Article

Thermodynamics and nonlocality in continuum physics

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Abstract

The paper is devoted to the modelling of nonlocality in continuum physics through constitutive functions that depend on suitable gradients. For definiteness the attention is addressed to elastic solids, heat conductors, and magnetic solids. Models are developed where both the requirements of the second law of thermodynamics and the balance equations are satisfied for the constitutive functions that involve gradients of strain, temperature, heat flux, and magnetization. Concerning elastic and magnetic solids, it is shown that, depending on the chosen variables, the standard symmetry property of the stress holds identically. The models so developed are free from any hyperstress tensor frequently considered in the literature.

Keywords: thermodynamic consistency; second law; entropy extra flux; nonlocal models; heat conduction; second-gradient elasticity; magnetization gradients

1. Introduction

The modelling of materials is often described by allowing for nonlocality in space in that the response at a point $\mathbf{x}$ is affected by appropriate fields in a neighbourhood of $\mathbf{x}$. The effect of the neighbourhood may be described by means of convolution integrals [1] or higher-order gradients (see [2] and refs therein). In periodic architectures (metamaterials) the nonlocality describes effective properties that go beyond those of the constituent parts [3,4]. According to the literature on nonlocality, in solids and fluids [2,5,6] there seems to be the need of new concepts for the description of the stress and strain states. It was pointed out in [2] that higher-order gradients involved in nonlocal thermodynamics represent micro-length effects and these phenomena are very important in nanostructures. Moreover, a second-gradient model offers the possibility of describing the adherence interaction of a three-dimensional viscous fluid with one-dimensional structures immersed in it, thanks to the non-classical structure of the stresses which act on the fluid [7]. Accordingly, nonlocal effects can also be viewed as corrections of local descriptions.

In this paper we investigate the thermodynamic admissibility of nonlocal models by looking for the requirements placed by the second law of thermodynamics. Despite the wide literature on nonlocality, thermodynamics needs corresponding developments also in connection with the balance of energy. About the balance of energy, most papers on nonlocality use a balance with an additional energy flux associated with the hyperstress. It seems a conceptual improvement to involve the stress through the Cauchy stress tensor and to avoid any recourse to ad-hoc energy fluxes.

For definiteness we look at nonlocality for elastic solids, heat conductors, and magnetic materials. The nonlocal description in terms of gradients (often referred to as weakly nonlocal approach) seems to be the most appropriate modelling and the wide variety of

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theories of this type in the literature substantiates this view. The key feature of higher-gradient theories is the presence of multiple length scales and the possibility of encompass non-standard interactions. Yet the more involved structure of gradient-type models places problems about the consistency with the balance equations. In particular, from the balance of angular momentum it follows a condition on the symmetry of the stress tensor and often this condition is ignored.

Relative to some schemes in the literature, we mention that Eringen [8] in passing from integral-type nonlocality to gradient-type nonlocality arrived to a constitutive equation of the form

$$\mathbf{T} := \mathbf{T}^{nl} - \ell^2 \Delta \mathbf{T}^{nl} = \mathbb{C}\nabla \mathbf{u}.$$

where $\mathbf{T}^{nl}$ is the nonlocal stress tensor, ℓ is a length scale parameter, $\mathbb{C}$ is the elasticity tensor, ∇ is the gradient operator, Δ is the Laplacian and $\mathbf{u}$ is the displacement field. Next Askes and Aifantis [9] arrived at the dual representation with equation of equilibrium

$$\mathbb{C}(\nabla\nabla\mathbf{u} - \ell^2 \nabla\nabla\Delta\mathbf{u}) + \mathbf{f} = \mathbf{0},$$

where $\mathbf{f}$ is the body force density.

Lately various approaches have modelled the internal structure by starting with a new form of the stress power. The non-classical form of the approach is made formal (see, e.g., [2,10,11]) by accounting for a hyperstress, say $\mathbf{G}$, characterized by the mechanical power in the Eulerian form

$$w = \mathbf{T} \cdot \mathbf{L} + \mathbf{G} \cdot (\nabla \mathbf{L}), \tag{1}$$

where $\mathbf{L} = \nabla \mathbf{v}$ is the velocity gradient. An analogous formulation in terms of the deformation gradient $\mathbf{F}$ has been applied in [11,12] in the Lagrangian form

$$w_R = \mathbf{T}_R \cdot \dot{\mathbf{F}} + \mathbf{G}_R \cdot \nabla_R \dot{\mathbf{F}}.$$

Furthermore similar formulations have been investigated in terms of strain rate [13] by assuming

$$w = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\Sigma} \cdot \nabla \dot{\boldsymbol{\varepsilon}},$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ denotes the infinitesimal strain. In the corresponding approaches, $\mathbf{T}, \mathbf{T}_R, \boldsymbol{\sigma}$ denote the stress and $\mathbf{G}, \mathbf{G}_R, \boldsymbol{\Sigma}$ the hyperstress (see the details in § 2.3).

More recently the internal structure has been framed within a Hamiltonian scheme with a strain energy density as a function of strain and strain gradients (see, e.g., [5] and [9] for an overview of formulations).

The purpose of this paper is to establish thermodynamically-consistent nonlocal models of elasticity, heat conduction, and magnetism without any appeal to hyperstresses and corresponding powers. The consistency is determined by deriving the restrictions placed by the Clausius-Duhem inequality as the statement of the second law. The novelty of our application of the Clausius-Duhem inequality is to regard the entropy production as a constitutive function as is the case for the entropy flux. Furthermore our procedure avoids the introduction of ad-hoc energy fluxes and merely assumes (for non-polar solids) $\mathbf{T} \cdot \mathbf{D}$ as the stress power, where $\mathbf{D}$ is the stretching tensor, as is the case for classical local theories.

Notation and definitions

Let $\Omega \subset \mathscr{E}^3$ be the time dependent region occupied by the body. The points of the body are labelled by the position vector in a reference configuration R. The motion of the points is described by a twice continuously differentiable function $\mathbf{x}(\mathbf{X}, t)$ on $\Omega \times \mathbb{R}$. The displacement of $\mathbf{X}$ is denoted by $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$. We let $\mathbf{F}$ denote the deformation gradient, $F_{i_K} = \partial_{X_K} x_i$, and let $J = \det \mathbf{F} > 0$ while $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1})$ is the Green-Lagrange strain tensor. The symbols ∇, ∇_R denote the gradient in Ω and in R. For any function $\phi(\mathbf{x}, t) = \phi(\mathbf{x}(\mathbf{X}, t), t)$ it is $\partial_{X_K}\phi = \partial_{x_i}\phi\partial_{X_K} x_i$ whence $\nabla_R\, \phi = \mathbf{F}^T\nabla\phi$. sym and skw denote the symmetric and skew-symmetric parts while Sym and Skw are the sets of symmetric and skew-symmetric tensors. The symbol $\mathbf{L}$ denotes the velocity gradient, $L_{ij} = \partial_{x_j} v_i$, while $\mathbf{D} = \mathrm{sym}\mathbf{L}$ and $\mathbf{W} = \mathrm{skw}\mathbf{L}$. $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{T}_{RR} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ is the second Piola stress tensor.

In dealing with magnetization in matter we use the magnetization $\mathbf{M}$, per unit volume, and the magnetic field $\mathbf{H}$, while μ_0 is the permeability of free space. The referential description is developed by using the Lagrangian fields $\mathfrak{M} = J\mathbf{F}^{-1}\mathbf{M}$ and $\mathfrak{H} = \mathbf{F}^T\mathbf{H}$.

2. Balance equations

Let ρ be the current mass density and ρ_R the mass density in the reference configuration. The conservation of mass results in

$$\rho J = \rho_R, \tag{2}$$

ρ_R being possibly dependent on the position $\mathbf{X}$. The balance equations are now revised by considering non-polar and polar media. In particular polar magnetic solids are considered.

2.1. Non-polar solids

We consider non-polar solids and then neither micro-structures nor surface and body couples occur. Hence by the well-known Cauchy theorem it follows the existence of a stress tensor $\mathbf{T}$ such that the motion is governed by

$$\rho\ddot{\mathbf{u}} = \nabla\cdot\mathbf{T} + \rho\mathbf{b}, \tag{3}$$

$\mathbf{b}$ being the body force. For non-polar solids the balance of angular momentum implies that

$$\mathbf{T} = \mathbf{T}^T. \tag{4}$$

Consequently, the balance of energy is assumed in the form

$$\rho\dot{\varepsilon} = \mathbf{T}\cdot\mathbf{D} - \nabla\cdot\mathbf{q} + \rho r, \tag{5}$$

where ε is the specific energy density, while $\mathbf{q}$ and r are surface and body terms of non-mechanical nature, say heat flux vector and energy supply.

Let η be the specific entropy density and θ the absolute temperature. The balance of entropy is assumed in the form

$$\rho\dot{\eta} + \nabla\cdot\mathbf{j} - \rho r/\theta = \rho\gamma \geq 0, \tag{6}$$

where $\mathbf{j}$ is the entropy flux. Consistent with the principle of increase of entropy, the quantity γ is called the (rate of) entropy production and is assumed to be non-negative. As is standard let

$$\mathbf{j} = \frac{\mathbf{q}}{\theta} + \mathbf{k},$$

with $\mathbf{k}$ the extra-entropy flux. Replacing $\nabla\cdot\mathbf{q} - \rho r$ from eq. (5) and considering the free energy

$$\psi = \varepsilon - \theta\eta$$

we can write eq. (6) in the form

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}\cdot\mathbf{D} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta + \theta\nabla\cdot\mathbf{k} = \rho\theta\gamma \geq 0. \tag{7}$$

Following the literature we refer to (7) as the Clausius-Duhem (CD) inequality. A thermodynamic process is the set of functions $\rho, \mathbf{u}, \mathbf{T}, \mathbf{b}, \epsilon, \mathbf{q}, r, \eta, \mathbf{k}, \gamma$ of $(\mathbf{x}, t) \in \Omega \times \mathbb{R}$ describing the evolution of the body; $\mathbf{b}$ and r are required to be given by (3) and (5). The second law of thermodynamics characterizes the physically admissible processes as follows.

Postulate. *For every admissible thermodynamic process the inequality $\gamma \geq 0$ is valid at any point $\mathbf{x} \in \Omega$ and time $t \in \mathbb{R}$.*

For later application we consider the referential version of (7). Define the referential fluxes

$$\mathbf{q}_R = J\mathbf{F}^{-1}\mathbf{q}, \quad \mathbf{k}_R = J\mathbf{F}^{-1}\mathbf{k} \tag{8}$$

and notice that (Cf. [14], §1.2.2)

$$\nabla_R \theta = \nabla\theta\, \mathbf{F}, \qquad J\mathbf{q} \cdot \nabla\theta = \mathbf{q}_R \cdot \nabla_R \theta, \qquad J\nabla \cdot \mathbf{k} = \nabla_R \cdot \mathbf{k}_R.$$

Notice that

$$\dot{\mathbf{E}} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{L}^T\mathbf{F} + \mathbf{F}^T\mathbf{L}\mathbf{F}) = \mathbf{F}^T\mathbf{D}\mathbf{F}.$$

Furthermore, since $\mathbf{T}_{RR} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ then

$$\mathbf{T} \cdot \mathbf{D} = J^{-1}\mathbf{T}_{RR} \cdot \dot{\mathbf{E}}.$$

Consequently, multiplication of (7) by J results in

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R \theta + \theta\nabla_R \cdot \mathbf{k}_R = \rho_R\theta\gamma \geq 0; \tag{9}$$

hereafter the requirement $\gamma \geq 0$ is understood and not written.

It is of interest that

$$\mathbf{T} \in \mathrm{Sym} \quad \Longleftrightarrow \quad \mathbf{T}_{RR} \in \mathrm{Sym}.$$

2.2. Polar magnetic solids

The mass conservation is expressed by eq. (2). The balance of linear momentum involves the mechanical body force density $\rho\mathbf{b}$ and the magnetic force density (per unit volume) $\mathbf{f}$ and the mechanical stress tensor $\mathbf{T}$. The corresponding equation of motion is

$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} + \mathbf{f} + \nabla \cdot \mathbf{T}.$$

Often $\mathbf{f}$ is assumed to be $\mu_0(\mathbf{M} \cdot \nabla)\mathbf{H}$, with $\mathbf{M}$ the magnetization (per unit volume), and $\mathbf{H}$ the magnetic field. The detailed expression of $\mathbf{f}$ is inessential to the present investigation.

Since $\mathbf{M}$ is not generally collinear to $\mathbf{H}$ a body couple vector $\mathbf{c}$ appears in the form

$$\mathbf{c} = \mu_0\mathbf{M} \times \mathbf{H}.$$

Hence the balance of angular momentum leads to the condition

$$\mathrm{skw}(\mathbf{T} + \mu_0\mathbf{H} \otimes \mathbf{M}) = \mathbf{0}. \tag{10}$$

Since $\mathbf{T} \notin \mathrm{Sym}$ then the stress power is $\mathbf{T} \cdot \mathbf{L}$. Hence the balance of energy is based on the mechanical power $\mathbf{T} \cdot \mathbf{L}$ and the magnetic power $\rho\mu_0\mathbf{H} \cdot \dot{\mathbf{m}}$, with $\mathbf{m} = \mathbf{M}/\rho$. It then follows the equation

$$\rho\dot{\epsilon} = \mathbf{T} \cdot \mathbf{L} + \rho\mu_0\mathbf{H} \cdot \dot{\mathbf{m}} - \nabla \cdot \mathbf{q} + \rho r. \tag{11}$$

The balance of entropy is assumed in the general form (6). Now, substitution of $\nabla \cdot \mathbf{q} - \rho r$ from (11) and using the free energy $\psi = \varepsilon - \theta\eta$ we obtain

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T} \cdot \mathbf{L} + \rho\mu_0 \mathbf{H} \cdot \dot{\mathbf{m}} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta + \theta\nabla \cdot \mathbf{k} = \rho\theta\gamma \geq 0. \tag{12}$$

Multiplying (12) by J and using the referential fluxes (8) we obtain

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + J\mathbf{T} \cdot \mathbf{L} + \rho_R\mu_0 \mathbf{H} \cdot \dot{\mathbf{m}} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\,\theta + \theta\nabla_R\,\cdot \mathbf{k}_R = \rho_R\theta\gamma \geq 0. \tag{13}$$

To determine the thermodynamic restrictions it is convenient to describe the magnetization through the Lagrangian field (see [15])

$$\mathfrak{M} = \rho_R \mathbf{F}^{-1}\mathbf{m}.$$

By direct calculation we find

$$\dot{\mathbf{m}} = (1/\rho_R)[\mathbf{L}\mathbf{F}\mathfrak{M} + \mathbf{F}\dot{\mathfrak{M}}]$$

and then the CD inequality (13) may be written in the form

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + \rho_R\mu_0\mathfrak{H} \cdot \dot{\mathfrak{M}} + J(\mathbf{T} + \mu_0\mathbf{H} \otimes \mathbf{M}) \cdot \mathbf{L} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\,\theta + \theta\nabla_R\,\cdot \mathbf{k}_R = \rho_R\theta\gamma \geq 0. \tag{14}$$

2.3. *Remarks on the expression of the stress power w*

Significant differences between the approaches to strain-gradient theories are due mainly to the form of the stress power w. In this paper $\mathbf{T}$ is the Cauchy stress. Consequently it is $\mathbf{T} \in \mathrm{Sym}$ for non-polar bodies and hence

$$w = \mathbf{T} \cdot \mathbf{D} \quad \text{or} \quad Jw = \mathbf{T}_{RR} \cdot \dot{\mathbf{E}}$$

according as we follow a Eulerian or a Lagrangian description. For polar bodies (e.g. magnetic) $\mathbf{T} \notin \mathrm{Sym}$ and then

$$w = \mathbf{T} \cdot \mathbf{L}.$$

Instead, as we see in a while, $\mathbf{T}$ is only a part of the stress effect and then the power is taken in various ways. In [11] about second-gradient continua, the power w is expressed via the stress tensor $\mathbf{T}$ and the hyperstress tensor $\mathbf{G}$ in the form

$$w = \mathbf{T} \cdot \mathbf{L} + \mathbf{G} \cdot (\nabla\nabla\mathbf{v});$$

the present power w is the internal power expenditure in [11]. Consequently there is a power per unit volume

$$(\nabla \cdot \tilde{\mathbf{T}}) \cdot \mathbf{v}, \qquad \tilde{\mathbf{T}} = \mathbf{T} - \nabla \cdot \mathbf{G},$$

and a boundary power density $(\mathbf{G}\mathbf{n}) \cdot \mathbf{L}$.

With some similarities, in [13] w is viewed as the internal mechanical power and assumed to be in the form

$$w = \mathbf{T} \cdot \mathbf{L} - \nabla \cdot \mathbf{N}, \tag{15}$$

where $\mathbf{N}$ is viewed as a interstitial work flux, first introduced in [16]. Next it is assumed that the stress $\mathbf{T} \notin \mathrm{Sym}$ consists of two parts,

$$\mathbf{T} = \sigma + \nabla \cdot \mathbf{\Sigma}$$

and the stress power is

$$w = \boldsymbol{\sigma} \cdot \mathbf{L} - \boldsymbol{\Sigma} \cdot (\nabla\nabla\mathbf{v}) = (\boldsymbol{\sigma} + \nabla \cdot \boldsymbol{\Sigma}) \cdot \mathbf{L} - \nabla \cdot (\boldsymbol{\Sigma}\mathbf{L})$$

so that eq. (15) holds with $\mathbf{N} = \boldsymbol{\Sigma}\mathbf{L}$.

The structure of the elastic stress $\mathbf{T}$ in the form $\mathbf{T} = \boldsymbol{\sigma} + \nabla \cdot \boldsymbol{\Sigma}$ is obtained in [17] by starting a CD inequality in the form

$$\mathbf{T} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{\psi} + P \geq 0$$

where P is the energy residual while $\psi = \psi(\boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon})$. Let $\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}}\psi, \boldsymbol{\Sigma} = \partial_{\nabla\boldsymbol{\varepsilon}}\psi$. Then

$$(\mathbf{T} - \partial_{\boldsymbol{\varepsilon}}\psi) \cdot \dot{\boldsymbol{\varepsilon}} - \partial_{\nabla\boldsymbol{\varepsilon}}\psi \cdot \nabla\dot{\boldsymbol{\varepsilon}} = (\mathbf{T} - \boldsymbol{\sigma} + \nabla \cdot \boldsymbol{\Sigma}) \cdot \dot{\boldsymbol{\varepsilon}} - \nabla \cdot (\boldsymbol{\Sigma}\dot{\boldsymbol{\varepsilon}}) + P \geq 0.$$

Hence it is concluded that, for elastic deformations,

$$\mathbf{T} = \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\Sigma}.$$

Hence in [13,17] the effective stress tensor is a joint effect of $\boldsymbol{\sigma}$ and the hyperstress $\boldsymbol{\Sigma}$.

3. Strain-gradient elasticity

With the purpose of extending the classical theory of elasticity, nonlocal properties are introduced in various ways in the literature. At first the elasticity theory has been improved by allowing for internal structures. In this sense deformable directors have been introduced in [18] thus allowing for a structured unit cell and generalizing the model of Cosserat continua. Along this line it is worth mentioning the works on linear elasticity with couple stresses where the model is improved by allowing for couple stress and body couple vectors [19]. In [20] the theory is free from couple stress and body couple and accounts for the internal structure by letting the relative deformation and the micro-deformation gradient be among the variables in addition to the macro-strain. The corresponding equations of motion are then derived through Hamilton's principle upon a proper definition of kinetic and potential energies.

Lately various approaches have modelled the internal structure by letting the stress tensor depend on strain gradients. The non-classical form of the approach is made formal by accounting for a hyperstress, say $\mathbf{G}$, dependent on the strain gradient and assuming the stress power to be in the form (1) (see, e.g., [2,10,11]). The corresponding elasticity-gradient theories are called Laplacian-based (see [9]). Other formulations have been developed in terms of the variable $\mathbf{F}$ [11,12] or $\boldsymbol{\varepsilon}$ [13,17]. More recently the internal structure has been framed within a Hamiltonian scheme with a strain energy density as a function of strain and strain gradients (see, e.g., [5] and [9] for an overview of formulations).

In this section we develop three thermodynamic schemes for strain-gradient elasticity without introducing any hyperstress tensor. The difference among the schemes is based on the variable representing the strain gradient, namely the Green-Lagrange strain $\mathbf{E}$, the deformation gradient $\mathbf{F}$, and the infinitesimal strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$.

3.1. Models involving the Green-Lagrange strain E

Models of nonlocal elastic solids are investigated by letting the stress tensor depend on strain gradients, in addition to temperature gradients. For definiteness we look for fourth-grade elastic solids in that the dependence is allowed up to fourth-order gradients[1]. Hence we let

$$\theta, \dot{\theta}, \nabla_R\, \theta, \nabla_R\, \dot{\theta}, \nabla_R\, \nabla_R\, \theta, \nabla_R\, \nabla_R\, \nabla_R\, \theta, \nabla_R\, \nabla_R\, \nabla_R\, \nabla_R\, \theta,$$
$$\mathbf{E}, \dot{\mathbf{E}}, \nabla_R\, \mathbf{E}, \nabla_R\, \dot{\mathbf{E}}, \nabla_R\, \nabla_R\, \mathbf{E}, \nabla_R\, \nabla_R\, \nabla_R\, \mathbf{E}, \nabla_R\, \nabla_R\, \nabla_R\, \nabla_R\, \mathbf{E},$$

be the variables and

$$\psi, \eta, \mathbf{T}_{RR}, \mathbf{q}_R, \mathbf{k}_R, \gamma$$

be the constitutive functions. Though we might proceed with a strict application of the rule of equipresence [21], for formal simplicity we let the free energy depend on gradients up to second order, namely

$$\psi = \psi(\theta, \mathbf{E}, \nabla_R\, \theta, \nabla_R\, \mathbf{E}, \nabla_R\, \nabla_R\, \theta, \nabla_R\, \nabla_R\, \mathbf{E}). \tag{16}$$

Assume the function ψ is continuously differentiable while $\eta, \mathbf{T}_{RR}, \mathbf{q}_R, \mathbf{k}_R, \gamma$ are continuous.

Using the Coleman-Noll procedure [22], we now establish the thermodynamic requirements for fourth-grade materials with a free energy in the form (16). can be proved. Upon computation of $\dot{\psi}$ and substitution in (9) we have

$$\begin{aligned} -\rho_R(\partial_\theta\psi + \eta)\dot{\theta} - \rho_R\partial_{\nabla_R\,\theta}\psi \cdot \nabla_R\,\dot{\theta} - \rho_R\partial_{\nabla_R\,\nabla_R\,\theta}\psi \cdot \nabla_R\,\nabla_R\,\dot{\theta} \\ -\rho_R\partial_{\mathbf{E}}\psi \cdot \dot{\mathbf{E}} - \rho_R\partial_{\nabla_R\,\mathbf{E}}\psi \cdot \nabla_R\,\dot{\mathbf{E}} - \rho_R\partial_{\nabla_R\,\nabla_R\,\mathbf{E}}\psi \cdot \nabla_R\,\nabla_R\,\dot{\mathbf{E}} + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} \\ -\frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\,\theta + \theta\nabla_R\, \cdot \mathbf{k}_R = \rho_R\theta\gamma \geq 0. \end{aligned} \tag{17}$$

The values of $\nabla_R\,\dot{\theta}, \nabla_R\,\nabla_R\,\theta, \nabla_R\,\dot{\mathbf{E}}, \nabla_R\,\nabla_R\,\dot{\mathbf{E}}$ cannot be regarded as (arbitrary and) independent of the other terms in the CD inequality, particularly in the expression of $\nabla_R\, \cdot \mathbf{k}_R$. Accordingly we divide eq. (17) by θ to have

$$\begin{aligned} -\frac{\rho_R}{\theta}(\partial_\theta\psi + \eta)\dot{\theta} - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi \cdot \nabla_R\,\dot{\theta} - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi \cdot \nabla_R\,\nabla_R\,\dot{\theta} \\ +\frac{1}{\theta}(\mathbf{T}_{RR} - \rho_R\partial_{\mathbf{E}}\psi) \cdot \dot{\mathbf{E}} - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathbf{E}}\psi \cdot \nabla_R\,\dot{\mathbf{E}} - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\mathbf{E}}\psi \cdot \nabla_R\,\nabla_R\,\dot{\mathbf{E}} \\ -\frac{1}{\theta^2}\mathbf{q}_R \cdot \nabla_R\,\theta + \nabla_R\, \cdot \mathbf{k}_R = \rho_R\gamma \geq 0. \end{aligned} \tag{18}$$

Next we consider the identities

$$\begin{aligned} -\frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi \cdot \nabla_R\,\dot{\theta} &= -\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi\dot{\theta}) + [\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi)]\dot{\theta}, \\ -\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi\nabla_R\,\nabla_R\,\dot{\theta} &= -\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi\nabla_R\,\dot{\theta}) + [\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi)] \cdot \nabla_R\,\dot{\theta} \\ &= -\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi\nabla_R\,\dot{\theta}) + \nabla_R\, \cdot \{[\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi)]\dot{\theta}\} \\ &\quad - [(\nabla_R\, \otimes \nabla_R\,) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi)]\dot{\theta} \end{aligned}$$

and the analogous ones with $\mathbf{E}$. Now, define

$$\begin{aligned} \delta_\theta^{(2)}\psi &= \partial_\theta\psi - \frac{\theta}{\rho_R}\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi) + \frac{\theta}{\rho_R}(\nabla_R\, \otimes \nabla_R\,) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\theta}\psi) \\ \delta_{\mathbf{E}}^{(2)}\psi &= \partial_{\mathbf{E}}\psi - \frac{\theta}{\rho_R}\nabla_R\, \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathbf{E}}\psi) + \frac{\theta}{\rho_R}(\nabla_R\, \otimes \nabla_R\,) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R\,\nabla_R\,\mathbf{E}}\psi). \end{aligned}$$

[1] Sometimes (see, e.g., [2]) the grade of the materials is defined to be the order of the highest velocity gradient in the internal power.

Remark 1. *The functions $\delta^{(2)}_\theta \psi$ and $\delta^{(2)}_{\mathbf{E}} \psi$ can be viewed as the extended versions, via the factor ρ_R/θ, of the variational derivatives of ψ, of order 2, with respect to θ and $\mathbf{E}$. If ρ_R and θ are constant then*

$$\delta^{(2)}_\theta \psi = \partial_\theta \psi - \nabla_R \cdot (\partial_{\nabla_R \theta} \psi) + (\nabla_R \otimes \nabla_R) \cdot (\partial_{\nabla_R \nabla_R \theta} \psi)$$

is the standard variational derivative, of order 2, of mathematical analysis [23].

Upon substitution of these identities in (18) we may write

$$-\frac{\rho_R}{\theta} \hat{\eta}\, \dot{\theta} + \frac{1}{\theta} \hat{\mathbf{T}}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta^2} \mathbf{q}_R \cdot \nabla_R \theta + \nabla_R \cdot (\mathbf{k}_R - \hat{\mathbf{k}}_\theta - \hat{\mathbf{k}}_{\mathbf{E}}) = \rho_R \gamma \geq 0, \tag{19}$$

where

$$\hat{\eta} = \eta + \delta^{(2)}_\theta \psi, \qquad \hat{\mathbf{T}}_{RR} = \mathbf{T}_{RR} - \rho_R \delta^{(2)}_{\mathbf{E}} \psi,$$

$$\hat{\mathbf{k}}_\theta = \Big[\frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi - \nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \nabla_R \theta} \psi) \Big] \dot{\theta} + \frac{\rho_R}{\theta} \partial_{\nabla_R \nabla_R \theta} \psi \nabla_R \dot{\theta},$$

$$\hat{\mathbf{k}}_{\mathbf{E}} = \Big[\frac{\rho_R}{\theta} \partial_{\nabla_R \mathbf{E}} \psi - [\nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \nabla_R \mathbf{E}} \psi) \Big] \dot{\mathbf{E}} + \frac{\rho_R}{\theta} \partial_{\nabla_R \nabla_R \mathbf{E}} \psi \nabla_R \dot{\mathbf{E}}$$

Sufficient conditions for the validity of (19) with $\gamma \geq 0$ determine particular thermodynamically consistent models. In this sense a simple case arises by letting

$$\mathbf{k}_R = \hat{\mathbf{k}}_\theta + \hat{\mathbf{k}}_{\mathbf{E}},$$

$$\hat{\eta} = -\eta_1 \dot{\theta}, \qquad \hat{\mathbf{T}}_{RR} = \mathbf{A} \dot{\mathbf{E}}, \qquad \mathbf{q}_R = -\mathbf{K} \nabla_R \theta$$

where $\eta_1 \geq 0$ and $\mathbf{A}, \mathbf{K}$ are semi-positive definite tensors of fourth and second order. Hence the corresponding expression of the entropy production is

$$\rho_R \gamma = \eta_1 \frac{\rho_R}{\theta} |\dot{\theta}|^2 + \frac{1}{\theta} \mathbf{A} \dot{\mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{1}{\theta^2} \mathbf{K} \nabla_R \theta \cdot \nabla_R \theta.$$

The vectors $\hat{\mathbf{k}}_\theta$ and $\hat{\mathbf{k}}_{\mathbf{E}}$ denote entropy fluxes, induced by the time derivatives $\dot{\theta}$ and $\dot{\mathbf{E}}$. Within the first order, the linear dependence of entropy fluxes on the time derivatives $\dot{\theta}$ or $\dot{\mathbf{E}}$ is common in the literature (see, e.g., [14, § 8.9.1] and [12,24]).

Restrict attention to the non-dissipative part of $\mathbf{T}_{RR}$, say

$$\mathbf{T}_{RR} = \rho_R \delta^{(2)}_{\mathbf{E}} \psi, \tag{20}$$

In suffix notation, the constitutive equation (20) for the Piola stress $\mathbf{T}_{RR}$ takes the form

$$(\mathbf{T}_{RR})_{HK} = \partial_{E_{HK}} \psi - \frac{\theta}{\rho_R} \partial_{X_P} (\frac{\rho_R}{\theta} \partial_{\partial_{X_P} E_{HK}} \psi) + \frac{\theta}{\rho_R} \partial_{X_P} \partial_{X_Q} (\frac{\rho_R}{\theta} \partial_{\partial_{X_P} \partial_{X_Q} E_{HK}} \psi)$$

By definition, the Cauchy stress $\mathbf{T}$ is then found to be

$$\mathbf{T} = J^{-1} \mathbf{F} \mathbf{T}_{RR} \mathbf{F}^T = \rho \mathbf{F} \delta^{(2)}_{\mathbf{E}} \psi \mathbf{F}^T;$$

in components

$$T_{ij} = \rho F_{iH} [\partial_{E_{HK}} \psi - \frac{\theta}{\rho_R} \partial_{X_P} (\frac{\rho_R}{\theta} \partial_{\partial_{X_P} E_{HK}} \psi) + \frac{\theta}{\rho_R} \partial_{X_P} \partial_{X_Q} (\frac{\rho_R}{\theta} \partial_{\partial_{X_P} \partial_{X_Q} E_{HK}} \psi)] F^T_{Kj}. \tag{21}$$

We mention that, within the first order, the occurrence of the variational derivative is common in the Ginzburg-Landau modelling of phase fields [25].

If θ and ρ_R are constant then

$$T_{ij} = \rho F_{iH}[\partial_{E_{HK}}\psi - \partial_{X_P}\partial_{\partial_{X_P}E_{HK}}\psi + \partial_{X_P}\partial_{X_Q}\partial_{\partial_{X_P}\partial_{X_Q}E_{HK}}\psi)]F^T_{Kj}.$$

As an example, if

$$\psi = \Psi(\theta) + \tfrac{1}{2}a|\mathbf{E}|^2 + \tfrac{1}{2}b|\nabla_R\,\mathbf{E}|^2 + \tfrac{1}{2}c|\nabla_R\,\nabla_R\,\mathbf{E}|^2, \tag{22}$$

then

$$T_{ij} = \rho F_{iH}(aE_{HK} - b\Delta_R\,E_{HK} + c\Delta_R\Delta_R E_{HK})F^T_{Kj}. \tag{23}$$

The symmetry of $\mathbf{T}_{RR}$ and $\mathbf{T}$ is apparent from the thermodynamic requirement (20) and, necessarily, in the selected examples.

The result (23) for the strain $\mathbf{E}$ and the stress $\mathbf{T}$ has a simple physical analogue within the one-dimensional model of particles and springs [9]. Denote by $x_n = nd + u_n$ the position of the n-th particle, with d the particle spacing at equilibrium. Letting K be the spring constant and M the mass of the particles we can write the force on the n-th particle in the form

$$F_n = K(u_{n+1} - 2u_n + u_{n-1}).$$

A formal Taylor's formula for u_{n+1} and u_{n-1} allows us to write

$$u_{n+1} = u_n + d\partial_x u + \tfrac{1}{2}d^2\partial_x^2 u + \tfrac{1}{6}d^3\partial_x^3 u + \tfrac{1}{24}d^4\partial_x^4 u + ...$$
$$u_{n-1} = u_n - d\partial_x u + \tfrac{1}{2}d^2\partial_x^2 u - \tfrac{1}{6}d^3\partial_x^3 u + \tfrac{1}{24}d^4\partial_x^4 u + ...$$

Hence it follows that

$$F = Kd^2(\partial_x^2 u + \tfrac{1}{12}d^2\partial_x^4 u + ...) \tag{24}$$

The correspondence $F \leftrightarrow \partial_x T$ and $\partial_x u \leftrightarrow E$ yields the formal correspondence between (24) and (23). The main difference between Eqs. (24) and (23) concerns the sign of the strain gradient terms (see [9] for an in-depth discussion).

Though the Lagrangian scheme for the strain-gradient is quite unusual in the literature, by analogy with other schemes one might say that $\partial_{\mathbf{E}}\psi$ represents the (local) stress, while $\partial_{\nabla_R\,\mathbf{E}}\psi$ and $\partial_{\nabla_R\,\nabla_R\,\mathbf{E}}\psi$ denote hyperstresses (of different order). Apart from the interpretation of the tensors $\partial_{\mathbf{E}}\psi$, $\partial_{\nabla_R\,\mathbf{E}}\psi$, and $\partial_{\nabla_R\,\nabla_R\,\mathbf{E}}\psi$ here we conclude that the effective stress, entering the equation of motion, involves the strain and even-order derivatives of strain.

It is worth remarking that if the dependence on the strain is through $\mathbf{E}$ and its gradients then eq. (21) is the form of thermodynamically-admissible stresses where the temperature gradient gives a further contribution.

Among the admissible models, we mention the case where

$$\hat{\eta} = 0, \quad \partial_{\nabla_R\,\theta}\psi = \mathbf{0}, \quad \partial_{\nabla_R\,\nabla_R\,\theta}\psi = \mathbf{0}$$

and hence

$$\eta = -\partial_\theta\psi.$$

If the free energy has the form (22) where a, b, c are independent of θ and $\Psi = \Psi(\theta)$ then

$$\eta = -\Psi'(\theta).$$

Consequently, the internal energy $\varepsilon = \psi + \theta\eta$ is given by

$$\varepsilon = \Psi - \theta\Psi'(\theta) + \tfrac{1}{2}a|\mathbf{E}|^2 + \tfrac{1}{2}b|\nabla_R\,\mathbf{E}|^2 + \tfrac{1}{2}c|\nabla_R\,\nabla_R\,\mathbf{E}|^2.$$

Otherwise, a free energy of the form (22) with a independent of temperature but b, c proportional to θ, say $b = b_0\theta, c = c_0\theta$, gives

$$\eta = -\Psi'(\theta) + \tfrac{1}{2}b_0|\nabla_R \mathbf{E}|^2 + \tfrac{1}{2}c_0|\nabla_R \nabla_R \mathbf{E}|^2$$

and then

$$\varepsilon = \Psi - \theta\Psi'(\theta) + \tfrac{1}{2}a|\mathbf{E}|^2$$

is independent of strain gradients.

3.2. Models involving the deformation gradient $\mathbf{F}$

The description of the material in terms of the deformation gradient $\mathbf{F}$ might seem similar to the previous one in terms of $\mathbf{E}$. However, as we can see, things are significantly different. The Cauchy stress $\mathbf{T}$ has to be symmetric due to the balance of angular momentum (4). For formal convenience we write the stress power in terms of $\dot{\mathbf{F}}$. Since $\mathbf{T} \in \mathrm{Sym}$ then

$$\mathbf{T}\cdot\mathbf{D} = \mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot(\dot{\mathbf{F}}\mathbf{F}^{-1}) = (\mathbf{T}\mathbf{F}^{-T})\cdot\dot{\mathbf{F}} = J^{-1}\mathbf{T}_R\cdot\dot{\mathbf{F}},$$

where $\mathbf{T}_R = J\mathbf{T}\mathbf{F}^{-T}$ is the first Piola stress. Relative to expected results for $\mathbf{T}_R$, the check of the symmetry is through the equality

$$\mathbf{T}_R\mathbf{F}^T = \mathbf{F}\mathbf{T}_R^T. \tag{25}$$

For technical convenience we follow the Lagrangian description. Hence we multiply the CD inequality (7) by J to obtain

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}_R\cdot\dot{\mathbf{F}} - \frac{1}{\theta}\mathbf{q}_R\cdot\nabla_R\,\theta + \theta\nabla_R\cdot\mathbf{k}_R = \rho_R\theta\gamma \geq 0. \tag{26}$$

Possible models of strain-gradient elastic solids are now investigated. Here too we look for fourth-grade models. By analogy with the dependence on $\mathbf{E}$ we let

$$\theta, \dot{\theta}, \nabla_R\,\theta, \nabla_R\,\dot{\theta}, \nabla_R\nabla_R\,\theta, \nabla_R\nabla_R\nabla_R\,\theta, \nabla_R\nabla_R\nabla_R\nabla_R\,\theta,$$
$$\mathbf{F}, \dot{\mathbf{F}}, \nabla_R\,\mathbf{F}, \nabla_R\,\dot{\mathbf{F}}, \nabla_R\nabla_R\,\mathbf{F}, \nabla_R\nabla_R\nabla_R\,\mathbf{F}, \nabla_R\nabla_R\nabla_R\nabla_R\,\mathbf{F}$$

be the variables and

$$\psi, \eta, \mathbf{T}_R, \mathbf{q}_R, \mathbf{k}_R, \gamma$$

the constitutive functions. For formal simplicity we consider the free energy in the form

$$\psi = \psi(\theta, \mathbf{F}, \nabla_R\,\theta, \nabla_R\,\mathbf{F}, \nabla_R\nabla_R\,\theta, \nabla_R\nabla_R\,\mathbf{F}); \tag{27}$$

the dependence on the gradients of $\mathbf{F}$ is considered, in a different way, in [26]. The function (27) is continuously differentiable while $\eta, \mathbf{T}_R, \mathbf{q}_R, \mathbf{k}_R, \gamma$ are continuous. Compute $\dot{\psi}$ and substitute in (26) to obtain

$$\begin{aligned}-\rho_R\partial_{\nabla_R\theta}\psi\cdot\nabla_R\,\dot{\theta} - \rho_R\partial_{\nabla_R\nabla_R\theta}\psi\cdot\nabla_R\nabla_R\,\dot{\theta} - \rho_R\partial_{\nabla_R\mathbf{F}}\psi\cdot\nabla_R\,\dot{\mathbf{F}} - \rho_R\partial_{\nabla_R\nabla_R\mathbf{F}}\psi\cdot\nabla_R\nabla_R\,\dot{\mathbf{F}}\\ -\rho_R(\partial_\theta\psi + \eta)\dot{\theta} + (\mathbf{T}_R - \rho_R\partial_{\mathbf{F}}\psi)\cdot\dot{\mathbf{F}} - \frac{1}{\theta}\mathbf{q}_R\cdot\nabla_R\,\theta + \theta\nabla_R\cdot\mathbf{k}_R = \rho_R\theta\gamma \geq 0.\end{aligned} \tag{28}$$

Since $\nabla_R \cdot \mathbf{k}_R$ comprises linear terms in $\nabla_R \nabla_R \dot{\theta}, \nabla_R \nabla_R \dot{\mathbf{F}}$ we need some rearrangements of (28). Divide by θ the remaining form of the CD inequality (28) to have

$$\begin{aligned}
-\frac{\rho_R}{\theta}(\partial_{\nabla_R \theta}\psi \cdot \nabla_R \dot{\theta} + \partial_{\nabla_R \nabla_R \theta}\psi \cdot \nabla_R \nabla_R \dot{\theta}) - \frac{\rho_R}{\theta}(\partial_{\nabla_R \mathbf{F}}\psi \cdot \nabla_R \dot{\mathbf{F}} + \partial_{\nabla_R \nabla_R \mathbf{F}}\psi \cdot \nabla_R \nabla_R \dot{\mathbf{F}}) \\
-\frac{\rho_R}{\theta}(\partial_\theta \psi + \eta)\dot{\theta} + \frac{1}{\theta}(\mathbf{T}_R - \rho_R \partial_{\mathbf{F}}\psi) \cdot \dot{\mathbf{F}} - \frac{1}{\theta^2}\mathbf{q}_R \cdot \nabla_R \theta + \nabla_R \cdot \mathbf{k}_R = \rho_R \gamma \geq 0. \quad (29)
\end{aligned}$$

Using the identities

$$-\frac{\rho_R}{\theta}\partial_{\nabla_R \theta}\psi \cdot \nabla_R \dot{\theta} = -\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \theta}\psi\, \dot{\theta}) + [\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \theta}\psi)]\dot{\theta},$$

$$\begin{aligned}
-\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi \cdot \nabla_R \nabla_R \dot{\theta} &= -\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi\, \nabla_R \dot{\theta}) + [\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi)] \cdot \nabla_R \dot{\theta} \\
&= -\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi \nabla_R \dot{\theta}) + \nabla_R \cdot \{[\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi)]\dot{\theta}\} \\
&\quad -[(\nabla_R \otimes \nabla_R) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi)]\dot{\theta}
\end{aligned}$$

and

$$-\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi \cdot \nabla_R \dot{\mathbf{F}} = -\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi\, \dot{\mathbf{F}}) + [\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi)]\dot{\mathbf{F}},$$

$$\begin{aligned}
-\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi \cdot \nabla_R \nabla_R \dot{\mathbf{F}} &= -\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi \nabla_R \dot{\mathbf{F}}) + \nabla_R \cdot \{[\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi)]\dot{\mathbf{F}}\} \\
&\quad -[(\nabla_R \otimes \nabla_R) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi)]\dot{\mathbf{F}}.
\end{aligned}$$

we can write eq. (29) in the form

$$\begin{aligned}
-\frac{\rho_R}{\theta}(\eta + \delta_\theta^{(2)}\psi)\dot{\theta} + \frac{1}{\theta}(\mathbf{T}_R - \rho_R \delta_{\mathbf{F}}^{(2)}\psi) \cdot \dot{\mathbf{F}} - \frac{1}{\theta^2}\mathbf{q}_R \cdot \nabla_R \theta \\
+\nabla_R \cdot (\mathbf{k}_R - \hat{\mathbf{k}}_\theta - \hat{\mathbf{k}}_{\mathbf{F}}) = \rho_R \gamma \geq 0. \qquad (30)
\end{aligned}$$

where

$$\delta_\theta^{(2)}\psi = \partial_\theta \psi - \frac{\theta}{\rho_R}\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \theta}\psi) + \frac{\theta}{\rho_R}(\nabla_R \otimes \nabla_R) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi),$$

$$\delta_{\mathbf{F}}^{(2)}\psi = \partial_{\mathbf{F}}\psi - \frac{\theta}{\rho_R}\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi) + \frac{\theta}{\rho_R}(\nabla_R \otimes \nabla_R) \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi),$$

and

$$\hat{\mathbf{k}}_\theta = \Big[\frac{\rho_R}{\theta}\partial_{\nabla_R \theta}\psi - \nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi)\Big]\dot{\theta} + \frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \theta}\psi\, \nabla_R \dot{\theta},$$

$$\hat{\mathbf{k}}_{\mathbf{F}} = \Big[\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi - \nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi)\Big]\dot{\mathbf{F}} + \frac{\rho_R}{\theta}\partial_{\nabla_R \nabla_R \mathbf{F}}\psi\, \nabla_R \dot{\mathbf{F}}.$$

The CD inequality (30) allows various models of strain-gradient elasticity. The simplest one restricts the dissipative effects to heat conduction and is expressed by the constitutive functions

$$\eta = -\delta_\theta(2)\psi, \qquad \mathbf{T}_R = \rho_R \delta_{\mathbf{F}}^{(2)}\psi,$$

$$\mathbf{k}_R = \hat{\mathbf{k}}_\theta + \hat{\mathbf{k}}_{\mathbf{F}}, \quad \mathbf{q}_R \cdot \nabla_R \theta = \rho_R \theta^2 \gamma.$$

The Piola stress $\mathbf{T}_R$ need not be symmetric but $\mathbf{T}$ and $\mathbf{T}_{RR}$ are required to be so. Hence we have to check the symmetry condition (25) whence

$$(\delta_{\mathbf{F}}^{(2)}\psi)\mathbf{F}^T \in \mathrm{Sym}. \qquad (31)$$

The dependence of ψ on $\mathbf{F}$ might be through the invariant quantities $J = \det \mathbf{F}$ and $\xi = \frac{1}{2}|\mathbf{F}|^2$. To begin with we check whether $(\partial_{\mathbf{F}}\psi)\mathbf{F}^T \in \mathrm{Sym}$. In suffix notation, since

$$\partial_{F_{iK}}\psi = \partial_J \psi J F^{-1}_{Ki} + \partial_\xi \psi F_{iK}$$

then

$$[(\partial_{\mathbf{F}}\psi)\mathbf{F}^T]_{ij} = \partial_{F_{iK}}\psi F^T_{Kj} = \partial_J\psi J F^{-1}_{Ki} F_{jK} + \partial_\xi \psi F_{iK}F_{jK} = \partial_J \psi J \delta_{ij} + \partial_\xi \psi F_{iK} F_{jK} \in \mathrm{Sym}.$$

This positive check is not surprising because

$$J = \det \mathbf{F} = [\det(\mathbf{F}^T\mathbf{F})]^{1/2} = [\det(2\mathbf{E} + \mathbf{1})]^{1/2},$$
$$|\mathbf{F}|^2 = F_{iK}F_{iK} = \mathrm{tr}\,(\mathbf{F}^T\mathbf{F}) = \mathrm{tr}\,(2\mathbf{E}+\mathbf{1}),$$

and the dependence on $\mathbf{E}$ provides admissible strains.

As to the dependence on $\nabla_R \mathbf{F}$ assume, for simplicity,

$$\psi = \Psi(\theta, \mathbf{F}) + \tfrac{1}{2}c|\nabla_R \mathbf{F}|^2.$$

Hence

$$(\partial_{\nabla_R \mathbf{F}}\psi)_{iKQ} = c\partial_{X_Q} F_{iP}, \quad \Big[\frac{\theta}{\rho_R}\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi)\Big]_{iK} = \Delta_R F_{iK} + \frac{\theta}{\rho_R}\partial_{X_Q}(\frac{\rho_R}{\theta})(\partial_{X_Q}F_{iK}).$$

Consequently,

$$\Big[\frac{\theta}{\rho_R}\nabla_R \cdot (\frac{\rho_R}{\theta}\partial_{\nabla_R \mathbf{F}}\psi)\Big]_{iK} F_{jK} = (\Delta_R F_{iK})F_{jK} + \frac{\theta}{\rho_R}\partial_{X_Q}(\frac{\rho_R}{\theta})(\partial_{X_Q}F_{iK})F_{jK},$$

is apparently non-symmetric. Likewise, the dependence on the second-order derivatives lead to non-symmetric terms. As a consequence the symmetry condition (31) is not fulfilled.

3.3. Models involving Eulerian variables

As with other models in continuum mechanics, a simpler and more direct description of the elastic properties of a body is thought as being given in the Eulerian description. As a first attempt we might consider the scheme arising from the choice of the displacement $\mathbf{u}$, the gradient $\nabla\mathbf{u}$, and the second gradient $\nabla\nabla\mathbf{u}$ as the variables along with the thermal variables [27].

The mathematical form of the total time derivative $(\nabla\mathbf{u})^{\cdot}$ much influences the thermodynamic consequences on the constitutive properties. For any continuously-differentiable function $g(\mathbf{x}, t)$ we have

$$(\partial_{x_i} g)^{\cdot} = \partial_t \partial_{x_i} g + v_k \partial_{x_k}\partial_{x_i} g = \partial_{x_i}\dot{g} - (\mathbf{L}^T\nabla)_i g. \tag{32}$$

If g is replaced with the displacement vector $\mathbf{u}$ we have

$$(\partial_{x_i} u_j)^{\cdot} = \partial_{x_i}\dot{u}_j - (\mathbf{L}^T\nabla)_i u_j = L_{ji} - (\mathbf{L}^T\nabla)_i u_j.$$

Hence we compute

$$\begin{aligned}\partial_{\nabla\mathbf{u}}\psi \cdot (\nabla\mathbf{u})^{\cdot} &= \partial_{\partial_{x_q}u_p}\psi[\partial_{x_q}\dot{u}_p - L^T_{qr}\partial_{x_r}u_p] = \partial_{\partial_{x_q}u_p}\psi[\delta_{rp}L_{rq} - L_{rq}\partial_{x_r}u_p] \\ &= [(\delta_{rp} - \partial_{x_r}u_p)\partial_{\partial_{x_q}u_p}\psi]L_{rq} = [(\mathbf{1} - \nabla\mathbf{u})\partial_{\nabla\mathbf{u}}\psi]\cdot\mathbf{L}\end{aligned}$$

where δ_{rp} denotes the Kronecker delta.

Then a repeated use of (32) and some rearrangements lead to

$$(\partial_{x_i}\partial_{x_j}u_p)^{\cdot} = \partial_{x_i}(\partial_{x_j}u_p)^{\cdot} - (\mathbf{L}^T\nabla)_i(\partial_{x_j}u_p) = \partial_{x_i}[\partial_{x_j}\dot{u}_p - (\mathbf{L}^T\nabla)_j u_p] - (\mathbf{L}^T\nabla)_i(\partial_{x_j}u_p)]$$
$$= (\delta_{rp} - \partial_{x_r}u_p)\partial_{x_i}L_{rj} - (L_{rj}\partial_{x_j} + L_{ri}\partial_{x_j})\partial_{x_r}u_p.$$

With these relations we might proceed with a model where elastic nonlocality is described by $\nabla\mathbf{u}$ and $\nabla\nabla\mathbf{u}$. However this would prevent the symmetry of the stress tensor.

An Eulerian description of the strain-gradient model, along with the need of a symmetric Cauchy stress, might suggest that we look for the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ as the mechanical deformation variable. Here we ascertain whether this dependence satisfies the requirements of thermodynamics and the symmetry of the Cauchy stress tensor.

Observe that, in suffix notation,

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}(\partial_{x_j}u_i + \partial_{x_i}u_j)^{\cdot} = \tfrac{1}{2}\{\partial_{x_j}\dot{u}_i - (\mathbf{L}^T\nabla)_j u_i + \partial_{x_i}\dot{u}_j - (\mathbf{L}^T\nabla)_i u_j\}$$
$$= D_{ij} - \tfrac{1}{2}(L_{rj}\partial_{x_r}u_i + L_{ri}\partial_{x_r}u_j)$$

As it must be, the expression of $\dot{\varepsilon}_{ij}$ is symmetric with respect to the indices i, j. For formal convenience, define the spatial deformation gradient $\boldsymbol{\xi}$,

$$\boldsymbol{\xi} = \nabla\mathbf{u}, \qquad \xi_{ij} := \partial_{x_j}u_i.$$

Hence we can write $\dot{\boldsymbol{\varepsilon}}$ in the form

$$\dot{\varepsilon}_{ij} = D_{ij} - \tfrac{1}{2}(\xi_{ir}L_{rj} + \xi_{jr}L_{ri}), \qquad \dot{\boldsymbol{\varepsilon}} = \mathbf{D} - \mathrm{sym}(\boldsymbol{\xi}\mathbf{L}).$$

Furthermore, for later purposes we consider some identities. First we notice that

$$\partial_{\boldsymbol{\varepsilon}}\psi\cdot\dot{\boldsymbol{\varepsilon}} = \partial_{\varepsilon_{ij}}\psi\, D_{ij} - (\partial_{x_r}u_i\partial_{\varepsilon_{ij}}\psi)L_{rj} = \partial_{\boldsymbol{\varepsilon}}\psi\cdot\mathbf{D} - (\boldsymbol{\xi}^T\partial_{\boldsymbol{\varepsilon}}\psi)\cdot\mathbf{L}. \tag{33}$$

Next, by

$$(\nabla\boldsymbol{\varepsilon})^{\cdot} = \nabla\dot{\boldsymbol{\varepsilon}} - (\mathbf{L}^T\nabla)\boldsymbol{\varepsilon}$$

we have

$$\partial_{\partial_{x_k}\varepsilon_{ij}}\psi(\partial_{x_k}\varepsilon_{ij})^{\cdot} = \partial_{\partial_{x_k}\varepsilon_{ij}}\psi(\partial_{x_k}\dot{\varepsilon}_{ij}) - (\partial_{x_q}\varepsilon_{ij}\partial_{\partial_{x_p}\varepsilon_{ij}}\psi)L_{qp}$$

or

$$\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\boldsymbol{\varepsilon})^{\cdot} = \partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\dot{\boldsymbol{\varepsilon}}) - (\nabla\varepsilon_{ij}\otimes\partial_{\nabla\varepsilon_{ij}}\psi)\cdot\mathbf{L}. \tag{34}$$

Let

$$\theta, \nabla\theta, \boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon}, \nabla\nabla\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \mathbf{L} \tag{35}$$

be the variables and

$$\psi, \eta, \mathbf{T}, \mathbf{q}, \mathbf{k}, \gamma$$

the constitutive functions. To save writing we let the free energy ψ take the form

$$\psi = \psi(\theta, \boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon});$$

the independence of ψ of $\nabla\theta, \nabla\nabla\boldsymbol{\varepsilon}, \boldsymbol{\xi}$ and $\mathbf{L}$, when the set of variables is (35), can be proved using the Coleman-Noll procedure. For the moment we ignore the requirement $\mathbf{T} = \mathbf{T}^T$ and then establish whether this follows from thermodynamics; accordingly we let $\mathbf{T}\cdot\mathbf{L}$, instead of $\mathbf{T}\cdot\mathbf{D}$, be the stress power. Consequently, the CD inequality (7) results in

$$-\rho(\partial_\theta\psi + \eta)\dot{\theta} - \rho\partial_{\boldsymbol{\varepsilon}}\psi\cdot\dot{\boldsymbol{\varepsilon}} - \rho\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\boldsymbol{\varepsilon})^{\cdot} + \mathbf{T}\cdot\mathbf{L} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta + \theta\nabla\cdot\mathbf{k} = \rho\theta\gamma \geq 0.$$

Using the identities (33) and (34) we have

$$\mathbf{T}\cdot\mathbf{L}-\rho\partial_{\boldsymbol{\varepsilon}}\psi\cdot\dot{\boldsymbol{\varepsilon}}-\rho\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\boldsymbol{\varepsilon})^{\cdot}=[\mathbf{T}-\rho\partial_{\boldsymbol{\varepsilon}}\psi-\rho\boldsymbol{\xi}^T\partial_{\boldsymbol{\varepsilon}}\psi+\rho(\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi)]\cdot\mathbf{L}-\rho\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\dot{\boldsymbol{\varepsilon}})$$

and then the CD inequality can be written in the form

$$-\rho(\partial_\theta\psi+\eta)\dot{\theta}+[\mathbf{T}-\rho\partial_{\boldsymbol{\varepsilon}}\psi-\rho\boldsymbol{\xi}^T\partial_{\boldsymbol{\varepsilon}}\psi+\rho(\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi)]\cdot\mathbf{L}-\rho\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\dot{\boldsymbol{\varepsilon}})$$
$$\frac{1}{\theta}\mathbf{q}\cdot\nabla\theta+\theta\nabla\cdot\mathbf{k}=\rho\theta\gamma\geq 0.$$

The linearity and arbitrariness of $\dot{\theta}$ implies that $\partial_\theta\psi+\eta=0$. Divide by θ the remaining relation to have

$$\frac{1}{\theta}[\mathbf{T}-\rho\partial_{\boldsymbol{\varepsilon}}\psi-\rho\boldsymbol{\xi}^T\partial_{\boldsymbol{\varepsilon}}\psi+\rho(\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi)]\cdot\mathbf{L}-\frac{1}{\theta}\rho\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\dot{\boldsymbol{\varepsilon}})$$
$$-\frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta+\nabla\cdot\mathbf{k}=\rho\gamma\geq 0.$$

Notice that

$$-\frac{1}{\theta}\rho\partial_{\nabla\boldsymbol{\varepsilon}}\psi\cdot(\nabla\dot{\boldsymbol{\varepsilon}})=-\nabla\cdot[\frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi\dot{\boldsymbol{\varepsilon}}]+\nabla\cdot[\frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi]\cdot\dot{\boldsymbol{\varepsilon}}$$
$$=-\nabla\cdot[\frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi(\mathbf{D}-\mathrm{sym}(\boldsymbol{\xi}\mathbf{L}))]+[\nabla\cdot(\frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi)]\cdot[\mathbf{D}-\mathrm{sym}(\boldsymbol{\xi}\mathbf{L})].$$

We might take $\mathbf{k}$ in the form

$$\mathbf{k}=\frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi(\mathbf{D}-\mathrm{sym}(\boldsymbol{\xi}\mathbf{L})). \tag{36}$$

Yet, since $\boldsymbol{\xi}\mathbf{L}=\boldsymbol{\xi}(\mathbf{D}+\mathbf{W})$, the condition (36) on $\mathbf{k}$ is allowed provided we let $\mathbf{k}$ be a non-objective quantity. Based on the assumed non-objectivity of $\mathbf{k}$, we can write the remaining CD inequality in the form

$$\boldsymbol{\mathfrak{T}}\cdot(\mathbf{D}+\mathbf{W})-\frac{1}{\theta}\mathbf{q}\cdot\nabla\theta=\rho\theta\gamma\geq 0, \tag{37}$$

where

$$\boldsymbol{\mathfrak{T}}:=\mathbf{T}-\rho\delta_{\boldsymbol{\varepsilon}}\psi-\rho\boldsymbol{\xi}^T\delta_{\boldsymbol{\varepsilon}}\psi+\rho\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi,\qquad \delta_{\boldsymbol{\varepsilon}}\psi=\partial_{\boldsymbol{\varepsilon}}\psi-\frac{\theta}{\rho}\nabla\cdot(\frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi)$$

The linearity and arbitrariness of $\mathbf{W}$ implies that

$$\mathrm{skw}\boldsymbol{\mathfrak{T}}=\mathbf{0}, \tag{38}$$

and then (37) reduces to

$$\boldsymbol{\mathfrak{T}}\cdot\mathbf{D}-\frac{1}{\theta}\mathbf{q}\cdot\nabla\theta=\rho\theta\gamma\geq 0. \tag{39}$$

While the reduced CD inequality (39) allows for non-dissipative properties modelled by the stress and the heat flux, the condition (38) yields

$$\mathrm{skw}\mathbf{T}=\rho\,\mathrm{skw}\{\boldsymbol{\xi}^T\delta_{\boldsymbol{\varepsilon}}\psi-\rho\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi\}.$$

The tensor $\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi$ is symmetric whenever $\partial_{\nabla\boldsymbol{\varepsilon}}\psi$ is a scalar times $\nabla\boldsymbol{\varepsilon}$. While this is true for a number of models, the requirement $\mathrm{skw}\{\boldsymbol{\xi}^T\delta_{\boldsymbol{\varepsilon}}\psi\}=\mathbf{0}$ looks very restrictive. Accordingly, strain-gradient elastic solids are not consistent, in general, with the joint validity of the second law of thermodynamics, expressed by the CD inequality, and the symmetry of the Cauchy stress tensor arising from the balance of angular momentum.

If, however, strain-gradient models are considered in the Eulerian description in terms of the infinitesimal strain $\boldsymbol{\varepsilon}$ then a justification might be the observation that for small strains ($|\boldsymbol{\varepsilon}| \ll 1$ and then $|\boldsymbol{\xi}| \ll 1$) in linear models, where

$$\delta_{\boldsymbol{\varepsilon}}\psi = \alpha_1\boldsymbol{\varepsilon} - \alpha_2\Delta\boldsymbol{\varepsilon}, \qquad \partial_{\nabla\boldsymbol{\varepsilon}}\psi = \alpha_3\nabla\boldsymbol{\varepsilon},$$

the constitutive function

$$\tilde{\mathbf{T}} = \rho\delta_{\boldsymbol{\varepsilon}}\psi - \rho\nabla\boldsymbol{\varepsilon}\otimes\partial_{\nabla\boldsymbol{\varepsilon}}\psi + \mathfrak{T},$$

where $\mathfrak{T}$ is symmetric and subject to (39), can be used as a symmetric approximation. With the same condition $|\boldsymbol{\xi}| \ll 1$, the approximate extra-entropy flux

$$\tilde{\mathbf{k}} = \frac{\rho}{\theta}\partial_{\nabla\boldsymbol{\varepsilon}}\psi\,\mathbf{D}$$

is objective.

We now examine why some approaches in the literature lead directly to results analogous to

$$\mathbf{T} = \rho\delta_{\varepsilon}\psi,$$

possibly to within a dissipative stress $\mathfrak{T}$.

3.4. Relation to other approaches

Other approaches in the literature involve modified general principles as the framework of the strain-gradient models. Here we sketch the main distinctions among some approaches.

• Balance equations through the hyperstress

The starting point is the generalization of the stress power. A possibly non-symmetric Cauchy stress $\mathbf{T}$ is considered and, in addition to the classical stress power $\mathbf{T}\cdot\mathbf{L}$, an analogous power $\mathbf{G}\cdot\nabla\mathbf{L}$ is considered, $\mathbf{G}$ being a third-order tensor (hyperstress). Hence the internal stress-power, say w, is assumed in the form [2]

$$w = \mathbf{T}\cdot\mathbf{L} + \mathbf{G}\cdot\nabla\mathbf{L}.$$

By using the principle of virtual power, whereby the external and internal powers are equal, upon a proper definition of both powers the balance of linear momentum is found to involve an effective stress tensor

$$\hat{\mathbf{T}} = \mathbf{T} - \nabla\cdot\mathbf{G}, \qquad \hat{T}_{ij} = T_{ij} - \partial_{x_k}G_{ijk},$$

with $\mathbf{T}\in \mathrm{Sym}$.

• Use of the infinitesimal strain ε or the displacement gradient $\nabla\mathbf{u}$ as a variable

The approach in [27] follows the main ideas in [2] though it addresses the attention to the Eulerian variable $\nabla\mathbf{u}$. The power w is taken in the form

$$\mathbf{T}\cdot\nabla\dot{\mathbf{u}} + \mathbf{G}\cdot\nabla\nabla\dot{\mathbf{u}}.$$

Furthermore, with $\psi(\nabla\mathbf{u}, \nabla\nabla\mathbf{u})$ the derivative $\dot{\psi}$ is taken in the form

$$\dot{\psi} = \partial_{\nabla\mathbf{u}}\psi\cdot\nabla\dot{\mathbf{u}} + \partial_{\nabla\nabla\mathbf{u}}\psi\cdot\nabla\nabla\dot{\mathbf{u}}$$

thus following the approximation $(\nabla \mathbf{u})^{\cdot} = \nabla \dot{\mathbf{u}}$, $(\nabla\nabla \mathbf{u})^{\cdot} = \nabla\nabla \dot{\mathbf{u}}$. Hence the terms $\boldsymbol{\xi}^T \mathbf{u}$ and $\boldsymbol{\xi}^T \nabla \mathbf{u}$ are ignored. Anyway, the stresses $\mathbf{T}$ or $\hat{\mathbf{T}}$ need not be symmetric.

In [17] different representations of the stress are considered as defined by

$$\sigma^{(0)} = \partial_{\boldsymbol{\varepsilon}} \psi, \qquad \boldsymbol{\sigma}^{(1)} = \partial_{\nabla \boldsymbol{\varepsilon}} \psi,$$

and letting the stress power have the form

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(0)}) \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^{(1)} \cdot (\nabla \dot{\boldsymbol{\varepsilon}}),$$

with $\boldsymbol{\sigma}$ the stress tensor. Again, by the definitions of $\sigma^{(0)}$ and $\boldsymbol{\sigma}^{(1)}$, the procedure amounts to the approximation $(\nabla \boldsymbol{\varepsilon})^{\cdot} \simeq \nabla \dot{\boldsymbol{\varepsilon}}$.

• Modelling based on the action principle

There are approaches where the constitutive equations are defined through a strain energy while the evolution (or balance) equations are established through the Euler-Lagrange equations of an appropriate action integral [5,6]. The Cauchy stress tensor $\boldsymbol{\sigma}$ and the hyperstress (or double stress tensor) $\boldsymbol{\tau}$ are defined through a strain energy density $\mathcal{W}(\boldsymbol{\varepsilon}, \nabla \boldsymbol{\varepsilon})$ in the form

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} \mathcal{W}, \qquad \boldsymbol{\tau} = \partial_{\nabla \boldsymbol{\varepsilon}} \mathcal{W}.$$

The evolution equation is the EL (Euler-Lagrange) equation associated with the Lagrangian density $\mathcal{L} = -\mathcal{W} - \mathcal{V}$ where $\mathcal{V} = -\mathbf{u} \cdot \mathbf{f}$ is the potential of the body force $\mathbf{f}$. The EL equation reads

$$\partial_{u_i} \mathcal{L} - \partial_{x_j} \partial_{\partial_{x_j} u_i} \mathcal{L} + \partial_{x_k} \partial_{x_j} \partial_{\partial_{x_k} \partial_{x_j} u_i} \mathcal{L} = 0$$

and results in the equilibrium condition

$$\partial_{x_j} (\sigma_{ij} - \partial_{x_m} \tau_{ijm}) + f_i = 0,$$

which indicates $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}$ as the effective stress. The choice of $\mathcal{W}$ specifies the equilibrium condition.

It is of interest that in general a variational approach leads to an EL equation that looks similar to what happens in thermodynamics thanks to the extra-entropy flux. Yet, in thermodynamics the variational derivative involves the temperature while the EL equations are not affected by the temperature.

4. Nonlocality in heat conduction

Heat conduction in solids may exhibit nonlocal effects in that the conduction is affected by higher-order gradients of the temperature. However nonlocality may occur and be suitably described by involving higher-order derivatives of the heat flux vector. Differently from the strain-gradient elasticity, we show that the nonlocal properties are induced directly by the entropy flux and the heat production rather then by the free energy. For simplicity we let the body be rigid and then the reference configuration is identified with the current configuration. The CD inequality simplifies to

$$-\rho(\dot{\psi} + \eta \dot{\theta}) - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta + \theta \nabla \cdot \mathbf{k} = \rho \theta \gamma \geq 0. \tag{40}$$

4.1. Nonlocality through temperature gradients

Consider a heat conductor described by the variables

$$\theta, \dot{\theta}, \nabla \theta, \nabla\nabla \theta, \nabla\nabla\nabla \theta.$$

Let $\psi, \eta, \mathbf{k}$ and $\gamma \geq 0$ be the constitutive functions. Upon computation of $\dot{\psi}$ and substitution into (40) we have

$$-\rho(\partial_\theta \psi + \eta)\dot{\theta} - \rho\partial_{\dot{\theta}}\psi\ddot{\theta} - \rho\partial_{\nabla\theta}\psi \cdot \nabla\dot{\theta} - \rho\partial_{\nabla\nabla\theta}\psi \cdot \nabla\nabla\dot{\theta} - \rho\partial_{\nabla\nabla\nabla\theta}\psi \cdot \nabla\nabla\nabla\dot{\theta}$$
$$-\frac{1}{\theta}\mathbf{q} \cdot \nabla\theta + \theta\nabla \cdot \mathbf{k} = \rho\theta\gamma \geq 0. \tag{41}$$

The linearity and arbitrariness of $\ddot{\theta}, \nabla\nabla\dot{\theta}$ and $\nabla\nabla\nabla\dot{\theta}$ imply that

$$\partial_{\dot{\theta}}\psi = 0, \quad \partial_{\nabla\nabla\theta}\psi = \mathbf{0}, \quad \partial_{\nabla\nabla\nabla\theta}\psi = \mathbf{0},$$

so that the free energy is allowed to depend on θ and $\nabla\theta$. Hence we divide the remaining equation by θ and use the identity

$$-\frac{\rho}{\theta}\partial_{\nabla\theta}\psi \cdot \nabla\dot{\theta} = -(\frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot{\theta}) + [\nabla \cdot (\frac{\rho}{\theta}\partial_{\nabla\theta}\psi)]\dot{\theta}$$

to obtain

$$-\frac{\rho}{\theta}(\eta + \delta_\theta\psi)\dot{\theta} - \frac{1}{\theta^2}\mathbf{q} \cdot \nabla\theta + \nabla \cdot (\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot{\theta}) = \rho\gamma \geq 0,$$

where

$$\delta_\theta\psi = \partial_\theta\psi - \frac{\theta}{\rho}\nabla \cdot (\frac{\rho}{\theta}\partial_{\nabla\theta}\psi).$$

Assume $\mathbf{k}, \mathbf{q}$, and γ are independent of $\dot{\theta}$. Hence the linearity and arbitrariness of $\dot{\theta}$ imply that

$$\eta = -\delta_\theta\psi. \tag{42}$$

Otherwise we might assume the constitutive equation (42) and examine the possible thermodynamic restrictions on the other constitutive functions.

Likewise we might take $\mathbf{k} = (\rho/\theta)\partial_{\nabla\theta}\psi\,\dot{\theta}$. This is a possible selection but, as we see in a moment, it would result quite restrictive.

Now we are left with the CD inequality in the form

$$-\frac{1}{\theta^2}\mathbf{q} \cdot \nabla\theta + \nabla \cdot (\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot{\theta}) = \rho\gamma \geq 0. \tag{43}$$

Also on the basis of the Guyer-Krumhansl model [28], we check the thermodynamic admissibility of the constitutive equation

$$\mathbf{q} = -\kappa(\theta)\nabla\theta + h_0\theta^2\nabla \cdot [\nabla\nabla\theta + 2\Delta\theta\mathbf{1}], \tag{44}$$

where $\kappa(\theta) > 0, h_0 > 0$. Notice that

$$\frac{1}{\theta^2}\mathbf{q} \cdot \nabla\theta = -\frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2 + h_0\nabla \cdot [(\nabla\theta \cdot \nabla)\nabla\theta + 2(\Delta\theta)\nabla\theta] - |\nabla\nabla\theta|^2 - 2|\Delta\theta|^2.$$

Substitution in (43) yields

$$\nabla \cdot \Big\{\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot{\theta} - h_0\big[(\nabla\theta \cdot \nabla)\nabla\theta + 2(\Delta\theta)\nabla\theta\big]\Big\}$$
$$+\frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2 + h_0|\nabla\nabla\theta|^2 + 2h_0|\Delta\theta|^2 = \rho\gamma \geq 0.$$

Consistent with the requirement $\gamma \geq 0$ we conclude that (44) satisfies the CD inequality with entropy production

$$\gamma = \frac{\kappa(\theta)}{\rho\theta^2}|\nabla\theta|^2 + \frac{h_0}{\rho}|\nabla\nabla\theta|^2 + \frac{2h_0}{\rho}|\Delta\theta|^2 \geq 0$$

and extra-entropy flux

$$\mathbf{k} = \frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot\theta + h_0[(\nabla\theta\cdot\nabla)\nabla\theta + 2(\Delta\theta)\nabla\theta].$$

In words, the function (44) is thermodynamically consistent in that the quantity $\mathbf{q}\cdot\nabla\theta/\theta^2$ is split in two parts, one providing an entropy flux, the other determining the entropy production.

We remark that, here and in the following, the free energy ψ is allowed to depend on the temperature gradient. Apparently this dependence contrasts with some classical results by Coleman et al. [21,29] where the CD inequality is used to show that the free energy and the entropy are independent of the temperature gradients. However, as devised by Müller in [30], we account for the dependence of the free energy on the temperature gradient by adding an extra term of entropy flux as constitutive quantity.

4.2. *Nonlocality through gradients of the heat flux*

To emphasize how the availability of $\mathbf{k}$ and γ as constitutive functions makes the thermodynamic analysis extremely flexible, we now investigate a model of heat conduction where the nonlocality is modelled through gradients of the heat flux [28,31].

Let

$$\theta, \dot\theta, \nabla\theta, \nabla\nabla\theta, \mathbf{q}, \nabla\mathbf{q}, \nabla\nabla\mathbf{q}$$

be the set of variables and $\psi, \eta, \dot{\mathbf{q}}, \mathbf{k}$, and γ the constitutive functions. Compute $\dot\psi$ and substitute in (40) to obtain

$$\begin{aligned} &-\rho(\partial_\theta\psi + \eta)\dot\theta - \rho\partial_{\dot\theta}\psi\ddot\theta - \rho\partial_{\nabla\theta}\psi\cdot\nabla\dot\theta - \rho\partial_{\nabla\nabla\theta}\psi\cdot\nabla\nabla\dot\theta - \rho\partial_{\mathbf{q}}\psi\cdot\dot{\mathbf{q}} \\ &-\rho\partial_{\nabla\mathbf{q}}\psi\cdot\nabla\dot{\mathbf{q}} - \rho\partial_{\nabla\nabla\mathbf{q}}\psi\cdot\nabla\nabla\dot{\mathbf{q}} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta + \theta\nabla\cdot\mathbf{k} = \rho\theta\gamma \geq 0. \end{aligned} \tag{45}$$

First we observe that $\ddot\theta$, $\nabla\nabla\dot\theta$, and $\nabla\nabla\dot{\mathbf{q}}$ occur linearly. Their arbitrariness implies that

$$\partial_{\dot\theta}\psi = 0, \quad \partial_{\nabla\nabla\theta}\psi = \mathbf{0}, \quad \partial_{\nabla\nabla\mathbf{q}}\psi = \mathbf{0}.$$

Again, divide by θ the remaining equation and notice that

$$-\frac{\rho}{\theta}\partial_{\nabla\theta}\psi\cdot\nabla\dot\theta = -\nabla\cdot(\frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot\theta) + [\nabla\cdot(\frac{\rho}{\theta}\partial_{\nabla\theta}\psi)]\dot\theta,$$

$$-\frac{\rho}{\theta}\partial_{\nabla\mathbf{q}}\psi\cdot\nabla\dot{\mathbf{q}} = -\nabla\cdot(\frac{\rho}{\theta}\partial_{\nabla\mathbf{q}}\psi\,\dot{\mathbf{q}}) + [\nabla\cdot(\frac{\rho}{\theta}\partial_{\nabla\mathbf{q}}\psi)]\cdot\dot{\mathbf{q}},$$

Consequently, we can write the CD inequality in the form

$$-\frac{\rho}{\theta}(\eta + \delta_\theta\psi)\dot\theta - \frac{\rho}{\theta}\delta_{\mathbf{q}}\psi\cdot\dot{\mathbf{q}} - \frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta + \nabla\cdot(\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot\theta - \frac{\rho}{\theta}\partial_{\nabla\mathbf{q}}\psi\,\dot{\mathbf{q}}) = \rho\gamma \geq 0. \tag{46}$$

For definiteness and based on the Guyer-Krumhansl model we consider the evolution equation

$$\dot{\mathbf{q}} = -\frac{1}{\tau(\theta)}\mathbf{q} - \frac{k(\theta)}{\tau(\theta)}\nabla\theta + \frac{h(\theta)}{\tau(\theta)}[\Delta\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})], \quad \tau > 0, \tag{47}$$

and wonder whether and how eq. (47) is consistent with (46). Upon substitution of $\dot{\mathbf{q}}$ in (46) we have

$$\frac{\rho}{\theta\tau}\delta_{\mathbf{q}}\psi\cdot\mathbf{q}+(-\frac{1}{\theta^2}\mathbf{q}+\frac{k\rho}{\tau\theta}\delta_{\mathbf{q}}\psi)\cdot\nabla\theta-\frac{h\rho}{\tau\theta}\delta_{\mathbf{q}}\psi\cdot[\Delta\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q})]$$
$$-\frac{\rho}{\theta}(\eta+\delta_\theta\psi)\dot{\theta}+\nabla\cdot(\mathbf{k}-\frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot{\theta}-\frac{\rho}{\theta}\partial_{\nabla\mathbf{q}}\psi\,\dot{\mathbf{q}})=\rho\gamma\geq 0. \tag{48}$$

Equation (48) is satisfied by requiring that the left-hand side be non-negative. Furthermore we follow some assumptions that make the results sufficient conditions for the validity of (48).

First we let

$$\eta=-\delta_\theta\psi;$$

as is common, we neglect a possible dissipative term $-\varkappa(\theta)\dot{\theta},\varkappa>0$, which would be thermodynamically admissible. Next we assume

$$\partial_{\nabla\mathbf{q}}\psi=\mathbf{0},$$

and hence $\delta_{\mathbf{q}}\psi=\partial_{\mathbf{q}}\psi$. Furthermore we assume

$$\frac{k\rho}{\tau\theta}\partial_{\mathbf{q}}\psi-\frac{1}{\theta^2}\mathbf{q}=\mathbf{0},$$

whence

$$\frac{\rho}{\theta\tau}\partial_{\mathbf{q}}\psi\cdot\mathbf{q}=\frac{1}{k\theta^2}|\mathbf{q}|^2.$$

Thus the remaining part of the CD inequality is

$$\frac{1}{k\theta^2}\mathbf{q}^2-\frac{h}{k\theta^2}\mathbf{q}\cdot[\Delta\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q})]+\nabla\cdot(\mathbf{k}-\frac{\rho}{\theta}\partial_{\nabla\theta}\dot{\theta})=\rho\gamma\geq 0. \tag{49}$$

The further assumption that

$$\alpha:=\frac{h}{k\theta^2}$$

be constant and the identity

$$\mathbf{q}[\Delta\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q})]=\nabla\cdot[(\nabla\mathbf{q}^2/2)+2(\nabla\cdot\mathbf{q})\mathbf{q}]-|\nabla\mathbf{q}|^2-2|\nabla\cdot\mathbf{q}|^2$$

allow us to write the CD inequality (49) in the form

$$\frac{1}{k\theta^2}\mathbf{q}^2+\alpha|\nabla\mathbf{q}|^2+2\alpha|\nabla\cdot\mathbf{q}|^2+\nabla\cdot[\mathbf{k}-\frac{\rho}{\theta}\partial_{\nabla\theta}\psi\dot{\theta}-\alpha\nabla\mathbf{q}^2/2-2\alpha(\nabla\cdot\mathbf{q})\mathbf{q}]=\rho\gamma\geq 0.$$

Consequently, it follows that $\gamma\geq 0$ is satisfied by letting

$$k>0,\quad \alpha\geq 0,\quad \mathbf{k}=\frac{\rho}{\theta}\partial_{\nabla\theta}\psi\dot{\theta}+\alpha\nabla\mathbf{q}^2/2+2\alpha(\nabla\cdot\mathbf{q})\mathbf{q}.$$

To sum up, the constitutive equation (47) is thermodynamically consistent if

$$\psi=\frac{\tau}{2k\rho\theta}\mathbf{q}^2+f(\theta,\nabla\theta),\quad k(\theta)>0,\quad \frac{h}{k\theta^2}>0 \text{ is constant.}$$

Remark 2. *Since* $\Delta\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q})=\nabla\cdot[\nabla\mathbf{q}+2(\nabla\cdot\mathbf{q})\mathbf{1}]$*, if* $\beta:=h(\theta)/\tau(\theta)$ *is constant then the evolution equation (47) may be written in the form*

$$\dot{\mathbf{q}}+\frac{1}{\tau}\mathbf{q}+\frac{k}{\tau}\nabla\theta=\nabla\cdot\mathbf{Q},\qquad Q_{ij}:=\beta\partial_{x_j}q_i+2\beta(\nabla\cdot\mathbf{q})\delta_{ij}.$$

This divergence form of the evolution equation is analogous to the structure of the balance equations in [32].

4.3. Nonlinear models of nonlocal heat conduction

Based on nonlocal models of heat conduction in nanosystems [33,34] we generalize the nonlocality of the Guyer-Krumhansl model by allowing for nonlinear effects. For definiteness we look for the thermodynamic consistency of the constitutive equation

$$\dot{\mathbf{q}} = -\frac{1}{\tau}\mathbf{q} - k\nabla\theta + \lambda[\Delta\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})] + \mu[2(\mathbf{q}\cdot\nabla)\mathbf{q} + (\nabla\cdot\mathbf{q})\mathbf{q}]. \tag{50}$$

By analogy with with the previous procedure we let

$$\theta, \dot{\theta}, \nabla\theta, \nabla\nabla\theta, \mathbf{q}, \nabla\mathbf{q}, \nabla\nabla\mathbf{q}$$

be the variables and $\psi, \eta, \dot{\mathbf{q}}, \mathbf{k}$, and γ the constitutive functions. Upon substitution of $\dot{\psi}$ in the CD inequality (45) we find that

$$\partial_{\dot{\theta}}\psi = 0, \qquad \partial_{\nabla\nabla\theta}\psi = \mathbf{0}, \qquad \partial_{\nabla\nabla\mathbf{q}}\psi = \mathbf{0},$$

and assume that $\partial_{\nabla\mathbf{q}}\psi = \mathbf{0}$. Hence the CD inequality simplifies to

$$-\frac{\rho}{\theta}(\eta + \delta_\theta\psi)\dot{\theta} - \frac{\rho}{\theta}\partial_{\mathbf{q}}\psi\cdot\dot{\mathbf{q}} - \frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta + \nabla\cdot(\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\dot{\theta}) = \rho\gamma \geq 0.$$

Substitution of $\dot{\mathbf{q}}$ from (50) results in

$$\begin{aligned}-\frac{\rho}{\theta}(\eta + \delta_\theta\psi)\dot{\theta} + \frac{\rho}{\theta\tau}\partial_{\mathbf{q}}\psi\cdot\mathbf{q} + (\frac{k\rho}{\theta}\partial_{\mathbf{q}}\psi - \frac{1}{\theta^2}\mathbf{q})\cdot\nabla\theta - \lambda\frac{\rho}{\theta}\partial_{\mathbf{q}}\psi\cdot[\Delta\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})]\\ -\mu\frac{\rho}{\theta}\partial_{\mathbf{q}}\psi\cdot[2(\mathbf{q}\cdot\nabla)\mathbf{q} + (\nabla\cdot\mathbf{q})\mathbf{q}] + \nabla\cdot(\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\dot{\theta}) = \rho\gamma.\end{aligned}$$

Again we let

$$\eta + \delta_\theta\psi = 0, \qquad \frac{k\rho}{\theta}\partial_{\mathbf{q}}\psi - \frac{1}{\theta^2}\mathbf{q} = \mathbf{0}, \tag{51}$$

whence

$$\begin{aligned}\frac{1}{k\tau\theta^2}\mathbf{q}^2 - \frac{\lambda}{k\theta^2}\mathbf{q}\cdot[\Delta\mathbf{q} + 2\nabla(\nabla\cdot\mathbf{q})] - \frac{\mu}{k\theta^2}\mathbf{q}\cdot[2(\mathbf{q}\cdot\nabla)\mathbf{q} + (\nabla\cdot\mathbf{q})\mathbf{q}]\\ +\nabla\cdot(\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\dot{\theta}) = \rho\gamma.\end{aligned} \tag{52}$$

We determine sufficient conditions on $\mathbf{k}$ and $\gamma \geq 0$ for the validity of (52) by assuming that

$$c_1 = \frac{\lambda}{k\theta^2}, \qquad c_2 = \frac{\mu}{k\theta^2}$$

are constant. Hence we observe that

$$\mathbf{q}\cdot\Delta\mathbf{q} = \nabla\cdot(\nabla\mathbf{q}^2/2) - |\nabla\mathbf{q}|^2, \qquad 2\mathbf{q}\cdot\nabla(\nabla\cdot\mathbf{q}) = 2\nabla\cdot(\mathbf{q}\nabla\cdot\mathbf{q}) - 2|\nabla\cdot\mathbf{q}|^2,$$
$$2\mathbf{q}\cdot(\mathbf{q}\cdot\nabla)\mathbf{q} + \mathbf{q}^2\nabla\cdot\mathbf{q} = \nabla\cdot(\mathbf{q}^2\mathbf{q}).$$

Equation (52) can then be given the form

$$\begin{aligned}\frac{1}{k\tau\theta^2}\mathbf{q}^2 + c_1(|\nabla\mathbf{q}|^2 + 2|\nabla\cdot\mathbf{q}|^2)\\ +\nabla\cdot\left(\mathbf{k} - \frac{\rho}{\theta}\partial_{\nabla\theta}\psi\,\dot{\theta} - \tfrac{1}{2}c_1\nabla\mathbf{q}^2 - 2c_1\mathbf{q}\nabla\cdot\mathbf{q} - c_2\mathbf{q}^2\mathbf{q}\right) = \rho\gamma.\end{aligned} \tag{53}$$

Hence it follows that $\gamma \geq 0$ is satisfied by letting

$$k, \tau > 0, \qquad c_1 \geq 0,$$
$$\mathbf{k} = \frac{\rho}{\theta} \partial_{\nabla\theta}\psi\, \dot{\theta} + \tfrac{1}{2} c_1 \nabla \mathbf{q}^2 + 2c_1 \mathbf{q} \nabla \cdot \mathbf{q} + c_2 \mathbf{q}^2 \mathbf{q}).$$

The second requirement in (51) is a relation between the conductivity k and the free energy ψ. For definiteness, select ψ in the form

$$\psi = \psi_0(\theta, \nabla\theta) + \Psi(\theta, \xi_n), \qquad \xi_n = |\mathbf{q}|^n, n \geq 2.$$

Hence (51) gives

$$\frac{1}{k} = \rho\theta \partial_{\xi_n} \Psi \, n |\mathbf{q}|^{n-2}.$$

When $n = 2$ the simple case

$$\rho\Psi(\theta, \xi) = \beta(\theta)\xi_2$$

results in

$$\frac{1}{k} = 2\theta\beta(\theta)$$

and establishes a relation in a finite form between k and β.

In an evolution problem we have to fix the fluxes at the boundary, namely $\mathbf{q} \cdot \mathbf{n}$ and $\mathbf{k} \cdot \mathbf{n}$. Letting $\partial_{\nabla\theta}\psi = \nu \nabla\theta$ we might have

$$\mathbf{q} \cdot \mathbf{n} = q_n, \qquad \nu \frac{\rho}{\theta} \dot{\theta} \partial_n \theta + \tfrac{1}{2} c_1 \partial_n \mathbf{q}^2 + [2c_1 \nabla \cdot \mathbf{q} + c_2 \mathbf{q}^2] \mathbf{q} \cdot \mathbf{n} = \varkappa_n,$$

at $\partial\Omega$, with $\partial_n = \mathbf{n} \cdot \nabla$, while the fluxes q_n and $\varkappa_n$ are assigned on the boundary.

5. Nonlocality in magnetization

We examine thermodynamic restrictions on the dependence of constitutive functions on gradient fields in polar media. For definiteness we consider effects of temperature and magnetization gradients in polar magnetic solids. In addition to be of interest on their own, these dependencies look of basic importance in detailed models of dynamics of magnetic domains.

We describe the continuum through the variables

$$\theta, \mathbf{F}, \mathfrak{M}, \dot{\theta}, \nabla_R\, \theta, \dot{\mathbf{F}}, \nabla_R\, \mathbf{F}, \nabla_R\, \mathfrak{M}, \nabla_R\, \nabla_R\, \theta, \nabla_R\, \nabla_R\, \mathfrak{M}$$

and let $\psi, \eta, \mathbf{T}, \mathbf{q}_R, \mathfrak{H}, \dot{\mathfrak{M}}, \mathbf{k}_R$, and γ be given by constitutive functions. The free energy, as well as η and γ, has to be Euclidean invariant and then so has to be the dependence on $\mathbf{F}, \nabla\mathbf{F}$.

Upon computation and substitution of $\dot{\psi}$ in (14) we find

$$\begin{aligned}
-\rho_R(\partial_\theta\psi + \eta)\dot{\theta} + (\mu_0\mathfrak{H} - \rho_R\partial_{\mathfrak{M}}\psi) \cdot \dot{\mathfrak{M}} - \rho_R\partial_{\dot{\mathbf{F}}}\psi \cdot \ddot{\mathbf{F}} - \rho_R\partial_{\dot{\theta}}\psi\ddot{\theta} + J(\mathbf{T} + \mu_0\mathbf{H} \cdot \mathbf{M}) \cdot \mathbf{W} \\
-\rho_R\partial_{\mathbf{F}}\psi \cdot \dot{\mathbf{F}} + J(\mathbf{T} + \mu_0\mathbf{H} \otimes \mathbf{M}) \cdot \mathbf{D} - \rho_R\partial_{\dot{\mathfrak{M}}}\psi \cdot \ddot{\mathfrak{M}} - \rho_R\partial_{\nabla_R\,\mathfrak{M}}\psi \cdot \nabla_R\,\dot{\mathfrak{M}} \\
-\rho_R\partial_{\nabla_R\,\nabla_R\,\theta}\psi \cdot \nabla_R\,\nabla_R\,\dot{\theta} - \rho_R\partial_{\nabla_R\,\nabla_R\,\mathfrak{M}}\psi \cdot \nabla_R\,\nabla_R\,\dot{\mathfrak{M}} - \rho_R\partial_{\nabla_R\,\theta}\psi \cdot \nabla_R\,\dot{\theta} - \rho_R\partial_{\nabla_R\,\mathbf{F}}\psi \cdot \nabla_R\,\dot{\mathbf{F}} \\
-\frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\,\theta + \theta\nabla_R\, \cdot \mathbf{k}_R = \rho_R\theta\gamma \geq 0
\end{aligned}$$

First we observe that $\ddot{\theta}, \ddot{\mathbf{F}}, \ddot{\mathfrak{M}}, \nabla_R\,\nabla_R\,\dot{\theta}, \nabla_R\,\nabla_R\,\dot{\mathfrak{M}}$ occur linearly and can take arbitrary values. Hence it follows

$$\partial_{\dot{\theta}}\psi = 0, \quad \partial_{\dot{\mathbf{F}}}\psi = \mathbf{0}, \quad \partial_{\dot{\mathfrak{M}}}\psi = \mathbf{0}, \quad \partial_{\nabla_R\,\nabla_R\,\theta}\psi = \mathbf{0}, \quad \partial_{\nabla_R\,\nabla_R\,\mathfrak{M}}\psi = \mathbf{0}.$$

The remaining equation has the form

$$\begin{aligned} &-\rho_R(\partial_\theta\psi+\eta)\dot\theta+(\mu_0\mathfrak{H}-\rho_R\partial_{\mathfrak{M}}\psi)\cdot\dot{\mathfrak{M}}+J(\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M})\cdot\mathbf{W}\\ &\quad-\rho_R\partial_{\mathbf{F}}\psi\cdot\dot{\mathbf{F}}+J(\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M})\cdot\mathbf{D}-\rho_R\partial_{\nabla_R\mathfrak{M}}\psi\cdot\nabla_R\dot{\mathfrak{M}}\\ &-\rho_R\partial_{\nabla_R\theta}\psi\cdot\nabla_R\dot\theta-\rho_R\partial_{\nabla_R\mathbf{F}}\psi\cdot\nabla_R\dot{\mathbf{F}}-\frac{1}{\theta}\mathbf{q}_R\cdot\nabla_R\theta+\theta\nabla_R\cdot\mathbf{k}_R=\rho_R\theta\gamma\geq 0. \end{aligned} \tag{54}$$

We now consider the three analogous terms in $\nabla_R\dot{\mathfrak{M}}, \nabla_R\dot\theta, \nabla_R\dot{\mathbf{F}}$. In this regard, divide (54) by θ an observe that

$$-\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi\cdot\nabla_R\dot{\mathbf{F}}=-\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi\dot{\mathbf{F}})+[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi)]\cdot\dot{\mathbf{F}}.$$

Analogous identities hold for the terms with $\nabla_R\theta$ and $\nabla_R\dot{\mathfrak{M}}$. Furthermore

$$[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi)]\cdot\dot{\mathbf{F}}=[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi)]\cdot(\mathbf{LF})=\{[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi)]\mathbf{F}^T\}\cdot(\mathbf{D}+\mathbf{W}).$$

Hence we can write (54), upon division by θ, in the form

$$\begin{aligned} &-\frac{\rho_R}{\theta}(\eta+\delta_\theta\psi)\dot\theta+\frac{1}{\theta}(\mu_0\mathfrak{H}-\rho_R\delta_{\mathfrak{M}}\psi)\cdot\dot{\mathfrak{M}}+\frac{J}{\theta}(\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M}-\rho\delta_{\mathbf{F}}\psi\,\mathbf{F}^T)\cdot\mathbf{W}\\ &\qquad+\frac{J}{\theta}(\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M}-\rho\delta_{\mathbf{F}}\psi\mathbf{F}^T)\cdot\mathbf{D}-\frac{1}{\theta^2}\mathbf{q}_R\cdot\nabla_R\theta\\ &+\nabla_R\cdot(\mathbf{k}_R-\frac{\rho_R}{\theta}\partial_{\nabla_R\theta}\psi\dot\theta-\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi\dot{\mathbf{F}}-\frac{\rho_R}{\theta}\partial_{\nabla_R\mathfrak{M}}\psi\,\dot{\mathfrak{M}})=\rho_R\gamma\geq 0, \end{aligned} \tag{55}$$

where

$$\delta_{\mathfrak{M}}\psi=\partial_{\mathfrak{M}}\psi-\frac{\theta}{\rho_R}\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\mathfrak{M}}\psi),$$

and the like for δ_θ and $\delta_{\mathbf{F}}$. The linearity and arbitrariness of $\mathbf{W}$ implies that

$$\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M}-\rho\delta_{\mathbf{F}}\psi\,\mathbf{F}^T\in\mathrm{Sym}. \tag{56}$$

Furthermore we let

$$\eta=-\delta_\theta\psi,\qquad \mu_0\mathfrak{H}=\rho_R\delta_{\mathfrak{M}}\psi$$

and

$$\mathbf{k}_R=\frac{\rho_R}{\theta}\partial_{\nabla_R\theta}\psi\dot\theta+\frac{\rho_R}{\theta}\partial_{\nabla_R\mathbf{F}}\psi\dot{\mathbf{F}}+\frac{\rho_R}{\theta}\partial_{\nabla_R\mathfrak{M}}\psi\,\dot{\mathfrak{M}}.$$

Dissipative effects are described in terms of $\mathbf{D}$ and $\nabla_R\theta$. Viscous effects are represented by letting

$$\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M}-\rho\delta_{\mathbf{F}}\psi\,\mathbf{F}^T=\mathbf{\Lambda}\mathbf{D},$$

where $\mathbf{\Lambda}$ is a positive definite tensor function. The heat flux $\mathbf{q}_R$ is then subject to

$$\mathbf{q}_R\cdot\nabla_R\theta=\rho_R\theta^2\gamma_q,\qquad \gamma_q=\gamma-\mathbf{D}\cdot\mathbf{\Lambda}\mathbf{D}.$$

We assume $\gamma_q\geq 0$ and express the function $\mathbf{q}_R$ through the representation formula,

$$\mathbf{q}_R=-\frac{\rho_R\theta^2\gamma}{|\nabla_R\theta|^2}\nabla_R\theta+(\mathbf{1}-\frac{\nabla_R\theta\otimes\nabla_R\theta}{|\nabla_R\theta|^2})\mathbf{g},$$

where $\mathbf{g}$ is a generic vector function of the variables under consideration.

The thermodynamic result (56) differs from (10) by the term $\mathrm{skw}(\rho\delta_{\mathbf{F}}\psi\mathbf{F}^T)$. In order to analyze this contribution, for definiteness we let ψ depend on $\mathbf{F}$ and $\nabla_R\,\mathbf{F}$ through

$$J = \det\mathbf{F}, \quad \xi = \tfrac{1}{2}|\mathbf{F}|^2, \quad \zeta = \tfrac{1}{2}|\nabla_R\,\mathbf{F}|^2.$$

The dependence on $\mathbf{F}$ through the Green-Lagrange tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1})$ will be commented upon later on.

Now, if ψ depends on $\mathbf{F}$ and $\nabla_R\,\mathbf{F}$ through J, ξ and ζ then

$$\delta_{\mathbf{F}}\psi = \partial_J\psi J\mathbf{F}^{-T} + \partial_\xi\psi\mathbf{F} - \frac{\theta}{\rho_R}[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_\zeta\psi\nabla_R\,\mathbf{F})]$$

whence

$$\delta_{\mathbf{F}}\psi\mathbf{F}^T = \partial_J\psi J\mathbf{1} + \partial_\xi\psi\mathbf{F}\mathbf{F}^T - \frac{\theta}{\rho_R}[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_\zeta\psi\nabla_R\,\mathbf{F})]\mathbf{F}^T.$$

Consequently

$$\mathrm{skw}(\delta_{\mathbf{F}}\psi\mathbf{F}^T) = -\frac{\theta}{\rho_R}\mathrm{skw}\{[\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_\zeta\psi\nabla_R\,\mathbf{F})]\mathbf{F}^T\} \neq \mathbf{0}.$$

We now show that the contrast with the condition (10) is removed if the free energy depends on $\mathbf{F}$ and $\nabla_R\,\mathbf{F}$ through $\mathbf{E}$ and $\nabla_R\,\mathbf{E}$ so that ψ depends on

$$\theta, \mathbf{E}, \mathfrak{M}, \dot\theta, \nabla_R\,\theta, \dot{\mathbf{E}}, \nabla_R\,\mathbf{E}, \nabla_R\,\mathfrak{M}, \nabla_R\,\nabla_R\,\theta, \nabla_R\,\nabla_R\,\mathfrak{M}.$$

We compute $\dot\psi$ and substitute in the CD inequality (14). The linearity and arbitrariness of $\ddot\theta, \ddot{\mathbf{E}}, \nabla_R\,\nabla_R\,\dot\theta, \nabla_R\,\nabla_R\,\dot{\mathfrak{M}}$ imply that

$$\partial_{\dot\theta}\psi = 0, \quad \partial_{\dot{\mathbf{E}}}\psi = \mathbf{0}, \quad \partial_{\nabla_R\,\nabla_R\,\theta}\psi = \mathbf{0}, \qquad \partial_{\nabla_R\,\nabla_R\,\mathfrak{M}}\psi = \mathbf{0}.$$

Divide by θ the remaining expression of the CD inequality to have

$$\begin{aligned} -\frac{\rho_R}{\theta}(\partial_\theta\psi + \eta)\dot\theta - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi\cdot\nabla_R\,\dot\theta - \frac{\rho_R}{\theta}\partial_{\mathbf{E}}\psi\cdot\dot{\mathbf{E}} - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathbf{E}}\psi\cdot\nabla_R\,\dot{\mathbf{E}} + \frac{\rho_R}{\theta}(\mu_0\mathfrak{H} - \partial_{\mathfrak{M}}\psi)\cdot\dot{\mathfrak{M}} \\ -\frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathfrak{M}}\psi\cdot\nabla_R\,\dot{\mathfrak{M}} + \frac{J}{\theta}(\mathbf{T} + \mu_0\mathbf{H}\otimes\mathbf{M})\cdot\mathbf{L} - \frac{1}{\theta^2}\mathbf{q}_R\cdot\nabla_R\,\theta + \nabla_R\cdot\mathbf{k}_R = \rho_R\gamma \geq 0. \end{aligned}$$

Using the standard identities for the terms with $\nabla_R\,\dot\theta$, $\nabla_R\,\dot{\mathbf{E}}$, and $\nabla_R\,\dot{\mathfrak{M}}$ we may write

$$\begin{aligned} -\frac{\rho_R}{\theta}(\eta + \delta_\theta\psi)\dot\theta - \frac{\rho_R}{\theta}\delta_{\mathbf{E}}\psi\cdot\dot{\mathbf{E}} + \frac{\rho_R}{\theta}(\mu_0\mathfrak{H} - \delta_{\mathfrak{M}}\psi)\cdot\dot{\mathfrak{M}} + \frac{J}{\theta}(\mathbf{T} + \mu_0\mathbf{H}\otimes\mathbf{M})\cdot\mathbf{L} \\ -\frac{1}{\theta^2}\mathbf{q}_R\cdot\nabla_R\,\theta + \nabla_R\cdot\tilde{\mathbf{k}}_R = \rho_R\gamma, \end{aligned} \tag{57}$$

where

$$\tilde{\mathbf{k}}_R = \mathbf{k}_R - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\theta}\psi\,\dot\theta - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathbf{E}}\psi\,\dot{\mathbf{E}} - \frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathfrak{M}}\psi\,\dot{\mathfrak{M}},$$

and, e.g.,

$$\delta_{\mathbf{E}}\psi = \partial_{\mathbf{E}}\psi - \frac{\theta}{\rho_R}\nabla_R\cdot(\frac{\rho_R}{\theta}\partial_{\nabla_R\,\mathbf{E}}\psi).$$

We then let

$$\tilde{\mathbf{k}}_R = \mathbf{0}.$$

Assuming η is independent of $\dot\theta$, by the linearity and arbitrariness of $\dot\theta$ we conclude that

$$\eta = -\delta_\theta\psi.$$

Furthermore we restrict attention to stationary conditions and then assume

$$\mu_0 \mathfrak{H} - \delta_{\mathfrak{M}} \psi = \mathbf{0}.$$

Since $\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{D} \mathbf{F}$ and

$$\mathbf{F}^T \delta_{\mathbf{E}} \psi \, \mathbf{F} \in \mathrm{Sym}$$

then the CD inequality (57) simplifies to

$$\frac{J}{\theta}(\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M}) \cdot \mathbf{W} + \frac{J}{\theta}(\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} - \rho \mathbf{F}^T \delta_{\mathbf{E}} \psi \, \mathbf{F}) \cdot \mathbf{D} - \frac{1}{\theta^2} \mathbf{q}_R \cdot \nabla_R \theta = \rho_R \gamma \geq 0.$$

The arbitrariness of $\mathbf{W}$ implies the validity of the condition (10),

$$\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} \in \mathrm{Sym}.$$

Furthermore

$$\mathbf{T} = -\mu_0 \mathbf{H} \otimes \mathbf{M} + \rho \mathbf{F}^T \delta_{\mathbf{E}} \psi \, \mathbf{F} + \mathfrak{T}$$

where $\mathfrak{T}$ is the possible dissipative stress subject to the reduced inequality

$$J \mathfrak{T} \cdot \mathbf{D} - \frac{1}{\theta} \mathbf{q}_R \cdot \nabla_R \theta = \rho_R \theta \gamma \geq 0.$$

Notice that $\nabla_R \dot{\theta}$ and $\nabla_R \dot{\mathfrak{M}}$ can occur also in $\nabla_R \cdot \mathbf{k}_R$ in that

$$\nabla_R \cdot \mathbf{k}_R = \partial_{\dot{\theta}} \mathbf{k}_R \cdot \nabla_R \dot{\theta} + \partial_{\dot{\mathfrak{M}}} \mathbf{k}_R \cdot \nabla_R \dot{\mathfrak{M}} + ...$$

the dots denoting terms originated by the other dependencies. To determine $\mathbf{k}_R$ it is convenient to divide the remaining equation by θ to have

$$\begin{aligned} &\frac{1}{\theta}\{-\rho_R(\partial_\theta \psi + \eta)\dot{\theta} + (\mu_0 \mathfrak{H} - \rho_R \partial_{\mathfrak{M}} \psi) \cdot \dot{\mathfrak{M}} - \rho_R \partial_{\nabla_R \theta} \psi \cdot \nabla_R \dot{\theta} - \rho_R \partial_{\nabla_R \mathfrak{M}} \psi \cdot \nabla_R \dot{\mathfrak{M}}\} \\ &\qquad + \frac{J}{\theta}(\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} - \rho \mathbf{F}^T \partial_{\mathbf{E}} \psi \mathbf{F}) \cdot \mathbf{D} - \frac{1}{\theta^2} \mathbf{q}_R \cdot \nabla_R \theta + \nabla_R \cdot \mathbf{k}_R = \rho_R \gamma \geq 0 \end{aligned} \tag{58}$$

We now use the identities

$$-\frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi \cdot \nabla_R \dot{\theta} = -\nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi \dot{\theta}) + [\nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi)] \dot{\theta},$$

$$-\frac{\rho_R}{\theta} \partial_{\nabla_R \mathfrak{M}} \psi \cdot \nabla_R \dot{\mathfrak{M}} = -\nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \mathfrak{M}} \psi \dot{\mathfrak{M}}) + [\nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \mathfrak{M}} \psi)] \cdot \dot{\mathfrak{M}},$$

and define

$$\delta_\theta \psi = \partial_\theta \psi - \frac{\theta}{\rho_R} \nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi), \qquad \delta_{\mathfrak{M}} \psi = \partial_{\mathfrak{M}} \psi - \frac{\theta}{\rho_R} \nabla_R \cdot (\frac{\rho_R}{\theta} \partial_{\nabla_R \mathfrak{M}} \psi).$$

Hence equation (58) may be given the form

$$\begin{aligned} &-\frac{\rho_R}{\theta}(\delta_\theta \psi + \eta)\dot{\theta} + \frac{1}{\theta}(\mu_0 \mathfrak{H} - \rho_R \delta_{\mathfrak{M}} \psi) \cdot \dot{\mathfrak{M}} + \frac{J}{\theta}(\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} - \rho \mathbf{F}^T \partial_{\mathbf{E}} \psi \mathbf{F}) \cdot \mathbf{D} \\ &\qquad - \frac{1}{\theta^2} \mathbf{q} \cdot \nabla_R \theta + \nabla_R \cdot (\mathbf{k}_R - \frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi \dot{\theta} - \frac{\rho_R}{\theta} \partial_{\nabla_R \mathfrak{M}} \psi \dot{\mathfrak{M}}) = \rho_R \gamma \geq 0. \end{aligned} \tag{59}$$

Consequently we let

$$\mathbf{k}_R = \frac{\rho_R}{\theta} \partial_{\nabla_R \theta} \psi \dot{\theta} + \frac{\rho_R}{\theta} \partial_{\nabla_R \mathfrak{M}} \psi \dot{\mathfrak{M}}.$$

The entropy η and the magnetic field $\mathfrak{H}$ might depend on $\dot\theta$ and $\dot{\mathfrak{M}}$ thus providing non-zero entropy production terms. Since we are eventually interested in stationary solutions it is appropriate to assume

$$\partial_{\dot\theta}\eta = 0, \quad \partial_{\dot{\mathfrak{M}}}\eta = \mathbf{0}, \quad \partial_{\dot\theta}\mathfrak{H} = \mathbf{0}, \quad \partial_{\dot{\mathfrak{M}}}\mathfrak{H} = \mathbf{0}.$$

Hence the linearity and arbitrariness of $\dot\theta, \dot{\mathfrak{M}}$ imply that

$$\eta = -\delta_\theta \psi, \qquad \mu_0 \mathfrak{H} = \rho_R \delta_{\mathfrak{M}} \psi. \tag{60}$$

Define

$$\boldsymbol{\mathcal{T}} := \mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} - \rho \mathbf{F}^T \partial_{\mathbf{E}} \psi \mathbf{F}.$$

Equation (59) then reduces to

$$J\boldsymbol{\mathcal{T}} \cdot \mathbf{D} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\, \theta = \rho_R \theta \gamma \geq 0.$$

Though $\boldsymbol{\mathcal{T}}$ and $\mathbf{q}_R$ might result in cross-coupling effects we assume $\boldsymbol{\mathcal{T}}$ is independent of $\nabla_R\, \theta$ and $\mathbf{q}_R$ is independent of $\mathbf{D}$. Hence we let

$$\gamma = \gamma_{\mathbf{D}} + \gamma_{\nabla_R \theta}, \qquad \gamma_{\mathbf{D}} \geq 0, \quad \gamma_{\nabla_R \theta} \geq 0,$$

where $\gamma_{\mathbf{D}}$ is independent of $\nabla_R\, \theta$ and $\gamma_{\nabla_R \theta}$ is independent of $\mathbf{D}$. It follows

$$\boldsymbol{\mathcal{T}} \cdot \mathbf{D} = \rho\theta\gamma_{\mathbf{D}}, \qquad \mathbf{q}_R \cdot \nabla_R\, \theta = \rho_R \theta^2 \gamma_{\nabla_R \theta}.$$

The representation formulae for $\boldsymbol{\mathcal{T}}$ and $\mathbf{q}_R$ read

$$\boldsymbol{\mathcal{T}} = \frac{\rho\theta\gamma_{\mathbf{D}}}{|\mathbf{D}|^2}\mathbf{D} + \Big(\mathbf{I} - \frac{\mathbf{D}\otimes\mathbf{D}}{|\mathbf{D}|^2}\Big)\mathbf{G}, \qquad \mathbf{q}_R = \frac{\rho_R \theta^2 \gamma_{\nabla_R \theta}}{|\nabla_R\, \theta|^2}\nabla_R\, \theta + \Big(\mathbf{1} - \frac{\nabla_R\, \theta \otimes \nabla_R\, \theta}{|\nabla_R\, \theta|^2}\Big)\mathbf{g},$$

where $\mathbf{G}$ is a second-order tensor function (independent of $\nabla_R\, \theta$) and $\mathbf{g}$ is a vector-valued function (independent of $\mathbf{D}$) . The expressions of $\boldsymbol{\mathcal{T}}$ and $\mathbf{q}_R$ simplify if the entropy production rates are selected in the form

$$\gamma_{\mathbf{D}} = \frac{1}{\rho\theta}\mathbf{G} \cdot \mathbf{D}, \qquad \gamma_{\nabla_R \theta} = \frac{1}{\rho\theta^2}\mathbf{g} \cdot \nabla_R\, \theta,$$

where the constitutive functions $\mathbf{G}$ and $\mathbf{g}$ are subject to

$$\mathbf{G} \cdot \mathbf{D} \geq 0, \quad \mathbf{g} \cdot \nabla_R\, \theta \geq 0.$$

The result would be $\boldsymbol{\mathcal{T}} = \mathbf{G}$ and $\mathbf{q}_R = \mathbf{g}$.

6. Conclusions

This paper is devoted to the modelling of nonlocality in continuum physics through constitutive functions that depend on suitable spatial gradients. For definiteness, attention is addressed to elastic solids, heat conductors, and magnetic solids. The presentation of the various models is based on the view that the set of constitutive functions obeys the requirements of the second law of thermodynamics and satisfies the balance equations.

As to the balance equations, the stress tensor, in elastic and magnetic solids, is subject to a well-known symmetry condition arising from the balance of angular momentum. While the symmetry is often ignored in the literature, here emphasis is given to the connection between the validity of the symmetry and the selection of independent variables. For nonpolar media no body couple vector occurs and then the stress tensor is taken to be symmetric. Instead, for polar materials the stress is subject to appropriate symmetry restrictions; for definiteness the modelling is established for magnetic materials in terms of the Lagrangian magnetic field $\mathfrak{H} = \mathbf{F}^T\mathbf{H}$.

Usually the approaches developed in the literature are based on different schemes. There are cases where a form of the principle of virtual power is used conceptually in place of the second law of thermodynamics. In many approaches the standard Cauchy stress is abandoned in favour of a pair of stress tensor $\mathbf{T}$ and hyperstress $\mathbf{G}$ with power in the form (1). In a deeply different approach the evolution equations are derived as the Euler-Lagrange equations of a suitable functional (of strain and strain gradient).

This paper provides models of nonlocality by following a systematic analysis, through the second law inequality, of functions of strain gradients or temperature gradients. It is then shown that nonlocal models that satisfy the balance equations can be established without any modification of the basic principles, namely within the balance equations and the second law inequality.

Among the results we mention that within the strain-gradient elasticity with a free energy function of the strain $\mathbf{E}$ and the gradients $\nabla_R \mathbf{E}$, $\nabla_R \nabla_R \mathbf{E}$, the Cauchy stress proves symmetric and involves gradients up to the fourth order. As to heat conduction, nonlocal models are given in terms of both temperature gradients and heat-flux gradients. Next models of magnetic materials are established with the magnetic field $\mathfrak{H}$ in terms of the gradients of the magnetization $\mathfrak{M}$ (eq. (60)).

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