

A New Nonlocal Temperature-Dependent Model for Adhesive Contact



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Abstract The aim of this note is twofold. First of all, we propose a very partial survey on the mathematical modeling and analysis of adhesive contact and delamination. Secondly, we advance a new model for adhesive contact with thermal effects that includes nonlocal adhesive forces and surface damage effects, as well as nonlocal heat flux contributions on the contact surface. In the derivation of the model, we follow the approach by M. Frémond applying it to nonlocal adhesive contact.

Keywords Adhesive contact · Delamination · Rate-dependent processes · Nonlocal effects · Temperature

1 Introduction

Adhesive contact and delamination have been intensively investigated in recent years both from the analytical and the mechanical viewpoint. First of all, a thorough understanding of these inelastic phenomena on surfaces plays an important role in the stability analysis for laminate structures, more and more used in industry. For instance, laminated materials enjoy remarkable energy absorption properties, and are therefore preferable to conventional metallic structures in designing energy-absorbing elements in vehicles, cf. e.g. [25]. Indeed, delamination is a progressive

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separation of bonded laminates, usually due to the degradation of the adhesive substance gluing them together. Since it is an inelastic process on a surface, from a mathematical standpoint adhesive contact and delamination can be modeled by resorting to a *surface damage theory*. From a broader perspective, models for joint volume and surface damage processes are and will be more and more relevant for the description of the degradation of monumental stones. Their deterioration is due to the harmful combination of environmental conditions and mechanical actions. In fact, physical and chemical mechanisms (such as, e.g., pollution) induce a progressive increase of the surface rugosity and the porosity of the external layers, with the formation of micro-cracks and fractures in crystal grains. A model describing this phenomenon has been first advanced and analyzed in the recent [13]. It would be interesting to gain further insight into this degradation process by mathematically modeling the damage of the external layers of monuments via a ‘surface damage approach’, in order to evaluate its influence on the behavior of the whole structure.

In this note we are going to introduce a model for nonlocal adhesive contact with thermal effects that pertains to a class of models for adhesive contact and delamination originating from the approach by M. FRÉMOND, cf. [29] and the pioneering paper [28], also in the frame of the theory of generalized standard materials [30]. Typically, one considers two elastic bodies Ω_+ , $\Omega_- \subset \mathbb{R}^3$ (throughout this note, we shall confine the discussion to the 3D case, meaningful for the applications), possibly subject to viscosity and inertia, bonded along a *prescribed* contact surface Γ_C ; we set $O := \Omega_+ \cup \Omega_- \cup \Gamma_C$. Neglecting thermal effects, the evolution, during a finite time interval $(0, T)$, of adhesive contact and delamination between Ω_+ and Ω_- is described in terms of an internal variable $\chi : \Gamma_C \times (0, T) \rightarrow [0, 1]$. The parameter χ has in fact the meaning of a damage variable, as it describes the fraction of fully effective molecular links in the bonding. Namely,

$$\chi(x, t) = \begin{cases} 1 \\ 0 \end{cases} \quad \text{means that the bonding is } \begin{cases} \text{fully intact} \\ \text{completely broken} \end{cases} \quad (1)$$

at the material point $x \in \Gamma_C$ and the process time $t \in (0, T)$, with $\chi(x, t) \in (0, 1)$ for the intermediate states. The momentum balance for the displacement field $\mathbf{u} : (\Omega_- \cup \Omega_+) \times (0, T) \rightarrow \mathbb{R}^3$ is thus coupled with the flow rule for χ . The resulting PDE system can be derived via a generalized version of the principle of virtual power (cf. also Sect. 2 below); it has the abstract structure of a *generalized gradient system*

$$\varrho \mathbf{u}_{tt} + \mu \partial \mathcal{V}(\mathbf{u}_t) + D_u \mathcal{E}(t, \mathbf{u}, \chi) \ni \mathbf{0} \quad \text{in } \mathbf{U}^* \quad \text{a.e. in } (0, T), \quad (2a)$$

$$\partial \mathcal{R}(\partial_t \chi) + D_\chi \mathcal{E}(t, \mathbf{u}, \chi) \ni 0 \quad \text{in } X^* \quad \text{a.e. in } (0, T). \quad (2b)$$

Typically, for the momentum balance (2a) the ambient space \mathbf{U} is $H^1(O \setminus \Gamma_C; \mathbb{R}^3)$ (or a subspace of the latter space, in order to account for Dirichlet conditions on the

displacement on a portion Γ_{Dir} of ∂O); the constant $\varrho \geq 0$ modulates the inertial term, while the dissipation potential $\mathcal{V} : \mathbf{U} \rightarrow [0, +\infty)$, given by

$$\mathcal{V}(\mathbf{u}_t) := \int_{O \setminus \Gamma_C} \frac{1}{2} \mathbf{V} \epsilon(\mathbf{u}_t) : \epsilon(\mathbf{u}_t) \, dx,$$

features the (positive definite, symmetric) viscosity tensor \mathbf{V} , ($\epsilon(\mathbf{u}_t)$ denoting the classical strain rate tensor), and $\partial \mathcal{V} : \mathbf{U} \rightrightarrows \mathbf{U}^*$ is its subdifferential in the sense of convex analysis. The ambient space X for the flow rule (2b) can be (formally) taken as $L^2(\Gamma_C)$; the dissipation potential $\mathcal{R} : X \rightarrow [0, +\infty]$, with convex subdifferential $\partial \mathcal{R} : X \rightrightarrows X^*$, typically consists of two contributions

$$\mathcal{R}(\chi_t) := \int_{\Gamma_C} \mathbf{R}(\chi_t) \, dS + \nu \int_{\Gamma_C} I_{(-\infty, 0]}(\chi_t) \, dS. \quad (3)$$

The indicator term $I_{(-\infty, 0]}$ enforces the constraint

$$\chi_t \leq 0 \quad \text{a.e. in } \Gamma_C \times (0, T), \quad (4)$$

translating the fact that the degradation of the adhesive bonds between Ω_+ and Ω_- is irreversible; the dissipation density $\mathbf{R} : \mathbf{R} \rightarrow [0, +\infty)$ is convex and will be discussed below. Finally, the driving energy functional $\mathcal{E} : (0, T) \times \mathbf{U} \times X \rightarrow (-\infty, +\infty]$ for the adhesive contact/delamination model generally takes the form

$$\begin{aligned} \mathcal{E}(t, \mathbf{u}, \chi) := & \int_{O \setminus \Gamma_C} \frac{1}{2} \mathbf{E} \epsilon(\mathbf{u}) : \epsilon(\mathbf{u}) \, dx + \lambda \int_{\Gamma_C} I_{(-\infty, 0]}([\mathbf{u}] \cdot \mathbf{n}) \, dS \\ & + \mathcal{E}_{\Gamma_C}(\chi) + \mathcal{E}_{\text{coup}}(\mathbf{u}, \chi) - \mathbf{U}^* \langle \mathbf{F}(t), \mathbf{u} \rangle_{\mathbf{U}}. \end{aligned} \quad (5)$$

Namely, the elastic energy contribution, featuring the (positive definite, symmetric) elasticity tensor \mathbf{E} , ($\epsilon(\mathbf{u})$ denoting the linearized strain tensor), is added with a term that involves the normal component of the displacement jump $[[\mathbf{u}]]$ across Γ_C (i.e., $[[\mathbf{u}]]$ is the difference of the traces on Γ_C of $\mathbf{u}|_{\Omega_+}$ and $\mathbf{u}|_{\Omega_-}$), as \mathbf{n} denoted the outward unit normal vector to the boundary $\partial \Omega$. The indicator function $I_{(-\infty, 0]}$ ensures that along a solution to system (2) the non-interpenetration constraint between the bodies Ω_+ and Ω_- , namely

$$[[\mathbf{u}]] \cdot \mathbf{n} \leq 0 \quad \text{a.e. in } \Gamma_C \times (0, T), \quad (6)$$

holds. The ‘‘coupling’’ functional $\mathcal{E}_{\text{coup}}$ is typically given by

$$\mathcal{E}_{\text{coup}}(\mathbf{u}, \chi) := \int_{\Gamma_C} J([\mathbf{u}], \chi) \, dS \quad (7)$$

with $J : \mathbb{R}^3 \times [0, 1] \rightarrow [0, +\infty]$ smooth, while the surface contribution \mathcal{E}_{Γ_C} may involve

- a gradient, regularizing term; in that case, it is customary to assume that Γ_C is a “flat surface”, embedded in \mathbb{R}^2 (cf. also Sect. 2), on which the Hausdorff measure coincides with the Lebesgue one. Therefore, one writes dx in place of dS for integrals on Γ_C , and for the gradient terms the usage of Laplace–Beltrami operators can be avoided. Hence, one can for instance choose a regularizing contribution of the form $\int_{\Gamma_C} \frac{1}{2} |\nabla \chi|^2 dx$;
- the indicator term $\int_{\Gamma_C} I_{[0,1]}(\chi) dx$ enforcing the constraint

$$\chi \in [0, 1] \quad \text{a.e. in } \Gamma_C \times (0, T); \quad (8)$$

- and, possibly, other smooth terms.

Finally, the function $\mathbf{F} : (0, T) \rightarrow \mathbf{U}^*$ subsumes volume forces and applied tractions on the ‘Neumann part’ Γ_{Neu} of ∂O . The coefficients μ , ν , λ in (2a)–(2b), (3), and (5) are all non-negative and may thus switch off, or on, the viscosity contribution to the momentum balance, the unidirectionality constraint (4) for the evolution of χ , and the non-intepetration condition (6), respectively. Similarly, inertial effects in (2a) are neglected if one takes $\varrho = 0$.

A major distinction has to be made between

- *rate-independent* models, in which the dissipation density R is positively homogeneous of degree 1, namely

$$R(\ell \dot{\chi}) = \ell R(\dot{\chi}) \quad \text{for all } \dot{\chi} \in \mathbb{R}, \ell \geq 0 \quad (9)$$

- and *rate-dependent* models, featuring a dissipation density with superlinear growth at infinity; typically, in adhesive contact models one considers a *quadratic* dissipation potential, i.e. (setting all physical constants equal to 1)

$$R(\dot{\chi}) = \frac{1}{2} |\dot{\chi}|^2 \quad \text{for all } \dot{\chi} \in \mathbb{R}. \quad (10)$$

Although the focus of this note is on a *rate-dependent* system for nonlocal, temperature-dependent adhesive contact, we will also very partially review the, intensively growing, literature on rate-independent models.

1.1 Rate-Independent Models

Adhesive contact and delamination as *rate-independent* processes have been actively studied over the last 15 years since the pioneering paper [31], which addressed a *quasistatic* (i.e. without inertial effects) model where viscosity in \mathbf{u} was

also neglected (viz., with $\varrho = \mu = 0$ in (2a)). In such a case, since the displacement variable is at equilibrium and the subdifferential operator $\partial\mathcal{R} : X \rightrightarrows X^*$ is 0-(positively) homogeneous, system (2) is invariant upon (increasing) time rescalings, which reflects the modeling ansatz that the system possesses no internal time scale. In the 1-homogeneous case, though, system (2) is only formally written as a subdifferential inclusion holding pointwise in time. Indeed, the dissipation density R has linear growth at infinity, and thus one can in general expect only BV-time regularity for χ . Thus, χ may have jumps as a function of time, and its derivative χ_t need not be defined. In view of this, it is necessary to formulate (2) in a suitably weak way. In [31] the authors resorted to the commonest solvability concept for rate-independent processes, namely the notion of *energetic solution* (cf. [32]). It consists of an energy-dissipation balance (which, of course, only features the 1-homogeneous dissipation potential \mathcal{R} , as viscosity and inertia in \mathbf{u} are neglected) and of a (global) stability condition that can be indeed recast as a (global) minimization problem. The main result of [31] is the existence of solutions to the Cauchy problem for the energetic formulation. Without entering into further details, let us only mention that [31] addressed the case in which the ‘coupling energy’ from (7) is given by

$$\mathcal{E}_{\text{coup}}(\mathbf{u}, \chi) = \int_{\Gamma_C} \frac{\kappa}{2} \chi |[[\mathbf{u}]]|^2 dx, \quad (11)$$

with $\kappa \geq 0$. This term penalizes displacement jumps in points with strictly positive χ but does not exclude them.

Obviously, one expects that the blow-up of the coefficient κ will lead to a model with the *brittle constraint*

$$\chi [[\mathbf{u}]] = 0 \quad \text{a.e. in } \Gamma_C \times (0, T), \quad (12)$$

that allows for displacement jumps (i.e., $[[\mathbf{u}]] \neq 0$) only at points where the bonding is completely broken (i.e., $\chi = 0$), and otherwise imposes the transmission condition $[[\mathbf{u}]] = 0$ on the displacements. Systems encompassing (12) are frequently referred to as *brittle delamination* models, as opposed to the *adhesive contact* models with the coupling energy from (11).

The convergence of energetic solutions of the adhesive contact system examined in [31] to the brittle delamination system was rigorously proved in [42], relying on Γ -convergence type arguments. In fact, the energetic formulation has an intrinsically variational character that allows for limiting procedures based on notions of variational convergence. This has been crucially exploited, for instance, for dimensional reduction analyses, cf. e.g. [33], dealing with the limit passage from bulk to surface damage as the thickness of an interface between two elastic bodies tends to zero, and [27], deriving models for both adhesive contact and brittle delamination in 2D plates as limits of delamination between 3D *thin* plates. More in general, the flexibility of the energetic concept allows for an easy coupling of delamination with other inelastic processes such as plasticity and phase transformations, as well as for

enhancements to cohesive-type or mixed mode models. We refer to [41] (and the references therein) for a comprehensive survey of these aspects.

The range of applicability of the energetic formulation can be broadened to encompass models coupling rate-independent evolution of the internal variable with rate-dependent evolution of the displacement, subject to viscosity and, possibly, inertia. This extended notion of energetic solution was exploited, for instance, in [40]. Relying on a further generalization of energetic solutions for rate-independent processes also subject to thermal effects (cf. [39]), in [37] a model for adhesive contact between two *thermoviscoelastic* bodies was analyzed; the extension to brittle delamination was carried out in [38]. The PDE systems analyzed in [37] and [38] couple (2) with the temperature equation in the *bulk domain* $O \setminus \Gamma_C$; in this case, the overall system has a more complex structure than that of a generalized gradient system and, accordingly, its energetics is more involved. A common feature of the coupled rate-independent/rate-dependent systems addressed in [37, 40] and [38] is that, due to the presence of inertial terms in the momentum balance, the unilateral non-interpenetration constraint (6) was not incorporated in the model and, accordingly, the contribution $\int_{\Gamma_C} I_{(-\infty, 0]}(\llbracket \mathbf{u} \rrbracket \cdot \mathbf{n}) \, dS$ was either neglected or replaced by a term only penalizing interpenetration, without excluding it.

1.2 Rate-Dependent Models

Also *rate-dependent* delamination systems, featuring the *quadratic* dissipation potential (10), have been intensively studied over the last two decades.

In [7] we first approached the study of FRÉMOND's model for adhesive contact, neglecting inertia (i.e. setting $\varrho = 0$ in (2a)), in favor of a more transparent formulation of the momentum balance that would account for the reaction forces associated with the constraints encompassed by the model. More precisely, we confined our analysis to the case of a single body Ω in adhesive contact with a rigid support, such that the contact surface Γ_C is a part of their common boundary. In this case, the jump of the displacement $\llbracket \mathbf{u} \rrbracket$ coincides with its trace on Γ_C , hereafter simply denoted by \mathbf{u} . We considered the driving energy functional

$$\begin{aligned} \mathcal{E}(t, \mathbf{u}, \chi) &:= \int_{\Omega} \frac{1}{2} \mathbf{E} \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) \, dx + \int_{\Gamma_C} I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n}) \, dx \\ &\quad + \int_{\Gamma_C} \left(\frac{1}{2} |\nabla \chi|^2 + I_{[0, 1]}(\chi) - w_s \chi \right) \, dx + \int_{\Gamma_C} \frac{\kappa}{2} \chi |\mathbf{u}|^2 \, dx \\ &\quad - \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{u} \, dx - \int_{\Gamma_{\text{Neu}}} \mathbf{g}(t) \cdot \mathbf{u} \, dx, \end{aligned}$$

in which the coupling contribution is given by (11), w_s is a positive constant related to the cohesion on the adhesive, \mathbf{f} is a volume force and \mathbf{g} a traction applied on the Neumann part Γ_{Neu} of the boundary $\partial\Omega = \overline{\Gamma_{\text{Dir}}} \cup \overline{\Gamma_{\text{Neu}}} \cup \overline{\Gamma_C}$. Imposing

zero Dirichlet boundary conditions on the Dirichlet boundary Γ_{Dir} we derived the following *quasistatic* PDE system for adhesive contact

$$-\operatorname{div}(\mathbb{V}\epsilon(\mathbf{u}_t) + \mathbb{E}\epsilon(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega \times (0, T), \quad (13a)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\text{Dir}} \times (0, T), \quad (13b)$$

$$(\mathbb{V}\epsilon(\mathbf{u}_t) + \mathbb{E}\epsilon(\mathbf{u}))\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\text{Neu}} \times (0, T), \quad (13c)$$

$$(\mathbb{V}\epsilon(\mathbf{u}_t) + \mathbb{E}\epsilon(\mathbf{u}))\mathbf{n} + \kappa\chi\mathbf{u} + \partial I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_{\text{C}} \times (0, T), \quad (13d)$$

$$\chi_t + \partial I_{(-\infty, 0]}(\chi_t) - \Delta\chi + \partial I_{[0, 1]}(\chi) \ni w_s - \frac{\kappa}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_{\text{C}} \times (0, T), \quad (13e)$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_{\text{C}} \times (0, T). \quad (13f)$$

Observe that (13d) generalizes the Signorini conditions from the basic unilateral contact theory, in that the term $\kappa\chi\mathbf{u}$ represents the resistance to tension related to the action of microscopic bonds between the surfaces of the adhering solids. Let us stress that in system (13) all constraints on the variables \mathbf{u} and χ are rendered by means of multivalued operators. This approach, though leading to some analytical difficulties, allows us to account for the internal reactions ‘activated’ by the constraints.

The main result of [7] states the existence of solution to the Cauchy problem for (13), with the flow rule (13e) formulated as a subdifferential inclusion holding a.e. on $\Gamma_{\text{C}} \times (0, T)$, and the momentum balance (13a) formulated with test functions from $H^1(\Omega; \mathbb{R}^3)$ having null trace on Γ_{Dir} . In particular, the weak formulation of (13a) with the boundary conditions (13b)–(13d) features a selection ξ from the (suitably realized) subdifferential $\partial I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$. This term has the meaning of a reaction force, activated ‘on the boundary’ of the non-intepetration constraint, namely when $\mathbf{u} \cdot \mathbf{n} = 0$. Dropping the inertial term in (13a) we were able to gain estimates for the term $\xi \in \partial I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$, and thus to encompass it in the momentum balance. Subsequently, in [45] (cf. also [43, 44]) an existence result was proved for a *dynamic* adhesive contact system with the non-interpenetration condition (6), in which the analytical difficulties attached to the coupling of inertia and a unilateral constraint were bypassed by resorting to a suitable weak solution concept introduced in [17].

The analysis in [7] has been extended in various directions. Indeed, the long-time behavior of system (13) has been addressed in [6], where the existence and properties of the associated ω -limit set have been investigated, while frictional effects have been included in the model studied in [9]. Furthermore, the coupling between adhesive contact and volume damage has been investigated in [16]. In the *rate-dependent* context, the (asymptotic) relation between bulk and surface models has been addressed in [4, 5]. Therein, by means of the method of asymptotic expansions, models of ‘imperfect interfaces’ have been derived as limits of processes in a thin adhesive substrate between two bodies that undergoes a degradation process, as its thickness vanishes. The limiting models couple the evolution of the (bulk) displace-

ment and temperature variables to that of a surface damage parameter, encompassing unilateral contact conditions along the interface. However, a rigorous dimensional reduction analysis from a model for bulk damage to a surface damage one, in the same spirit as [33], is still missing. Likewise, the limit passage from adhesive contact to brittle delamination (namely, the asymptotic analysis of the adhesive contact system (13) as the parameter κ blows up) is still open. In fact, for rate-dependent systems it is more challenging to perform this kind of analyses, for they would involve non-trivial limit passages in the momentum balance and in the flow rule.

The model introduced in [7] has been extended in [8] to the temperature-dependent framework, considering heat generation effects in the phenomenon of adhesive contact, too. The thermal evolution of the system is assumed to be ruled by the heat exchange between the body and the adhesive substance through the contact surface. In particular, allowing for different temperatures in the bulk domain and on the contact surface, we assumed that temperature variables θ and θ_s are governed by two distinct entropy balance laws. The analysis carried out in [8] leads to an existence result for (the initial-value problem associated with) the following PDE system:

$$\partial_t(\ln(\theta)) - \operatorname{div}(\mathbf{u}_t) - \Delta\theta = h \quad \text{in } \Omega \times (0, T), \quad (14a)$$

$$\partial_n\theta = \begin{cases} 0 & \text{in } (\partial\Omega \setminus \Gamma_C) \times (0, T), \\ -k(\chi)(\theta - \theta_s) & \text{in } \Gamma_C \times (0, T), \end{cases} \quad (14b)$$

$$\partial_t(\ln(\theta_s)) - \partial_t(\lambda(\chi)) - \Delta\theta_s = k(\chi)(\theta - \theta_s) \quad \text{in } \Gamma_C \times (0, T), \quad (14c)$$

$$\partial_{\mathbf{n}_s}\theta_s = 0 \quad \text{in } \partial\Gamma_C \times (0, T), \quad (14d)$$

$$-\operatorname{div}(\mathbf{E}\epsilon(\mathbf{u}) + \mathbf{V}\epsilon(\mathbf{u}_t) + \theta\mathbf{I}) = \mathbf{f} \quad \text{in } \Omega \times (0, T), \quad (14e)$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\text{Dir}} \times (0, T), \quad (14f)$$

$$(\mathbf{E}\epsilon(\mathbf{u}) + \mathbf{V}\epsilon(\mathbf{u}_t) + \theta\mathbf{I})\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\text{Neu}} \times (0, T), \quad (14g)$$

$$\begin{aligned} (\mathbf{E}\epsilon(\mathbf{u}) + \mathbf{V}\epsilon(\mathbf{u}_t) + \theta\mathbf{I})\mathbf{n} + \kappa\chi\mathbf{u} \\ + \partial I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \ni \mathbf{0} \end{aligned} \quad \text{in } \Gamma_C \times (0, T), \quad (14h)$$

$$\begin{aligned} \chi_t - \Delta\chi + \partial I_{[0, 1]}(\chi) + \gamma'(\chi) \\ \ni -\lambda'(\chi)(\theta_s - \theta_{\text{eq}}) - \frac{\kappa}{2}|\mathbf{u}|^2 \end{aligned} \quad \text{in } \Gamma_C \times (0, T), \quad (14i)$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{in } \partial\Gamma_C \times (0, T), \quad (14j)$$

where λ , k , and γ are sufficiently smooth functions, whereas $\theta_{\text{eq}} > 0$ is a phase transition temperature, κ a positive constant, and \mathbf{I} the identity matrix. Let us comment that a peculiarity of the system is that, the evolution of the temperature variables θ and θ_s is governed by *entropy*, in place of *internal energy*, balance equations. While referring to, e.g., [14, 15] for a more accurate illustration of this approach, we may mention here that the entropy equations are recovered by rescaling the internal energy balance equations, neglecting some higher order

dissipative terms under the small perturbation assumption. As shown in [10, Sec. 2], this leads to a thermodynamically consistent model, where the strict positivity of θ and θ_s is enforced by the very form of the equations, cf. (14a) and (14c). On the other hand, the singular character of the terms $\partial_t(\ln(\theta))$ and $\partial_t(\ln(\theta_s))$ occurring in (14a) and (14c) does not allow but for poor time-regularity for θ and θ_s . Again, we note that the evolution of the temperatures is essentially governed by the heat exchange throughout the contact surface. More precisely, the entropy flux through Γ_C (namely the term $k(\chi)(\theta - \theta_s)$ in (14b)) plays the role of a source of entropy in (14c). From the analytical point of view, this results in a nonlinear coupling between (14a)–(14b) and (14c) and gives rise to further technical difficulties.

The thermodynamical modeling approach from [8] was also adopted in [10], where frictional contributions are further encompassed in a temperature-dependent model through a regularization of the classical Coulomb law, here generalized to the case of adhesive contact and assuming thermal dependence of the friction coefficient. The main result in [10] states the existence of solutions to a temperature-dependent system for adhesive contact with friction, in which all the constraints on the internal variables, as well as the unilateral contact conditions and the friction law, are rendered by means of subdifferential operators, in accordance with the approach developed in [7–9].

Concerning frictional contact problems, with or without adhesion, we recall an alternative approach which replaces the unilateral contact conditions (the *Signorini contact conditions*) rendered by (13d) with a *normal compliance* condition, allowing for the interpenetration of the surface asperities and thus for dispensing with the unilateral constraint on $\mathbf{u} \cdot \mathbf{n}$. Analytically, the normal compliance law corresponds to a penalization of the subdifferential operator $\partial I_{(-\infty, 0]}$ in (13d). In this connection, we refer e.g. to [1], analyzing a *dynamic* model for frictional contact in thermoviscoelasticity with a power-law normal compliance condition, and the corresponding generalization of Coulomb's law of friction. Contact with a deformable foundation is considered in [2] as well, where a dynamic contact problem for a thermoviscoelastic body, with frictional and wear effects on the contact surface, is investigated. A wide class of *dynamic* frictional contact problems in thermoelasticity and thermoviscoelasticity is also tackled in [24], with contact rendered by means of a normal compliance law. In the context of contact with a deformable foundation, we quote [35], where a dynamic frictional contact problem with adhesion is formulated by coupling a hyperbolic hemivariational inequality for the displacement and a first-order ODE for the adhesive field. By means of abstract results on variational-hemivariational inequalities, the existence and the regularity of a solution were proved (for these techniques and their applications in Contact Mechanics, see e.g. the monograph [34]).

A different approach to frictional contact problems was developed in [23] and in [21] (cf. also the references in the monographs [22, 46]), where contact with a rigid foundation is modeled by the Signorini conditions in velocity form, that are indeed expressed in terms of \mathbf{u}_t rather than of \mathbf{u} . In this context, existence results were obtained for contact problems including frictional and thermal effects.

We also recall the contributions [19, 20, 36] on the analysis of systems coupling friction, adhesion and *unilateral contact* (modeled via the classical Signorini conditions), akin to FRÉMOND's model (see [28, 29]). In [36], a consistent model describing unilateral contact, adhesion and friction was derived and the related quasistatic problem was written in terms of two variational inequalities and a first-order ODE (indeed, local interactions in the adhesive substance were neglected); the main result therein provides the existence of solutions for an incremental formulation of the problem; some numerical schemes are also given. In [19, 20], based on [36], contact problems with adhesion and friction were considered in the quasistatic elastic case and in the dynamic viscoelastic case, respectively. Let us stress that, differently from our approach, in all of these contributions the existence of solutions is proved for formulations of the related PDE systems that involve variational inequalities, and not evolutionary differential inclusions, for the displacement.

We conclude this short review on frictional contact problems with adhesion returning to the temperature-dependent case in order to illustrate our paper [11]. Differently from [10], therein the temperature equations for the bulk and the surface temperatures θ and θ_s are derived from the *internal energy* balance, without neglecting any higher order dissipative contribution. The presence of quadratic terms on their right-hand side that are only estimated in $L^1(\Omega \times (0, T))$ and in $L^1(\Gamma_C \times (0, T))$, respectively, leads to considerable difficulties. By resorting to Boccardo-Gallouët [3] type estimates and under suitable growth conditions on the heat capacity and the heat conductivity of the system, the existence of a solution for the related initial-boundary value problem has been proved in [11].

1.3 Adhesive Contact with Nonlocal Effects

In [12] we have further extended the analysis of the (isothermal) adhesive contact model by FRÉMOND, by assuming that also nonlocal forces act on the contact surface. More precisely, in addition to the interactions, on the contact surface, between damage (of the adhesive substance) at a point and damage in its neighborhood, in the model in [12] we have encompassed a nonlocal interaction among the adhesive substance, the body Ω and its rigid support. This results in an integral term further contributing to the resistance to tension in the generalized Signorini conditions (13d), and in a second integral term, coupled to it, in the flow rule for the adhesion parameter. The motivation for this enhancement of the model comes from experiments showing that elongation, i.e. a variation of the distance of two distinct points on the contact surface, may have damaging effects on the adhesive substance, cf. [26].

The aim of this note is to introduce a model further encompassing thermal effects in the nonlocal system studied in [12], still within the assumption that the temperatures in the body and in the adhesive substance are a priori different, as in the models from [8, 10, 11]. As we have seen, this ansatz leads to a different

approach to the modeling of the heat exchange between the body and the glue located on the contact surface. In particular, we are now going to assume a nonlocal interaction between the body and the adhesive as far as it concerns heat transfer, as well. Accordingly, the related PDE system (cf. (54) ahead) will contain integral terms in the boundary condition for the bulk absolute temperature on the contact surface, and in the equation for the surface temperature, too. We will derive it in the upcoming Sect. 2.

2 The Model and the PDE System

In this section we present the modeling approach leading to the PDE system (54) ahead for an adhesive contact process in the presence of nonlocal thermo-mechanical effects. In the derivation of system (54), we shall refer to the theory introduced by FRÉMOND [29] and developed in [12, 26] for nonlocal adhesive contact models in the isothermal case. As in [12], we will confine the discussion to the reduced case in which only one body is considered in adhesive contact with a rigid support. We observe that this choice has the advantage of simplifying the exposition in comparison to the two-body case, without affecting the relevance of the model.

During a time interval $(0, T)$, $T > 0$, we consider a thermoviscoelastic body located in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ and lying on a rigid support on a part of its boundary, on which some adhesive substance is present. The contact surface Γ_C between the body and the support is part of the boundary of Ω , given by $\partial\Omega = \overline{\Gamma}_{\text{Dir}} \cup \overline{\Gamma}_{\text{Neu}} \cup \overline{\Gamma}_C$. Here Γ_{Dir} , Γ_{Neu} , and Γ_C are open subsets in the relative topology of $\partial\Omega$, each of them with a smooth boundary and disjoint one from each other. Without loss of generality, we suppose that Γ_C is a *flat* surface and identify it with a subset of \mathbb{R}^2 . That is why, all integrals on Γ_C will involve the Lebesgue measure, that coincides with the Hausdorff measure by the flatness requirement. We prescribe zero Dirichlet boundary conditions on Γ_{Dir} , while we assume that a traction is applied on Γ_{Neu} .

2.1 The State and Dissipative Variables

The phenomenon of adhesive contact is modeled by state and dissipative variables, describing the thermomechanical equilibrium of the system and its evolution, respectively.

In the bulk domain Ω the state variables are the absolute temperature θ and the symmetric linearized strain tensor $\epsilon(\mathbf{u})$

$$(\theta, \epsilon(\mathbf{u})) \quad \text{in } \Omega, \quad (15)$$

while the dissipative variables are

$$(\nabla\theta, \epsilon(\mathbf{u}_t)) \quad \text{in } \Omega. \quad (16)$$

On the contact surface Γ_C the state variables are

$$(\theta_s, \chi, \nabla\chi, \mathbf{u}) \quad \text{on } \Gamma_C, \quad (17)$$

where θ_s is the absolute temperature of the adhesive substance, χ is the surface damage-type parameter from (1), its gradient $\nabla\chi$ accounts for interactions of the degradation of the adhesive substance in a point, with the degradation in the neighborhood of that point and, with slight abuse of notation, we denote by \mathbf{u} the trace of \mathbf{u} on Γ_C . The surface dissipative variables are

$$(\nabla\theta_s, \chi_t, \theta - \theta_s) \quad \text{on } \Gamma_C, \quad (18)$$

where $\theta - \theta_s$ represents the thermal gap on the contact surface (still denoting by θ the trace of the bulk absolute temperature on Γ_C).

Finally, on Γ_C we also consider state and dissipative variables describing the interaction between the body and the support along the contact surface: in order to distinguish between those describing local interactions and those rendering nonlocal interactions, we will now make explicit their dependence on the variables $x \in \Gamma_C$ and $(x, y) \in \Gamma_C \times \Gamma_C$, respectively. In particular, the state variables attached to local interactions, defined pointwise in Γ_C , are

$$(\chi(x), \mathbf{u}(x)), \quad x \in \Gamma_C, \quad (19)$$

while the state variable describing nonlocal damaging effects is defined in $\Gamma_C \times \Gamma_C$ by

$$\mathbf{g}(x, y) := 2(x - y)\mathbf{u}(x), \quad (x, y) \in \Gamma_C \times \Gamma_C. \quad (20)$$

Analogously, we consider as dissipative variables the surface thermal gap defined pointwise in Γ_C , i.e.

$$(\theta(x) - \theta_s(x)), \quad x \in \Gamma_C, \quad (21)$$

as well as a dissipative variable defined in $\Gamma_C \times \Gamma_C$ by

$$\mathbf{G}(x, y) := 2(x - y)(\theta(x) - \theta_s(y)), \quad (x, y) \in \Gamma_C \times \Gamma_C. \quad (22)$$

Let us point out that the terms $\mathbf{g}(x, y) := 2(x - y)\mathbf{u}(x)$ and $\mathbf{G}(x, y) := 2(x - y)(\theta(x) - \theta_s(y))$ render the nonlocal contributions to the degradation process of the adhesive. In particular, as analyzed in [26], $\mathbf{g}(x, y)$ takes into account the elongation

as a source of damage to the adhesive substance, while $\mathbf{G}(x, y)$ describes the effects due to the evolution of the thermal gap between two different points on the contact surface. These terms lead to integral contributions both in the normal reaction and in the flow rule for χ , and to a nonlocal heat flux contribution on the contact surface (see (34), (36), (47), and (51) below).

2.2 The Free Energy

The free energy \mathcal{F} of the system is given by the sum of three contributions:

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(\theta, \epsilon(\mathbf{u}), \theta_s, \chi, \nabla\chi, \mathbf{u}, \mathbf{g}) \\ &:= \mathcal{F}_\Omega(\theta, \epsilon(\mathbf{u})) + \mathcal{F}_{\Gamma_C}(\theta_s, \chi, \nabla\chi) + \mathcal{F}_{\text{inter}}(\chi, \mathbf{u}, \mathbf{g}), \end{aligned} \quad (23)$$

with \mathcal{F}_Ω , \mathcal{F}_{Γ_C} , and $\mathcal{F}_{\text{inter}}$ the bulk, surface, and interaction free energies. We prescribe the bulk free energy as

$$\begin{aligned} \mathcal{F}_\Omega(\theta, \epsilon(\mathbf{u})) &:= \int_\Omega \Psi_\Omega(\theta(x), \epsilon(\mathbf{u}(x))) \, dx \quad \text{with} \\ \Psi_\Omega(\theta, \epsilon(\mathbf{u})) &:= \theta - \theta \log(\theta) + \theta \text{tr}(\epsilon(\mathbf{u})) + \frac{1}{2} \epsilon(\mathbf{u}) \mathbf{E} \epsilon(\mathbf{u}), \end{aligned} \quad (24)$$

(recall that \mathbf{E} is the elasticity tensor). The surface free energy is given by

$$\begin{aligned} \mathcal{F}_{\Gamma_C}(\theta_s, \chi, \nabla\chi) &:= \int_{\Gamma_C} \Psi_{\Gamma_C}(\theta_s(x), \chi(x), \nabla\chi(x)) \, dx \quad \text{with} \\ \Psi_{\Gamma_C}(\theta_s, \chi, \nabla\chi) &= \theta_s - \theta_s \log(\theta_s) \\ &\quad + I_{[0,1]}(\chi) + \gamma(\chi) + \frac{1}{2} |\nabla\chi|^2 + \lambda(\chi)(\theta_s - \theta_{\text{eq}}), \end{aligned} \quad (25)$$

where the indicator function $I_{[0,1]}$ of the interval $[0, 1]$ imposes the physical constraint $\chi \in [0, 1]$, since $I_{[0,1]}(\chi) = 0$ if $\chi \in [0, 1]$ and $I_{[0,1]}(\chi) = +\infty$ otherwise. The function λ is related to the latent heat and the constant $\theta_{\text{eq}} > 0$ is a phase transition temperature. Moreover, the function γ , sufficiently smooth and possibly nonconvex, describes non-monotone dynamics for χ (it may model some cohesion in the material). Finally, we prescribe the interaction free energy as a sum between local and nonlocal contributions

$$\mathcal{F}_{\text{inter}}(\chi, \mathbf{u}, \mathbf{g}) := \mathcal{F}_{\text{inter}}^{\text{l}}(\chi, \mathbf{u}) + \mathcal{F}_{\text{inter}}^{\text{nl}}(\chi, \mathbf{g}), \quad (26a)$$

where the local contribution $\mathcal{F}_{\text{inter}}^1$ is given by

$$\begin{aligned} \mathcal{F}_{\text{inter}}^1(\chi, \mathbf{u}) &:= \int_{\Gamma_C} \Psi_{\text{inter}}^1(\chi(x), \mathbf{u}(x)) \, dx \quad \text{with} \\ \Psi_{\text{inter}}^1(\chi, \mathbf{u}) &= I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n}) + \frac{\kappa}{2} \chi |\mathbf{u}|^2 \end{aligned} \quad (26b)$$

and the nonlocal one consists of the integral functional on $\Gamma_C \times \Gamma_C$

$$\begin{aligned} \mathcal{F}_{\text{inter}}^{\text{nl}}(\chi, \mathbf{g}) &:= \iint_{\Gamma_C \times \Gamma_C} \Psi_{\text{inter}}^{\text{nl}}(\chi(x), \chi(y), \mathbf{g}(x, y)) \, dx \, dy \quad \text{with} \\ \Psi_{\text{inter}}^{\text{nl}}(\chi(x), \chi(y), \mathbf{g}(x, y)) &= \frac{1}{2} \mathbf{g}^2(x, y) H(\chi(x), \chi(y)) e^{-\frac{|x-y|^2}{d^2}}, \end{aligned} \quad (26c)$$

where d a given constant and the interaction function H is assumed to be symmetric, i.e. $H(x, y) = H(y, x)$.

2.3 The Dissipation Potential

We follow the approach proposed by J.J. Moreau and prescribe the dissipated energy by means of a so-called pseudo-potential of dissipation, which is a convex, nonnegative functional, attaining its minimal value 0 at $\mathbf{0}$. More precisely, the dissipation potential is again given by the sum of a bulk, a surface, and an interaction part, i.e.

$$\begin{aligned} \mathcal{P} &= \mathcal{P}(\nabla\theta, \epsilon(\mathbf{u}_t), \nabla\theta_s, \chi_t, \theta - \theta_s, \mathbf{G}) \\ &:= \mathcal{P}_{\Omega}(\nabla\theta, \epsilon(\mathbf{u}_t)) + \mathcal{P}_{\Gamma_C}(\nabla\theta_s, \chi_t) + \mathcal{P}_{\text{inter}}(\theta - \theta_s, \mathbf{G}). \end{aligned} \quad (27)$$

The volume part is given by

$$\begin{aligned} \mathcal{P}_{\Omega}(\nabla\theta, \epsilon(\mathbf{u}_t)) &:= \int_{\Omega} \Phi_{\Omega}(\nabla\theta(x), \epsilon(\mathbf{u}_t(x))) \, dx \quad \text{with} \\ \Phi_{\Omega}(\nabla\theta, \epsilon(\mathbf{u}_t)) &= \frac{\alpha_b(\theta)}{2\theta} |\nabla\theta|^2 + \frac{1}{2} \epsilon(\mathbf{u}_t) \mathbf{V} \epsilon(\mathbf{u}_t), \end{aligned} \quad (28)$$

(recall that \mathbf{V} is the viscosity tensor), with α_b the bulk heat conductivity coefficient. The surface contribution \mathcal{P}_{Γ_C} to the pseudo-potential of dissipation is

$$\begin{aligned} \mathcal{P}_{\Gamma_C}(\nabla\theta_s, \chi_t) &:= \int_{\Gamma_C} \Phi_{\Gamma_C}(\nabla\theta_s(x), \chi_t(x)) \, dx \quad \text{with} \\ \Phi_{\Gamma_C}(\nabla\theta_s, \chi_t) &= \frac{\alpha_s(\theta_s)}{2\theta_s} |\nabla\theta_s|^2 + \frac{1}{2} |\chi_t|^2, \end{aligned} \quad (29)$$

where α_s denotes the surface heat conductivity coefficient. Finally, the interaction contribution $\mathcal{P}_{\text{inter}}$ also consists of a local and a nonlocal term, i.e.

$$\mathcal{P}_{\text{inter}}(\theta - \theta_s, \mathbf{G}) := \mathcal{P}_{\text{inter}}^{\text{l}}(\theta - \theta_s) + \mathcal{P}_{\text{inter}}^{\text{nl}}(\mathbf{G}). \quad (30a)$$

The local contribution is given by

$$\begin{aligned} \mathcal{P}_{\text{inter}}^{\text{l}}(\theta - \theta_s) &:= \int_{\Gamma_{\text{C}}} \Phi_{\text{inter}}^{\text{l}}(\theta(x) - \theta_s(x)) \, dx && \text{with} \\ \Phi_{\text{inter}}^{\text{l}}(\theta - \theta_s) &= \frac{k(\chi)}{2} (\theta - \theta_s)^2, \end{aligned} \quad (30b)$$

where the positive (and smooth) function k is a surface thermal diffusion coefficient. The nonlocal contribution is

$$\begin{aligned} \mathcal{P}_{\text{inter}}^{\text{nl}}(\mathbf{G}) &:= \iint_{\Gamma_{\text{C}} \times \Gamma_{\text{C}}} \Phi_{\text{inter}}^{\text{nl}}(\mathbf{G}(x, y)) \, dx \, dy && \text{with} \\ \Phi_{\text{inter}}^{\text{nl}}(\mathbf{G}(x, y)) &= \frac{1}{2} |\mathbf{G}(x, y)|^2 K(\chi(x), \chi(y)) e^{-\frac{|x-y|^2}{d^2}}. \end{aligned} \quad (30c)$$

Here, the interaction function K is assumed to be symmetric, i.e. $K(x, y) = K(y, x)$. For simplicity, from now on for the functions H and K in (26c) and (30c) we set

$$H(\chi(x), \chi(y)) = K(\chi(x), \chi(y)) := \chi(x)\chi(y).$$

We note that in (26c) and (30c) the exponential term describes the attenuation of nonlocal interactions as the distance $|x - y|$ between two points x and y on the contact surface increases.

2.4 The Balance Equations and the Constitutive Laws

Now, we recover the equations of the system, written in the bulk domain and on the contact surface, by the general laws of Thermomechanics with the free energy (23) and the potential of dissipation (27).

We derive the momentum balance equation for macroscopic movements (33) and the flow rule for the adhesive parameter (36) from the principle of virtual power, in which local and nonlocal microscopic forces responsible for the degradation of the adhesive substance are included. More precisely, for any virtual bulk velocity \mathbf{v} with $\mathbf{v} = \mathbf{0}$ on Γ_{Dir} and for any virtual microscopic velocity w on the contact surface, we

define the power of the internal forces in Ω and Γ_C as follows

$$\begin{aligned} \mathcal{P}_{\text{int}} := & - \int_{\Omega} \Sigma \epsilon(\mathbf{v}) \, d\Omega - \int_{\Gamma_C} (Bw + \mathbf{H}\nabla w) \, dx + \int_{\Gamma_C} \mathbf{R}\mathbf{v} \, dx \\ & + \iint_{\Gamma_C \times \Gamma_C} 2\mathbf{m}(x, y)(x - y)\mathbf{v}(x) \, dx \, dy \\ & + \iint_{\Gamma_C \times \Gamma_C} (B_{\text{nl}}^1(x, y)w(x) + B_{\text{nl}}^2(x, y)w(y)) \, dx \, dy. \end{aligned} \quad (31)$$

Here, Σ is the Cauchy stress tensor, \mathbf{R} the classical macroscopic reaction on the contact surface, B and \mathbf{H} are local interior forces, responsible for the degradation of the adhesive bonds between the body and the support. The terms $\mathbf{m}(x, y)$ and $B_{\text{nl}}^i(x, y)$, $i = 1, 2$, are new scalar nonlocal contributions: they stand for internal microscopic nonlocal forces on the contact surface and describe the effects of the elongation as a source of damage. The power of the external forces is given by

$$\mathcal{P}_{\text{ext}} := \int_{\Omega} \mathbf{f}\mathbf{v} \, d\Omega + \int_{\Gamma_{\text{Neu}}} \mathbf{h}\mathbf{v} \, d\Gamma, \quad (32)$$

where \mathbf{f} is a bulk known external force, while \mathbf{h} is a given traction on Γ_{Neu} . Note that here we have neglected any microscopic external force and any acceleration power.

The principle of virtual power, holding for every virtual microscopic and macroscopic velocities and every subdomain in Ω , leads to the quasistatic momentum balance

$$- \operatorname{div} \Sigma = \mathbf{f} \quad \text{in } \Omega, \quad (33)$$

supplemented by the following boundary conditions

$$\Sigma \mathbf{n}(x) = \mathbf{R}(x) + \int_{\Gamma_C} 2(x - y)\mathbf{m}(x, y) \, dy \quad \text{in } \Gamma_C, \quad (34)$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\text{Dir}}, \quad \Sigma \mathbf{n} = \mathbf{h} \quad \text{in } \Gamma_{\text{Neu}}. \quad (35)$$

Observe that in (34) the boundary condition for the stress tensor on the contact surface combines a local contribution involving the (pointwise) reaction $\mathbf{R}(x)$ and a nonlocal force (defined in terms of the new variable $\mathbf{m}(x, y)$), related to the elongation.

Again, the principle of virtual power leads to a micro-force balance on the contact surface given by

$$\begin{aligned} B(x) - \operatorname{div} \mathbf{H}(x) &= \int_{\Gamma_C} (B_{\text{nl}}^1(x, y) + B_{\text{nl}}^2(y, x)) \, dy \quad \text{in } \Gamma_C, \\ \mathbf{H} \cdot \mathbf{n}_s &= 0 \quad \text{on } \partial\Gamma_C, \end{aligned} \quad (36)$$

where \mathbf{n}_s denotes the outward unit normal vector to $\partial\Gamma_C$.

Constitutive relations for Σ , \mathbf{R} , B , \mathbf{H} , \mathbf{m} , and B_{nl}^i , $i = 1, 2$, are given in terms of the free energies and the pseudo-potentials of dissipation. More precisely, the constitutive relation for the stress tensor Σ is

$$\Sigma := \frac{\partial\Psi_\Omega}{\partial\epsilon(\mathbf{u})} + \frac{\partial\Phi_\Omega}{\partial\epsilon(\mathbf{u}_t)} = E\epsilon(\mathbf{u}) + \theta\mathbf{I} + \mathbf{V}\epsilon(\mathbf{u}_t) \quad (37)$$

with \mathbf{I} the identity matrix, while the local reaction is

$$\mathbf{R} := -\frac{\partial\Psi_{\text{inter}}^1}{\partial\mathbf{u}} = -\kappa\chi\mathbf{u} - \partial I_{(-\infty,0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \quad (38)$$

combined with

$$\mathbf{m}(x, y) := -\frac{\partial\Psi_{\text{inter}}^{\text{nl}}}{\partial\mathbf{g}}(x, y) = -\mathbf{g}(x, y)\chi(x)\chi(y)e^{-\frac{|x-y|^2}{d^2}}. \quad (39)$$

Concerning the microscopic forces B and \mathbf{H} , we prescribe

$$B := \frac{\partial\Psi_{\Gamma_C}}{\partial\chi} + \frac{\partial\Psi_{\text{inter}}^1}{\partial\chi} + \frac{\partial\Phi_{\Gamma_C}}{\partial\chi_t} = \partial I_{[0,1]}(\chi) + \gamma'(\chi) + \lambda'(\chi)(\theta_s - \theta_{\text{eq}}) + \frac{\kappa}{2}|\mathbf{u}|^2 + \chi_t \quad (40)$$

and let \mathbf{H} be

$$\mathbf{H} := \frac{\partial\Psi_{\Gamma_C}}{\partial\nabla\chi} = \nabla\chi, \quad (41)$$

while the terms B_{nl}^1 and B_{nl}^2 are (formally) defined as derivatives of $\Psi_{\text{inter}}^{\text{nl}}$ with respect to the values of the surface damage parameter in x and $y \in \Gamma_C$, respectively, as follows

$$B_{\text{nl}}^1(x, y) := -\frac{\partial\Psi_{\text{inter}}^{\text{nl}}}{\partial\chi(x)} = -\frac{1}{2}\mathbf{g}^2(x, y)\chi(y)e^{-\frac{|x-y|^2}{d^2}}, \quad (42)$$

$$B_{\text{nl}}^2(x, y) := -\frac{\partial\Psi_{\text{inter}}^{\text{nl}}}{\partial\chi(y)} = -\frac{1}{2}\mathbf{g}^2(x, y)\chi(x)e^{-\frac{|x-y|^2}{d^2}}. \quad (43)$$

The equations for the temperature variables are recovered from the first principle of thermodynamics, i.e. the internal energy balance written in the bulk domain and on the contact surface. The internal energy balance equation in Ω reads

$$\theta_{s_t} + \text{div}\mathbf{q} = \frac{\partial\Phi_\Omega}{\partial\epsilon(\mathbf{u}_t)}\epsilon(\mathbf{u}_t) + h \quad \text{in } \Omega, \quad (44)$$

which h an external heat source. Here the entropy s is defined by the constitutive relation

$$s := -\frac{\partial \Psi_\Omega}{\partial \theta} = \log(\theta) - \operatorname{div} \mathbf{u}, \quad (45)$$

and the heat flux \mathbf{q} is given by

$$\mathbf{q} := -\theta \frac{\partial \Phi_\Omega}{\partial \nabla \theta} = -\alpha_b(\theta) \nabla \theta \quad (46)$$

(recall that α_b is the heat conductivity coefficient in the bulk domain). The boundary conditions on \mathbf{q} are prescribed on $\Gamma_{\text{Dir}} \cup \Gamma_{\text{Neu}}$ as $\mathbf{q} \cdot \mathbf{n} = 0$, and on Γ_C as

$$(\mathbf{q} \cdot \mathbf{n})(x) = \theta(x) \left(F^1(x) + \int_{\Gamma_C} 2(x-y) \mathbf{M}(x, y) dy \right), \quad x \in \Gamma_C. \quad (47)$$

Here, the term

$$F(x) := F^1(x) + \int_{\Gamma_C} 2(x-y) \mathbf{M}(x, y) dy, \quad x \in \Gamma_C, \quad (48)$$

split into local and nonlocal contributions, represents the total entropy flux through the contact surface. In particular, we prescribe

$$F^1(x) := \frac{\partial \Phi_{\text{inter}}^1}{\partial (\theta - \theta_s)}(x) = k(\chi(x))(\theta(x) - \theta_s(x)) \quad (49)$$

and

$$\mathbf{M}(x, y) := \frac{\partial \Phi_{\text{inter}}^{\text{nl}}}{\partial \mathbf{G}}(x, y) = \mathbf{G}(x, y) \chi(x) \chi(y) e^{-\frac{|x-y|^2}{d^2}}. \quad (50)$$

Finally, the internal energy balance on Γ_C is written as

$$\begin{aligned} \theta_s(x) \partial_t s_s(x) + \operatorname{div} \mathbf{q}_s(x) &= \theta_s(x) \left(F^1(x) + \int_{\Gamma_C} 2(y-x) \mathbf{M}(y, x) dy \right) \\ &\quad + \frac{\partial \Psi_{\Gamma_C}}{\partial \chi_t}(x) \chi_t(x), \quad x \in \Gamma_C \end{aligned} \quad (51)$$

supplemented by no-flux boundary conditions $\mathbf{q}_s \cdot \mathbf{n}_s = 0$ on $\partial \Gamma_C$. Here

$$s_s := -\frac{\partial \Psi_{\Gamma_C}}{\partial \theta_s} = \log(\theta_s) - \lambda(\chi) \quad (52)$$

denotes the entropy on the contact surface and

$$\mathbf{q}_s := -\theta_s \frac{\partial \Phi_{\Gamma_C}}{\partial \nabla \theta_s} = -\alpha_s(\theta_s) \nabla \theta_s \quad (53)$$

the heat flux (recall that α_s is the heat conductivity coefficient on Γ_C). We note that, like for the entropy flux through the contact surface (the term F in (48)), also for the entropy source involved on the right-hand side of (51), we have distinguished local and nonlocal contributions.

With these choices, combining the previous constitutive relations with the balance laws, we derive the PDE system (54) below.

2.5 The PDE System

By the previous constitutive relations and balance laws, we obtain the following boundary value problem

$$\begin{aligned} \theta_t - \theta \operatorname{div}(\mathbf{u}_t) - \operatorname{div}(\alpha_b(\theta) \nabla \theta) \\ = \epsilon(\mathbf{u}_t) \mathbf{V} \epsilon(\mathbf{u}_t) + h, \end{aligned} \quad \text{in } \Omega \times (0, T), \quad (54a)$$

$$\alpha_b(\theta) \nabla \theta \cdot \mathbf{n} = 0, \quad \text{in } (\Gamma_{\text{Dir}} \cup \Gamma_{\text{Neu}}) \times (0, T), \quad (54b)$$

$$\begin{aligned} \alpha_b(\theta) \nabla \theta \cdot \mathbf{n} = -\theta \left(k(\chi)(\theta - \theta_s) \right. \\ \left. + \int_{\Gamma_C} 2(x-y) \mathbf{G}(x, y) \chi(x) \chi(y) e^{-\frac{|x-y|^2}{d^2}} dy \right), \end{aligned} \quad \text{in } \Gamma_C \times (0, T), \quad (54c)$$

$$-\operatorname{div}(\mathbf{E}\epsilon(\mathbf{u}) + \mathbf{V}\epsilon(\mathbf{u}_t) + \theta \mathbf{I}) = \mathbf{f}, \quad \text{in } \Omega \times (0, T), \quad (54d)$$

$$\mathbf{u} = \mathbf{0}, \quad \text{in } \Gamma_{\text{Dir}} \times (0, T), \quad (54e)$$

$$(\mathbf{E}\epsilon(\mathbf{u}) + \mathbf{V}\epsilon(\mathbf{u}_t) + \theta \mathbf{I}) \mathbf{n} = \mathbf{h}, \quad \text{in } \Gamma_{\text{Neu}} \times (0, T), \quad (54f)$$

$$\begin{aligned} (\mathbf{E}\epsilon(\mathbf{u}) + \mathbf{V}\epsilon(\mathbf{u}_t) + \theta \mathbf{I}) \mathbf{n} \\ + \kappa \chi \mathbf{u} + \partial I_{(-\infty, 0]}(\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \\ + \int_{\Gamma_C} 2(x-y) \mathbf{g}(x, y) \chi(x) \chi(y) e^{-\frac{|x-y|^2}{d^2}} dy \ni \mathbf{0}, \end{aligned} \quad \text{in } \Gamma_C \times (0, T), \quad (54g)$$

$$\begin{aligned} \partial_t \theta_s - \theta_s \lambda'(\chi) \chi_t - \operatorname{div}(\alpha_s(\theta_s) \nabla \theta_s) = |\chi_t|^2 \\ + \theta_s \left(k(\chi)(\theta - \theta_s) \right) \end{aligned}$$

$$\begin{aligned}
& + \int_{\Gamma_C} 2(y-x)\mathbf{G}(y,x)\chi(x)\chi(y)e^{-\frac{|x-y|^2}{d^2}} dy, & \text{in } \Gamma_C \times (0, T), & (54h) \\
\alpha_s(\theta_s)\nabla\theta_s \cdot \mathbf{n}_s = 0, & & \text{in } \partial\Gamma_C \times (0, T), & (54i)
\end{aligned}$$

$$\begin{aligned}
& \chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) + \gamma'(\chi) \\
& + \lambda'(\chi)(\theta_s - \theta_{\text{eq}}) \ni -\frac{\kappa}{2}|\mathbf{u}|^2 \\
& - \frac{1}{2} \int_{\Gamma_C} (\mathbf{g}^2(x,y) + \mathbf{g}^2(y,x)) \chi(y)e^{-\frac{|x-y|^2}{d^2}} dy, & \text{in } \Gamma_C \times (0, T), & (54j)
\end{aligned}$$

$$\partial_{\mathbf{n}_s}\chi = 0, \quad \text{in } \partial\Gamma_C \times (0, T), \quad (54k)$$

where we have written explicitly the dependence of the unknowns $(\theta, \theta_s, \mathbf{u}, \chi)$ on the variable $x \in \Gamma_C$ only in the nonlocal terms involving integrals (with respect to the spatial variable $y \in \Gamma_C$), cf. (54c), (54g), (54h), and (54j).

Let us stress that, with respect to the ‘standard’ Frémond system for adhesive contact (see e.g. (14)), (54) encompasses integral terms in the flux boundary conditions (54c) for θ , in Eq. (54h), in the normal reaction (54g), and in the flow rule (54j) for χ . In particular, the thermal evolution of the system depends on a local contribution related to $\theta - \theta_s$ evaluated at the same point $x \in \Gamma_C$, and on a nonlocal one described by the function \mathbf{G} from (22) that involves the thermal gap between different points x and y on the contact surface. Analogously, the source of damage on the right-hand side of (54j) features local and nonlocal terms and, in particular, it may be different from zero even if $\mathbf{u} = \mathbf{0}$, due to the integral contribution (in terms of the variable \mathbf{g} from (20)) that renders the damaging effects of elongation.

2.6 Outlook to the Analysis

The analysis of system (54) will be carried out in the forthcoming [18]. There, taking into account the L^1 -character of the right-hand sides of the temperature equations (54a) and (54h), we will address the existence of a weak solution for (54).

References

1. Andrews, K.T., Kuttler, K.L., Shillor, M.: On the dynamic behaviour of a thermoviscoelastic body in frictional contact with a rigid obstacle. *Eur. J. Appl. Math.* **8**, 417–436 (1997)
2. Andrews, K.T., Shillor, M., Wright, S., Klarbring, A.: A dynamic thermoviscoelastic contact problem with friction and wear. *Int. J. Eng. Sci.* **14**, 1291–1309 (1997)
3. Boccardo, L., Gallouët, T.: Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* **87**, 149–169 (1989)
4. Bonetti, E., Bonfanti, G., Lebon, F., Rizzoni, R.: A model of imperfect interface with damage. *Meccanica* **52**, 1911–1922 (2017)
5. Bonetti, E., Bonfanti, G., Lebon, F.: Derivation of imperfect interface models coupling damage and temperature. *Comput. Math. Appl.* **77**, 2906–2916 (2019)
6. Bonetti, E., Bonfanti, G., Rossi, R.: Well-posedness and long-time behaviour for a model of contact with adhesion. *Indiana Univ. Math. J.* **56**, 2787–2819 (2007)
7. Bonetti, E., Bonfanti, G., Rossi, R.: Global existence for a contact problem with adhesion. *Math. Methods Appl. Sci.* **31**, 1029–1064 (2008)
8. Bonetti, E., Bonfanti, G., Rossi, R.: Thermal effects in adhesive contact: modelling and analysis. *Nonlinearity* **22**, 2697–2731 (2009)
9. Bonetti, E., Bonfanti, G., Rossi, R.: Analysis of a unilateral contact problem taking into account adhesion and friction. *J. Differ. Equ.* **253**, 438–462 (2012)
10. Bonetti, E., Bonfanti, G., Rossi, R.: Analysis of a temperature-dependent model for adhesive contact with friction. *Phys. D* **285**, 42–62 (2014)
11. Bonetti, E., Bonfanti, G., Rossi, R.: Modeling via internal energy balance and analysis of adhesive contact with friction in thermoviscoelasticity. *Nonlinear Anal. Real World Appl.* **22**, 473–507 (2015)
12. Bonetti, E., Bonfanti, G., Rossi, R.: Global existence for a nonlocal model for adhesive contact. *Appl. Anal.* **97**, 1315–1339 (2018)
13. Bonetti, E., Cavaterra, C., Freddi, F., Grasselli, M., Natalini, R.: A nonlinear model for marble sulphation including surface rugosity: theoretical and numerical results. *Commun. Pure Appl. Anal.* **18**, 977–998 (2019)
14. Bonetti, E., Colli, P., Fabrizio, M., Gilardi, G.: Global solution to a singular integrodifferential system related to the entropy balance. *Nonlinear Anal.* **66**, 1949–1979 (2007)
15. Bonetti, E., Colli, P., Fabrizio, M., Gilardi, G.: Modelling and long-time behaviour for phase transitions with entropy balance and thermal memory conductivity. *Discrete Contin. Dyn. Syst. Ser. B* **6**, 1001–1026 (2006)
16. Bonetti, E., Frémond, M.: Analytical results on a model for damaging in domains and interfaces. *ESAIM Control Optim. Calc. Var.* **17**, 955–974 (2011)
17. Bonetti, E., Rocca, E., Scala, R., Schimperna, G.: On the strongly damped wave equation with constraint. *Commun. Partial Differ. Equ.* **42**, 1042–1064 (2017)
18. Bonfanti, G., Colturato, M., Rossi, R.: Existence results for a nonlocal temperature-dependent system modelling adhesive contact (2020). In preparation
19. Cocou, M., Rocca, R.: Existence results for unilateral quasistatic contact problems with friction and adhesion. *M2AN Math. Model. Numer. Anal.* **34**, 981–1001 (2000)
20. Cocou, M., Schryve, M., Raous, M.: A dynamic unilateral contact problem with adhesion and friction in viscoelasticity. *Z. Angew. Math. Phys.* **61**, 721–743 (2010)
21. Eck, C.: Existence of solutions to a thermo-viscoelastic contact problem with Coulomb friction. *Math. Models Methods Appl. Sci.* **12**, 1491–1511 (2002)
22. Eck, C., Jarušek, J., Krbeč, M.: *Unilateral Contact Problems*. Pure and Applied Mathematics vol. 270, Chapman & Hall/CRC, Boca Raton (2005)
23. Eck, C., Jařusek, J.: The solvability of a coupled thermoviscoelastic contact problem with small Coulomb friction and linearized growth of frictional heat. *Math. Meth. Appl. Sci.* **22**, 1221–1234 (1999)

24. Figueiredo, I., Trabucho, L.: A class of contact and friction dynamic problems in thermoelasticity and in thermoviscoelasticity. *Int. J. Eng. Sci.* **1**, 45–66 (1995)
25. Fleming, D.C., Vizzini, A.: The energy absorption of composite plates under off-axis loads. *J. Compos. Mater.* **30**, 1977–1995 (1996)
26. Freddi, F., Frémond, M.: Damage in domains and interfaces: a coupled predictive theory. *J. Mech. Mater. Struct.* **1**, 1205–1233 (2006)
27. L. Freddi, R. Paroni, Roubíček, T., Zanini, C.: Quasistatic delamination models for Kirchhoff–Love plates. *ZAMM Z. Angew. Math. Mech.* **91**, 845–865 (2011)
28. Frémond, M., Nedjar, B.: Damage, gradient of damage and principle of virtual power. *Int. J. Solids Struct.* **33**, 1083–1103 (1996)
29. Frémond, M.: *Non-Smooth Thermomechanics*. Springer, Berlin (2002)
30. Halphen, B., Nguyen, Q.S.: Sur les matériaux standards généralisés. *J. Mécanique* **14**, 39–63 (1975)
31. Kočvara, M., Mielke, A., Roubíček, T.: A rate-independent approach to the delamination problem. *Math. Mech. Solids* **11**, 423–447 (2006)
32. Mielke, A., Roubíček, T.: *Rate-Independent Systems. Theory and Application*. Applied Mathematical Sciences, vol. 193. Springer, New York (2015)
33. Mielke, A., Roubíček, T., Thomas, M.: From damage to delamination in nonlinearly elastic materials at small strains. *J. Elast.* **109**, 235–273 (2012)
34. Migórski, S., Ochal, A., Sofonea, M.: *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*. Advances in Mechanics and Mathematics, vol. 26. Springer, New York (2013)
35. Migórski, S., Zeng, S.: Hyperbolic hemivariational inequalities controlled by evolution equations with application to adhesive contact model. *Nonlinear Anal. Real World Appl.* **43**, 121–143 (2018)
36. Raous, M., Cangémi, L., Cocu, M.: A consistent model coupling adhesion, friction, and unilateral contact. *Comput. Methods Appl. Mech. Eng.* **177**, 383–399 (1999)
37. Rossi, R., Roubíček, T.: Thermodynamics and analysis of rate-independent adhesive contact at small strains. *Nonlinear Anal.* **74**, 3159–3190 (2011)
38. Rossi, R., Thomas, M.: From an adhesive to a brittle delamination model in thermo-viscoelasticity. *ESAIM Control Optim. Calc. Var.* **21**, 1–59 (2015)
39. Roubíček, T.: Thermodynamics of rate-independent processes in viscous solids at small strains. *SIAM J. Math. Anal.* **42**, 256–297 (2010)
40. Roubíček, T.: Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity. *SIAM J. Math. Anal.* **45**, 101–126 (2013)
41. Roubíček, T., Kružík, M., Mantič, V., Panagiotopoulos, C.G., Vodička, R., Zeman, J.: Delamination and adhesive contacts, their mathematical modeling and numerical treatment. To appear as Chap.11 In: Mantič, V. (ed.), *Mathematical Methods and Models in Composites*, 2nd edn. Imperial College Press, London
42. Roubíček, T., Scardia, L., Zanini, C.: Quasistatic delamination problem. *Contin. Mech. Thermodyn.* **21**, 223–235 (2009)
43. Scala, R.: Limit of viscous dynamic processes in delamination as the viscosity and inertia vanish. *ESAIM Control Optim. Calc. Var.* **23**, 593–625 (2017)
44. Scala, R.: A weak formulation for a rate-independent delamination evolution with inertial and viscosity effects subjected to unilateral constraint. *Interfaces Free Bound.* **19**, 79–107 (2017)
45. Scala, R., Schimperna, G.: A contact problem for viscoelastic bodies with inertial effects and unilateral boundary constraints. *Eur. J. Appl. Math.* **28**, 91–122 (2017)
46. Shillor, M., Sofonea, M., Telega, J.J.: *Models and Analysis of Quasistatic Contact*. Lecture Notes in Physics, vol. 655. Springer, Berlin (2004)