



Nontransitive indifference preferences in bi-multi-objective optimization problems

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Abstract

In this paper, we introduce and discuss a generalization of the classical multi-objective optimization problem employing pairs of functions. More specifically, this procedure is referred to as bi-multi-objective optimization. A justification of this general optimization procedure is presented, related both to multi-objective optimization under ambiguity concerning individual preferences and to Pareto optimality for a family of preferences with nontransitive indifference. Incidentally, the binary relation naturally associated to a bi-multi-objective optimization problem is represented by a finite bi-multi-utility, which generalizes the classical finite multi-utility representation to the nontransitive case. Finally, we present an important application in the context of Markowitz portfolio selection under ambiguity concerning both the vector of returns and the covariance matrix.

Keywords Pareto optimal element · Bi-multi-objective optimization · Interval order · Ambiguity · Markowitz portfolio selection

JEL Classification C60

1 Introduction

The *multi-objective optimization* (see e.g. Miettinen (1999) and Ehrgott (2005)) is the most popular tool which allows to choose among different options in the presence

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of many *agents* (or *criteria*) represented by finitely many real-valued functions u_i ($i \in \{1, \dots, m\}$), which have to be maximized all at the same time by following the approach introduced in Pareto (1896), and therefore called *Pareto optimality*. In particular, such a notion appears in Debreu (1954) and Mas-Colell et al. (1995) in the framework of general equilibrium. It is worth underlying that the multi-objective optimization appears in different applications of mathematics, such as insurance theory (see e.g. Asimit et al. (2017)), design engineering (see e.g. Das (1999)), economics and risk-sharing (see e.g. Chateauneuf et al. (2015); Chanas and Kuchta (1996) and Barrieu and Scandolo (2008)), and portfolio selection (see e.g. Fliege and Werner (2014) and Xidonas et al. (2017)).

Recently, in Bevilacqua et al. (2018), the authors have tackled the classical multi-objective optimization problem by referring to the naturally associated *preorders*, in such a way that, since the Pareto optimal elements are precisely the maximal elements of such preorders, the classical results concerning the existence of maximal elements for not necessarily total preorders on compact topological spaces can be used (see e.g. the famous results of Bergstrom (1975); Ward (1954) and Rodríguez-Palmero and García-Lapresta (2002), and the recent results in Bosi and Zuanon (2017) and Bosi and Sbaiz (2025)).

In this paper we introduce a generalization of multi-objective optimization, called *bi-multi-objective optimization*. Such a problem has the form

$$\max_{x \in X} [(u_1(x), v_1(x)), \dots, (u_m(x), v_m(x))] = \max_{x \in X} (\mathbf{u}(x), \mathbf{v}(x)), \quad m \geq 2, \quad (1.1)$$

where the pair (u_i, v_i) of real-valued functions on the set X associated with the i -th individual satisfies, for every $i \in \{1, \dots, m\}$, the condition $u_i \leq v_i$.

We start by presenting the *Pareto optimality* condition as follows.

Definition 1.1 A point $x_0 \in X$ is said to be *Pareto optimal* if for no $x \in X$ it occurs that $u_i(x_0) \leq v_i(x)$ for all $i \in \{1, \dots, m\}$ and $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for at least one index $\bar{i}$.

It is worth noting that bi-multi-objective optimization coincides with the classical multi-objective optimization when $u_i = v_i$ for all $i \in \{1, \dots, m\}$.

Now, we present an interpretation of bi-multi-objective optimization based on *decision theory*, since to every pair (u_i, v_i) we can associate the *interval order* $\preceq_i$ on X , defined for all $x \in X$ by

$$x \preceq_i y \Leftrightarrow u_i(x) \leq v_i(y).$$

Hence, the appearance and use of bi-multi-objective optimization can be related to the *intransitivity of the indifference* of the individual interval orders (see e.g. Bosi and Zuanon (2014a) and Bosi and Zuanon (2014b)).

Incidentally, we observe that we can associate to every bi-multi-objective optimization problem

$$\max_{x \in X} (\mathbf{u}(x), \mathbf{v}(x))$$

the reflexive binary relation $\preceq$ on X defined by

$$x \preceq y \Leftrightarrow u_i(x) \leq v_i(y), \quad \forall i \in \{1, \dots, m\}.$$

This is a finite *bi-multi-utility representation* of a not necessarily transitive binary relation $\preceq$ on a set X , which is performed by means of a finite family $\{(u_i, v_i)\}_{i=1, \dots, m}$ of pairs (u_i, v_i) of real-valued functions on X such that $u_i \leq v_i$ for all $i \in \{1, \dots, m\}$. Moreover, this bi-multi-utility representation in the nontransitive case generalizes the classical finite *multi-utility representation* $\{u_i\}_{i=1, \dots, m}$ of a preorder $\preceq$ on X (see e.g. Bevilacqua et al. (2018) and Kaminski (2007)), according to which

$$x \preceq y \Leftrightarrow u_i(x) \leq u_i(y), \quad \forall i \in \{1, \dots, m\}.$$

Another interpretation of the bi-multi-utility optimization is linked to the concept of *ambiguity*, concerning the individual preferences, in the spirit of Haskell et al. (2016). Indeed, $(\mathbf{u}, \mathbf{v})$ can be interpreted as a range of *utility functions* $\mathbf{w}$ such that $\mathbf{u} \leq \mathbf{w} \leq \mathbf{v}$. The set of all weak Pareto optimal solutions to the bi-multi-objective optimization problem corresponding to the pair $(\mathbf{u}, \mathbf{v})$, includes all the weak Pareto optimal solutions to the multi-objective optimization problems corresponding to all functions $\mathbf{w}$ between $\mathbf{u}$ and $\mathbf{v}$ (see Theorem 3.1 below).

Compared to other interval optimization methods proposed in the literature (see e.g. Ishibuchi and Tanaka (1990); Chateauneuf et al. (2015) and Wu (2009)), our approach appears simpler, since it does not require any particular choice among the possible orderings of intervals. On the other hand, bi-multi-objective optimization, which can be also applied to portfolio choice, is close to multi-objective optimization with interval-valued objective functions (see e.g. Wu et al. (2013)).

We conclude this introductory part by presenting the structure of the paper. Section 2 contains the notation, the basic definitions and the preliminary results. Section 3 presents the characterization of the solutions to the bi-multi-objective optimization problem in terms of the individual interval orders and the representation of the maximal elements of the naturally associated reflexive binary relation. Section 4 is devoted to an application to Markowitz portfolio selection under ambiguity. Section 5 concludes, presenting also some future research directions.

2 Notation and preliminaries

Now, we are going to present some classical definitions relative to multi-objective optimization. The interested reader could refer to Miettinen (1999) and Ehrgott (2005).

Definition 2.1 The *Multi-objective Optimization Problem* (MOP) is formulated by means of the standard notation

$$\max_{x \in X} [u_1(x), \dots, u_m(x)] = \max_{x \in X} \mathbf{u}(x), \quad m \geq 2, \quad (2.1)$$

where X is the *choice set* (or the *decision space*), u_i is the *decision function* (in this case a *utility function*) associated with the i -th individual (or criterion), and $\mathbf{u} : X \rightarrow \mathbb{R}^m$ is the vector-valued function defined by $\mathbf{u}(x) = (u_1(x), \dots, u_m(x))$ for all $x \in X$.

Remark 2.2 *Mutatis mutandis*, the formulation (2.1) of the multi-objective optimization problem is equivalent to

$$\min_{x \in X} [f_1(x), \dots, f_m(x)] = \min_{x \in X} \mathbf{f}(x), \quad m \geq 2.$$

Definition 2.3 Consider the multi-objective optimization problem (2.1). Then a point $x_0 \in X$ is said to be

1. *Pareto optimal with respect to the function* $\mathbf{u} = (u_1, \dots, u_m) : X \rightarrow \mathbb{R}^m$ if for no $x \in X$ it occurs that $u_i(x_0) \leq u_i(x)$ for all $i \in \{1, \dots, m\}$ and at the same time $u_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for at least one index $\bar{i}$;
2. *weakly-Pareto optimal with respect to the function* $\mathbf{u} = (u_1, \dots, u_m) : X \rightarrow \mathbb{R}^m$ if for no $x \in X$ it occurs that $u_i(x_0) < u_i(x)$ for all $i \in \{1, \dots, m\}$.

Following the Definition 2.3 of Bevilacqua et al. (2018), the set of all (weakly) Pareto optimal elements with respect to the function $\mathbf{u} = (u_1, \dots, u_m) : X \rightarrow \mathbb{R}^m$ will be denoted by $X_{\mathbf{u}}^{Par}$ ($X_{\mathbf{u}}^{w-Par}$, respectively). It is clear that $X_{\mathbf{u}}^{Par} \subset X_{\mathbf{u}}^{w-Par}$ for every positive integer m , for every nonempty set X and for every function $\mathbf{u} = (u_1, \dots, u_m) : X \rightarrow \mathbb{R}^m$.

We now introduce the *bi-multi-objective optimization problem*, and then the associated concepts of *Pareto optimal* and *weakly-Pareto optimal point*.

Definition 2.4 The *Bi-Multi-objective Optimization Problem* (BIMOP) is formulated as follows:

$$\max_{x \in X} [(u_1(x), v_1(x)), \dots, (u_m(x), v_m(x))] = \max_{x \in X} (\mathbf{u}(x), \mathbf{v}(x)), \quad m \geq 2, \quad (2.2)$$

where the pair (u_i, v_i) of decision functions associated with the i -th individual satisfies, for every $i \in \{1, \dots, m\}$, the condition $u_i \leq v_i$.

Definition 2.5 Consider the bi-multi-objective optimization problem (2.2). Then a point $x_0 \in X$ is said to be

1. *Pareto optimal with respect to the function* $(\mathbf{u}, \mathbf{v}) : X \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ if for no $x \in X$ it occurs that $u_i(x_0) \leq v_i(x)$ for all $i \in \{1, \dots, m\}$ and at the same time $v_{\bar{i}}(x_0) < v_{\bar{i}}(x)$ for at least one index $\bar{i}$;
2. *weakly-Pareto optimal with respect to the function* $(\mathbf{u}, \mathbf{v}) : X \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ if for no $x \in X$ it occurs that $v_i(x_0) < v_i(x)$ for all $i \in \{1, \dots, m\}$.

Definition 2.6 The set of all (weakly) Pareto optimal elements with respect to the function $(\mathbf{u}, \mathbf{v}) : X \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ will be denoted by $X_{(\mathbf{u}, \mathbf{v})}^{Par}$ ($X_{(\mathbf{u}, \mathbf{v})}^{w-Par}$, respectively).

Remark 2.7 It is clear that $X_{(\mathbf{u}, \mathbf{v})}^{w-Par} \supset X_{(\mathbf{u}, \mathbf{v})}^{Par}$. Indeed, consider any element $x_0 \notin X_{(\mathbf{u}, \mathbf{v})}^{w-Par}$. Then, it happens that, for some element $x \in X$, $u_i(x_0) \leq v_i(x_0) < u_i(x) \leq v_i(x)$.

$v_i(x)$ for all $i \in \{1, \dots, m\}$, which clearly implies that $x_0 \notin X_{(\mathbf{u}, \mathbf{v})}^{Par}$, since $u_i(x_0) < v_i(x)$ for all $i \in \{1, \dots, m\}$ and at the same time $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for some $\bar{i}$.

Remark 2.8 It is straightforward to check that

$$\max_{x \in X} \mathbf{u}(x) = \max_{x \in X} (\mathbf{u}(x), \mathbf{u}(x)),$$

i.e., problem (2.2) coincides with problem (2.1) in case that $u_i = v_i$ for every $i \in \{1, \dots, m\}$.

Example 2.9 Consider the case when $X = [0, 2]$ (i.e., X is the closed real interval with extremes 0 and 2), $m = 2$, $u_1(x) = x$, $v_1(x) = x + 1$, $u_2(x) = x^2 - 2$, $v_2(x) = x$. Observe that $u_i \leq v_i$ for $i = 1, 2$, so that we can consider the bi-multi-objective optimization problem

$$\max_{x \in [0, 2]} [(u_1(x), v_1(x)), (u_2(x), v_2(x))] = \max_{x \in [0, 2]} [(x, x + 1), (x^2 - 2, x)].$$

We have that 2 is the unique Pareto optimal point, since for every $x_0 \in [0, 2[$ it happens that

$$u_1(x_0) = x_0 \leq 2 + 1 = 3 = v_1(2), \quad u_2(x_0) = x_0^2 - 2 \leq 2 = v_2(2)$$

and on the other hand

$$v_2(x_0) = x_0 < 2^2 - 2 = 2 = u_2(2).$$

The set of all weakly-Pareto optimal points is $[1, 2]$, since $v_1(x_0) = x_0 + 1 < 2 = u_1(2)$ if and only if $x_0 < 1$, while $v_2(x_0) = x_0 < 2^2 - 2 = u_2(2)$ for every $x_0 < 2$.

3 Bi-multi-objective optimization

Here below we establish an important result, which relates the weakly-Pareto optimal solutions to the bi-multi-objective optimization problem with respect to the function $(\mathbf{u}(x), \mathbf{v}(x))$ and the weakly-Pareto optimal solutions to the multi-objective optimization problem with respect to any function $\mathbf{w} = (w_1, \dots, w_m)$ such that $u_i \leq w_i \leq v_i$ for every $i \in \{1, \dots, m\}$.

Theorem 3.1 Consider the bi-multi-objective optimization problem (2.2) with respect to the function $(\mathbf{u}(x), \mathbf{v}(x))$ and let $\mathbf{w} : X \rightarrow \mathbb{R}^m$ be any function such that $u_i \leq w_i \leq v_i$ for every $i \in \{1, \dots, m\}$. Then $X_{\mathbf{w}}^{w-Par} \subset X_{(\mathbf{u}, \mathbf{v})}^{w-Par}$.

Proof Consider the function $(\mathbf{u}(x), \mathbf{v}(x)) : X \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ with $u_i \leq v_i$ for every $i \in \{1, \dots, m\}$, and let $\mathbf{w} : X \rightarrow \mathbb{R}^m$ be any function such that $u_i \leq w_i \leq v_i$ for every $i \in \{1, \dots, m\}$. By contraposition, if $x_0 \in X$ is such that $x_0 \notin X_{(\mathbf{u}, \mathbf{v})}^{w-Par}$, then, according to Definition 2.5, it happens that there exists $x \in X$ such that, for every

$i \in \{1, \dots, m\}$, $w_i(x_0) \leq v_i(x_0) < u_i(x) \leq w_i(x)$, implying that $w_i(x_0) < w_i(x)$ for every $i \in \{1, \dots, m\}$. This means that $x_0 \notin X_{\mathbf{w}}^{w-Par}$ (see Definition 2.3). This consideration completes the proof. $\square$

Remark 3.2 We do not think that analogous considerations may concern the case of Pareto optimal elements. However, we shall consider restrictive conditions under which this may happen in a future paper.

Remark 3.3 The above Theorem 3.1 can be interpreted in terms of *ambiguity* concerning the criteria to be adopted in a classical multi-objective optimization problem. Indeed, the weakly-Pareto optimal solutions to a bi-multi-objective optimization problem (2.2) include all the weakly-Pareto optimal solutions to every multi-objective optimization problem (2.1) corresponding to a set of utilities $\mathbf{w}$ in the assigned *range* $(\mathbf{u}, \mathbf{v})$.

Remark 3.4 Since in problem (2.2), we require that $u_i \leq v_i$ for every $i \in \{1, \dots, m\}$, we have that the m (possibly degenerate) closed real intervals $[u_i(x), v_i(x)]$ are naturally associated to every $x \in X$.

In order to present a characterization and interpretation of the bi-multi-objective optimization problem (2.2) based on decision theory, we introduce some classical definitions relative to binary relations and in particular to *interval orders*. Since these definitions are standard (see e.g. Fishburn (1985)), the reader can refer for example to Bosi and Zuanon (2014a) and Bosi and Zuanon (2014b) for a deeper discussion concerning the existence of representations by means of pairs of *upper semicontinuous* real-valued functions.

In the sequel, the symbol $\preceq$ will stand for a *reflexive* binary relation on a set X (i.e., $x \preceq x$ for every $x \in X$). For all $x, y \in X$, $x \preceq y$ has to be read as “the alternative y is at least as preferable as the alternative x ”.

The *strict part* (or *asymmetric part*) of a binary relation $\preceq$ will be denoted by $<$, i.e., for all $x, y \in X$, $x < y$ if and only if $(x \preceq y)$ and *not* $(y \preceq x)$.

The *indifference relation* $\sim$ associated to $\preceq$ is defined, for all $x, y \in X$, as $x \sim y$ if and only if $(x \preceq y)$ and $(y \preceq x)$. Notice that $\sim$ is an *equivalence* on X when $\preceq$ is *transitive*, i.e., for all $x, y, z \in X$, $(x \preceq y)$ and $(y \preceq z) \Rightarrow x \preceq z$.

Definition 3.5 A *preorder* $\preceq$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive.

Definition 3.6 An *interval order* $\preceq$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \preceq z) \text{ and } (y \preceq w) \Rightarrow (x \preceq w) \text{ or } (y \preceq z).$$

This latter property is often referred to the *Ferrers property* (see e.g. Bosi and Zuanon (2014a)).

Since it is trivial that an interval order is *total* (i.e., for all $x, y \in X$ either $x \preceq y$ or $y \preceq x$), we have that actually $x < y$ if and only if *not* $(y \preceq x)$ ($x, y \in X$) when $\preceq$ is

an interval order. It is well-known that an interval order $\preceq$ is not transitive in general, while its strict part $<$ is always transitive.

We recall that a total preorder $\preceq$ on a set X is *represented* by a real-valued function u on X if, for all $x \in X$,

$$x \preceq y \Leftrightarrow u(x) \leq u(y).$$

In this case u is said to be a *utility function* for $\preceq$.

Definition 3.7 A pair (u, v) of real-valued functions on X is said to *represent* an interval order $\preceq$ on X if, for all $x, y \in X$,

$$x \preceq y \Leftrightarrow u(x) \leq v(y).$$

It is clear that, if (u, v) is a representation of an interval order $\preceq$ on X , then $u(x) \leq v(x)$ for every $x \in X$ due to the fact that $\preceq$ is reflexive. Hence, we have that $u \leq v$.

At this point, the classical definition of a *maximal element* is needed.

Definition 3.8 Let $\preceq$ be a reflexive binary relation on a set X . A point $x_0 \in X$ is said to be a *maximal element* of $(X, \preceq)$ if for no $x \in X$ it happens that $x_0 < x$.

The set of maximal elements for a reflexive binary relation $\preceq$ on a set X will be denoted by $X_M^{\preceq}$.

It should be noted that actually a point $x_0 \in X$ is a maximal element for an interval order $\preceq$ on a set X if and only if $x \preceq x_0$ for every $x \in X$.

Next, we introduce the concept of Pareto optimality with respect to a (finite) family of preferences (see e.g. Aspremont and Gevers (2002)).

Definition 3.9 A point $x_0 \in X$ is said to be *Pareto optimal with respect to the family* $\{\preceq_i\}_{i \in \{1, \dots, m\}}$ *of interval orders* if for no point $x \in X$ it occurs that $x_0 \preceq_i x$ for all $i \in \{1, \dots, m\}$, with at least one index $\bar{i}$ such that $x_0 <_{\bar{i}} x$.

The set of Pareto optimal maximal elements for a family $\{\preceq_i\}_{i \in \{1, \dots, m\}}$ of interval orders on a set X will be denoted by $X_{\{\preceq_i\}_{i \in \{1, \dots, m\}}}^{Par}$.

We are now ready to present a characterization of the solutions to the bi-multi-objective optimization problem.

Theorem 3.10 Consider the bi-multi-objective optimization problem (2.2) with respect to the function $(\mathbf{u}, \mathbf{v}) : X \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ with $u_i \leq v_i$ for every $i \in \{1, \dots, m\}$. Then, the following conditions are equivalent on a point $x_0 \in X$:

1. $x_0 \in X_{(\mathbf{u}, \mathbf{v})}^{Par}$;
2. $x_0 \in X_{\{\preceq_i\}_{i \in \{1, \dots, m\}}}^{Par}$ and the family $\{\preceq_i\}_{i \in \{1, \dots, m\}}$ of interval orders on X is such that, for every $i \in \{1, \dots, m\}$, $\preceq_i$ is represented by the pair (u_i, v_i) ;
3. $x_0 \in X_M^{\preceq}$, where $\preceq = \bigcap_{i=1}^m \preceq_i$ and, for every $i \in \{1, \dots, m\}$, the interval order $\preceq_i$ is represented by the pair (u_i, v_i) ;

4. $x_0 \in X_M^{\preceq}$, where the binary relation $\preceq$ on X defined as follows, for all $x, y \in X$:

$$x \preceq y \Leftrightarrow u_i(x) \leq v_i(y), \quad \forall i \in \{1, \dots, m\}. \quad (3.1)$$

Proof In order to show that 1. $\Rightarrow$ 2., we consider that if x_0 is not Pareto optimal with respect to the given family $\{\preceq_i\}_{i \in \{1, \dots, m\}}$ of interval orders on X , then there exists $x \in X$ such that $u_i(x_0) \leq v_i(x)$ for all $i \in \{1, \dots, m\}$, with at least one index $\bar{i}$ such that $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$, which implies that $x_0 \notin X_{(\mathbf{u}, \mathbf{v})}^{Par}$.

The proofs that 2. $\Rightarrow$ 3. and 3. $\Rightarrow$ 4. are rather simple and therefore they are left to the reader.

Finally, to show that 4. $\Rightarrow$ 1., we consider that if $x_0 \notin X_{(\mathbf{u}, \mathbf{v})}^{Par}$, then the existence of $x \in X$, such that $u_i(x_0) \leq v_i(x) \Leftrightarrow x_0 \preceq_i x$ for all $i \in \{1, \dots, m\}$ and at the same time $v_{\bar{i}}(x_0) < u_{\bar{i}}(x) \Leftrightarrow x_0 \prec_{\bar{i}} x$ for at least one index $\bar{i}$, precisely means that x_0 is not a maximal element of the binary relation $\preceq$ defined in (3.1). This consideration completes the proof. $\square$

Remark 3.11 It is clear that the finite representation (3.1) of a (reflexive) binary relation $\preceq$ generalizes the classical finite *multi-utility representation of a preorder* (see e.g. Kaminski (2007) and Bevilacqua et al. (2018)) according to which, for a given function $\mathbf{u} : X \rightarrow \mathbb{R}^m$ and for all $x, y \in X$,

$$x \preceq y \Leftrightarrow u_i(x) \leq u_i(y), \quad \forall i \in \{1, \dots, m\}. \quad (3.2)$$

Compared to this kind of representation, which is possible only when $\preceq$ is a preorder, the representation (3.1) has the advantage that it is compatible with *intransitive indifference*, i.e. situations when there exist $x, y, z \in X$ such that $x \sim y$ and $y \sim z$, but *not* ($x \sim z$). Moreover, representation (3.1) coincides with the representation of an interval order by means of two real-valued functions in the case when $m = 1$ (see Definition 3.7).

As an immediate corollary of Theorem 3.10 (see Remark 2.8), we get a characterization of the solutions to the multi-objective optimization problem.

Corollary 3.12 Consider the multi-objective optimization problem (2.1) with respect to the function $\mathbf{u} : X \rightarrow \mathbb{R}^m$. Then, the following conditions are equivalent on a point $x_0 \in X$:

1. $x_0 \in X_{\mathbf{u}}^{Par}$;
2. x_0 is Pareto optimal with respect to the family $\{\preceq_i\}_{i \in \{1, \dots, m\}}$ of total preorders on X such that, for every $i \in \{1, \dots, m\}$, $\preceq_i$ is represented by the function u_i ;
3. x_0 is a maximal element of the binary relation $\preceq = \bigcap_{i=1}^m \preceq_i$ where, for every $i \in \{1, \dots, m\}$, the total preorder $\preceq_i$ is represented by function u_i ;
4. x_0 is a maximal element for the preorder $\preceq$ on X defined as follows, for all $x, y \in X$:

$$x \preceq y \Leftrightarrow u_i(x) \leq u_i(y), \quad \forall i \in \{1, \dots, m\}. \quad (3.3)$$

Remark 3.13 In Theorem 3.10 and Corollary 3.12, some results concerning Pareto optimal solutions are given. We do not think that analogous considerations may concern the case of weakly Pareto optimal elements. However, we will investigate restrictive conditions under which this may happen in a future paper.

We conclude this section by presenting a topological condition guaranteeing the existence of solutions to the bi-multi-objective optimization problem.

We first recall that a real-valued function u on a topological space (X, τ) is said to be *upper semicontinuous* if $u^{-1}([-\infty, \alpha]) = \{x \in X : u(x) < \alpha\}$ is an open set for all $\alpha \in \mathbb{R}$. The well-known *Weierstrass extreme value theorem* guarantees that an upper semicontinuous real-valued function attains its maximum on a *compact* topological space. The previous topological definitions and results are found, for example, in Engelking (1989).

We now present a sufficient condition for the existence of a Pareto optimal solution for the bi-multi-objective optimization problem (2.2).

Theorem 3.14 $X_{(\mathbf{u}, \mathbf{v})}^{Par} \neq \emptyset$ provided that X is endowed with a compact topology τ and one of the following conditions is verified:

1. the functions u_i ($i \in \{1, \dots, m\}$) are upper semicontinuous;
2. the functions v_i ($i \in \{1, \dots, m\}$) are upper semicontinuous.

Proof 1. For every $i \in \{1, \dots, m\}$, consider a point $x_i \in \arg \max u_i$. These points exist since τ is a compact topology on X , and all the functions u_i are upper semicontinuous on the topological space (X, τ) . Further, let the index $i^* \in \{1, \dots, m\}$ be such that $u_{i^*}(x_{i^*}) = \max\{u_1(x_1), u_2(x_2), \dots, u_m(x_m)\}$. We have that $x_{i^*} \in X_{(\mathbf{u}, \mathbf{v})}^{Par}$, since for no $x \in X$ and $i \in \{1, \dots, m\}$ it may happen that $u_{i^*}(x_{i^*}) \leq v_{i^*}(x_{i^*}) < u_i(x)$.

2. We proceed in a perfectly analogous way, by defining the index $i^* \in \{1, \dots, m\}$ in such a way that $v_{i^*}(x_{i^*}) = \max\{v_1(x_1), v_2(x_2), \dots, v_m(x_m)\}$, with $x_i \in \arg \max v_i$ for every $i \in \{1, \dots, m\}$. Therefore, we have that $x_{i^*} \in X_{(\mathbf{u}, \mathbf{v})}^{Par}$. $\square$

As a corollary, we obtain a well-known result concerning the existence of a Pareto optimal solution to the multi-objective optimization problem (see Theorem 2.19 of Ehrgott (2005) in this respect).

Corollary 3.15 $X_{\mathbf{u}}^{Par} \neq \emptyset$ provided that X is endowed with a compact topology τ and the real-valued functions u_i ($i \in \{1, \dots, m\}$) are all upper semicontinuous.

Proof This is a particular application of the above Theorem 3.14, when $\mathbf{u} = \mathbf{v}$. $\square$

4 An application to Markowitz portfolio selection

The portfolio selection problem in the absence of short sales appears in the following form when expressed in terms of a multi-objective optimization problem (see e.g. Markowitz (1952)):

$$\max_{x \in X} [\mu^T x, -x^T \Sigma x], \quad (4.1)$$

where $X = \{x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$ is the set of all portfolios, $n > 0$ is the number of risky assets considered, $\mu \in \mathbb{R}^n$ is a vector of expected returns, and Σ is a covariance matrix of the returns (see e.g. Fliege and Werner (2014)). It is worth noticing that here the number of criteria is $m = 2$.

Therefore, problem (4.1) can be generalized by the following bi-multi-objective optimization problem:

$$\max_{x \in X} [(\mu_1^T x, \mu_2^T x), (-x^T \Sigma_2 x, -x^T \Sigma_1 x)], \quad (4.2)$$

where $\mu_1^T \leq \mu_2^T$ are two vectors of returns, and $\Sigma_1 \leq \Sigma_2$ are two covariance matrices (see also Wu et al. (2013)).

From Theorem 3.1, we can now state the following proposition.

Proposition 4.1 *Given two vectors $\mu_1^T \leq \mu_2^T$ of expected returns, and given two covariance matrices $\Sigma_1 \leq \Sigma_2$, consider the Markowitz bi-multi-objective optimization problem*

$$\max_{x \in X} [(\mu_1^T x, \mu_2^T x), (-x^T \Sigma_2 x, -x^T \Sigma_1 x)], \quad X = \{x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}. \quad (4.3)$$

We have that, for every vector of expected returns $\mu_1^T \leq \mu^T \leq \mu_2^T$ and for every covariance matrix $\Sigma_1 \leq \Sigma \leq \Sigma_2$, the weakly-Pareto optimal solutions to the corresponding multi-objective optimization problem

$$\max_{x \in X} [\mu^T x, -x^T \Sigma x], \quad X = \{x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}, \quad (4.4)$$

are also weakly-Pareto optimal for the above bi-multi-objective problem.

Remark 4.2 It is worth stressing the fact that it is easy to incorporate ambiguity concerning both the vector of returns and the covariance matrix in the classical Markowitz portfolio selection problem, since our bi-multi-objective optimization model takes into account a range of expected returns and covariance matrices.

5 Conclusions

In this paper, we have presented a bi-multi-objective optimization problem which is an extension of the classical multi-objective optimization problem. On one hand, it is simple to implement, and on the other hand it is a useful tool for incorporating ambiguity in the individual preferences.

In particular, we have presented the basic theory concerning this optimization problem, whose associated Pareto optimal elements can be interpreted as the maximal elements of the intersection of the interval orders representing the individual preferences.

Hopefully, we shall develop the full potential of our proposal in a future paper.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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