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On the stability of elastic structures subjected to follower forces

F. Levi · A. Carini

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Abstract Within the framework of a second-order theory, this study revisits classical stability problems characterized by critical loads associated with dynamic instability, previously explored by Feriani and Carini (in: Zingoni (ed) Insights and innovations in structural engineering, mechanics and computation, 2016) . The primary focus remains on systems exclusively comprising a single lumped mass. This simplification, together with the assumption of a negligible axial strain and the adoption of a second-order theory, reduces the analysed systems to problems featuring a single Lagrangian coordinate. Consequently, static methods become applicable, facilitating the derivation of analytical expressions for stiffness coefficients and enabling an investigation into dynamic stability. The study begins by examining a well-established case: a cantilever beam with a lumped mass positioned at its free end, subjected to a follower load, as presented in Panovko and Gubanova (Stability and oscillations of elastic systems: models, paradoxes and errors, Nunka Press, Moscow, 1967). In this current work, a novel lumped mass system is explored. The system consists of a straight-axis beam characterized by a constant cross-section area and stiffness. The distributed mass of the beam is neglected and the beam is hinged at one end, simply supported at an intermediate point, and left free at the other end, where a lumped mass is introduced. Various loading scenarios are scrutinized, including: (a) The application of a follower force to the free end; (b) The imposition of two forces—one conservative and one follower—both applied to the free end; and (c) The application of a uniformly distributed follower force along the length of the beam. As seen in the examples introduced in Feriani and Carini (2016), the new examples considered in the present paper reveal that the first asymptote of the stiffness coefficient corresponds to the critical load. This critical load corresponds to the phenomenon known as *divergence at infinity*, as described by Felippa (Nonlinear finite element methods, 2014). It is also confirmed that, in all cases, the first dynamical critical load equals the minimum value between the static buckling load of the original structure and the static buckling load of an auxiliary structure. This last differs from the original one because the concentrated mass is replaced by a constraint fixing the corresponding Lagrangian coordinate.

Keywords Dynamic stability · Follower forces · Divergence at infinity

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1 Introduction

The study of the stability of elastic systems subjected to non-conservative forces, such as follower and aerodynamic forces, has been the focus of extensive theoretical and applied research for many years (see, e.g., [3–9], [18], [22–31], [36–38]).

Recently, this field has been the subject of renewed interest, notably due to the influential work of Prof. Bigoni's school on the concept of follower forces, a concept that has faced occasional scrutiny and skepticism [10–16, 34].

It is widely acknowledged that the presence of non-conservative forces typically prevents the association of stability problems with functionals only dependent on generalized displacements, thus precluding the formulation of stability problems in classical variational terms. Leipholz devoted a considerable effort "to reconcile the fact that structural problems with follower forces are nonconservative with the fact that in some how a variational approach should yet be possible" (Leipholz [28], p. 45).

In the wake of the results established by Gurtin and Tonti [21, 35], Leipholz [28] introduced a convolutive functional, with respect to the spatial coordinates, whose quadratic part turns out to be undefined, providing a variational principle of stationarity, though not of extremality, applicable to non-conservative systems with uniformly distributed mass such as the Beck rod.

Pflüge [33] explored cases where the bar had a concentrated mass M at the end, in addition to the distributed mass m. Numerical solutions of the resulting equations are well-documented in the literature on structural stability. The two extreme cases, $m = 0$ or $M = 0$, yield closely related critical loads. When $M = 0$, the case of Beck's rod, with a critical load $P = 20.05EJ/l^2$, is recovered. When the distributed mass of the rod is negligible, then the critical load becomes $P = 20.19EJ/l^2$. In [20] the latter case was discussed. It regarded the classical stability problem of a cantilever beam with a straight axis, no damping, a concentrated mass M at a free end, and subjected to various follower loads (see Figs. 1a, b, c). The axial strain was disregarded and small displacements were assumed.

The present paper explores a novel lumped mass system that consists of a straight-axis beam with a constant cross-section area and stiffness. It is devoid of distributed mass, hinged at one end, simply supported at an intermediate point at an arbitrary distance a from the hinge, and free at the other end, where a lumped mass is present. Three different load cases are considered: (a) A follower force P at the free end (Fig. 2a); (b) A conservative force Q and a follower force P at the free end (Fig. 2b), and (c) A uniformly distributed follower force p (Fig. 2c). In the case of a uniformly distributed mass, Zorii and Chernukha [39] initially studied the system subjected to a follower force P at the free end, later followed by Elishakoff and Hollkamp [17]. It is intuitively apparent that for a $a \to 0$, the results align with those obtained in [20], while for a $a \to l$, the Eulerian critical load is reestablished. This suggests the existence of a transition value a^*, such that for $a < a^*$, a specific dynamic instability, called *instability due to divergence at*

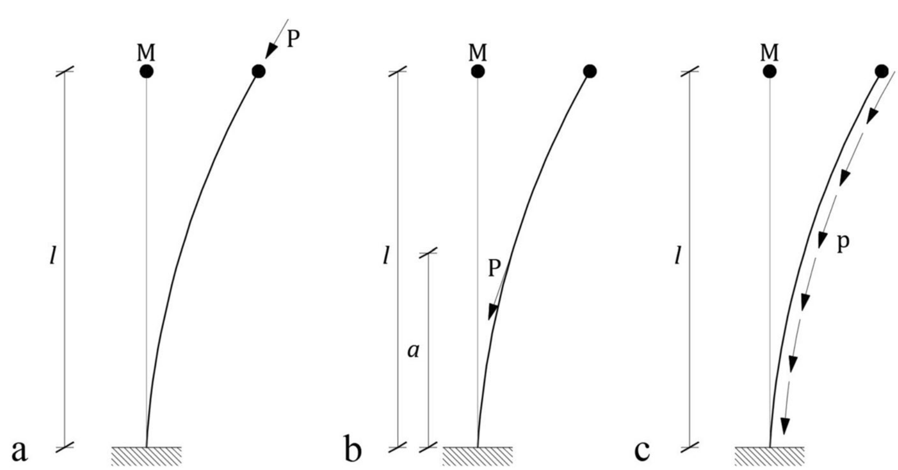


Fig. 1 Cantilever beam with a lumped mass on which **a** a follower force P acts at the free end, **b** a follower force P acts at an intermediate point, and **c** a uniformly distributed follower force p acts

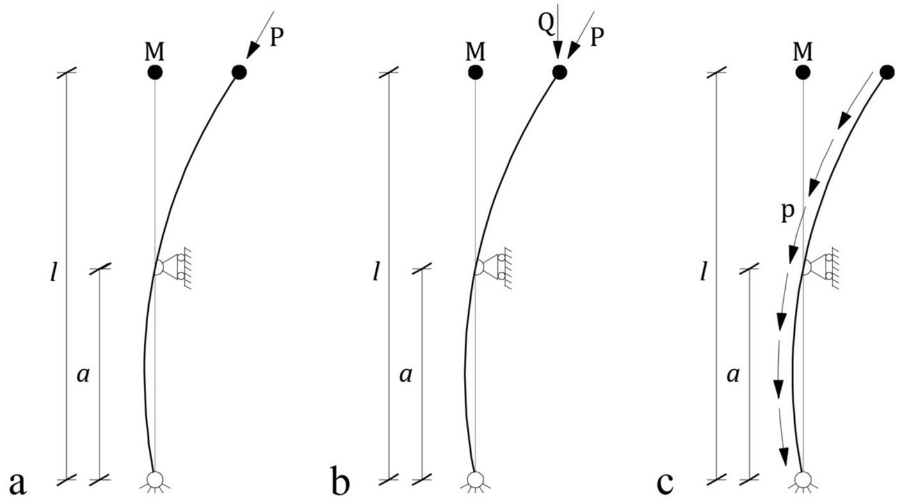


Fig. 2 Beam hinged at one end, free at the other, simply supported at an intermediate point, with mass concentrated at the free end and subject to **a** a follower force *P* at the free end, **b** a conservative force *Q* and a follower force *P* at the free end, and **c** a uniformly distributed follower force *p*

infinity, occurs, while for $a > a^*$, an Eulerian instability due to divergence emerges.

Numerous other intriguing problems await future research, unaddressed in this study. These include structures with two or more concentrated masses or systems featuring both distributed and concentrated masses.

2 An energetic approach to dynamic stability of systems with a single Lagarangian coordinate

Consider an elastic system with a concentrated mass *M* subjected to follower forces. Suppose the system has only one Lagrangian coordinate *u* within a small displacement regime (Fig. 3a). Thus, the equation of motion takes the form:

$$M\ddot{u} + Ku = 0 \tag{1}$$

where the stiffness coefficient *K* accounts for second-order effects. Equation (1) characterises a dynamic system that can be modeled as a lumped mass constrained by a linear spring (see Fig. 3b). The system stability hinges on the sign of the stiffness coefficient. If *K* is positive, the motion is oscillatory and bounded; if negative, the motion is unbounded and non-oscillatory (divergence). Since the system possesses just one Lagrangian coordinate, unbounded oscillatory motion (flutter) is precluded. Indeed, a dynamical system represented as a lumped mass

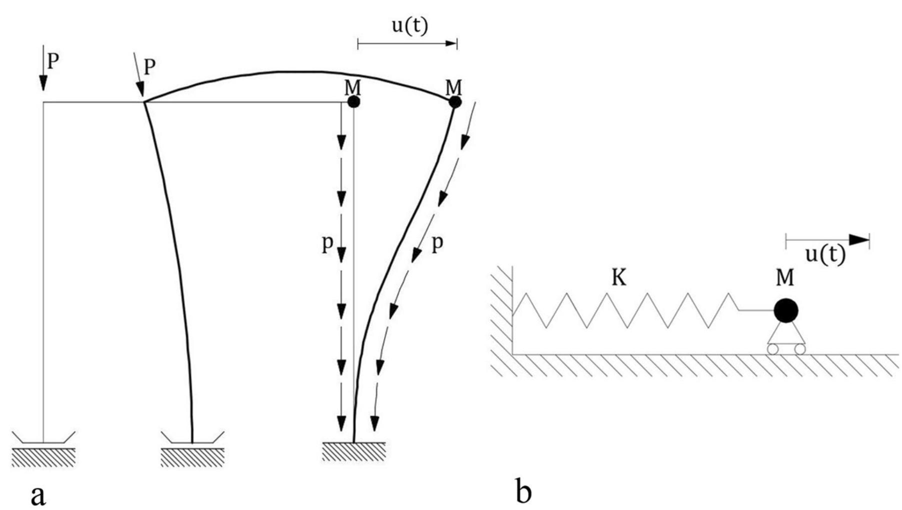


Fig. 3 **a** Elastic system with a concentrated mass *M* and with a single Lagrangian coordinate u in small displacement regime, subjected to follower forces; **b** equivalent dynamic system consisting of a lumped mass constrained by a linear spring

constrained by a linear spring necessarily qualifies as a conservative system with a total energy $\mathcal{E}$ (comprising kinetic and potential components) given by:

$$\mathcal{E} = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}Ku^2. \tag{2}$$

This energy remains conserved during the motion. The stability of the system can be statically examined by applying Dirichlet's theorem which leads to determine the sign of K. A positive K implies that the potential energy in the undeformed initial configuration is minimum, indicating system stability. Conversely, a negative K implies that the potential energy in the undeformed initial configuration is maximum, signalling instability.

For instance, consider the system in Fig. 1a. The stiffness coefficient's dependence on the applied load is non-linear, potentially changing sign due to the presence of an asymptote. As the compressive load gradually increases from zero, the stiffness coefficient increases and, consequently, the vibration frequency rises as well. When the load approaches a critical value, the stiffness coefficient grows without bounds, and the mass displacement shrinks to zero [32]. When the critical load is slightly exceeded, the other branch of the function is involved and the stiffness coefficient becomes negative, indicative of a non-oscillatory, unbounded motion.

Such phenomenon is named *divergence at infinity* [19]. It is a dynamic instability, since it depends on the mass properties of the structure. Surprisingly, despite its dynamic nature, divergence at infinity in single DOF dynamical systems can be studied by analysing the sign of the stiffness coefficient, which therefore makes possible the use of a static approach. The divergence at infinity phenomenon also manifests itself in the structures shown in Figs. 1b and c.

It may seem strange that a structure subjected to non-conservative forces can be associated with a potential. Yet this association holds true for the systems under examination because they have only one degree of freedom. For systems with two or more degrees of freedom (two or more Lagrangian coordinates), an energy formulation in the classical sense becomes impossible because, in these systems, the equations of motion entail asymmetric stiffness matrices, precluding energy formulations. An energy formulation is only feasible when the stiffness matrix is reduced to an order-1 matrix, as in the presence of a single Lagrangian coordinate. This allows us to associate an energy to the problem and employ an energy-based approach to determine the critical load. In the case of a single Lagrangian coordinate, the dynamic instability typical of systems subjected to follower forces shifts from *flutter* to *divergence at infinity*.

3 Appication of a static method for detecting dynamic instability in single-degree-of-freedom systems

3.1 The case of a follower force applied at the free end

Consider the example in Fig. 2a: an elastic straight-axis beam with length l, constant cross-section, moment of inertia J, negligible distributed mass, hinged at one end, simply supported at an intermediate point at a distance a from the hinge, and free at the other end. A concentrated mass M is present at the free end, and a follower force P acts upon it. The equation of motion, arising from an initial perturbation, takes the form of Eq. (1). Here, we determine the stiffness coefficient K by neglecting the mass and applying a transverse static force F at the free end, producing a unit displacement in the direction of F. By choosing the hinged end as the origin for the z-coordinate along the beam axis, denoting Young's modulus as E, the deflected curve $v(z)$ can be obtained by prescribing:

$$EJv_i'''' + Pv_i'' = 0 \quad \text{with} \quad i = 1 \quad \text{for} \quad 0 \le z \le a \quad \text{and} \quad i = 2 \quad \text{for} \quad a \le z \le l \tag{3}$$

with boundary conditions:

$$v_1(0) = 0, \quad v_1''(0) = 0, \quad v_2(l) = 1, \quad v_2''(l) = 0 \tag{4}$$

and with continuity conditions:

$$v_1(a) = v_2(a) = 0, \quad v_1'(a) = v_2'(a), \quad v_1''(a) = v_2''(a). \tag{5}$$

By solving the equation of the deflected curve, it is possible to determine K:

$$K = -EJv_2'''(l) = P\alpha \frac{a\sin(\alpha a)}{\alpha a^2\sin(\alpha l) - \alpha al\sin(\alpha l) + l\sin(\alpha a)[\cos(\alpha a)\sin(\alpha l) - \cos(\alpha l)\sin(\alpha a)]} \tag{6}$$

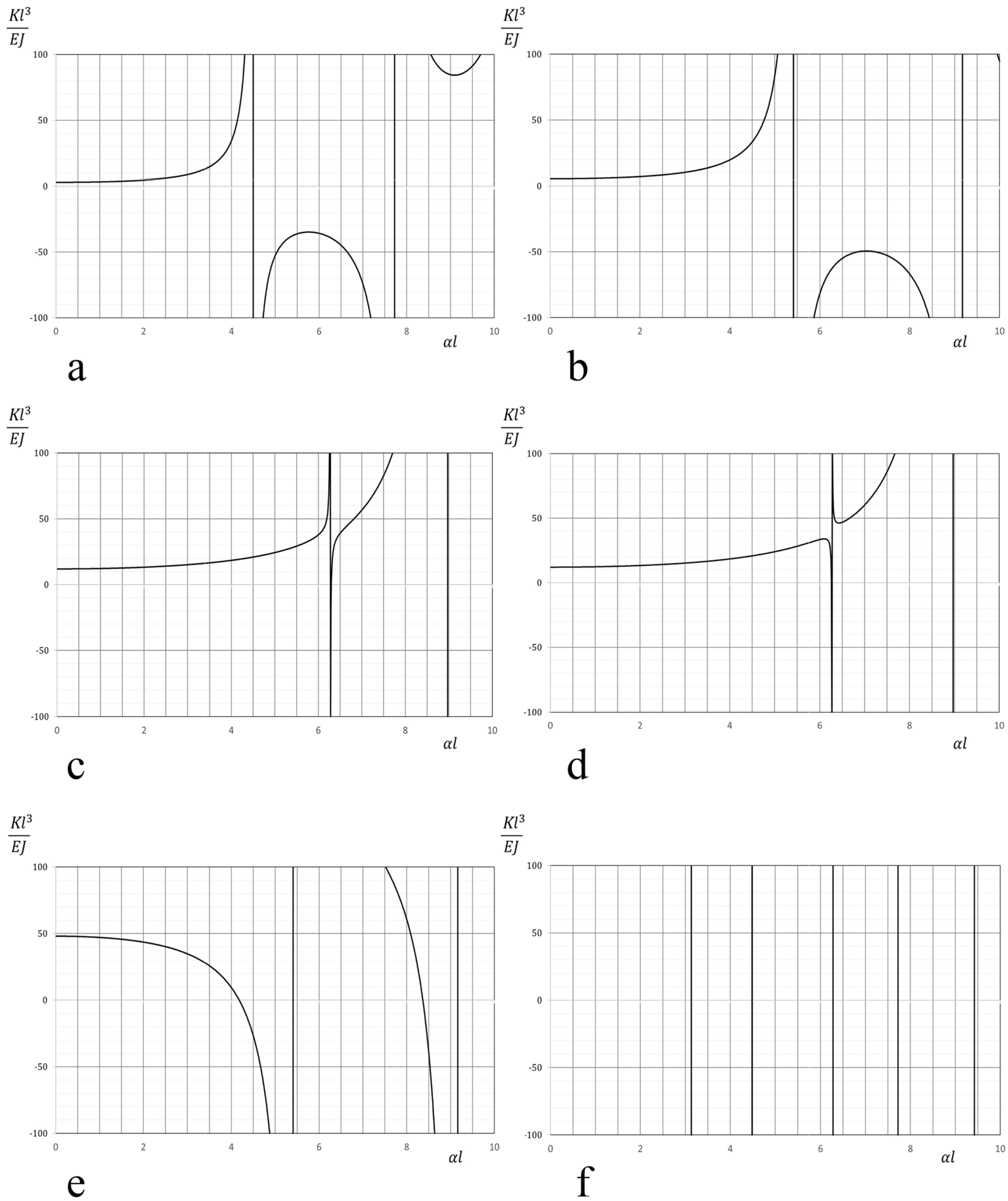


Fig. 4 Dimensionless stiffness Kl^3/EJ versus αl when **a** $c = 0$, **b** $c = 0.25$, **c** $c = 0.5^-$, **d** $c = 0.5^+$, **e** $c = 0.75$, **f** $c = 1$

with $\alpha^2 = P/EJ$. Assuming $c = a/l$, Fig. 4 shows the dimensionless stiffness coefficient Kl^3/EJ as a function of αl for various values of c. When $c = 0$, a well known case is found [32]. In this case, asymptotes occur as

$$\tan \alpha l = \alpha l. \tag{7}$$

The first asymptote corresponds to $Kl^3/EJ = 20.1934$, indicating divergence at infinity. When $c = 1$, another familiar case is obtained: the pinned-pinned Euler rod to which the Eulerian critical load $Kl^3/EJ = 9.8696$ corresponds. Analysing Fig. 4 for increasing values of the ratio c from zero to one, it emerges that the type of stability goes from divergence at infinity to Eulerian divergence, the shift occurring at $c = 0.5$.

3.2 Comparison of the critical load due to divergence at infinity with the Eulerian critical load due to divergence of an auxiliary structure

Consider $c = 0$. As widely recognized, Eq. 7 matches the equation determining the Euler buckling load of a clamped-pinned beam. This correspondence arises because as the critical load is approached by gradually increasing the applied follower load, the stiffness coefficient approaches infinity (Fig. 4a), and the displacement of the free end tends to zero. This phenomenon prevails for all c values between 0 and 0.5, where divergence at infinity occurs. For $c \geq 0.5$, the dynamical critical load coincides with the Eulerian critical load of the massless beam.

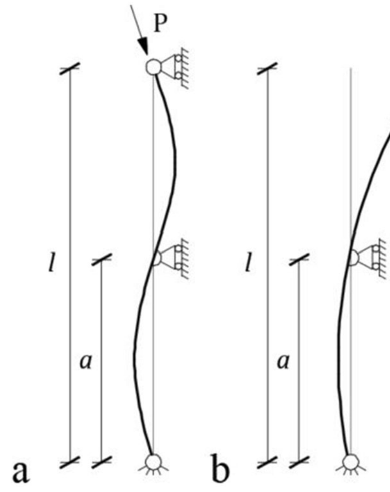


Fig. 5 **a** Auxiliary structure, **b** real beam without lumped mass

A general conjecture highlighted in [20] is here confirmed: when instability arises due to divergence at infinity, in order to compute the associated load it is possible to consider an auxiliary structure differing from the original one in that the lumped mass is replaced by a constraint fixing the corresponding Lagrangian coordinate; a statical calculation can then be performed. If the critical load of this new structure is due to divergence, meaning independent of mass

Table 1 Beam of Fig. 2a. Critical loads for different values of c for **a** the real beam, **b** the auxiliary beam, and **c** the real beam without the lumped mass

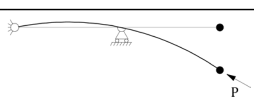
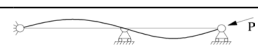
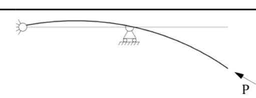
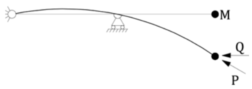
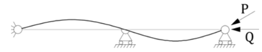
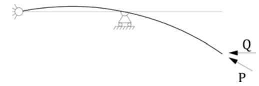
$c = \frac{a}{l}$			
0	20.19342097	**20.19342097**	∞
0.1	23.22477944	**23.22477944**	986.96044011
0.2	27.05331941	**27.05331941**	246.74011003
0.3	31.75504645	**31.75504645**	109.66227112
0.4	36.79994680	**36.79994680**	61.68502751
0.5	39.47841374	**39.47841760**	**39.47841760**
0.6	27.41556778	36.79994680	**27.41556778**
0.7	20.14204980	31.75504645	**20.14204980**
0.8	15.42125688	27.05331941	**15.42125688**
0.9	12.18469679	23.22477944	**12.18469679**
1	9.86960440	20.19342097	**9.86960440**

Table 2 Beam of Fig. 2b. Case $P = Q$. Critical loads for different values of c for **a** the real beam, **b** the auxiliary beam, and **c** the real beam without lumped mass

$c = \frac{a}{l}$ $P = Q$			
0	$\frac{(P+Q)_{cr} l^2}{EJ}$ =9.866812	20.193421	**9.866812**
0.1	10.557424	23.224779	**10.557424**
0.2	11.285784	27.053319	**11.285784**
0.3	12.004017	31.755046	**12.004017**
0.4	12.621084	36.799947	**12.621084**
0.5	13.003321	39.478418	**13.003321**
0.6	13.012767	36.799947	**13.012767**
0.7	12.593363	31.755046	**12.593363**
0.8	11.825351	27.053319	**11.825351**
0.9	10.869127	23.224779	**10.869127**
1	9.869604	20.193421	**9.869604**

distribution, then it coincides with the critical load by divergence at infinity of the original structure.

This finding holds true for the beams in Figs. 2b and c. Hence, to determine the dynamic critical load of a linear elastic structure subjected to follower forces in the small displacements regime, it is generally sufficient to identify the minimum critical load between the one obtained from the introduced auxiliary structure (Fig. 5a) and the statically derived critical load (looking for equilibrium configurations other than the trivial) of the actual structure (Fig. 5b). Table 1 presents the critical load values derived from this approach. It can be seen that for $0 \le c \le 0.5$ the correct critical loads are those of the auxiliary structure, while for $0.5 \le c \le 1$ the correct critical loads are those statically derived from the actual structure.

3.3 The case of a conservative force and a follower force both applied at the free end

Consider the case of the combined action of a follower force P and a conservative "dead" force Q (see Fig. 2b). As in the previous case, the stiffness

Table 3 Beam of Fig. 2b. Case $P = 1.5Q$. Critical loads for different values of c for **a** the real beam, **b** the auxiliary beam, and **c** the real beam without the lumped mass

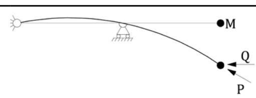
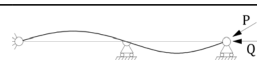
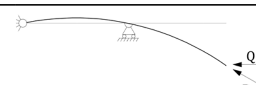
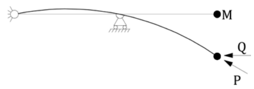
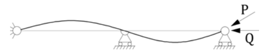
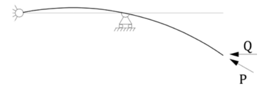
$c = \frac{a}{l}$ $P = 1.5Q$			
0	$\frac{(P+Q)_{cr} l^2}{EJ}$ =20.190719	**20.193421**	1.013006×10^{15}
0.1	23.224779	**23.224779**	986.960440
0.2	27.053319	**27.053319**	115.479420
0.3	31.755046	**31.755046**	94.568172
0.4	36.799947	**36.799947**	76.718834
0.5	18.714629	39.478418	**18.714629**
0.6	15.813116	36.799947	**15.813116**
0.7	14.060247	31.755046	**14.060247**
0.8	12.528197	27.053319	**12.528197**
0.9	11.128291	23.224779	**11.128291**
1	9.869604	20.193421	**9.869604**

Table 4 Beam of Fig. 2b. Case $P = 10Q$. Critical loads for different values of c for **a** the real beam, **b** the auxiliary beam, and **c** the real beam without the lumped mass

$c = \frac{a}{l}$ $P = 10Q$			
0	$\frac{(P+Q)_{cr} l^2}{EJ}$ =20.193420	**20.193421**	1.013006×10^{15}
0.1	23.224779	**23.224779**	986.960440
0.2	27.053319	**27.053319**	246.740110
0.3	31.755046	**31.755046**	104.574488
0.4	36.799947	**36.799947**	65.450961
0.5	39.478418	**39.478418**	39.478418
0.6	25.757606	36.799947	**25.757606**
0.7	18.921044	31.755046	**18.921044**
0.8	14.778474	27.053319	**14.778474**
0.9	11.943393	23.224779	**11.943393**
1	9.869604	20.193421	**9.869604**

coefficient K is determined by applying a transversal static force F at the free end of the beam that produces a unit displacement in the direction of F. In this case, by following the procedure described in Sect. 3.1, the following expression of the stiffness coefficient K is obtained:

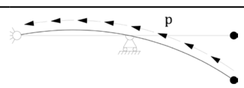
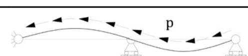
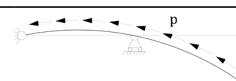
$$K = \frac{P\alpha a \sin(\alpha a) + Q[\alpha a \sin(\alpha l) - \sin(\alpha a)(\cos(\alpha a)\sin(\alpha l) - \cos(\alpha l)\sin(\alpha a))]}{\alpha a^2 \sin(\alpha l) - \alpha a l \sin(\alpha l) + l \sin(\alpha a)(\cos(\alpha a)\sin(\alpha l) - \cos(\alpha l)\sin(\alpha a))}. \quad (8)$$

For $Q = 0$, the case analysed in Sect. 3.1 is recovered. The critical load is reached when the stiffness K is no longer positive. The critical loads are given in the first column of Tables 2, 3, and 4 for different P/Q ratios, particularly for $P/Q = 1$, 1.5, and 10. The critical loads for the case of a hinged free end (auxiliary beam) and an unrestrained massless free end beam are derived by solving the equations:

$$EJv_i'''' + (P+Q)v_i'' = 0 \quad \text{with} \quad i = 1 \quad \text{for} \quad 0 \le z \le a \quad \text{and} \quad i = 2 \quad \text{for} \quad a \le z \le l \quad (9)$$

with, for the first case (auxiliary beam), the boundary and continuity conditions, respectively:

Table 5 Beam of Fig. 2c. Critical loads for different values of c for **a** the real beam, **b** the auxiliary beam, and **c** the real beam without the lumped mass

$c = \frac{a}{l}$			
0	$(p_{cr} l^3)/EJ$ =57.0162	**57.0162**	∞
0.1	69.4495	**69.4495**	1262.9830
0.2	82.4450	**82.4450**	276.1083
0.3	88.7741	**88.7741**	129.0508
0.4	76.9404	80.0298	**76.9404**
0.5	52.3994	65.6796	**52.3994**
0.6	38.8634	53.8126	**38.8634**
0.7	30.6110	45.1049	**30.6110**
0.8	25.2281	38.7692	**25.2281**
0.9	21.5499	34.0918	**21.5499**
1	18.9567	30.5741	**18.9567**

$$v_1(0) = 0, \quad v_1''(0) = 0, \quad v_2(l) = 0, \quad v_2''(l) = 0$$
$$v_1(a) = v_2(a) = 0, \quad v_1'(a) = v_2'(a), \quad v_1''(a) = v_2''(a) \tag{10}$$

while, for the second case (unrestrained massless free end), the fourth condition should be replaced by $EJv_2'''(l) = -Qv_2'(l)$. The values of the critical loads are given in the second and third columns of Tables 2, 3 and 4, respectively. As highlighted by the bold shaded areas, the dynamic instability is either an instability due to divergence at infinity or an Eulerian instability. More precisely, as in the previous case, the dynamic critical load is the minimum between the two static critical loads by divergence.

3.4 The case of a uniformly distibuted follower force

The same analysis applied to the problems in Fig. 2a and b can be extended to the problem in Fig. 2c. Unfortunately, this last problem requires a numerical analysis, typically involving the finite element method. The beam is subdivided into standard straight finite elements with two nodes, neglecting the axial strain. Each node has only two degrees of freedom (transverse displacement and rotation), with cubic interpolation functions. Each element features constant cross-section and homogeneous material. Stiffness and mass matrices are computed dividing the beam into 10 finite elements to ensure accuracy.

In Table 5 the results of the dynamic analyses are compared to the results obtained by the static analyses of the auxiliary beam and of the real beam without the lumped mass. When $0 \leq c \leq 0.3$, the correct critical loads correspond to those of the auxiliary structure, while for $0.4 \leq c \leq 1$, the correct critical loads are statically derived from the actual structure.

4 Conclusions

Within the context of second-order theory, a static method is detected in order to determine the dynamic critical loads in single-degree-of-freedom systems subjected to follower forces, with mass modelled as a lumped mass.

This study focuses on a specific lumped mass system: a straight-axis beam with a constant cross-section area and stiffness, devoid of distributed mass, hinged at one end, simply supported at an intermediate point, and free at the other end, where a lumped mass is present. Various loading scenarios are considered: (a) A follower force applied to the free end; (b) Two forces, one conservative and one follower, both acting at the free end; and (c) A uniformly distributed follower force on the beam.

It is shown that, in all cases, the critical load equals the minimum between the buckling load of the given structure and the buckling load of an auxiliary structure, where, in this last case, the concentrated mass is replaced by a constraint fixing the corresponding Lagrangian coordinate.

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Data availability No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Alliney S, Tralli A (1984) Extended variational formulation and F.E. models for nonlinear beams under nonconservative loading. Compt Meths Appl Mech Engrg 46:177–194
2. Alliney S, Tralli A (1985) Extended variational formulation and F.E. models for nonlinear nonconservative mechanical problems. Compt Meths Appl Mech Engrg 51:209–219

3. Altman W, Marmo de Oliveira A (1984) Application of the quadratic functional to non conservative problems of elastic stability. Comput Struct 18:141–145
4. Argyris JH, Symeonidis S (1981) Nonlinear finite element analysis of elastic systems under nonconservative loading-Natural formulation. Part I. Quasistatic problems. Comput Meths Appl Mech Engrg 26:75–123
5. Argyris JH, Symeonidis S (1981) A sequel to: nonlinear finite element analysis of elastic systems under nonconservative loading-Natural formulation. Part I. Quasistatic problems. Comput Meths Appl Mech Engrg 26:377–383
6. Argyris JH, Staub K, Symeonidis S (1981) Nonlinear finite element analysis of elastic systems under nonconservative loading-Natural formulation. Part II. Dynamic problems. Comput Meths Appl Mech Engrg 28:241–258
7. Ballio G (1967) Formulazione variazionale del problema dell'asta caricata in punta da forze non conservative. Costruzioni metalliche 4:1–7 ((**in italian**))
8. Ballio G (1967) Sulla trave alta sollecitata da carichi di tipo non conservativo. Rendiconti dell'Istituto Lombardo di Scienze e Lettere 101:307–330 (**in italian**)
9. Barsoum RS (1971) Finite element method applied to the problem of stability of a non-conservative system. Internat J Numer Meths Engrg 3:63–87
10. Bigoni D, Noselli G (2011) Experimental evidence of flutter and divergence instabilities induced by dry friction. J Mech Phys Solids 59:2208–2226
11. Bigoli D, Kirillov O, Misseroni D, Noselli G, Tommasini M (2018) Flutter and divergence instability in the Pfluger column: experimental evidence of the Ziegler destabilization paradox. J Mech Phys Solids 116:99–116
12. Bigoni D, Misseroni D, Tommasini M, Kirillov O, Noselli G (2018) Detecting singular weak-dissipation limit for flutter onset in reversible systems. Phys Rev E 97:023003
13. Bigoni D (2019) Flutter from friction in solids and structures. In: Bigoni D, Kirillov ON (eds) Dynamic stability and bifurcation in nonconservative mechanics, Chap 1. CISM Lecture Notes No. 586. Springer
14. Bigoni D, Misseroni D (2020) Structures loaded with a force acting along a fixed straight line, or the "Reut's column problem". J Mech Phys Solids 134:103741
15. Cazzolli A, Dal Corso F, Bigoni D (2020) Non-holonomic constraints inducing flutter instability in structures under conservative loadings. J Mech Phys Solids 138:103919
16. Cazzolli A, Dal Corso F, Bigoni D (2021) Flutter instability and Ziegler destabilization paradox for elastic rods subject to non-holonomic constraints. J Appl Mech 88(3):031003
17. Elishakoff I, Hollkamp J (1987) Computerized symbolic solution for a nonconservative system in which instability occurs by flutter in one range of a parameter and by divergence in another. Comput Meths Appl Mech Engrg 62:27–46
18. Elishakoff I, Pellegrini F (1988) Exact solutions for buckling of some divergence-type nonconservative systems in terms of Bessel and Lommel functions. Comput Meths Appl Mech Engrg 66:107–119
19. Felippa C (2014) *Nonlinear Finite Element Methods*. Department of Aerospace Engineering Sciences, University of Colorado at Boulder, Spring, www.colorado.edu
20. Feriani A, Carini A (2016) Some considerations on dynamic stability, in: A. Zingoni (Ed.), *Insights and Innovations in Structural Engineering, Mechanics and Computation*, Proceedings of the Sixth International Conference on Structural Engineering, Mechanics and Computation, Cape Town, South Africa, 5-7 September 2016
21. Gurtin ME (1964) Variational principles for linear initial-value problems. Quart Appl Math 22:252–256
22. Hibbitt HD (1979) Some follower forces and load stiffeness. Internat J Num Meths Engrg 14:937–941
23. Inman DJ (1983) Dynamic of asymmetric nonconservative systems. J Appl Mech 50:199–203
24. Inman DJ, Olsen CL (1988) Dynamic of symmetrizable nonconservative systems. J Appl Mech 55:206–212
25. Leipholz H (1969) Application of Liapunov's direct method to the stability problem of rods subject to follower forces. *Instability of Continuum systems*, Symposium Herrenald (Germany), 8-12 september, 1969, pp. 1-10, Ed. H. Leipholz
26. Leipholz H (1974) On conservative elastic system of the first and second kind. Ingenieur-Archiv 43:255–271
27. Leipholz H (1974) On a generalization of the concept of selfadjointness and of Reyleigh's Quotient. Mech Res Comm 2:72–76
28. Leipholz H (1978) On a variational principle for Beck's rod. Mech Res Comm 5(1):45–49
29. Leipholz H (1981) On the generalization of the lower bound theorem for elastic rods and plates subjected to compressive follower forces. Comput Meths Appl Mech Engrg 21:101–120
30. Leipholz H (1985) On the application of the energy method to continuous follower load systems. Comput Meths Appl Mech Engrg 53:259–276
31. Leipholz H (1986) On principles of stationarity for non-selfadjoint rod problems. Comput Meths Appl Mech Engrg 59:215–226
32. Panovko Y, Gubanova I (1967) Stability and oscillations of elastic systems: models, paradoxes and errors, 2nd edn. Nunka Press, Moscow
33. Pflüger A (1950) Stabilitätsprobleme der Elastostatik. Springer-Verlag, Berlin
34. Tommasini M, Kirillov ON, Misseroni D, Bigoni D (2016) The destabilizing effect of external damping: Singular flutter boundary for the Pfluger column with vanishing external dissipation. J Mech Phys Solids 91:204–215
35. Tonti E (1973) On the variational formulation for linear initial value problems. *Annali di Matematica Pura ed Applicata*, Serie Quarta XCV, pp 331-359
36. Ziegler H (1953) Linear elastic stability. *Z. angew Math. Phys.***4**, 89-121 and 168-185
37. Ziegler H (1956) On the concept of elastic stability. Andvn Appl Mech 4:351–403
38. Ziegler H (1968) Principles of structural stability of structures. Blaisdell Publishing Company, Waltham, Massachusetts
39. Zorii LM, Chernukha YuA (1971) Influence of supports on the dynamic stability of elastic column. Prikl Makh 7(2):134–136 ((**in Russian**))