*

i) $s \in C_T^r(s)$.
ii) $r^2(t) - r^2(s) \leq ||t-s||^2$, *for all* $t \in T$.

Classical Voronoi cells satisfy the obvious equality $V_T(s) \cap T = \{s\}$. The situation for general power sets is quite different, as shown, for instance, by Proposition 2.4. Our next result provides an upper estimate for $C_T^r(s) \cap T$.

Proposition 2.5 *One has*

$$C_T^r(s) \cap T \subseteq \{t \in T : ||s-t||^2 \leq r^2(s) - r^2(t)\}.$$

Proof Let $t \in C_T^r(s) \cap T$. Then $\langle t-s, t\rangle \leq b_r(t,s)$, which, using (3), is easily seen to be equivalent to the inequality $||s-t||^2 \leq r^2(s) - r^2(t)$. □

Corollary 2.1 *Let* $T \subseteq \mathbb{R}^n$. *If*

$$r^2(s) - r^2(t) < ||s-t||^2, \text{ for all } t \in T \setminus \{s\},$$

then

$$C_T^r(s) \cap T \subseteq \{s\}.$$

Combining Corollary 2.1 with Proposition 2.4, we obtain the following corollary.

Corollary 2.2 *Let* $T \subseteq \mathbb{R}^n$. *If*

$$|r^2(s) - r^2(t)| < ||s-t||^2, \text{ for all } t \in T \setminus \{s\},$$

then

$$C_T^r(s) \cap T = \{s\}.$$

The recession cone of a nonempty closed convex set $X \subseteq \mathbb{R}^n$ is

$$0^+X := \left\{d \in \mathbb{R}^n : d + X \subseteq X\right\},$$

and the normal cone to X at $x \in X$ is

$$N_X(x) := \left\{ x^* \in \mathbb{R}^n : \left\langle y - x, x^* \right\rangle \leq 0 \text{ for all } y \in X \right\}.$$

The negative polar cone of $X \subseteq \mathbb{R}^n$ is

$$X^0 := \{x^* \in \mathbb{R}^n : \langle x, x^* \rangle \leq 0, \text{ for all } x \in X\}.$$

From the representation (2) of $C_T^r(s)$ as the solution set of a linear inequality system, one immediately gets the following result.

Proposition 2.6 *If $C_T^r(s) \neq \emptyset$, then $0^+ C_T^r(s) = (T - s)^0 = N_{\mathrm{clconv}T}(s)$.*

In view of Proposition 2.6, the recession cone $0^+ C_T^r(s)$ does not depend on r, as long as $C_T^r(s) \neq \emptyset$. In particular, one has $0^+ C_T^r(s) = 0^+ V_T(s)$ if $C_T^r(s) \neq \emptyset$.

Corollary 2.3 *If $C_T^r(s) \neq \emptyset$, then $C_T^r(s)$ is bounded if and only if $s \in \operatorname{int} \operatorname{conv} T$.*

The particular case of Corollary 2.3 corresponding to ordinary Voronoi cells was given in [10, Proposition 5].

As we already observed in the preceding section, $C_T^r(s)$ is a convex polyhedron if T is finite. More generally, we have the following result.

Proposition 2.7 *If $\mathrm{cl} K_T^r(s)$ is polyhedral, then $C_T^r(s)$ is a polyhedron.*

Proof It is sufficient to observe that

$$C_T^r(s) = \{x \in \mathbb{R}^n : \langle x, x^* \rangle \leq \beta, \quad \text{for all } (x^\star, \beta) \in \mathrm{cl} K_T^r(s)\}.$$

Indeed, if $\mathrm{cl} K_T^r(s)$ is polyhedral, then it is the cone generated by a finite number of vectors $(x_1^\star, \beta_1), \ldots, (x_m^\star, \beta_m)$, and it is easy to see that

$$C_T^r(s) = \{x \in \mathbb{R}^n : \langle x, x_i^\star \rangle \leq \beta_i, \quad i = 1, \ldots, m\}.$$

□

The converse implication in Proposition 2.7 does not hold, as the following example shows.

Example 2.1 Let

$$S := \{(\alpha, \beta) \in \mathbb{R}^2 : (\alpha - 1)^2 + (\beta - 1)^2 = 1, \ \alpha \leq 1, \ \beta \leq 1\},$$

and define

$$T := \{(0, 0)\} \cup \left\{ \frac{2}{\alpha^2 + \beta^2} (\alpha, \beta) : (\alpha, \beta) \in S \right\}.$$

It is not difficult to see that

$$\begin{aligned} V_T((0,0)) &= \{(x,y) \in \mathbb{R}^2 : \alpha x + \beta y \le 1, \text{ for all } (\alpha,\beta) \in S\} \\ &= \{(x,y) \in \mathbb{R}^2 : (1-\cos\phi)x + (1-\sin\phi)y \le 1, \text{ for all } \phi \in [0, \frac{\pi}{2}]\} \\ &= \{(x,y) \in \mathbb{R}^2 : \min_{\phi \in [0,\frac{\pi}{2}]} \{(\cos\phi)x + (\sin\phi)y\} \ge x + y - 1\}. \end{aligned}$$

To compute $\min_{\phi \in [0,\frac{\pi}{2}]}\{(\cos\phi)x + (\sin\phi)y\}$, we distinguish three cases:

- $(x,y) \in \mathbb{R}^2_{++}$. In this case,

$$\min_{\phi \in [0,\frac{\pi}{2}]} \{(\cos\phi)x + (\sin\phi)y\} = \min\{x, y, \sqrt{x^2+y^2}\} = \min\{x, y\}.$$

- $(x,y) \in \mathbb{R}^2_{--}$. In this case,

$$\min_{\phi \in [0,\frac{\pi}{2}]} \{(\cos\phi)x + (\sin\phi)y\} = \min\{x, y, -\sqrt{x^2+y^2}\} = -\sqrt{x^2+y^2}.$$

- $(x,y) \notin \mathbb{R}^2_{++} \cup \mathbb{R}^2_{--}$. In this case,

$$\min_{\phi \in [0,\frac{\pi}{2}]} \{(\cos\phi)x + (\sin\phi)y\} = \min\{x, y\}.$$

Using these computations, it is easy to see that (2.1) yields

$$V_T((0,0)) = \{(x,y) \in \mathbb{R}^2 : x \le 1, y \le 1\},$$

so that $C^0_T((0,0)) = V_T((0,0))$ is a polyhedron.

On the other hand, one has

$$\begin{aligned} \mathrm{cl}K^r_T(s) &= \mathrm{clcone}\left\{\frac{2}{\alpha^2+\beta^2}(\alpha,\beta,1) : (\alpha,\beta) \in S\}\right\} \\ &= \mathrm{clcone}\,\{(\alpha,\beta,1) : (\alpha,\beta,1) \in S \times \{1\}\} \\ &= \mathrm{clcone}(S \times \{1\}) = \mathrm{cl}\mathbb{R}_+\mathrm{conv}(S \times \{1\}) \\ &= \mathrm{cl}\mathbb{R}_+\,(\mathrm{conv}(S) \times \{1\})\,; \end{aligned}$$

hence $\left(\mathrm{cl}K^r_T(s)\right) \cap \left(\mathbb{R}^2 \times \{1\}\right) = \mathrm{conv}(S) \times \{1\}$, which, since

$$\mathrm{conv}\,(S) = \{(\alpha,\beta) \in \mathbb{R}^2 : (\alpha-1)^2 + (\beta-1)^2 \le 1,\ \alpha \le 1,\ \beta \le 1\},$$

shows that the cone $\mathrm{cl}K^r_T(s)$ is not polyhedral.

We have the following generalization of a result on classical Voronoi cells [10, Proposition 8], establishing a necessary condition for polyhedrality of power cells; our proof is much simpler than that of [10, Proposition 8].

Proposition 2.8 *If $C_T^r(s)$ is a polyhedron, then* $\operatorname{clcone}(T-s)$ *is polyhedral.*

Proof If $C_T^r(s)$ is a polyhedron, then its recession cone is polyhedral; since, by Proposition 2.6, one has $\operatorname{clcone}(T-s) = (T-s)^{00} = \left(0^+ C_T^r(s)\right)^0$, and the result follows from the fact that the negative polar of a polyhedral cone is polyhedral, too. □

Notice that the converse implication fails. Let $n \geq 2$, and consider any non-empty and non polyhedral closed convex set in $\mathbb{R}^n$ with polyhedral recession cone. Then, by [10, Theorem 2], it can be written as $V_T(s)$ for some $T \subseteq \mathbb{R}^n$ and some $s \in T$, but $\operatorname{clcone}(T-s) = \left(0^+ V_T(s)\right)^0$ (see the proof of Proposition 2.8), so that $\operatorname{clcone}(T-s)$ is polyhedral.

For the validity of the converse to Proposition 2.7, one needs extra assumptions on T and r, as the ones considered in the following proposition.

Proposition 2.9 *If T is unbounded and $\liminf_{t\in T,\ \|t\|\to\infty} \frac{r(t)}{||t||} = 0$, then $\operatorname{cl}K_T^r(s)$ is polyhedral if and only if $C_T^r(s)$ is a polyhedron.*

Proof By Proposition 2.7, if $\operatorname{cl}K_T^r(s)$ is polyhedral then $C_T^r(s)$ is a polyhedron. For the converse implication, we will use [9, Theorem 5.13(i)], according to which, if $C_T^r(s)$ is a polyhedron, then $\operatorname{clcone}\left(\{(t-s, b_r(t,s)) | t \in T\} \cup \{(\mathbf{0}, 1)\}\right)$ is polyhedral, so the thesis will follow from the equality

$$\operatorname{cl}K_T^r(s) = \operatorname{clcone}\left(\{(t-s, b_r(t,s)) : t \in T\} \cup \{(\mathbf{0}, 1)\}\right), \tag{5}$$

which we will now prove. To this aim, take a sequence $t_k \in T$ such that $||t_k|| \to \infty$ and $\frac{r(t_k)}{||t_k||} \to 0$. Since

$$\frac{2}{||t_k - s||^2} b_r(t_k, s) = \frac{1 - \frac{||s||^2}{||t_k||^2} + \frac{r^2(s)}{||t_k||^2} - \frac{r^2(t_k)}{||t_k||^2}}{1 + \frac{||s||^2}{||t_k||^2} - \frac{2\langle t_k, s\rangle}{||t_k||^2}},$$

we have

$$\frac{2}{||t_k - s||^2}(t_k - s, b_r(t_k, s)) \to (\mathbf{0}, 1).$$

This proves (5), as we needed. □

Let us recall the definition of the Bouligand tangent cone of a set $X \subseteq \mathbb{R}^n$ at $x^0 \in X$:

$$\mathcal{B}(S, x^0) := \{d \in \mathbb{R}^n : \exists\{x_k\} \subset X, x_k \to x^0, \exists\{\alpha_k\}, \alpha_k \to +\infty : \alpha_k(x_k - x^0) \to d\}.$$

The following results partially extend [10, Proposition 10].

Proposition 2.10 *If $\limsup_{t\in T, t\to s} \frac{r^2(s) - r^2(t)}{||s-t||^2} < +\infty$, then $C_T^r(s) \subseteq s + \mathcal{B}(T, s)^0$.*

Proof Let $x \in C_T^r(s)$ and $d \in \mathcal{B}(T, s)$. Then $d = \lim \lambda_k (t_k - s)$ for sequences $t_k \in T$, $t_k \to s$, and $\lambda_k \to +\infty$. We have

$$\langle t_k - s, x \rangle \leq b_r(t_k, s) = \frac{1}{2}\left(||t_k||^2 - ||s||^2 + r^2(s) - r^2(t_k)\right);$$

hence

$$\langle t_k - s, x - s \rangle \leq \frac{1}{2}\left(||t_k - s||^2 + r^2(s) - r^2(t_k)\right).$$

Multiplying by λ_k and passing to the limit for $t_k \to s$, $\lambda_k \to +\infty$, we obtain

$$\langle d, x - s \rangle \leq \frac{1}{2} \lim \lambda_k \left(||t_k - s||^2 + r^2(s) - r^2(t_k)\right).$$

By the hypothesis, there exists $M \geq 0$ such that, for k large enough,

$$r^2(s) - r^2(t_k) \leq M||s - t||^2.$$

Then

$$\langle d, x - s \rangle \leq \frac{1}{2} \lim \lambda_k \left(||t_k - s||^2 + M||s - t_k||^2\right) = \frac{1+M}{2} \lim \lambda_k ||t_k - s||^2 = 0,$$

which proves that $x - s \in \mathcal{B}(T, s)^0$ or, equivalently, $x \in s + \mathcal{B}(T, s)^0$. □

Corollary 2.4 *If* $\limsup_{t \in T, t \to s} \frac{r^2(s) - r^2(t)}{||s-t||^2} < +\infty$*, then*

i) $\dim C_T^r(s) \leq n - \dim lin\mathcal{B}(T, s)$.
ii) If, in addition, $\mathcal{B}(T, s) = \mathbb{R}^n$*, then* $C_T^r(s) \subseteq \{s\}$.

Proof Inequality i) follows from the inclusions

$$C_T^r(s) \subseteq s + \mathcal{B}(T, s)^0 \subseteq s + (\text{lin}\mathcal{B}(T, s))^\perp.$$

Statement ii) follows directly from Proposition 2.10 and i). □

According to [10, Theorem 2], every nonempty closed convex set $F \subseteq \mathbb{R}^n$ is a Voronoi cell; more specifically, for every $s \in F$ there exists a (closed) set $T \subseteq \mathbb{R}^n$ such that $s \in T$ and $V_T(s) = F$. The situation is quite different for general power sets, when r is not constant. Observe, for instance, that, using (2), it is easy to see that, for $T \subseteq \mathbb{R}^n$ and $s \in T$, one has, $C_T^{\|\cdot\|_{|T}}(s) = (T - s)^0$. Consequently, for a nonempty closed convex set $F \subseteq \mathbb{R}^n$, a necessary and sufficient condition for the existence of $T \subseteq \mathbb{R}^n$ and $s \in T$ such that $C_T^{\|\cdot\|_{|T}}(s) = F$ is that F be a cone. Indeed, if F is a closed convex cone, one has $\mathbf{0} \in F^0$ and $C_{F^0}^{\|\cdot\|_{|F^0}}(0) = F^{00} = F$. We will next address the problem of finding sufficient conditions for a nonempty closed convex set to be a power cell. The following translation result will be a useful tool.

Proposition 2.11 *Let $q \in \mathbb{R}^n$ and $T_q := T - q$, and define $r_q : T_q \to \mathbb{R}_+$ by $r_q(t) := r(t+q)$. Then*

$$C_{T_q}^{r_q}(s-q) = C_T^r(s) - q.$$

Proof It is a routine exercise, using the identity

$$b_{r_q}(t-q, s-q) + \langle t-s, q\rangle = b_r(t,s).$$

$\square$

Our next lemma gives a sufficient condition, in terms of support functions, for a closed convex set to be a power cell. We recall that the support function of $F \subseteq \mathbb{R}^n$ is $\sigma_F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, defined by $\sigma_F(x^*) := \sup_{x \in F}\langle x, x^*\rangle$. The condition provided by the lemma is very difficult to check in practice, but it will be useful to obtain the much easier to check sufficient condition given in the subsequent theorem.

Lemma 2.1 *Let $F \subsetneq \mathbb{R}^n$ be nonempty, convex and closed, $\phi : \mathbb{R}^n \to \mathbb{R}_+$, and $s \in \mathbb{R}^n$. If, for every $x^\star \in \mathbf{S} \cap \sigma_F^{-1}(\mathbb{R})$, there exists $\alpha_{x^\star} > 0$ such that*

$$\alpha_{x^\star}(\sigma_F(x^\star) - \langle x^\star, s\rangle) = \frac{1}{2}\left(\alpha_{x^\star}^2 + \phi^2(s) - \phi^2(\alpha_{x^\star}x^\star + s)\right), \tag{6}$$

then there exists $T \subseteq \mathbb{R}^n$ such that $s \in T$ and $C_T^{\phi|_T}(s) = F$.

If, moreover, F is a polyhedron, then we can take T finite.

Proof We first consider the case when $s = \mathbf{0}$. Since F is closed and convex, we have

$$F = \{x \in \mathbb{R}^n : \langle x^\star, x\rangle \le \sigma_F(x^\star), \text{ for all } x^\star \in U\}$$

for some $U \subseteq \mathbf{S} \cap \sigma_F^{-1}(\mathbb{R})$. Multiplying both sides of the inequality $\langle x^*, x\rangle \le \sigma_F(x*)$ with the strictly positive number α_{x^*} defined by (6) and setting $t_{x^*} := \alpha_{x^*}x^*$, we see that that inequality can be equivalently written $\langle t_{x^*}, x\rangle \le \frac{1}{2}\left(\|t_{x^*}\|^2 + \phi^2(0) - \phi^2(t_{x^*})\right)$. Hence, by (2), we have $F = C_T^\phi(0)$ for

$$T := \{\mathbf{0}\} \cup \{t_{x^\star} : x^\star \in U\}.$$

We now consider an arbitrary $s \in \mathbb{R}^n$, and define $F_s := F - s$ and $\phi_s : \mathbb{R}^n \to \mathbb{R}_+$ by $\phi_s(t) := r(t+s)$. Then, (6) yields $\alpha_{x^\star}\sigma_{F_s}(x^\star) = \frac{1}{2}\left(\alpha_{x^\star}^2 + \phi_s^2(\mathbf{0}) - \phi_s^2(\alpha_{x^\star}x^\star)\right)$, so we obtain the existence of $T' \subseteq \mathbb{R}^n$ such that $\mathbf{0} \in T'$ and $C_{T'}^{\phi_s}(\mathbf{0}) = F_s$. Setting $T := T' + s$, we have $T' = T_s$, so that, by Proposition 2.11, we have $C_T^\phi(s) = C_{T'}^{\phi_s}(\mathbf{0}) + s = F_s + s = F$.

Notice that if F is a polyhedron and U is minimal in the definition of F, then U is finite and, as a consequence, T is finite. $\square$

Our main sufficient condition for a nonempty closed convex set F to be a power cell will be expressed in terms of the oriented distance function. The oriented distance function to a nonempty proper subset $F \subseteq \mathbb{R}^n$ is the function $\Delta_F : \mathbb{R}^n \to \mathbb{R}$ defined by $\Delta_F(x) := d_F(x) - d_{\mathbb{R}^n \setminus F}(x)$, with $d_F : \mathbb{R}^n \to \mathbb{R}$ denoting the Euclidean distance function to F, given by $d_F(x) := \inf_{s \in F} \|x - s\|$. It is well known that, if F is a convex set, then Δ_F is a convex function [11, Proposition 4]. Furthermore, as proved in [6, (10) and Remark 4.2], one has $\Delta_F(x) = \max_{x^\star \in \mathbf{S}} \{\langle x, x^* \rangle - \sigma_F(x^\star)\}$.

Theorem 2.1 *Let $F \subseteq \mathbb{R}^n$ be convex and closed, $s \in \mathbb{R}^n$, and $\phi : \mathbb{R}^n \to \mathbb{R}_+$. If ϕ is continuous and satisfies*

$$\liminf_{t \to s} \frac{\phi^2(t) - \phi^2(s)}{||t - s||} > 2\Delta_F(s) \tag{7}$$

and

$$\limsup_{||t|| \to \infty} \frac{\phi(t)}{||t||} < 1,$$

then there exists $T \subseteq \mathbb{R}^n$ such that $s \in T$ and $C_T^{\phi|_T}(s) = F$.

If, moreover, F is a polyhedron, then one can take T finite.

Proof We first consider the case when $s = \mathbf{0}$. According to Lemma 2.1, it will be enough to prove that, for every $x^\star \in \mathbf{S}$, the equation

$$f(\alpha) = \sigma_F(x^\star), \tag{8}$$

with $f(\alpha) := \frac{1}{2}\left(\alpha + \frac{\phi^2(\mathbf{0}) - \phi^2(\alpha x^\star)}{\alpha}\right)$, has a strictly positive solution $\alpha > 0$. Because of (7), we have

$$\begin{aligned}\limsup_{\alpha \to 0^+} f(\alpha) &= \frac{1}{2} \limsup_{\alpha \to 0^+} \frac{\phi^2(\mathbf{0}) - \phi^2(\alpha x^\star)}{\alpha} \le \frac{1}{2} \limsup_{t \to \mathbf{0}} \frac{\phi^2(\mathbf{0}) - \phi^2(t)}{||t||} \\ &< -\Delta_F(\mathbf{0}) = \min_{y^\star \in \mathbf{S}} \sigma_F(y^\star) \le \sigma_F(x^\star).\end{aligned}$$

On the other hand, from

$$\liminf_{\alpha \to +\infty} \left(1 - \frac{\phi^2(\alpha x^\star)}{\alpha^2}\right) = 1 - \limsup_{\alpha \to +\infty} \frac{\phi^2(\alpha x^\star)}{\alpha^2} \ge 1 - \left(\limsup_{||t|| \to \infty} \frac{\phi(t)}{||t||}\right)^2 > 0$$

we deduce that, for α large enough, $1 - \frac{\phi^2(\alpha x^\star)}{\alpha^2}$ is bounded below by a positive number; hence

$$\liminf_{\alpha \to +\infty} f(\alpha) = \frac{1}{2} \liminf_{\alpha \to +\infty} \alpha \left(1 - \frac{\phi^2(\alpha x^\star)}{\alpha^2}\right) = +\infty.$$

Since ϕ is continuous, from the inequalities

$$\limsup_{\alpha \to 0^+} f(\alpha) < \sigma_F(x^\star) < +\infty = \lim_{\alpha \to +\infty} f(\alpha),$$

using the imtermediate value theorem we deduce the existence of a solution $\alpha > 0$ to equation (8), as we wanted to prove.

For the general case when $s \in \mathbb{R}^n$ is arbitrary, by replacing s, ϕ and F with $\mathbf{0}$, $\phi_s : \mathbb{R}^n \to \mathbb{R}_+$ defined by $\phi_s(t) := \phi(t+s)$ and $F - s$, respectively, we obtain the existence of $T' \subseteq \mathbb{R}^n$ such that $\mathbf{0} \in T'$ and $C_{T'}^{\phi_s}(\mathbf{0}) = F - s$.; then, setting $T := T' + s$, by Proposition 2.11 we get $C_T^{\phi}(s) = C_{T'}^{\phi_s}(\mathbf{0}) + s = F$.

As in the proof of Lemma 2.1, notice that if F is a polyhedron and U is minimal in the definition of F, then U is finite and, as a consequence, T is finite. □

Remark 2.1 The assumptions of Theorem 2.1 are satisfied when $s \in \mathrm{int}F$ and r is constant; therefore, one obtains the particular case when $s \in \mathrm{int}F$ of [10, Theorem 2] as a direct consequence of Theorem 2.1.

3 The Interior of Power Cells

The following expression for the interior of power cells is an easy consequence of (2).

Proposition 3.1 *One has*

$$\mathrm{int}C_T^r(s) = \left\{ x \in \mathbb{R}^n : \inf_{t \in T \setminus \{s\}} \frac{b_r(t,s) - \langle t - s, x\rangle}{||t-s||} > 0 \right\}.$$

As in the case of Theorem 2.1, a straightforward application of Gale Theorem [9, Propositions 1 and 2 in Table 3.1] yields the following characterization of the nonemptiness of the interior of power cells.

Proposition 3.2 *The following statements are equivalent:*

i) $\mathrm{int}C_T^r(s) \neq \emptyset$.
ii) $(\mathbf{0}, -1) \notin \bigcap_{\epsilon > 0} \mathrm{clcone}\{(t - s, b_r(t,s) - \epsilon||t-s||) \mid t \in T \setminus \{s\}\}$.

A proof almost identical to that of Proposition 2.2, with obvious changes and using Proposition 3.2, yields the following sufficient condition for the nonemptiness of the interior of power cells.

Proposition 3.3 *If* $s \notin \mathrm{clconv}(T \setminus \{s\})$ *and there exists* $\epsilon > 0$ *such that the function* $b_r(\cdot, s) - \epsilon \|\cdot - s\|$ *is bounded, then* $\mathrm{int}C_T^r(s) \neq \emptyset$.

Corollary 3.1 *If* T *is finite and* s *is an extreme point of* $\mathrm{conv}T$*, then*

$$\mathrm{int}C_T^r(s) \neq \emptyset.$$

Proof Since s is an extreme point of conv$T$ and T is finite, we have

$$s \notin \text{conv}\left((\text{conv}T) \setminus \{s\}\right) \supseteq \text{conv}(T \setminus \{s\}) = \text{clconv}(T \setminus \{s\}),$$

so that the assumptions of Proposition 3.3 are satisfied. □

By strengthening the inequalities in Proposition 2.4.ii, one gets a characterization of the inclusion $s \in \text{int}C_T^r(s)$.

Proposition 3.4 *One has* $s \in \text{int}C_T^r(s)$ *if and only if there exists* $\epsilon > 0$ *such that*

$$r^2(t) - r^2(s) \leq ||t-s||^2 - \epsilon||t-s||, \textit{ for all } t \in T \tag{9}$$

More specifically, the inequalities (9) hold if and only if

$$s + \frac{\epsilon}{2}\mathbf{B} \subseteq C_T^r(s).$$

Proof The inclusion $s \in \text{int}C_T^r(s)$ is equivalent to the existence of $\delta > 0$ such that $s + \delta y \in C_T^r(s)$ for all $y \in \mathbf{B}$, which means that

$$\langle t-s, s\rangle + \delta\langle t-s, y\rangle \leq b_r(t,s), \text{ for all } t \in T \setminus \{s\} \text{ and } y \in \mathbf{B},$$

and this can be easily seen to be equivalent to (9) with $\epsilon := 2\delta$. □

Condition (9) can be weakened when $K_T^r(s)$ is closed. The following result generalizes [10, Proposition 15(ii)].

Proposition 3.5 *If* $r^2(t) - r^2(s) < ||t-s||^2$ *for all* $t \in T\setminus\{s\}$ *and* $K_T^r(s)$ *is closed, then* $s \in \text{int}C_T^r(s)$.

Proof It is easy to check that the inequalities $r^2(t) - r^2(s) < ||t-s||^2$ for all $t \in T\setminus\{s\}$ mean that s satisfies with strict inequality all the inequalities in the representation (2) except the trivial one corresponding to $t = s$, that is, s is a Slater point. Hence, since $K_T^r(s)$ being closed implies that the linear inequality system in the representation (2) is locally Farkas-Minkowski in the sense of [16], by [16, Corollary 4.1(i)] we conclude that $s \in \text{int}C_T^r(s)$. □

The assumption that $K_T^r(s)$ is closed is not superfluous in Proposition 3.5. Consider, for instance, the case when $T := \mathbb{R}^n$; then, for every $s \in \mathbb{R}^n$ one has $V_T(s) = \{s\}$. Notice that, in this case, $K_T^r(\mathbf{0}) = (\mathbb{R}^n \times (0, +\infty)) \cup \{(\mathbf{0}, 0)\}$.

4 Conditions for Power Cells to Make a Tessellation of $\mathbb{R}^n$

A covering of $\mathbb{R}^n$ is said to be a tesselation if every two members of the covering have their intersection contained in the intersection of their boundaries. We begin this section by proving that power cells satisfy this intersection condition.

Proposition 4.1 *If* $s_1, s_2 \in T$ *with* $s_1 \neq s_2$, *then* $C_T^r(s_1) \cap \operatorname{int} C_T^r(s_2) = \emptyset$.

Proof If $x \in C_T^r(s_1) \cap \operatorname{int} C_T^r(s_2)$, then, by (2), we have $\langle s_2 - s_1, x\rangle \leq b_r(s_2, s_1)$ and $\langle s_1 - s_2, x\rangle < b_r(s_1, s_2)$, which contradicts the equality

$$b_r(s_2, s_1) + b_r(s_1, s_2) = 0.$$

□

Recalling that $\operatorname{int} A \cap \operatorname{int} B = \operatorname{int}(A \cap B)$, one obtains the following corollary.

Corollary 4.1 *If* $s_1, s_2 \in T$ *with* $s_1 \neq s_2$, *then* $\operatorname{int}\left(C_T^r(s_1) \cap C_T^r(s_2)\right) = \emptyset$.

Proof We have

$$\operatorname{int}\left(C_T^r(s_1) \cap C_T^r(s_2)\right) = \operatorname{int} C_T^r(s_1) \cap \operatorname{int} C_T^r(s_2) \subseteq C_T^r(s_1) \cap \operatorname{int} C_T^r(s_2) = \emptyset.$$

□

Recalling that a real-valued function is upper semi-continuous (u.s.c.) when its hypograph is closed, the following results give sufficient conditions for a point to belong to some power cell.

Lemma 4.1 *Let* T *be unbounded and closed,* r *be u.s.c., and* $x \in \mathbb{R}^n$. *If*

$$||x|| < \frac{1}{2} \liminf_{\substack{t \in T \\ ||t|| \to \infty}} \left(||t|| - \frac{r^2(t)}{||t||}\right),$$

then $x \in \bigcup_{s \in T} \operatorname{int} C_T^r(s)$.

Proof Notice that, for $s \in T$, we have $x \in C_T^r(s)$ if and only if s is a global minimum of $p_r(x, \cdot)$ over T, and that, if x satisfies the inequality in the statement, so does every point sufficiently close to x. Thus, it will suffice to prove that, if the limit condition in the statement holds, then $p_r(x, \cdot)$ is coercive. This coercivity condition is proved as follows.
Take $\alpha \in \left(||x||, \frac{1}{2} \liminf_{t \in T,\, ||t|| \to \infty} \left(||t|| - \frac{r^2(t)}{||t||}\right)\right)$. We have $p_r(x, t) = \|x\|^2 + \|t\| \left(-2\left\langle x, \frac{t}{\|t\|}\right\rangle + \|t\| - \frac{r^2(t)}{||t||}\right)$; hence, if $||t||$ is large enough,

$$\begin{aligned} p_r(x, t) &> \|x\|^2 + \|t\| \left(-2\left\langle x, \frac{t}{\|t\|}\right\rangle + 2\alpha\right) \geq \|x\|^2 - 2\|x\|\,\|t\| + 2\alpha\,\|t\| \\ &= \|x\|^2 + 2(\alpha - \|x\|)\,\|t\|. \end{aligned}$$

Therefore, if $x \neq \mathbf{0}$, we have $\lim_{t \in T,\, ||t|| \to \infty} p_r(x, t) = +\infty$, as we had to prove. On the other hand, $p_r(\mathbf{0}, t) = ||t|| \left(||t|| - \frac{r^2(t)}{||t||}\right)$; hence, since for $||t||$ large enough we have $||t|| - \frac{r^2(t)}{||t||} > \beta$ for some $\beta > 0$, we deduce that

$$\lim_{t\in T,\ ||t||\to\infty} p_r\left(\mathbf{0}, t\right) = +\infty,$$

that is, $p_r\left(\mathbf{0}, \cdot\right)$ is coercive. This ends the proof. □

Proposition 4.2 *Let T be unbounded and closed, r be u.s.c., and $q \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ satisfies*

$$||x - q|| < \frac{1}{2}\liminf_{t\in T-q,\ ||t||\to\infty}\left(||t+q|| - \frac{r^2(t+q)}{||t+q||}\right), \tag{10}$$

then

$$x \in \bigcup_{s\in T} \mathrm{int} C_T^r(s). \tag{11}$$

Proof Set $T' := T + q$, and define $r' : \mathbb{R}^n \to \mathbb{R}_+$ by $r'(t) := r(t+q)$. Then, making the change of variable $t' := t + q$ in (10), we obtain

$$||x - q|| < \frac{1}{2}\liminf_{t'\in T,\ ||t'||\to\infty}\left(||t'|| - \frac{r'^2(t')}{||t'||}\right);$$

hence, by Lemma 4.1, we have $x - q \in \bigcup_{s\in T} \mathrm{int} C_{T'}^{r'}(s' - q) = \bigcup_{s\in T} \mathrm{int} C_T^r(s) - q$, which proves (11). □

Corollary 4.2 *Let T be unbounded and closed, r be u.s.c., and $q \in \mathbb{R}^n$. If*

$$\liminf_{t\in T-q,\ ||t||\to\infty}\left(||t+q|| - \frac{r^2(t+q)}{||t+q||}\right) > 0,$$

then $q \in \bigcup_{s\in T} \mathrm{int} C_T^r(s)$.

Our last results generalize [10, Proposition 1] .

Theorem 4.1 *Let T be closed and r be u.s.c. If either T is bounded or*

$$\lim_{t\in T,\ ||t||\to\infty}\left(||t|| - \frac{r^2(t)}{||t||}\right) = +\infty,$$

then $\{C_T^r(s)\}_{s\in T}$ is a tesselation of $\mathbb{R}^n$.

Proof This is an immediate consequence of Lemma 4.1 and Proposition 4.1. □

Corollary 4.3 *Let T be closed and r be u.s.c. If T is unbounded and*

$$\limsup_{t\in T,\ ||t||\to\infty} \frac{r(t)}{||t||} < 1,$$

then $\{C_T^r(s)\}_{s\in T}$ is a tesselation of $\mathbb{R}^n$.

Proof It follows directly from Theorem 4.1, noticing that

$$||t|| - \frac{r^2(t)}{||t||} = ||t|| \left(1 - \frac{r^2(t)}{||t||^2}\right).$$

□

5 Conclusions

There is an extensive literature on power cells (see, e.g. [2]), but, to the best of our knowledge, a systematic study of their fundamental properties has not been carried out so far. In the current paper, we have undertaken such a study, focusing mainly on obtaining conditions on the weight function r under which power cells preserve the main properties that classical Voronoi cells have. As is the case of Voronoi cells induced by the Euclidean distance, power cells corresponding to the Euclidean norm are closed convex sets. We have obtained conditions for their nonemptiness as well as for that of their interiors, and we have studied other properties, including boundedness and polyhedrality. We have also proved that, under simple assumptions (for instance, when the set of sites is finite), the power cells make a tessellation of the space. We have illustrated the fact that power cells may behave quite differently from their classical counterparts; for example, even when they are nonempty, they do not necessarily contain the site they are associated with.

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