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General renewal equations motivated by biology and epidemiology

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Abstract

We present a unified framework ensuring well posedness and providing stability estimates to a class of Initial – Boundary Value Problems for renewal equations comprising a variety of biological or epidemiological models. This versatility is achieved considering fairly general – possibly non linear and/or non local – interaction terms, allowing both low regularity assumptions and independent variables with or without a boundary. In particular, these results also apply, for instance, to a model for the spreading of a Covid like pandemic or other epidemics. Further applications are shown to be covered by the present setting.

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Keywords: IBVP for renewal equations; Well posedness of epidemiological models; Differential equations in epidemic modeling; Age and space structured SIR models

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1. Introduction

In a variety of biological models, different species are typically described through their densities $u^1, u^2, \ldots, u^k$ and, in general, each u^h depends on time $t \in \mathbb{R}_+$, on age $a \in \mathbb{R}_+$, on a spatial coordinate in $\mathbb{R}^2$ or $\mathbb{R}^3$ and possibly also on some structural variables. Thus, a unified treatment of these models finds its natural setting in the following general mixed Initial – Boundary Value Problem (IBVP) in $\mathcal{X} = \mathbb{R}^m_+ \times \mathbb{R}^n$

$$\begin{cases} \partial_t u^h + \operatorname{div}_x \left(v^h(t,x)\, u^h\right) = g^h\left(t,x,u(t,x),u(t)\right) & (t,x) \in \mathbb{R}_+ \times \mathcal{X} \\ u^h(t,\xi) = u^h_b\left(t,\xi,u(t)\right) & (t,\xi) \in \mathbb{R}_+ \times \partial\mathcal{X} \\ u^h(0,x) = u^h_o(x) & x \in \mathcal{X}, \end{cases} \tag{1}$$

where $h = 1, \ldots, k$. Aiming at a rather general setting while keeping sharp estimates, without any loss in generality, we write (1) in the form

$$\begin{cases} \partial_t u^h + \operatorname{div}_x \left(v^h(t,x)\, u^h\right) = p^h\left(t,x,u(t)\right) u^h + q^h\left(t,x,u,u(t)\right) & (t,x) \in I \times \mathcal{X} \\ u^h(t,\xi) = u^h_b\left(t,\xi,u(t)\right) & (t,\xi) \in I \times \partial\mathcal{X} \\ u^h(0,x) = u^h_o(x) & x \in \mathcal{X}, \end{cases} \tag{2}$$

where $h = 1, \ldots, k$. Note that the decomposition of the source term g^h in (1) into p^h and q^h is neither unique nor in any sense restrictive.

We stress that both in (1) and in (2) the term $u(t)$ appearing in the right hand sides is understood as a *function*, so that both the source and boundary terms in (1), besides being *non linear*, also comprise quite general *non local*, i.e., *functional*, dependencies.

The current literature comprehends a multitude of well known models fitting into (1): we recall here for instance [1–9], leaving to Section 3 the highlighting of specific aspects of (1) in other recent or classical models. In particular, the well posedness and stability theorems below apply also to model (11) which, to our knowledge, does not fully fit into other well posedness results in the literature. At the same time, the literature covering particular instances of (1) dates back to classical milestones, such as [10–13]. Moreover, various textbooks introduce to the analytical study of models fitting into (1), see for instance [14–17,9,18].

A multitude of compartmental models share the key features of the chosen framework (1): they are the domain $\mathcal{X}$ of the x variable and the coexistence of rather general local and non local terms. Indeed, under the choice of $\mathcal{X}$ above, we comprise also bounded space/age domains [6], half lines [19], full vector spaces [7] as well as their combinations [3,5,20,21]. In all these cases, rather general conditions are assigned along the different types of boundaries that fit into (1), such as, for instance, natality terms [3,20,21]. The biological meaning imposes that these boundary terms, as well as the sources in (1), may contain both local and non local terms. The former ones comprehend, for instance, mortality terms [4,5], while the latter can be motivated by natality [3,20], predation [22] or interaction between populations [4], e.g., the propagation of an infection [5].

We underline that the present framework does not rely on any regularizing effect of diffusion. The general non local terms here considered need not have any smoothing effect, and can also be absent. The lack of diffusion operators ensures that any movement or evolution described by (1) propagates with a *finite* speed. In particular, the present approach is consistent with deterministic modeling, while the Laplace operator may also serve to describe various sorts of random effects, see for instance [23,24].

Within this general framework, we first prove well posedness, i.e., local existence, uniqueness and continuous dependence of the solution to (1) on the initial datum. Then, we provide conditions ensuring the global in time existence and the stability with respect to functions and parameters defining (1). Throughout, the functional setting is provided by $\mathbf{L^1}$ and the distance between solutions is always evaluated through the $\mathbf{L^1}$ norm. As a consequence, we can deal with non smooth solutions, a necessary feature in view of control problems. Moreover, the boundedness neither of the total variation nor of the $\mathbf{L^\infty}$ norm of the data is required. Indeed, among the different notions of solutions to IBVPs for renewal equations, we choose to establish our framework on that introduced in [25,26]. This definition not only is stated in terms of integral inequalities, more convenient in any limiting procedure, but remarkably it does not require any notion of trace, allowing us to deal with merely $\mathbf{L^1}$ solutions.

Remark that in (1) both the source terms and the boundary terms are non linear. Thus, a key tool in the proofs is Banach Contraction Theorem, based on precise estimates on scalar equations. Merely requiring some sort of local Lipschitz regularity does not rule out the possibility of finite time blow ups (in any norm), as shown below by explicit examples. We thus resort to a Gronwall type argument to obtain global in time existence. As a byproduct, we also record a uniqueness result in the general setting of (1) based, as in the classical Kružkov case, on a carefully chosen definition of solution, see § 2.1.

We also note that particular instances of equations falling within (1) can be studied through other techniques, such as, for instance, analytic semigroup theory, generalized entropy methods or Laplace transform. We refer, for instance, to [14–16,9].

The present results, besides unifying the treatment of various models, provide tools useful in tackling control/optimization problems based on (1). Indeed, the stability estimates proved in Theorem 2.5 ensure that general integral functionals defined on the solutions are Lipschitz continuous functions of the data and parameters characterizing (1). A further direction that can be pursued using the present results is that of inverse problems, i.e., exhibiting conditions ensuring that an optimal choice of data and parameters in (1) is possible, in order to best fit sets of given experimental data.

This paper is organized as follows. In Section 2 we provide the basic well posedness and stability results. Then, Section 3 is devoted to specific applications that fit into (1). The technical analytic proofs are deferred to the final Section 4.

2. Assumptions, definitions and results

Throughout, we set $\mathbb{R}_+ = [0, +\infty[$,

$$I = \mathbb{R}_+ \quad \text{or} \quad I = [0, T] \quad \text{and} \quad \mathcal{X} = \mathbb{R}_+^m \times \mathbb{R}^n \tag{3}$$

for a positive T.

First, we state what we mean by *solution* to (1). To this aim, we extend to the present case the definitions in [25,26], see in particular [27, Definition 3.5].

Definition 2.1. A map $u_* \in \mathbf{C^0}(I; \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k))$ is a *solution* to (1) if setting for $h = 1, \ldots, k$, $t \in I$, $x \in \mathcal{X}$ and $\xi \in \partial\mathcal{X}$

$$\mathcal{G}^h(t, x) = g^h\left(t, x, u_*(t, x), u_*(t)\right) \quad \text{and} \quad \mathcal{U}_b^h(t, \xi) = u_b^h\left(t, \xi, u_*(t)\right),$$

for $h = 1, \ldots, k$ the map u_*^h is a semi–entropy solution to the IBVP

$$\begin{cases} \partial_t u + \operatorname{div}_x \left(v^h(t,x)\, u \right) = \mathcal{G}^h(t,x) & (t,x) \in I \times \mathcal{X} \\ u(t,\xi) = \mathcal{U}_b^h(t,\xi) & (t,\xi) \in I \times \partial\mathcal{X} \\ u(0,x) = u_o^h(x) & x \in \mathcal{X}. \end{cases}$$

We recall in Definition 2.6 below the notion of semi-entropy solution.

The main result of this paper concerns the well posedness of the Cauchy Problem (2).

Theorem 2.2. *Use the notation* (3) *and let the following assumptions hold:*

(V) $v \in (\mathbf{C^1} \cap \mathbf{L}^\infty)(I \times \mathcal{X}; \mathbb{R}^{k\times(n+m)})$, $\operatorname{div}_x v^h \in \mathbf{L^1_{loc}}(I; \mathbf{L}^\infty(\mathcal{X}; \mathbb{R}))$ *for* $h = 1, \ldots, k$ *and there exists a positive* V *such that*

$$\left(v^h(t,x) \right)_i > V \qquad \forall\, (t,x) \in I \times \partial\mathcal{X} \text{ and for } \begin{array}{l} h = 1, \ldots, k; \\ i = 1, \ldots, m. \end{array}$$

(P) *For all* $w \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$*, the map* $(t,x) \to p(t,x,w)$ *is in* $\mathbf{C^0}(I \times \mathcal{X}; \mathbb{R}^k)$ *and there exist positive* P_1 *and* P_2 *such that for* $t \in I$*,* $x \in \mathcal{X}$*,* $w, w' \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$

$$\begin{aligned} \|p(t,x,w)\| &\le P_1 + P_2 \|w\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}; \\ \left\| p(t,x,w) - p(t,x,w') \right\| &\le P_2 \left\| w - w' \right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}. \end{aligned}$$

(Q) *For all* $w \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$*, the map* $(t,x,u) \to q(t,x,u,w)$ *is in* $\mathbf{C^0}(I \times \mathcal{X} \times \mathbb{R}^k; \mathbb{R}^k)$ *and there exist positive* Q_1 *and* Q_3 *and a function* $Q_2 \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathcal{X}; \mathbb{R}_+)$ *such that for* $t \in I$*,* $x \in \mathcal{X}$*,* $u, u' \in \mathbb{R}^k$*,* $w, w' \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$*:*

$$\begin{aligned} \|q(t,x,u,w)\| &\le Q_1 \|u\| + Q_2(x) \|w\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} + Q_3 \|u\| \|w\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}; \\ \left\| q(t,x,u,w) - q(t,x,u',w') \right\| &\le Q_1 \left\| u - u' \right\| + Q_3 \|w\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \left\| u - u' \right\| \\ &\quad + Q_3 \left\| u' \right\| \left\| w - w' \right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}. \end{aligned}$$

(BD) $u_b \colon \mathbb{R}_+ \times \partial\mathcal{X} \times \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k) \to \mathbb{R}^k$ *is such that for any* $w \in \mathbf{L^1}(\partial\mathcal{X}; \mathbb{R}^k)$*, the map* $(t,\xi) \to u_b(t,\xi,w)$ *is measurable. Moreover, there exists a function* $B \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\partial\mathcal{X}; \mathbb{R}_+)$ *such that for every* $t \in I$*,* $\xi \in \partial\mathcal{X}$*,* $w, w' \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$*,*

$$\begin{aligned} \|u_b(t,\xi,w)\| &\le B(\xi) \left(1 + \|w\|_{\mathbf{L^1}(\mathcal{X},\mathbb{R}^k)} \right) \\ \left\| u_b(t,\xi,w) - u_b(t,\xi,w') \right\| &\le B(\xi) \left\| w - w' \right\|_{\mathbf{L^1}(\mathcal{X},\mathbb{R}^k)}. \end{aligned}$$

(ID) $u_o \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$.

Then,

(WP.1) *There exists a positive* $T_* \in I$ *such that, setting* $I_* = [0, T_*]$*, the IBVP* (2) *admits a solution in the sense of Proposition* 2.1 *defined on* I_**.*

(WP.2) *Assume u_1 and u_2 solve (2) in the sense of Definition 2.1 with $u_1, u_2 \in \mathbf{L}^\infty(I \times \mathcal{X}; \mathbb{R}^k)$. Then, $u_1 = u_2$.*

(WP.3) *Let $\hat u_o, \check u_o \in \mathbf{L}^1(\mathcal{X}; \mathbb{R}^k)$. If $\hat u \colon \hat I \to \mathbb{R}^k$, respectively $\check u \colon \check I \to \mathbb{R}^k$, solve (2) in the sense of Definition 2.1 with initial datum $u_o = \hat u_o$, respectively $u_o = \check u_o$, then there exists a function $\mathcal{L} \in \mathbf{L}^\infty_{\mathbf{loc}}(\hat I \cap \check I; \mathbb{R})$ such that for all $t \in \hat I \cap \check I$*

$$\left\| \hat u(t) - \check u(t) \right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)} \le \mathcal{L}(t) \left\| \hat u_o - \check u_o \right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}.$$

The proof is deferred to Section 4.

In several applications it is of interest to guarantee that each component in the solution attains non negative values. To this aim, we state the following Corollary.

Corollary 2.3. *Let the same assumptions of Theorem 2.2 hold and assume moreover that for an index $h \in \{1, \dots, k\}$*

(Q+) *For $t \in I$, a.e. $x \in \mathcal{X}$, $u \in \mathbb{R}^k_+$, $w \in \mathbf{L}^1(\mathcal{X}; \mathbb{R}^k_+)$, $q^h(t, x, u, w) \ge 0$.*

(BD+) *For $t \in I$, $\xi \in \partial\mathcal{X}$ and $w \in \mathbf{L}^1(\mathcal{X}; \mathbb{R}^k)$, $u_b^h(t, \xi, w) \ge 0$.*

(ID+) *For a.e. $x \in \mathcal{X}$, $u_o^h(x) \ge 0$.*

Then the unique solution u to (2) also satisfies for every $t \in I_*$ and for a.e. $x \in \mathcal{X}$.

$$u^h(t, x) \ge 0. \tag{4}$$

The proof is deferred to Section 4.

The above result is of a local nature and, without further assumptions, it can not be extended to a global result, as the following examples show. Consider the Cauchy Problem (2) with $k = 1$, $m = 0$, $n = 1$, $\mathcal{X} = \mathbb{R}$, $p(t, x, w) = \int_0^1 w(x)\mathrm{d}x$, $q \equiv 0$, which results in

$$\begin{cases} \partial_t u = u \int_0^1 u(t, x)\mathrm{d}x \\ u(0, x) = \chi_{[0,1]}(x) \end{cases} \qquad \text{solved by} \qquad u(t, x) = \frac{1}{1-t}\, \chi_{[0,1]}(x).$$

Note that **(P)** holds with $P_1 = 0$ and $P_2 = 1$. Clearly, u blows up in any norm at $t = 1$.

Similarly, setting $k = 1$, $m = 1$, $n = 0$, $\mathcal{X} = \mathbb{R}_+$, $p(t, x, w) = \int_{\mathbb{R}_+} w(x)\mathrm{d}x$, $q \equiv 0$ in (2), which satisfies **(P)** with $P_1 = 0$ and $P_2 = 1$, leads to the Cauchy Problem

$$\begin{cases} \partial_t u + \partial_x u = u \int_{\mathbb{R}_+} u(t, x)\mathrm{d}x \\ u(t, 0) = 0 \\ u(0, x) = \chi_{[0,1]}(x), \end{cases} \qquad \text{solved by} \qquad u(t, x) = \frac{1}{1-t}\, \chi_{[t,t+1]}(x).$$

Again, the solution blows up in any norm at $t = 1$.

Typical biological/epidemiological models have further properties ensuring that solutions are defined globally in time. In particular, the model described in § 3.3 displays a quadratic right hand side similar to those in the examples above, differing in the sign. Nevertheless, in this example, well posedness holds globally in time. Indeed, in general, a lower bound on the solutions is available since Corollary 2.3 ensures that the components of the solution attain non negative values. An upper bound, preventing finite time blow up, is obtained through assumption **(BD)** on the boundary datum and a further condition, see (5) below, that bounds the overall growth.

Corollary 2.4. *Let $I = \mathbb{R}_+$. Let the assumptions of Corollary 2.3 hold for all $h = 1, \ldots, k$. Assume moreover that for suitable $C_1 \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+; \mathbf{L}^1(\mathcal{X}; \mathbb{R}))$ and $C_2 \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+; \mathbb{R})$,*

$$\sum_{h=1}^{k} p^h(t,x,w)\, u^h + q^h(t,x,u,w) \le C_1(t,x) + C_2(t) \sum_{h=1}^{k} u^h \tag{5}$$

for all $t \in \mathbb{R}_+$, a.e. $x \in \mathcal{X}$, $u, w \in \mathbb{R}^k$. Then, the solution to (2) is defined for all $t \in \mathbb{R}_+$.

Finally, we provide the stability estimates essential to tackle, for instance, control problems. To this aim, we need to slightly specialize the functional dependence of p, q and u_b on $u(t)$. We thus obtain sufficient conditions to apply Theorem 2.2 and get stability estimates.

Theorem 2.5. *Let assumptions* **(V)** *and* **(ID)** *hold. Assume that in (2), for $t \in I$, $x \in \mathcal{X}$, $u \in \mathbb{R}^k$, $w \in \mathbf{L}^1(\mathcal{X}; \mathbb{R}^k)$,*

$$\begin{aligned} p^h(t,x,w) &= P^h\left(t, x, \int_{\mathcal{X}} \mathcal{K}^h_p(t,x,x')\, w(x')\mathrm{d}x'\right) \\ q^h(t,x,u,w) &= Q^h\left(t, x, u, \int_{\mathcal{X}} \mathcal{K}^h_q(t,x,x')\, w(x')\mathrm{d}x'\right) \\ u^h_b(t,\xi,w) &= U^h_b\left(t, \xi, \int_{\mathcal{X}} \mathcal{K}^h_u(t,\xi,x')\, w(x')\mathrm{d}x'\right), \end{aligned} \tag{6}$$

where the functions above satisfy:

($\bar{\mathbf{P}}$) *There exist $\bar P_1 \ge 0$ and $\bar P_2 \ge 0$ such that, for every $h = 1, \ldots, k$, the function $P^h : I \times \mathcal{X} \times \mathbb{R}^{k_p} \to \mathbb{R}$ ($k_p \ge 1$) satisfies*

$$\left|P^h(t,x,\eta)\right| \le \bar P_1 + \bar P_2 \|\eta\| \quad \text{and} \quad \left|P^h(t,x,\eta_1) - P^h(t,x,\eta_2)\right| \le \bar P_2 \|\eta_1 - \eta_2\|$$

for every $t \in I$, $x \in \mathcal{X}$, $\eta, \eta_1, \eta_2 \in \mathbb{R}^{k_p}$; $\mathcal{K}^h_p \in \mathbf{L}^\infty(I \times \mathcal{X}^2; \mathbb{R}^{k_p k})$.

($\bar{\mathbf{Q}}$) *There exist $\bar Q_1, \bar Q_3 \ge 0$ and $\bar Q_2 \in \left(\mathbf{L}^1 \cap \mathbf{L}^\infty\right)\left(\mathcal{X}; \mathbb{R}^+\right)$ such that, for every $h = 1, \ldots, k$, the function $Q^h : I \times \mathcal{X} \times \mathbb{R}^k \times \mathbb{R}^{k_p} \to \mathbb{R}^+$ ($k_q \ge 1$) satisfies*

$$\begin{aligned} \left|Q^h(t,x,u,\eta)\right| &\le \bar Q_1 \|u\| + \bar Q_2(x)\|\eta\| + \bar Q_3 \|u\|\|\eta\| \\ \left|Q^h(t,x,u_1,\eta_1) - Q^h(t,x,u_2,\eta_2)\right| &\le \bar Q_1 \|u_1 - u_2\| + \bar Q_3 \|\eta_1\|\|u_1 - u_2\| \\ &\quad + \bar Q_3 \|u_2\|\|\eta_1 - \eta_2\| \end{aligned}$$

for every $t \in I$, $x \in \mathcal{X}$, $u, u_1, u_2 \in \mathbb{R}^k$, $\eta, \eta_1, \eta_2 \in \mathbb{R}^{k_q}$; $\mathcal{K}^h_q \in \mathbf{L}^\infty(I \times \mathcal{X}^2; \mathbb{R}^{k_q k})$.

($\overline{\mathbf{BD}}$) *There exists $\bar{B} \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\partial\mathcal{X}; \mathbb{R}_+)$ such that for every $h = 1, \ldots, k$, the function $U_b^h : I \times \partial\mathcal{X} \times \mathbb{R}^{k_u} \to \mathbb{R}_+$ satisfies*

$$\left|U_b^h(t,\xi,\eta)\right| \le \bar{B}(\xi)\,(1+\|\eta\|) \quad \text{and} \quad \left|U_b^h(t,\xi,\eta_1) - U_b^h(t,\xi,\eta_2)\right| \le \bar{B}(\xi)\,\|\eta_1-\eta_2\|$$

for every $t \in I$, $\xi \in \partial\mathcal{X}$ and $\eta, \eta_1, \eta_2 \in \mathbb{R}^{k_u}$; $\mathcal{K}_u^h \in \mathbf{L}^\infty(I \times \partial\mathcal{X} \times \mathcal{X}; \mathbb{R}^{k_u k})$.

Then, Theorem 2.2 applies. Moreover, if both systems

$$\begin{cases} \partial_t u^h + \operatorname{div}_x\left(v^h(t,x)\,u^h\right) = \hat{p}^h\,(t,x,u(t))\,u^h + \hat{q}^h\,(t,x,u,u(t)) & (t,x) \in I \times \mathcal{X} \\ u^h(t,\xi) = \hat{u}_b^h\,(t,\xi,u(t)) & (t,\xi) \in I \times \partial\mathcal{X} \\ u^h(0,x) = \hat{u}_o^h(x) & x \in \mathcal{X}, \end{cases} \tag{7}$$

$$\begin{cases} \partial_t u^h + \operatorname{div}_x\left(v^h(t,x)\,u^h\right) = \check{p}^h\,(t,x,u(t))\,u^h + \check{q}^h\,(t,x,u,u(t)) & (t,x) \in I \times \mathcal{X} \\ u^h(t,\xi) = \check{u}_b^h\,(t,\xi,u(t)) & (t,\xi) \in I \times \partial\mathcal{X} \\ u^h(0,x) = \check{u}_o^h(x) & x \in \mathcal{X}, \end{cases} \tag{8}$$

satisfy the assumptions above, then the following stability estimates hold:

$$\begin{aligned} &\left\|\hat{u}(t) - \check{u}(t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \\ &\le \mathcal{O}(1)\Big[\left\|\hat{P} - \check{P}\right\|_{\mathbf{L}^\infty\left([0,t]\times\mathcal{X}\times\mathbb{R}^{k_p};\mathbb{R}^k\right)} + \left\|\hat{\mathcal{K}}_p - \check{\mathcal{K}}_p\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{k_p k^2})} \\ &\qquad + \left\|\hat{Q} - \check{Q}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbf{L}^\infty(\mathbb{R}^k\times\mathbb{R}^{k_q};\mathbb{R}^k))} + \left\|\hat{\mathcal{K}}_q - \check{\mathcal{K}}_q\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{k_q k^2})} \\ &\qquad + \left\|\hat{U}_b - \check{U}_b\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbf{L}^\infty(\mathbb{R}^{k_u};\mathbb{R}^k))} + \left\|\hat{\mathcal{K}}_u - \check{\mathcal{K}}_u\right\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X}\times\mathcal{X};\mathbb{R}^{k_u k^2})}\Big] e^{\mathcal{O}(1)t} \end{aligned}$$

for every t such that $\hat{u}$ and $\check{u}$ are defined on $[0,t]$ and where the Landau symbol $\mathcal{O}(1)$ denotes a constant independent of the initial data.

The proof is deferred to Section 4.

Finally, we note that **(V)** and Definition 2.1 allow to immediately extend all results in the present section to the case $\mathcal{X} = \left(\prod_{i=1}^m I_i\right) \times \mathbb{R}^n$, as soon as $I_1, \ldots, I_m$ are (non trivial) real intervals bounded below. In particular, any of the I_i may well be bounded also above.

2.1. The definition of semi–entropy solution ensures uniqueness

This paragraph provides a definition of solution and the consequent uniqueness statement in a setting more general than the one usually found in the literature. In particular, it extends the results in [25, Section 3] to the slightly more general case of the unbounded domain $\mathcal{X}$. Indeed, with the notation (3), consider the fully nonlinear IBVP

$$\begin{cases} \partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u) & (t,x) \in I \times \mathcal{X} \\ u(t,\xi) = u_b(t,\xi) & (t,\xi) \in I \times \partial\mathcal{X} \\ u(0,x) = u_o(x) & x \in \mathcal{X}. \end{cases} \tag{9}$$

The following definition is the extension to (9) of [27, Definition 3.5], see also [25,26].

Definition 2.6. A *semi-entropy solution* to the IBVP (9) on the real interval I is a map $u \in \mathbf{L}^{\infty}_{\mathbf{loc}}(I; \mathbf{L}^{\mathbf{1}}(\mathcal{X}; \mathbb{R}))$ such that for any $\kappa \in \mathbb{R}$ and for any test function $\varphi \in \mathbf{C}^{\mathbf{1}}_{\mathbf{c}}(]-\infty, \sup I[\times \mathbb{R}^{n+m}; \mathbb{R}_+)$

$$\begin{aligned}
&\int_I \int_{\mathcal{X}} (u(t,x)-\kappa)^{\pm}\, \partial_t \varphi(t,x) \mathrm{d}x \mathrm{d}t \\
&+ \int_I \int_{\mathcal{X}} \mathrm{sgn}^{\pm}(u(t,x)-\kappa)\, (f(t,x,u)-f(t,x,\kappa)) \cdot \mathrm{grad}_x \varphi(t,x) \mathrm{d}x \mathrm{d}t \\
&+ \int_I \int_{\mathcal{X}} \mathrm{sgn}^{\pm}(u(t,x)-\kappa)\, [g(t,x,u(t,x)) - \mathrm{div}_x f(t,x,\kappa)]\, \varphi(t,x) \mathrm{d}x \mathrm{d}t \\
&+ \int_{\mathcal{X}} (u_o(x)-\kappa)^{\pm}\, \varphi(0,x) \mathrm{d}x \\
&+ \mathbf{Lip}(f) \int_I \int_{\partial\mathcal{X}} (u_b(t,\xi)-\kappa)^{\pm}\, \varphi(t,\xi) \mathrm{d}\xi \mathrm{d}t \geq 0
\end{aligned} \tag{10}$$

where $\mathbf{Lip}(f)$ is a Lipschitz constant of the map $u \to f(t,x,u)$, uniform in $(t,x) \in I \times \mathcal{X}$.

Above, we use the notation $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$.

A key feature of (10) is its ensuring uniqueness, which we detail in the next Proposition to ease comparisons with the current literature.

Proposition 2.7. *Consider the general scalar IBVP* (9) *under the assumptions*

(f) $f \in \mathbf{C}^{\mathbf{0}}(I \times \mathcal{X} \times \mathbb{R}; \mathbb{R}^{n+m})$ *admits continuous derivatives* $\partial_u f$, $\partial_u \mathrm{grad}_x f$, $D^2_{xx} f$ *with* $\partial_u f$ *and* $\mathrm{grad}_x f$ *bounded in* $(t,x) \in I \times \mathbb{R}_+$ *locally in* $u \in \mathbb{R}$*;* $\partial_u \mathrm{grad}_x f$ *is bounded.*

(g) $g, \partial_u g, \partial_{x_i} g \in \mathbf{C}^{\mathbf{0}}(I \times \mathcal{X} \times \mathbb{R}; \mathbb{R})$ *and for all* $(t,x) \in I \times \mathcal{X}$, $|g(t,x,u)| \leq G(u)$ *for a map* $G \in \mathbf{L}^{\infty}_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}_+)$ *and* $\partial_u g$ *is bounded.*

(bd) *The boundary datum satisfies* $u_b \in \mathbf{L}^{\infty}(I \times \partial\mathcal{X}; \mathbb{R})$.

(id) *The initial datum satisfies* $u_o \in \mathbf{L}^{\infty}(\mathcal{X}; \mathbb{R})$.

If $u_1, u_2 \in \mathbf{L}^{\infty}(I \times \mathcal{X}; \mathbb{R})$ *both satisfy* (10)*, then they coincide.*

This Proposition slightly extends [25, Theorem 18]. However, its proof relies on merely technical modifications to [25, Lemma 16 and Lemma 17], due to the present unboundedness of the domain $\mathcal{X}$. Very similar techniques are employed also in [28, § 2.6 and § 2.7], which is devoted to a hyperplane.

3. Sample applications

The structure of (1) is sufficiently flexible to comprise a variety of applications of mathematics to biology, in particular to epidemiology. The general results in the preceding section can be applied to well known models in the literature, see for instance [1,4,29,9]. In the next paragraphs, we select sample applications based on analytic structure that differ in the number of equations, in the number of independent variables, in the presence of (partial) boundaries and in the role of non local terms. In particular, § 3.1 deals with a recently proposed model, see [5], while the subsequent ones refer to other classical models that fit into (1).

3.1. The spreading of an epidemic

During the spreading of an epidemic, within a population we distinguish among individuals that are Susceptible, Infective, Hospitalized or Recovered, see [5]. Each of these populations is described through its time, age and space dependent density: $S = S(t, a, y)$, $I = I(t, a, y)$, $H = H(t, a, y)$ and $R = R(t, a, y)$, respectively. Remark that the distinction between I and H consists in the H individuals that, being hospitalized or quarantined, do not infect anyone although being ill. In its most general form, the model presented in [5, § 2] to describe the evolution of these populations, reads

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_y(v_S\, S) + \mu_S\, S = -(\rho \otimes I) S \\ \partial_t I + \partial_a I + \operatorname{div}_y(v_I\, I) + \mu_I\, I = (\rho \otimes I) S - \kappa\, I - \vartheta\, I \\ \partial_t H + \partial_a H + \mu_H\, H = +\kappa\, I - \eta\, H \\ \partial_t R + \partial_a R + \operatorname{div}_y(v_R\, R) + \mu_R\, R = +\vartheta\, I + \eta\, H \end{cases} \qquad \begin{array}{l} t \in \mathbb{R}_+ \\ a \in \mathbb{R}_+ \\ y \in \mathbb{R}^2 \end{array} \tag{11}$$

where the propagation of the infection is described by

$$(\rho \otimes I(t))\,(a, y) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \rho(a, a', y, y')\, I(t, a', y')\, \mathrm{d}y'\, \mathrm{d}a' . \tag{12}$$

Here, the function ρ plays the key role of describing how infective individuals infect others, at which distance and with which dependence on age or time, see [5] for more details. In (11), $v_S = v_S(t, a, y)$, $v_I = v_I(t, a, y)$ and $v_R = v_R(t, a, y)$ describe the time, age and, possibly, space dependent movements of the S, I and R individuals, while $\mu_S = \mu_S(t, a, y)$, $\mu_I = \mu_I(t, a, y)$, $\mu_H = \mu_H(t, a, y)$ and $\mu_R = \mu_R(t, a, y)$ are the mortalities. The term $\kappa = \kappa(t, a, y)$ describes how quickly infected individuals are confined to quarantine; $\vartheta = \vartheta(t, a, y)$, respectively $\eta = \eta(t, a, y)$, quantifies the speed at which infected, respectively quarantined, individuals recover.

System (11) needs to be supplemented by boundary and initial data:

$$\begin{cases} S(t, a = 0, y) = S_b(t, y) \\ I(t, a = 0, y) = 0 \\ H(t, a = 0, y) = 0 \\ R(t, a = 0, y) = 0 \end{cases} \quad \text{and} \quad \begin{cases} S(t = 0, a, y) = S_o(a, y) \\ I(t = 0, a, y) = I_o(a, y) \\ H(t = 0, a, y) = H_o(a, y) \\ R(t = 0, a, y) = R_o(a, y) . \end{cases} \tag{13}$$

Note that a more precise boundary term, though not amenable to be used in the short term, might be a natality term of the form

$$S(t, a=0, y) = \int_{\mathbb{R}_+} b(t, a', y)\, S(t, a', y) \mathrm{d}a'$$

which also fits in the framework of Theorem 2.2 and Theorem 2.5. Note that (11)–(12)–(13) is a system with independent variables (a, y) where a is bounded below while y is in $\mathbb{R}^2$ and no second order differential operator is present. The model (11)–(12)–(13) fits into (2) in the form (6) setting $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}^2$, $x = (a, y)$, $\xi = (0, y)$ and

$$\begin{array}{cccc}
k=4 & m=1 & n=2 & \\
u^1 = S & u^2 = I & u^3 = H & u^4 = R \\
w^1 = S(t) & w^2 = I(t) & w^3 = H(t) & w^4 = R(t) \\
v^1 = \begin{bmatrix} 1 \\ v_S \end{bmatrix} & v^2 = \begin{bmatrix} 1 \\ v_I \end{bmatrix} & v^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} & v^4 = \begin{bmatrix} 1 \\ v_R \end{bmatrix} \\
u_b^1 = S_b & u_b^2 = 0 & u_b^3 = 0 & u_b^4 = 0 \\
u_o^1 = S_o & u_o^2 = I_o & u_o^3 = H_o & u_o^4 = R_o
\end{array}$$

$$\begin{array}{rclrcl}
p^1(t,x,\Lambda) &=& -\mu_S - \Lambda & q^1(t,x,u,\Lambda) &=& 0 \\
p^2(t,x,\Lambda) &=& -\mu_I - \kappa - \vartheta & q^2(t,x,u,\Lambda) &=& \Lambda\, u_1 \\
p^3(t,x,\Lambda) &=& -\mu_H - \eta & q^3(t,x,u,\Lambda) &=& \kappa\, u_2 \\
p^4(t,x,\Lambda) &=& -\mu_R & q^4(t,x,u,\Lambda) &=& \vartheta\, u_2 + \eta\, u_3
\end{array}$$

and the only 2 non zero entries in $\mathcal{K}_p$ and $\mathcal{K}_q$ are valued ρ, so that

$$\int_{\mathcal{X}} \mathcal{K}_p^1 \left(t, (a, y), (a', y')\right)\, w(a', y') \mathrm{d}a' \mathrm{d}y' = \left(\rho \otimes I(t)\right)(a, y),$$

$$\int_{\mathcal{X}} \mathcal{K}_q^2 \left(t, (a, y), (a', y')\right)\, w(a', y') \mathrm{d}a' \mathrm{d}y' = \left(\rho \otimes I(t)\right)(a, y).$$

Proposition 3.1. *Set $\mathcal{I} = [0, T]$ or $\mathcal{I} = \mathbb{R}_+$. Let $v_S, v_I, v_R \in (\mathbf{C^1} \cap \mathbf{L}^\infty)(\mathcal{I} \times \mathcal{X}; \mathbb{R}^2)$ with divergence in $\mathbf{L^1}(\mathcal{I}; \mathbf{L}^\infty(\mathcal{X}; \mathbb{R}))$; $\rho \in \mathbf{L}^\infty(\mathbb{R}_+^2 \times \mathbb{R}^4; \mathbb{R})$ and $S_b \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathcal{I} \times \mathbb{R}^2; \mathbb{R})$. Let μ_S, μ_I, μ_H, μ_R, ϑ, η and κ be positive and in $\mathbf{L}^\infty$. Fix an initial datum (S_o, I_o, H_o, R_o) in $\mathbf{L^1}(\mathcal{X}; \mathbb{R}^4)$. Then:*

1. *Problem (11)–(12)–(13) fits into Theorem 2.2 and Theorem 2.5 and hence it admits a solution $(S, I, H, R) \in \mathbf{C^0}\left([0, T_*]; \mathbf{L^1}(\mathcal{X}; \mathbb{R}^4)\right)$, for a $T_* > 0$.*
2. *If the initial and boundary data (S_o, I_o, H_o, R_o) and S_b are non negative, if $\rho \geq 0$ and if the constants κ, η, θ are non negative, then Corollary 2.3 applies, ensuring that the solution is non negative: $(S, I, H, R)(t) \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}_+^4)$, for all $t \in [0, T_*]$.*
3. *If, in addition to what required at 2., the mortalities $\mu_S, \mu_I, \mu_H, \mu_R$ are non negative, then Proposition 2.4 applies, so that the solution is defined globally in time.*
4. *If, in addition to what required at 3., (S_o, I_o, H_o, R_o) in $\mathbf{L}^\infty(\mathcal{X}; \mathbb{R}_+^4)$, then the solution is locally bounded: $(S, I, H, R) \in \mathbf{L}^\infty(\mathcal{J} \times \mathcal{X}; \mathbb{R}_+^4)$, for any bounded interval $\mathcal{J} \subseteq \mathcal{I}$. Hence, (S, I, H, R) is the unique solution to (11) in the sense of Definition 2.1.*

The proof is deferred to Section 4.

As pointed out in (11), a natural control parameter is the coefficient $\kappa = \kappa(t, a, y)$, which determines how quickly infective individuals are isolated in quarantine.

A first natural choice for a *cost* to be minimized by a careful choice of κ is the total number of deaths on the time interval $[0, T]$, namely

$$\mathcal{D}(\kappa) = \int_0^T \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \left(\mu_I(t, a, y)\, I(t, a, y) + \mu_H(t, a, y)\, H(t, a, y)\right) \mathrm{d}y\, \mathrm{d}a\, \mathrm{d}t\,.$$

Proposition 3.1 ensures that the cost $\mathcal{D}$ is a continuous function of κ. Hence, standard compactness arguments, for instance in the case of a constant κ, ensure the existence of an optimal control. Moreover, the Lipschitz continuity, again ensured by Proposition 3.1, allows to use standard optimization algorithms to actually find near–to–optimal controls.

A second reasonable choice is to minimize the maximal number of infected individuals $\|I\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}_+\times\mathbb{R}^2;\mathbb{R})}$, aiming at minimizing the maximal stress on the health care system. Again, the continuity proved in Proposition 3.1 allows to use Weierstrass type arguments to exhibit the existence of optimal controls, thanks to the lower semicontinuity of the $\mathbf{L}^\infty$ norm with respect to the $\mathbf{L}^1$ distance.

3.2. Cell growth and division

Consider the classical model [3, Formula (2)] devoted to the description of cell growth and cell division, as extended in [21, Formulæ (1.5)–(1.7)]:

$$\begin{cases} \partial_t N + \partial_a N + \operatorname{div}_y (V(a, y)\, N) = -\lambda(a, y)\, N \\ N(t, 0, y) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \beta\left((a', y'), y, N(t, a', y')\right) \mathrm{d}y' \mathrm{d}a' \end{cases} \tag{14}$$

where $t \in \mathbb{R}_+$ is time, $a \in \mathbb{R}_+$ is age, $(y_1, \ldots, y_n) \in \mathbb{R}^n$ is an n–tuple of structure variables, $\lambda = \lambda(a, y)$ is the age– and state–specific loss rate, $N = N(t, a, y)$ is the population density and $V = V(a, y)$ is the (time independent) individual cell's growth rate. Therefore, (14) fits into (2) setting

$$k = 1\,, \quad n \in \mathbb{N}\,, \quad m = 1\,, \quad \mathcal{X} = \mathbb{R}_+ \times \mathbb{R}^n\,, \quad x = (a, y)\,, \quad \xi = (0, y)\,, \quad u = N\,, \quad w = N(t)\,,$$

$$v\,(t, (a, y)) = V(a, y)\,, \quad p\,(t, (a, y), N(t)) = -\lambda(a, y)\,, \quad q\,(t, (a, y), N, N(t)) = 0\,,$$

$$u_b(t, y, N, N(t)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \beta\left((a', y'), y, N(t, a', y')\right) \mathrm{d}a' \mathrm{d}y'\,.$$

Concerning the assumptions of Theorem 2.2, we have that **(V)** is satisfied as soon as $V \in (\mathbf{C^1} \cap \mathbf{L}^\infty)(\mathcal{X}; \mathbb{R}^n)$ and $\operatorname{div} V \in \mathbf{L^1}(I; \mathbf{L}^\infty(\mathcal{X}; \mathbb{R}))$. Condition **(P)** is met whenever $\lambda \in \mathbf{C^0} \cap \mathbf{L}^\infty$, with $P_1 = \|\lambda\|_{\mathbf{L}^\infty(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R})}$ and $P_2 = 0$. Assumption **(Q)** trivially holds. To comply with **(BD)**, we need β to be Lipschitz continuous and sublinear in its fourth argument, i.e., $\beta((a', y'), y, w) \leq B(y)\,(1 + |w|)$ for a suitable $B \in \mathbf{L^1} \cap \mathbf{L}^\infty$. Under these assumptions, Theorem 2.2 applies to (14).

As soon as $\beta \geq 0$ and the initial datum is non negative, also Corollary 2.3 applies, ensuring the solution is non negative. It is reasonable to assume from the biological point of view that $\lambda \geq 0$, so that also Corollary 2.4 applies (with $C_1 = 0$, $C_2 = 0$), ensuring that the solution is globally defined in time. It is straightforward to see that, as soon as β is linear in its third argument, it is possible to apply also Theorem 2.5.

3.3. An age and phenotypically structured population model

Within the general form (1) we recover also the recent model [20, Formula (1)], namely

$$\begin{cases} \varepsilon\, \partial_t M_\varepsilon + \partial_a \left(A(a,y)\, M_\varepsilon\right) = -\left(\displaystyle\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} M_\varepsilon(t,a',y')\mathrm{d}a'\mathrm{d}y' + d(a,y)\right) M_\varepsilon \\ M_\varepsilon(t,a=0,y) = \dfrac{1}{A(a=0,y)\,\varepsilon^n} \displaystyle\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \mathcal{M}\left(\frac{y'-y}{\varepsilon}\right) b(a',y')\, M_\varepsilon(t,a',y')\mathrm{d}a'\mathrm{d}y' \\ M_\varepsilon(t=0,a,y) = M_\varepsilon^0(a,y)\,. \end{cases} \tag{15}$$

Here, the dependent variable $M_\varepsilon = M_\varepsilon(t,a,y)$ describes the population density at time t, of age $a \in \mathbb{R}_+$ and trait $x \in \mathbb{R}^n$, so that $\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} M_\varepsilon(t,a,y)\mathrm{d}a\mathrm{d}x$ is the total population. The growth function $A = A(a,y)$ describes the age and trait dependent aging. The mortality, on the right hand side of the first equation in (15), both depends on the crowding, due to intraspecies competition, and on a given mortality $d = d(a,y)$. The function $b = b(a,y)$ quantifies the natality and is modulated by the mutation probability kernel $\mathcal{M}$, both defining the boundary term along $a = 0$, see also [30].

Note that the IBVP (15) can be seen as a prototype equation for various other similar models, see for instance [8, Formula (2.8)].

The above system (15) fits into (2) setting $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}^n$ and

$$\begin{aligned} & k = 1\,, \quad m = 1\,, \quad n \geq 1\,, \quad x = (a,y)\,, \quad \xi = (0,y)\,, \quad u = M_\varepsilon\,, \quad w = M_\varepsilon(t)\,, \\ & v = \begin{bmatrix} A(a,y)/\varepsilon \\ 0 \end{bmatrix}, \qquad p(t,x,w) = -\frac{1}{\varepsilon}\int_{\mathbb{R}^n} w(x)\mathrm{d}x - \frac{d(x)}{\varepsilon}\,, \qquad q(t,x,u,w) = 0\,, \\ & \qquad u_b(t,y,w) = \frac{1}{A(a=0,y)\,\varepsilon^n}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \mathcal{M}\left(\frac{y'-y}{\varepsilon}\right) b(a',y')\, w(a',y')\mathrm{d}a'\mathrm{d}y'\,. \end{aligned} \tag{16}$$

Proposition 3.2. *Let $A \in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(\mathcal{X};\mathbb{R})$ with $\inf A > 0$ and $\operatorname{div}_{a,y} A \in \mathbf{L}^\infty(\mathcal{X};\mathbb{R})$. Let $d \in \mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})$, $\mathcal{M} \in \mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})$ such that $\mathcal{M}(\eta) = 0$ whenever $\|\eta\| \geq r$, for a fixed $r > 0$. Moreover, $b \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}^n;\mathbb{R})$ such that $|b(a,y)| \leq (1 + \|y\|)^{-(n+1)}$. Then, for any initial datum $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathcal{X};\mathbb{R})$, Theorem 2.2 applies to the Cauchy Problem for (15) with datum u_o. If moreover $u_o \geq 0$, $A(0,y) \geq 0$, $\mathcal{M} \geq 0$ and $b \geq 0$, Corollary 2.3 and Corollary 2.4 apply, ensuring that the solution is non negative and defined on all $\mathbb{R}_+$.*

The proof is deferred to Section 4. Thus, the above result ensures existence on $[0,+\infty[$ as soon as all the assumptions are available therein, recovering the well posedness results in [30,20].

3.4. Further applications

We briefly recall here further models considered in the literature that fit within (1). In each of the cases below, we refer to the original sources for detailed descriptions of the modeling environments.

The model presented in [7, Formula (5)], devoted to the modeling of leukemia development, reads (here, $i = 2, \ldots, M-1$ for a fixed $M \in \mathbb{N}$, $M \geq 3$):

$$\begin{cases} \partial_t n_1 = \left(\dfrac{2\, a_1(x)}{1 + K \int_0^1 n_M(t,x') \mathrm{d}x'} - 1 \right) p_1(x)\, n_1 \\ \partial_t n_i = 2 \left(1 - \dfrac{a_{i-1}(x)}{1 + K \int_0^1 n_M(t,x') \mathrm{d}x'} \right) p_{i-1}(x)\, n_{i-1} + \left(\dfrac{2 a_i(x)}{1 + K \int_0^1 n_M(t,x') \mathrm{d}x'} - 1 \right) p_i(x)\, n_i \\ \partial_t n_M = 2 \left(1 - \dfrac{a_{M-1}(x)}{1 + K \int_0^1 n_M(t,x') \mathrm{d}x'} \right) p_{M-1}(x)\, n_{M-1} - d\, n_M \\ n_i(0,x) = n_i^o(x)\,. \end{cases} \tag{17}$$

Remark that (17) can be seen as a system of ordinary differential equations on functions defined on [0, 1] or, alternatively, as a system of ordinary differential equations coupled also through a non local dependence on the x variable. Nevertheless, it fits within (1): indeed, set $k = M$, $m = 0$, $n = 1$, $\mathcal{X} = \mathbb{R}$, $u = (n_1, \ldots, n_M)$, $v \equiv 0$, the other terms being obviously chosen.

It is worth noting that the recent model [2, Formula (13)], though devoted to an entirely different scenario, is analytically analogous to (17) and also fits within the framework formalized in Section 2. The use of Theorem 2.2 and Theorem 2.5 thus extends the results in [2,7] comprehending $\mathbf{L^1}$ solutions and providing a full set of stability estimates.

Another example is the model recently presented in [6, Formula (1.1)], devoted to an age–structured population described by the time, age and space dependent density $u = u(t, a, y)$:

$$\begin{cases} \partial_t u + \partial_a u = d(J * u(t) - u) + G\,(u(t)) \\ u(t, 0, y) = F\,(u(t)) \\ u(0, a, y) = \Phi(a, y) \end{cases} \tag{18}$$

considered in [6] for $a \in [0, a^+]$ and $y \in \Omega$, where $a^+ \in \,]0, +\infty[$ and $\Omega \subseteq \mathbb{R}^N$ are given. Above, J is a convolution kernel, while the functionals F and G are locally Lipschitz continuous with respect to the $\mathbf{L^1}$ norm. Model (18) fits into (1) setting $k = 1$, $m = 1$, $n = N$, $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}^N$, $x = (a, y)$, $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the choice of the other terms being immediate. The results in Section 2 immediately apply even if the age interval $[0, a^+]$ and the space domain are bounded, thanks to the generality of the assumptions required on v. This allows to have qualitative information on the dependence of the solutions exhibited in [6] on the various parameters and functions defining (18).

We recall also the following competitive population model with age structure as an example of a system of equations. It was introduced and studied from the optimal management point of view in [19, Formula (1.1)]:

$$\begin{cases} \partial_t u^1 + \partial_a u^1 = -\mu_1(a,u^1)\,u^1 - f^1(t,a)\,u^1 - u^1 \displaystyle\int_0^A c_1(a',a)\,u^2(t,a')\mathrm{d}a' \\ \partial_t u^2 + \partial_a u^2 = -\mu_2(a,u^2)\,u^2 - f^2(t,a)\,u^2 - u^2 \displaystyle\int_0^A c_2(a',a)\,u^1(t,a')\mathrm{d}a' \\ u^1(t,0) = \displaystyle\int_0^A \beta_1(a')\,u^1(t,a')\mathrm{d}a' \\ u^2(t,0) = \displaystyle\int_0^A \beta_2(a')\,u^2(t,a')\mathrm{d}a' \\ u^1(0,a) = u_o^1(a) \\ u^2(0,a) = u_o^2(a)\,. \end{cases} \tag{19}$$

Here, we have $k = 2$, $m = 1$, $n = 0$, $\mathcal{X} = \mathbb{R}_+$, $v = 1$. Under the assumptions of Theorem 2.2 and Theorem 2.5 we recover the continuity of the profit functional [19, Formula (1.2)]

$$J(f) = \int_0^T \int_0^A \Big(K_1(a)\, f^1(t,a)\,u^1(t,a) + K_2(a)\, f^2(t,a)\,u^2(t,a)\Big)\,\mathrm{d}a\mathrm{d}t\,,$$

now also in the setting of $\mathbf{L^1}$ solutions.

4. Analytic proofs

4.1. The scalar case

We now consider in detail the affine scalar case, namely (9) with $f(t,x,u) = v(t,x)\,u$ and $g(t,x,u) = p(t,x)\,u + q(t,x)$, i.e.,

$$\begin{cases} \partial_t u + \mathrm{div}_x\,(v(t,x)u) = p(t,x)\,u + q(t,x) & (t,x) \in \mathbb{R}_+ \times \mathcal{X} \\ u(t,\xi) = u_b(t,\xi) & (t,\xi) \in \mathbb{R}_+ \times \partial\mathcal{X} \\ u(0,x) = u_o(x) & x \in \mathcal{X}\,. \end{cases} \tag{20}$$

Recall the following standard notation. A *characteristic* of (20) is the solution $t \to X(t;t_o,x_o)$ to the following Cauchy Problem for the system of ordinary differential equations

$$\begin{cases} \dot{x} = v(t,x) & (t,x) \in I \times \mathcal{X} \\ x(t_o) = x_o\,. & (t_o,x_o) \in I \times \mathcal{X}\,. \end{cases} \tag{21}$$

For $\tau, t \in I$ and for $x \in \mathcal{X}$, define

$$\mathcal{E}(\tau,t,x) = \exp\left(\int_\tau^t \left(p\,(s, X(s;t,x)) - \mathrm{div}_x v\,(s, X(s;t,x))\right)\mathrm{d}s\right) \tag{22}$$

and for all $(t,x) \in I \times \mathcal{X}$, if $x \in X(t;[0,t[,\partial\mathcal{X})$, we set

$$T(t,x) = \inf\{s \in [0,t[\colon X(s;t,x) \in \mathcal{X}\}\ . \tag{23}$$

With the notation introduced above, we recall the well known formula

$$u(t,x) = \begin{cases} u_o\left(X(0;t,x)\right)\mathcal{E}(0,t,x) & \\ \qquad + \displaystyle\int_0^t q\left(\tau, X(\tau;t,x)\right)\mathcal{E}(\tau,t,x)\mathrm{d}\tau & x \in X(t;0,\mathcal{X}) \\ u_b\left(T(t,x), X\left(T(t,x);t,x\right)\right)\mathcal{E}\left(T(t,x),t,x\right) & \\ \qquad + \displaystyle\int_{T(t,x)}^t q\left(\tau, X(\tau;t,x)\right)\mathcal{E}(\tau,t,x)\mathrm{d}\tau & x \in X(t;[0,t[,\partial\mathcal{X}) \end{cases} \tag{24}$$

obtained from the integration along characteristics, a standard tool at least since the classical paper [10]. The following relations are of use below, for a proof see for instance [31, Chapter 3],

$$\partial_t X(t;t_o,x_o) = v\left(t, X(t;t_o,x_o)\right) \tag{25}$$

$$\partial_{t_o} X(t;t_o,x_o) = -v(t_o,x_o)\ \exp\int_{t_o}^t \mathrm{div}_x v\left(s; X(t,t_o,x_o)\right)\mathrm{d}s \tag{26}$$

$$D_{x_o} X(t;t_o,x_o) = M(t), \text{ the matrix } M \text{ solves } \begin{cases} \dot M = D_x v\left(t, X(t;t_o,x_o)\right) M \\ M(t_o) = \mathbf{Id}\ . \end{cases} \tag{27}$$

In order to prove that (24) solves (20) in the sense of Definition 2.6 and to provide the basic well posedness estimates, a few technical lemmas are in order. First introduce the following notation: where misunderstandings might arise, we use the positional notation for derivatives. For instance, with reference to the map $(t;t_o,x_o) \to X(t;t_o,x_o)$, we denote

$$\partial_2 X(t;t_o,x_o) = \partial_{t_o} X(t;t_o,x_o) = \lim_{\tau\to 0}\frac{X(t;t_o+\tau,x_o) - X(t;t_o,x_o)}{\tau}\ .$$

We also set $X = (X_1,\ldots,X_{m+n})$, with $X_i = X \cdot e_i$, where $(e_1,\ldots,e_{m+n})$ is the canonical base of $\mathbb{R}^{m+n}$. Recall also that $\partial_l X_i = \partial_l(X\cdot e_i) = (\partial_l X)\cdot e_i$, for $l = 1,2,3$ and $i = 1,\ldots,m+n$.

Lemma 4.1. *Under assumption* **(V)** *with $k = 1$, the map in* (23)

$$\begin{array}{rccl} T\colon & \{(t,x) \in \mathbb{R}_+ \times \mathcal{X}\colon x \in X(t;[0,t[\,,\partial\mathcal{X})\} & \to & \mathbb{R}_+ \\ & (t,x) & \mapsto & \inf\{s \in [0,t[\colon X(s;t,x) \in \mathcal{X}\} \end{array} \tag{28}$$

is well defined. Moreover, for all $t \in \mathbb{R}_+$ and a.e. $x \in \mathcal{X}$ such that $x \in X(t;[0,t[,\partial\mathcal{X})$, there exists a unique $i \in \{1,\ldots,m\}$, depending on t and x, such that

$$X_i(T(t,x);t,x) = 0. \tag{29}$$

Given $t \in \mathbb{R}_+$, for $i \in \{1, \dots, m\}$, call $\mathbb{X}_i^t$ the set of $x \in \mathcal{X}$ such that i is the unique *index satisfying* (29)*. Then, the map*

$$
\begin{array}{cccc} M_i : & \mathbb{X}_i^t & \to & \mathbb{R}_+ \times \mathbb{R}^{n+m-1} \\ & x & \mapsto & \Big(T(t,x), \big(X_j(T(t,x),t,x)\big)_{j\neq i}\Big) \end{array} \tag{30}
$$

is a local diffeomorphism. The derivatives of the function T are given by

$$
\partial_t T(t,x) = -\frac{\partial_2 X_i(T(t,x);t,x)}{v_i\,(T(t,x), X(T(t,x);t,x))} \tag{31}
$$

$$
\partial_{x_\ell} T(t,x) = -\frac{\partial_{3_\ell} X_i\,(T(t,x);t,x)}{v_i\,(T(t,x), X\,(T(t,x);t,x))} \qquad \ell = 1, \dots, n+m\,. \tag{32}
$$

Finally the absolute value of the determinant of the Jacobian matrix DM_i at x is

$$
\frac{1}{v_i\,(T(t,x), X(T(t,x);t,x))} \exp \int_t^{T(t,x)} \sum_{j=1}^{m+n} \partial_{x_j} v_j\,(s, X\,(s;t,x))\,\mathrm{d}s. \tag{33}
$$

Proof. By **(V)**, the usual Cauchy Theorem for systems of ordinary differential equations ensures that, for all $(t_o, x_o) \in \mathbb{R}_+ \times \mathcal{X}$, the Cauchy Problem (21) admits a unique solution defined on a maximal interval $[T_{(t_o,x_o)}, +\infty[$, with $T_{(t_o,x_o)} \in [0, t_o]$. Then, the map T defined in (23) can be written $T(t,x) = T_{(t,x)}$ whenever $T_{(t,x)} > 0$ and $T(t,x) = 0$ otherwise. Hence, the map (28) is well defined.

Once $x \in X(t; [0,t[, \partial\mathcal{X})$, it is clear that there exists at least one index i such that (29) holds. The uniqueness follows, since $X(t; \cdot, \cdot)$ is a diffeomorphism.

Fix $t > 0$, $i \in \{1, \dots, m\}$, and $x \in \mathbb{X}_i^t$. Locally around (t,x), the constraint (29) remains valid. To compute the derivatives of the map $(t,x) \to T(t,x)$, differentiating (29) with respect to t yields

$$
\partial_1 X_i\,(T(t,x);t,x)\;\partial_t T(t,x) + \partial_2 X_i\,(T(t,x);t,x) = 0
$$

and so, using (25),

$$
v_i\,(T(t,x), X\,(T(t,x);t,x))\,\partial_t T(t,x) + \partial_2 X_i\,(T(t,x);t,x) = 0
$$

which proves (31), while a differentiation with respect to x_ℓ ($\ell \in \{1, \dots, m+n\}$) yields

$$
\partial_1 X_i\,(T(t,x);t,x)\;\partial_{x_\ell} T(t,x) + \partial_{3_\ell} X_i\,(T(t,x);t,x) = 0
$$

and so, using (25),

$$
v_i\,(T(t,x), X\,(T(t,x);t,x))\;\partial_{x_\ell} T(t,x) + \partial_{3_\ell} X_i\,(T(t,x);t,x) = 0,
$$

which proves (32).

Consider the $(n+m) \times (n+m)$ Jacobian matrix DM_i. By (32), the first row is

$$\left(\partial_{x_1} T(t,x), \cdots, \partial_{x_{n+m}} T(t,x)\right) = \left(-\frac{\partial_{3_1} X_i}{v_i}, \cdots, -\frac{\partial_{3_{n+m}} X_i}{v_i}\right),$$

where, for simplicity, we omitted the arguments of the functions X_i and v_i. The remaining rows, indexed by $j \in \{1, \ldots, n+m\}$, $j \neq i$, of DM_i are given by

$$\begin{aligned}&\left(\partial_{x_1} X_j(T(t,x);t,x), \cdots, \partial_{x_{n+m}} X_j(T(t,x);t,x)\right)\\ &= \left(-v_j \frac{\partial_{3_1} X_i}{v_i} + \partial_{3_1} X_j, \cdots, -v_j \frac{\partial_{3_{n+m}} X_i}{v_i} + \partial_{3_{n+m}} X_j\right).\end{aligned}$$

We compute the determinant of DM_i using Gauss method. We modify all the rows, except the first one, by adding to each row a multiple of the first one. In this way the determinant of DM_i equals the determinant of the matrix

$$\begin{pmatrix} -\frac{\partial_{3_1} X_i}{v_i} & -\frac{\partial_{3_2} X_i}{v_i} & \cdots & -\frac{\partial_{3_{n+m}} X_i}{v_i} \\ \partial_{3_1} X_1 & \partial_{3_2} X_1 & \cdots & \partial_{3_{n+m}} X_1 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{3_1} X_{n+m} & \partial_{3_2} X_{n+m} & \cdots & \partial_{3_{n+m}} X_{n+m} \end{pmatrix}$$

in the case $i \neq 1, n+m$, the other cases being entirely similar. Therefore $|\det(DM_i)| = \frac{1}{v_i} |\det(D_3 X)|$. Using (27) and Liouville Theorem [32, Theorem 1.2, Chapter IV], we deduce

$$\begin{aligned}|\det(DM_i(x))| &= \frac{1}{v_i\,(T(t,x); X(T(t,x);t,x))} \exp \int_t^{T(t,x)} \mathrm{tr}\,(D_x v\,(s, X\,(s;t,x)))\,\mathrm{d}s \\ &= \frac{1}{v_i\,(T(t,x); X(T(t,x);t,x))} \exp \int_t^{T(t,x)} \sum_{j=1}^{m+n} \partial_{x_j} v_j\,(s, X\,(s;t,x))\,\mathrm{d}s\end{aligned}$$

which proves (33). □

The next two lemmas provide the basic *a priori* and stability estimates on (20).

Lemma 4.2. *Let* **(V)** *with* $k=1$ *hold, let* $p \in \mathbf{L}^\infty(I \times \mathcal{X}; \mathbb{R})$, $q \in \mathbf{L^1}(I \times \mathcal{X}; \mathbb{R})$, $u_b \in \mathbf{L^1}(I \times \partial\mathcal{X}; \mathbb{R})$ *and* $u_o \in \mathbf{L^1}(\mathcal{X}; \mathbb{R})$. *Then, for every* $t \in I$ *the solution to problem* (20) *defined through formula* (24) *satisfies the following* a priori *estimates:*

$$\begin{aligned}\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \leq{}& \left(\|q\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})} + \|u_o\|_{\mathbf{L^1}(\mathcal{X})}\right) e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t} \\ &+ \left(\sum_{i=1}^{m} \iint_{\Gamma_i} |u_b(\tau,\xi)|\, v_i(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi\right) e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t},\end{aligned} \tag{34}$$

where $\Gamma_i = M_i(\mathbb{X}_i^t)$ *with* M_i *as in* (30) *and* $\mathbb{X}_t^i$ *is as in Lemma* 4.1. *If moreover* $q \in \mathbf{L^1}(I; \mathbf{L}^\infty(\mathcal{X}; \mathbb{R}))$, $u_o \in \mathbf{L}^\infty(\mathcal{X}; \mathbb{R})$, *and* $u_b \in \mathbf{L}^\infty(I \times \partial\mathcal{X}; \mathbb{R})$, *then*

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^\infty(\mathcal{X};\mathbb{R})} \le & \left(\|u_o\|_{\mathbf{L}^\infty(\mathcal{X};\mathbb{R})} + \|u_b\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X};\mathbb{R}))} + \|q\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\right) \\ & \times \exp\left(\int_0^t \left(\|p(\tau)\|_{\mathbf{L}^\infty(\mathcal{X};\mathbb{R})} + \|\mathrm{div}_x v(\tau)\|_{\mathbf{L}^\infty(\mathcal{X};\mathbb{R})}\right) \mathrm{d}\tau\right).\end{aligned} \tag{35}$$

Proof. The proof of the $\mathbf{L}^\infty$ bound directly follows from

$$\mathcal{E}(\tau, t, x) \le \exp\left(\|p\|_{\mathbf{L^1}([\tau,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} + \|\mathrm{div}_x v\|_{\mathbf{L^1}([\tau,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\right),$$

and (24). In order to get the $\mathbf{L^1}$ bound, observe that $\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} = \|u(t)\|_{\mathbf{L^1}(X(t;0,\mathcal{X});\mathbb{R})} + \|u(t)\|_{\mathbf{L^1}(X(t;[0,t[,\partial\mathcal{X});\mathbb{R})}$. We thus consider two cases and apply a suitable change of variable.

By (24), for $t \in I$, we have that

$$\begin{aligned}\int_{X(t;0,\mathcal{X})} |u(t,x)|\mathrm{d}x \le & \int_{X(t;0,\mathcal{X})} |u_o(X(0;t,x))|\,\mathcal{E}(0,t,x)\,\mathrm{d}x \\ & + \int_{X(t;0,\mathcal{X})} \int_0^t |q(\tau, X(\tau;t,x))|\,\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau\mathrm{d}x.\end{aligned} \tag{36}$$

Consider the first term in the right hand side of (36). Using Liouville Theorem [32, Theorem 1.2, Chapter IV], the change of variables $\xi = X(0;t,x)$ and the assumptions on p,

$$\begin{aligned}\int_{X(t;0,\mathcal{X})} |u_o(X(0;t,x))|\mathcal{E}(0,t,x)\,\mathrm{d}x = & \int_{\mathcal{X}} |u_o(\xi)| \exp\left(\int_0^t p(s, X(s;0,\xi))\,\mathrm{d}s\right) \mathrm{d}\xi \\ \le & \|u_o\|_{\mathbf{L^1}(\mathcal{X})}\, e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})} t}.\end{aligned}$$

Consider the second term in the right hand side of (36). Using the change of variable $\xi = X(\tau;t,x)$,

$$\begin{aligned}& \int_{X(t;0,\mathcal{X})} \int_0^t |q(\tau, X(\tau;t,x))|\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau\mathrm{d}x \\ = & \int_0^t \int_{X(\tau;0,\mathcal{X})} |q(\tau,\xi)| \exp\left(\int_\tau^t p(s, X(s;\tau,\xi))\,\mathrm{d}s\right) \mathrm{d}\xi\mathrm{d}\tau \\ \le & \|q\|_{\mathbf{L^1}(X([0,t];0,\mathcal{X});\mathbb{R})} e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})} t}.\end{aligned}$$

Therefore, using (36), for $t \in I$, we deduce

$$\int_{X(t;0,\mathcal{X})} |u(t,x)|\mathrm{d}x \le \left(\|u_o\|_{\mathbf{L^1}(\mathcal{X})} + \|q\|_{\mathbf{L^1}(X([0,t];0,\mathcal{X});\mathbb{R})}\right) e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t}. \tag{37}$$

To estimate now the term depending on the boundary conditions, for $t \in I$, use (24):

$$\begin{aligned}\int_{X(t;[0,t[,\partial\mathcal{X})} |u(t,x)|\mathrm{d}x &= \sum_{i=1}^{m} \int_{\mathbb{X}_i^t} |u(t,x)|\mathrm{d}x \\ &\le \sum_{i=1}^{m} \int_{\mathbb{X}_i^t} |u_b\left(T(t,x), X\left(T(t,x);t,x\right)\right)| \,\mathcal{E}\left(T(t,x),t,x\right)\mathrm{d}x \\ &\quad + \sum_{i=1}^{m} \int_{\mathbb{X}_i^t} \int_{T(t,x)}^{t} |q\left(\tau, X\left(\tau;t,x\right)\right)| \,\mathcal{E}\left(\tau,t,x\right)\mathrm{d}\tau\mathrm{d}x.\end{aligned} \tag{38}$$

For $i \in \{1,\ldots,m\}$, use the diffeomorphism M_i in (30) as change of variables, i.e., $\tau = T(t,x)$, $\xi = X\left(T(t,x);t,x\right)$ and we set $\Gamma_i = M_i(\mathbb{X}_i^t)$. Thus, we have

$$\begin{aligned}&\int_{\mathbb{X}_i^t} |u_b\left(T(t,x), X\left(T(t,x);t,x\right)\right)| \,\mathcal{E}\left(T(t,x),t,x\right)\mathrm{d}x \\ &= \iint_{\Gamma_i} |u_b(\tau,\xi)| \exp\left(\int_\tau^t p\left(s, X(s;\tau,\xi)\right)\mathrm{d}s\right) v_i(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \\ &\le e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t} \iint_{\Gamma_i} |u_b(\tau,\xi)|\, v_i(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi.\end{aligned}$$

For $i \in \{1,\ldots,m\}$, using again the change of variables $\xi = X\left(\tau;t,x\right)$, define

$$\Xi_t^i = \left\{(\tau,\xi) \in \mathbb{R}^{1+m+n} \colon \tau \in [t, T(t,x)],\ x \in \mathbb{X}_t^i,\ \xi = X(\tau;t,x)\right\} \tag{39}$$

and we have

$$\begin{aligned}&\int_{\mathbb{X}_t^i} \int_{T(t,x)}^{t} |q\left(\tau, X\left(\tau;t,x\right)\right)| \,\mathcal{E}\left(\tau,t,x\right)\mathrm{d}\tau\mathrm{d}x \\ &= \iint_{\Xi_t^i} |q(\tau,\xi)| \exp\left(\int_\tau^t p\left(s, X(s;\tau,\xi)\right)\mathrm{d}s\right)\mathrm{d}\tau\mathrm{d}\xi \\ &\le \|q\|_{\mathbf{L^1}(\Xi_t^i;\mathbb{R})}\, e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t}.\end{aligned}$$

Therefore, using (38), for $t \in I$, we deduce

$$\int\limits_{X(t;[0,t[,\partial\mathcal{X})} |u(t,x)|\mathrm{d}x \le e^{\|p\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t} \sum_{i=1}^m \left[\iint\limits_{\Gamma_i} |u_b(\tau,\xi)|v_i(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \|q\|_{\mathbf{L}^1(\Xi_t^i;\mathbb{R})} \right].$$

This concludes the proof. □

Lemma 4.3. *Fix v satisfying* **(V)** *with $k=1$. Let $p_1, p_2 \in \mathbf{L}^\infty(I\times\mathcal{X};\mathbb{R})$, $q_1, q_2 \in \mathbf{L}^1(I\times\mathcal{X};\mathbb{R})$ with $u_{b,1}$ and $u_{b,2}$ as in Proposition 4.2 and let $u_{o,1}, u_{o,2}$ satisfy* **(ID)**. *Define u_1 and u_2 respectively the solutions to*

$$\begin{cases} \partial_t u_1 + \mathrm{div}_x(v\,u_1) = p_1\,u_1 + q_1 \\ u_1(t,\xi) = u_{b,1}(t,\xi) \\ u_1(0,x) = u_{o,1}(x) \end{cases} \quad \textit{and} \quad \begin{cases} \partial_t u_2 + \mathrm{div}_x(v\,u_2) = p_2\,u_2 + q_2 \\ u_2(t,\xi) = u_{b,2}(t,\xi) \\ u_2(0,x) = u_{o,2}(x). \end{cases}$$

Then, for every $t \in I$, the following stability estimate holds

$$\begin{aligned} &\|u_1(t) - u_2(t)\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} \\ \le\, &\mathcal{P}(t)\left\|u_{o,1} - u_{o,2}\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} \\ &+\mathcal{P}(t)\,\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|u_{b,1} - u_{b,2}\right\|_{\mathbf{L}^1(I\times\partial\mathcal{X};\mathbb{R})} \\ &+\mathcal{P}(t)\,\|q_1 - q_2\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R})} \\ &+\mathcal{P}(t)\left(\left\|u_{o,1}\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} + \|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|u_{b,2}\right\|_{\mathbf{L}^1([0,t]\times\partial\mathcal{X};\mathbb{R})}\right)\|p_1 - p_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} \\ &+\mathcal{P}(t)\,\|q_2\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R})}\,\|p_1 - p_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\,, \end{aligned} \tag{40}$$

where $\mathcal{P}(t) = \exp\left(t\,\max\left\{\|p_1\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}, \|p_2\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}\right\}\right)$.

Proof. Consider u_1 and u_2 the solutions to the two systems and fix $t \in I$. Define for $i = 1, 2$

$$\mathcal{E}_i(\tau,t,x) = \exp\left(\int_\tau^t \left(p_i\left(s, X(s;t,x)\right) - \mathrm{div}_x v\left(s, X(s;t,x)\right)\right)\mathrm{d}s\right).$$

We have the decomposition

$$\|u_1(t) - u_2(t)\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} = \int\limits_{X(t;0,\mathcal{X})} |u_1(t) - u_2(t)|\mathrm{d}x + \int\limits_{X(t;[0,t[,\partial\mathcal{X})} |u_1(t) - u_2(t)|\mathrm{d}x\,. \tag{41}$$

We treat the two terms in the right hand side of (41) separately. The first one is dealt with the explicit formula (24):

$$
\begin{aligned}
&\int_{X(t;0,\mathcal{X})} |u_1(t) - u_2(t)| \mathrm{d}x \\
\le& \int_{X(t;0,\mathcal{X})} \left| u_{o,1}\left(X\left(0;t,x\right)\right) \mathcal{E}_1\left(0,t,x\right) - u_{o,2}\left(X\left(0;t,x\right)\right) \mathcal{E}_2\left(0,t,x\right) \right| \mathrm{d}x \\
&+ \int_{X(t;0,\mathcal{X})} \int_0^t \left| q_1\left(\tau, X\left(\tau;t,x\right)\right) \mathcal{E}_1\left(\tau,t,x\right) - q_2\left(\tau, X\left(\tau;t,x\right)\right) \mathcal{E}_2\left(\tau,t,x\right) \right| \mathrm{d}\tau \mathrm{d}x \\
\le& \int_{X(t;0,\mathcal{X})} \mathcal{E}_1\left(0,t,x\right) \left| u_{o,1}\left(X\left(0;t,x\right)\right) - u_{o,2}\left(X\left(0;t,x\right)\right) \right| \mathrm{d}x \\
&+ \int_{X(t;0,\mathcal{X})} \left| u_{o,2}\left(X\left(0;t,x\right)\right) \right| \left| \mathcal{E}_1\left(0,t,x\right) - \mathcal{E}_2\left(0,t,x\right) \right| \mathrm{d}x \\
&+ \int_{X(t;0,\mathcal{X})} \int_0^t \mathcal{E}_1\left(\tau,t,x\right) \left| q_1\left(\tau, X\left(\tau;t,x\right)\right) - q_2\left(\tau, X\left(\tau;t,x\right)\right) \right| \mathrm{d}\tau \mathrm{d}x \\
&+ \int_{X(t;0,\mathcal{X})} \int_0^t \left| q_2\left(\tau, X\left(\tau;t,x\right)\right) \right| \left| \mathcal{E}_1\left(\tau,t,x\right) - \mathcal{E}_2\left(\tau,t,x\right) \right| \mathrm{d}\tau \mathrm{d}x .
\end{aligned}
$$

Using the two changes of variable $\xi = X\left(0;t,x\right)$ and $\xi = X\left(\tau;t,x\right)$, we obtain that

$$
\begin{aligned}
&\int_{X(t;0,\mathcal{X})} |u_1(t) - u_2(t)| \mathrm{d}x \\
\le& \int_{\mathcal{X}} \exp\left(\int_0^t p_1\left(s, X(s;0,\xi)\right) \mathrm{d}s \right) \left| u_{o,1}\left(\xi\right) - u_{o,2}\left(\xi\right) \right| \mathrm{d}\xi \\
&+ \int_{\mathcal{X}} \left| u_{o,2}\left(\xi\right) \right| \left| \exp\left(\int_0^t p_1\left(s, X(s;0,\xi)\right) \mathrm{d}s \right) - \exp\left(\int_0^t p_2\left(s, X(s;0,\xi)\right) \mathrm{d}s \right) \right| \mathrm{d}\xi \\
&+ \int_0^t \int_{X(\tau;0,\mathcal{X})} \left| q_1\left(\tau,\xi\right) - q_2\left(\tau,\xi\right) \right| \exp\left(\int_\tau^t p_1\left(s, X(s;\tau,\xi)\right) \mathrm{d}s \right) \mathrm{d}\xi \, \mathrm{d}\tau \\
&+ \int_0^t \int_{X(\tau;0,\mathcal{X})} \left| q_2\left(\tau,\xi\right) \right| \\
&\qquad \times \left| \exp\left(\int_\tau^t p_1\left(s, X(s;\tau,\xi)\right) \mathrm{d}s \right) - \exp\left(\int_\tau^t p_2\left(s, X(s;\tau,\xi)\right) \mathrm{d}s \right) \right| \mathrm{d}\xi \, \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le \mathcal{P}(t)\left(\left\|u_{o,1}-u_{o,2}\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})}+\|q_1-q_2\|_{\mathbf{L}^1\left(X([0,t];0,\mathcal{X});\mathbb{R}\right)}\right)\\
&\quad+\mathcal{P}(t)\left\|u_{o,2}\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})}\|p_1-p_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\\
&\quad+\mathcal{P}(t)\|q_2\|_{\mathbf{L}^1\left(X([0,t];0,\mathcal{X});\mathbb{R}\right)}\|p_1-p_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))},
\end{aligned}
$$

where we set

$$
\mathcal{P}(t)=\exp\left(\max\left\{\|p_1\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t\,,\ \|p_2\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}t\right\}\right). \tag{42}
$$

Pass now to the second term in the right hand side of (41), splitting among the different faces $\mathbb{X}_i^t$ for $i\in\{1,\dots,m\}$ as defined in (30):

$$
\int_{X(t;[0,t[,\partial\mathcal{X})}|u_1(t)-u_2(t)|\mathrm{d}x=\sum_{i=1}^m\int_{\mathbb{X}_i^t}|u_1(t)-u_2(t)|\mathrm{d}x.
$$

Fix $i\in\{1,\dots,m\}$, i.e. consider each term in the sum separately:

$$
\begin{aligned}
&\int_{\mathbb{X}_i^t}|u_1(t)-u_2(t)|\mathrm{d}x\\
&\le\int_{\mathbb{X}_i^t}\Big|u_{b,1}\left(T(t,x),X\left(T(t,x);t,x\right)\right)\mathcal{E}_1\left(T(t,x),t,x\right)\\
&\qquad\qquad-u_{b,2}\left(T(t,x),X\left(T(t,x);t,x\right)\right)\mathcal{E}_2\left(T(t,x),t,x\right)\Big|\,\mathrm{d}x\\
&\quad+\int_{\mathbb{X}_i^t}\int_{T(t,x)}^t\left|q_1\left(\tau,X\left(\tau;t,x\right)\right)\mathcal{E}_1\left(\tau,t,x\right)-q_2\left(\tau,X\left(\tau;t,x\right)\right)\mathcal{E}_2\left(\tau,t,x\right)\right|\mathrm{d}\tau\,\mathrm{d}x\\
&\le\int_{\mathbb{X}_i^t}\mathcal{E}_1\left(T(t,x),t,x\right)\\
&\qquad\qquad\times\Big|u_{b,1}\left(T(t,x),X\left(T(t,x);t,x\right)\right)-u_{b,2}\left(T(t,x),X\left(T(t,x);t,x\right)\right)\Big|\mathrm{d}x\\
&\quad+\int_{\mathbb{X}_i^t}\Big|u_{b,2}\left(T(t,x),X\left(T(t,x);t,x\right)\right)\Big|\left|\mathcal{E}_1\left(T(t,x),t,x\right)-\mathcal{E}_2\left(T(t,x),t,x\right)\right|\mathrm{d}x\\
&\quad+\int_{\mathbb{X}_i^t}\int_{T(t,x)}^t\mathcal{E}_1\left(\tau,t,x\right)\left|q_1\left(\tau,X\left(\tau;t,x\right)\right)-q_2\left(\tau,X\left(\tau;t,x\right)\right)\right|\mathrm{d}\tau\,\mathrm{d}x\\
&\quad+\int_{\mathbb{X}_i^t}\int_{T(t,x)}^t\left|q_2\left(\tau,X\left(\tau;t,x\right)\right)\right|\left|\mathcal{E}_1\left(\tau,t,x\right)-\mathcal{E}_2\left(\tau,t,x\right)\right|\mathrm{d}\tau\,\mathrm{d}x.
\end{aligned}
$$

We now use the diffeomorphism M_i as defined in (30), for $i \in \{1, \ldots, m\}$, and we use the set Ξ_t^i as in (39). We thus obtain, using (42), that

$$
\begin{aligned}
&\int_{\mathbb{X}_i^t} |u_1(t,x) - u_2(t,x)| \mathrm{d}x \\
&\le \iint_{\Gamma_i} \exp\left(\int_\tau^t p_1\left(s, X(s;\tau,\xi)\right) \mathrm{d}s\right) \left|u_{b,1}(\tau,\xi) - u_{b,2}(\tau,\xi)\right| v_i(\tau,\xi) \mathrm{d}\xi \mathrm{d}\tau \\
&\quad + \iint_{\Gamma_i} \left|u_{b,2}(\tau,\xi)\right| \\
&\qquad \times \left|\exp\left(\int_\tau^t p_1\left(s, X(s;\tau,\xi)\right) \mathrm{d}s\right) - \exp\left(\int_\tau^t p_2\left(s, X(s;\tau,\xi)\right) \mathrm{d}s\right)\right| v_i(\tau,\xi) \mathrm{d}\xi \mathrm{d}\tau \\
&\quad + \iint_{\Xi_t^i} \exp\left(\int_\tau^t p_1\left(s, X(s;\tau,\xi)\right) \mathrm{d}s\right) |q_1(\tau,\xi) - q_2(\tau,\xi)| \mathrm{d}\tau \mathrm{d}\xi \\
&\quad + \iint_{\Xi_t^i} |q_2(\tau,\xi)| \\
&\qquad \times \left|\exp\left(\int_\tau^t p_1\left(s, X(s;\tau,\xi)\right) \mathrm{d}s\right) - \exp\left(\int_\tau^t p_2\left(s, X(s;\tau,\xi)\right) \mathrm{d}s\right)\right| \mathrm{d}\tau \mathrm{d}\xi \\
&\le \mathcal{P}(t) \left\|v\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})} \left\|u_{b,1} - u_{b,2}\right\|_{\mathbf{L}^1(\Gamma_i;\mathbb{R})} \\
&\quad + \mathcal{P}(t) \left\|v\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})} \left\|u_{b,2}\right\|_{\mathbf{L}^1(\Gamma_i;\mathbb{R})} \left\|p_1 - p_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} \\
&\quad + \mathcal{P}(t) \left\|q_1 - q_2\right\|_{\mathbf{L}^1(\Xi_t^i;\mathbb{R})} \\
&\quad + \mathcal{P}(t) \left\|q_2\right\|_{\mathbf{L}^1(\Xi_t^i;\mathbb{R})} \left\|p_1 - p_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} \\
&\le \mathcal{P}(t) \left(\left\|v\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})} \left\|u_{b,1} - u_{b,2}\right\|_{\mathbf{L}^1(\Gamma_i;\mathbb{R})} + \left\|q_1 - q_2\right\|_{\mathbf{L}^1(\Xi_t^i;\mathbb{R})}\right) \\
&\quad + \mathcal{P}(t) \left\|v\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})} \left\|u_{b,2}\right\|_{\mathbf{L}^1(\Gamma_i;\mathbb{R})} \left\|p_1 - p_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} \\
&\quad + \mathcal{P}(t) \left\|q_2\right\|_{\mathbf{L}^1(\Xi_t^i;\mathbb{R})} \left\|p_1 - p_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} .
\end{aligned}
$$

Therefore, using (41), we deduce that

$$
\begin{aligned}
&\left\|u_1(t) - u_2(t)\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} \\
&\le \mathcal{P}(t) \left(\left\|u_{o,1} - u_{o,2}\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} + \left\|q_1 - q_2\right\|_{\mathbf{L}^1(\mathcal{X};([0,t];0,\mathcal{X});\mathbb{R})}\right) \\
&\quad + \mathcal{P}(t) \left\|u_{o,2}\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} \left\|p_1 - p_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{P}(t)\|q_2\|_{\mathbf{L^1}(X;([0,t];0,\mathcal{X});\mathbb{R})}\|p_1-p_2\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\\
&+\sum_{i=1}^m \mathcal{P}(t)\left(\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|u_{b,1}-u_{b,2}\right\|_{\mathbf{L^1}(\Gamma_i;\mathbb{R})}+\|q_1-q_2\|_{\mathbf{L^1}(\Xi_t^i;\mathbb{R})}\right)\\
&+\sum_{i=1}^m \mathcal{P}(t)\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|u_{b,2}\right\|_{\mathbf{L^1}(\Gamma_i;\mathbb{R})}\|p_1-p_2\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\\
&+\sum_{i=1}^m \mathcal{P}(t)\|q_2\|_{\mathbf{L^1}(\Xi_t^i;\mathbb{R})}\|p_1-p_2\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\\
\le\ &\mathcal{P}(t)\left\|u_{o,1}-u_{o,2}\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}\\
&+\mathcal{P}(t)\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|u_{b,1}-u_{b,2}\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbb{R})}\\
&+\mathcal{P}(t)\|q_1-q_2\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}\\
&+\mathcal{P}(t)\left(\|u_o\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}+\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|u_{b,2}\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbb{R})}\right)\|p_1-p_2\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\\
&+\mathcal{P}(t)\|q_2\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}\|p_1-p_2\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))},
\end{aligned}
$$

proving (40). □

Proposition 4.4. *Let v satisfy* **(V)** *with $k=1$, $p\in\mathbf{L}^\infty(I\times\mathcal{X};\mathbb{R})$, $q\in\mathbf{L^1}(I\times\mathcal{X};\mathbb{R})$, $u_b\in\mathbf{L^1}(I\times\partial\mathcal{X};\mathbb{R})$ and u_o satisfy* **(ID)** *with $k=1$. Then, formula (24) defines a solution $u=u(t,x)$ to (20) in the sense of Definition 2.6. Moreover, $u\in\mathbf{C^0}(I;\mathbf{L^1}(\mathcal{X};\mathbb{R}))$.*

Proof. The first part of the proof amounts to a careful piecing together various proofs found in the literature. In particular, the part of the solution depending on the initial data is dealt with exactly as in [22, Lemma 2.7] and [33, Lemma 5.1]. The part depending on the boundary datum is treated in the same way, exploiting the change of variables detailed in Lemma 4.1.

To prove the $\mathbf{C^0}$ regularity of the solution with respect to time, fix a $\bar t\in I$ and a sequence t_h, with $t_h\in I$, converging to $\bar t$. Then, assuming first that $t_h>t$, we have

$$
\begin{aligned}
\left\|u(t_h)-u(\bar t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}=&\int_{X(t_h;0,\mathcal{X})}\left|u(t_h,x)-u(\bar t,x)\right|\mathrm{d}x\\
&+\int_{\mathcal{X}\setminus(X(t_h;0,\mathcal{X})\cup X(\bar t;[0,\bar t[,\partial\mathcal{X}))}\left|u(t_h,x)-u(\bar t,x)\right|\mathrm{d}x\\
&+\int_{X(\bar t;[0,\bar t[,\partial\mathcal{X})}\left|u(t_h,x)-u(\bar t,x)\right|\mathrm{d}x\,.
\end{aligned}
$$

The second term vanishes as $h\to+\infty$, since it is the integral of a bounded quantity over a set of vanishing measure. Consider now the first term, the third one can be treated similarly.

$$
\int_{X(t_h;0,\mathcal{X})}\left|u(t_h,x)-u(\bar t,x)\right|\mathrm{d}x
$$

$$
\begin{aligned}
&= \int_{\mathcal{X}} \left| u(t_h, x) - u(\bar t, x) \right| \chi_{X(t_h;0,\mathcal{X})}(x)\, \mathrm{d}x \\
&\le \int_{\mathcal{X}} \left| u_o\left(X(0;t_h,x)\right) \mathcal{E}(\tau, t_h, x) - u_o\left(X(0;\bar t,x)\right) \mathcal{E}(\tau,\bar t,x) \right| \chi_{X(t_h;0,\mathcal{X})}(x) \mathrm{d}x \\
&+ \int_{\mathcal{X}} \left| \int_0^{t_h} q\left(\tau, X(\tau;t_h,x)\right) \mathcal{E}(\tau,t_h,x)\, \mathrm{d}\tau \right. \\
&\qquad \left. - \int_0^{\bar t} q\left(\tau, X(\tau;\bar t,x)\right) \mathcal{E}\left(\tau,\bar t,x\right) \mathrm{d}\tau \right| \chi_{X(t_h;0,\mathcal{X})}(x) \mathrm{d}x
\end{aligned}
$$

As $h \to +\infty$, we have that

$$
\begin{aligned}
u_o\left(X(0;t_h,x)\right) \mathcal{E}(\tau,t_h,x) &\to u_o\left(X(0;\bar t,x)\right) \mathcal{E}(\tau,\bar t,x) \\
\int_0^{t_h} q\left(\tau, X(\tau;t_h,x)\right) \mathcal{E}(\tau,t_h,x)\, \mathrm{d}\tau &\to \int_0^{\bar t} q\left(\tau, X(\tau;\bar t,x)\right) \mathcal{E}\left(\tau,\bar t,x\right) \mathrm{d}\tau
\end{aligned}
$$

for a.e. $x \in \mathcal{X}$, so that the corresponding integrals vanish by Lebesgue Dominated Convergence Theorem, which we can apply thanks to the $\mathbf{L^1}$ *a priori* bound (34). □

4.2. The general case of a system

Below, in the various estimates we use the following norms:

$$
\|u\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} = \sum_{h=1}^k \int_{\mathcal{X}} \left|u^h(x)\right| \mathrm{d}x \qquad \|u\|_{\mathbf{L^\infty}(I\times\mathcal{X};\mathbb{R}^k)} = \sum_{h=1}^k \left\|u^h\right\|_{\mathbf{L^\infty}(I\times\mathcal{X};\mathbb{R})}
$$
$$
\|u\|_{\mathbf{L^\infty}(I;\mathbf{L^1}(\mathcal{X};\mathbb{R}^k))} = \sum_{h=1}^k \left\|u^h\right\|_{\mathbf{L^\infty}(I;\mathbf{L^1}(\mathcal{X};\mathbb{R}))}.
$$

Proof of Theorem 2.2. The proof is divided in several steps. Let $I = [0, T]$ for $T > 0$.

Construction of the Operator $\mathcal{T}$. In the Banach space $\mathbf{C^0}(I; \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k))$, for

$$
M > \|u_o\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} + 1, \tag{43}
$$

introduce the closed subset X and the norm $\|\cdot\|_X$:

$$
X = \left\{ w \in \mathbf{C^0}(I; \mathbf{L^1}(\mathcal{X};\mathbb{R}^k)) \colon \|w\|_{\mathbf{L^\infty}(I;\mathbf{L^1}(\mathcal{X};\mathbb{R}^k))} \le M \right\}, \tag{44}
$$

$$
\|w\|_X = \sum_{h=1}^k \left\|w^h\right\|_{\mathbf{L^\infty}(I;\mathbf{L^1}(\mathcal{X};\mathbb{R}))}. \tag{45}
$$

Define the operator

$$\begin{array}{rccc} \mathcal{T}: & X & \longrightarrow & X \\ & w & \longmapsto & u \equiv \left(u^1, \ldots, u^k\right) \end{array} \tag{46}$$

where, for every $h \in \{1, \ldots, k\}$, u^h solves

$$\begin{cases} \partial_t u^h + \operatorname{div}_x \left(v^h(t,x) u^h\right) = p^h\left(t, x, w(t)\right) u^h \\ \qquad\qquad\qquad\qquad + q^h\left(t, x, w(t,x), w(t)\right) & (t,x) \in I \times \mathcal{X} \\ u^h(t,\xi) = u_b^h\left(t, \xi, w(t)\right) & (t,\xi) \in I \times \partial\mathcal{X} \\ u^h(0,x) = u_o^h(x) & x \in \mathcal{X}. \end{cases} \tag{47}$$

$\mathcal{T}$ is Well Defined. We prove that, for $w \in X$ and $h \in \{1, \ldots, k\}$, the source term in (47)

$$\mathcal{G}^h(t,x,u^h) = \mathcal{P}^h(t,x)\, u^h + \mathcal{Q}^h(t,x) \quad \text{where} \quad \begin{array}{l} \mathcal{P}^h(t,x) = p^h\left(t, x, w(t)\right) \\ \mathcal{Q}^h(t,x) = q^h\left(t, x, w(t,x), w(t)\right) \end{array}$$

is such that $\mathcal{P}^h \in \mathbf{L}^\infty(I \times \mathcal{X}; \mathbb{R})$ and $\mathcal{Q}^h \in \mathbf{L}^1(I \times \mathcal{X}; \mathbb{R})$.

By **(P)**, for every $t \in I$ and $x \in \mathcal{X}$, using also (44), we have

$$\begin{aligned} \left|\mathcal{P}^h(t,x)\right| = \left|p^h\left(t,x,w(t)\right)\right| \; &\leq \; P_1 + P_2 \left\|w(t)\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}; \\ \left\|\mathcal{P}^h\right\|_{\mathbf{L}^\infty(I\times\mathcal{X};\mathbb{R})} &\leq P_1 + P_2\, M, \end{aligned} \tag{48}$$

proving that $(t,x) \mapsto \mathcal{P}^h(t,x)$ is in $\mathbf{L}^\infty(I \times \mathcal{X}; \mathbb{R})$. On the other hand, by **(Q)** we have

$$\begin{aligned} &\left\|\mathcal{Q}^h\right\|_{\mathbf{L}^1([0,T]\times\mathcal{X};\mathbb{R})} \\ &= \int_0^T \int_{\mathcal{X}} \left|\mathcal{Q}^h(t,x)\right| \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathcal{X}} \left|q^h\left(t,x,w(t,x),w(t)\right)\right| \mathrm{d}x\,\mathrm{d}t \\ &\leq Q_1 \int_0^T \int_{\mathcal{X}} \left\|w(t,x)\right\| \mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathcal{X}} Q_2(x) \left\|w(t)\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)} \mathrm{d}x\,\mathrm{d}t + Q_3 \int_0^T \int_{\mathcal{X}} \left\|w(t,x)\right\| \left\|w(t)\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)} \mathrm{d}x\,\mathrm{d}t \\ &\leq Q_1 T \left\|w\right\|_X + \left\|Q_2\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})} T \left\|w\right\|_X + Q_3 T \left\|w\right\|_X^2, \end{aligned} \tag{49}$$

proving that $(t,x) \mapsto \mathcal{Q}^h(t,x)$ is in $\mathbf{L}^1(I \times \mathcal{X}; \mathbb{R})$.

Now we prove that, for every $w \in X$ and $h \in \{1, \ldots, k\}$, the boundary term $\mathcal{U}_b^h(t,\xi) = u_b^h(t,\xi,w(t))$ in (47) satisfies $\mathcal{U}_b^h \in \mathbf{L^1}(I \times \partial\mathcal{X}; \mathbb{R})$. By **(BD)** we have

$$
\begin{aligned}
\left\|\mathcal{U}_b^h\right\|_{\mathbf{L^1}(I\times\partial\mathcal{X};\mathbb{R})} &= \int_0^T\int_{\partial\mathcal{X}} \left|u_b^h(t,\xi,w(t))\right| \mathrm{d}\xi\,\mathrm{d}t \\
&\le \int_0^T\int_{\partial\mathcal{X}} B(\xi)\|w(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}\mathrm{d}\xi\,\mathrm{d}t + \int_0^T\int_{\partial\mathcal{X}} B(\xi)\mathrm{d}\xi\,\mathrm{d}t \\
&\le \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}\left(\|w\|_X + 1\right) T\,.
\end{aligned}
$$

Hence Proposition 4.4 applies to (47). To conclude this step, we show that the solution $u(t,x) \equiv \left(u^1(t,x),\ldots,u^k(t,x)\right)$ belongs to X in (44). By (34), (48), (49) and since $w \in X$, for $t \in I$,

$$
\begin{aligned}
\left\|u^h(t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} &\le e^{(P_1+P_2M)t}\left(\left\|Q^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})} + \left\|u_o^h\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}\right) \\
&\quad + e^{(P_1+P_2M)t}\sum_{i=1}^m \iint_{\Gamma_i} \left|u_b^h(\tau,\xi,w(\tau))\right| v_i^h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&\le \Big[\left(Q_1 + \|Q_2\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} + Q_3\|w\|_X\right) T\,\|w\|_X + \left\|u_o^h\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \\
&\quad + \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}\,\|v\|_{\mathbf{L^\infty}(I\times\mathcal{X};\mathbb{R}^{k\times(n+m)})}\,T\left(\|w\|_X + 1\right)\Big] e^{(P_1+P_2M)t} \\
&\le \Big[\left(Q_1 + \|Q_2\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} + Q_3 M\right) T\,M + \left\|u_o^h\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \\
&\quad + \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}\,\|v\|_{\mathbf{L^\infty}(I\times\mathcal{X};\mathbb{R}^{k\times(n+m)})}\,T(M+1)\Big] e^{(P_1+P_2M)t} \\
&\le \left(\left\|u_o^h\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} + \frac{1}{2k}\right) e^{(P_1+P_2M)T},
\end{aligned}
$$

whence $\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \le M$, once T is sufficiently small, thanks to the choice (43) of M.

$\mathcal{T}$ is a Contraction. Fix $\hat{w}$ and $\check{w}$ in X_M and call $\hat{u} = \mathcal{T}\hat{w}$, $\check{u} = \mathcal{T}\check{w}$. Use the notation

$$
\begin{aligned}
&\hat{\mathcal{P}}^h(t,x) = p^h\left(t,x,\hat{w}(t)\right), \ \hat{\mathcal{Q}}^h(t,x) = q^h\left(t,x,\hat{w}(t,x),\hat{w}(t)\right), \ \hat{\mathcal{U}}_b^h(t,\xi) = u_b^h\left(t,\xi,\hat{w}(t)\right), \\
&\check{\mathcal{P}}^h(t,x) = p^h\left(t,x,\check{w}(t)\right), \ \check{\mathcal{Q}}^h(t,x) = q^h\left(t,x,\check{w}(t,x),\check{w}(t)\right), \ \check{\mathcal{U}}_b^h(t,\xi) = u_b^h\left(t,\xi,\check{w}(t)\right).
\end{aligned}
$$

Then, by Lemma 4.3 and by (48), we have:

$$
\begin{aligned}
&\left\|\hat{u}^h(t) - \check{u}^h(t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \\
&\le e^{(P_1+P_2M)t}\,\|v\|_{\mathbf{L^\infty}([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|\hat{\mathcal{U}}_b^h - \check{\mathcal{U}}_b^h\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbb{R})} \\
&\quad + e^{(P_1+P_2M)t}\left\|\hat{\mathcal{Q}}^h - \check{\mathcal{Q}}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(M+\|v\|_{\mathbf{L}^\infty(I\times\mathcal{X};\mathbb{R}^{k\times(n+m)})}\left\|\check{\mathcal{U}}_b^h\right\|_{\mathbf{L}^1([0,t]\times\partial\mathcal{X};\mathbb{R})}+\left\|\check{\mathcal{Q}}^h\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R})}\right)\\
&\quad\times e^{(P_1+P_2M)t}\left\|\hat{\mathcal{P}}^h-\check{\mathcal{P}}^h\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}.
\end{aligned}
\tag{50}
$$

By **(P)** we have:

$$
\begin{aligned}
\left\|\hat{\mathcal{P}}^h-\check{\mathcal{P}}^h\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}&\le\int_0^t\left\|\hat{\mathcal{P}}^h(s)-\check{\mathcal{P}}^h(s)\right\|_{\mathbf{L}^\infty(\mathcal{X};\mathbb{R})}\mathrm{d}s\\
&\le P_2\int_0^t\left\|\hat{w}(s)-\check{w}(s)\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}\mathrm{d}s\\
&\le P_2\left\|\hat{w}-\check{w}\right\|_X T.
\end{aligned}
\tag{51}
$$

By **(Q)** we have:

$$
\begin{aligned}
&\left\|\hat{\mathcal{Q}}^h-\check{\mathcal{Q}}^h\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R})}\\
&\le Q_1\left\|\hat{w}-\check{w}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R}^k)}+Q_3\left\|\hat{w}\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathcal{X};\mathbb{R}^k))}\left\|\hat{w}-\check{w}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R}^k)}\\
&\quad+Q_3\left\|\check{w}\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathcal{X};\mathbb{R}^k))}\left\|\hat{w}-\check{w}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R}^k)}\\
&\le(Q_1+2\,M\,Q_3)\left\|\hat{w}-\check{w}\right\|_X T.
\end{aligned}
\tag{52}
$$

Similarly, by **(BD)**, we have:

$$
\left\|\hat{\mathcal{U}}_b^h-\check{\mathcal{U}}_b^h\right\|_{\mathbf{L}^1([0,t]\times\partial\mathcal{X};\mathbb{R})}\le\|B\|_{\mathbf{L}^1(\partial\mathcal{X};\mathbb{R})}\left\|\hat{w}-\check{w}\right\|_X T.
\tag{53}
$$

Therefore $\mathcal{T}$ is a contraction as soon as T is sufficiently small.

Existence of a Solution for Small Times. Proving that the unique fixed point of $\mathcal{T}$ solves (1) in the sense of Definition 2.1 amounts to pass to the limit in the integral inequality (10). This is possible thanks to the strong convergence ensured by the choice (45) of the norm in X. The proof of **(WP.1)** is completed.

Uniqueness. Assume that (2) admits the solutions $\hat{u}$ and $\check{u}$ in the sense of Definition 2.1. Then, their difference $\delta=\hat{u}-\check{u}$ solves

$$
\begin{cases}
\partial_t\delta^h+\operatorname{div}\left(v^h(t,x)\,\delta^h\right)=\hat{\mathcal{G}}^h(t,x)-\check{\mathcal{G}}^h(t,x)\\
\delta^h(t,\xi)=\hat{\mathcal{U}}_b^h(t,\xi)-\check{\mathcal{U}}_b^h(t,\xi)\\
\delta^h(0,x)=0
\end{cases}
$$

in the sense of Definition 2.1, where

$$
\begin{aligned}
\hat{\mathcal{G}}^h(t,x)&=p^h\left(t,x,\hat{u}(t)\right)\hat{u}^h+q^h\left(t,x,\hat{u},\hat{u}(t)\right) &\qquad \hat{\mathcal{U}}_b^h(t,\xi)&=\hat{u}_b^h\left(t,\xi,\hat{u}(t)\right);\\
\check{\mathcal{G}}^h(t,x)&=p^h\left(t,x,\check{u}(t)\right)\check{u}^h+q^h\left(t,x,\check{u},\check{u}(t)\right) &\qquad \check{\mathcal{U}}_b^h(t,\xi)&=\check{u}_b^h\left(t,\xi,\check{u}(t)\right).
\end{aligned}
$$

A straightforward application of the classical doubling of variable method [34], see [25, Lemma 16, Lemma 17], [28, Theorem 7.28], and also [22, Proposition 2.8], leads to the stability estimate

$$\left\|\delta^h(t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \le \int_0^t \left\|\hat{\mathcal{G}}^h(\tau)-\check{\mathcal{G}}^h(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \mathrm{d}\tau + \left\|v^h\right\|_{\mathbf{L}^\infty(I\times\mathcal{X};\mathbb{R}^{n+m})} \int_0^t \left\|\hat{\mathcal{U}}_b^h(\tau)-\check{\mathcal{U}}_b^h(\tau)\right\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})} \mathrm{d}\tau\,.$$

The assumptions **(P)** and **(Q)** allow now to use Gronwall Lemma, proving that $\delta \equiv 0$.

Continuous Dependence on the Initial Datum. With the notation in **(WP.3)**, define

$$\begin{aligned}
&\hat{\mathcal{P}}^h(t,x)=p^h\left(t,x,\hat{u}(t)\right), \; \hat{\mathcal{Q}}^h(t,x)=q^h\left(t,x,\hat{u}(t,x),\hat{u}(t)\right), \; \hat{\mathcal{U}}_b^h(t,\xi)=u_b^h\left(t,\xi,\hat{u}(t)\right),\\
&\check{\mathcal{P}}^h(t,x)=p^h\left(t,x,\check{u}(t)\right), \; \check{\mathcal{Q}}^h(t,x)=q^h\left(t,x,\check{u}(t,x),\check{u}(t)\right), \; \check{\mathcal{U}}_b^h(t,\xi)=u_b^h\left(t,\xi,\check{u}(t)\right),
\end{aligned}$$

for $t \in I$ and $h \in \{1,\ldots,k\}$. A further application of Lemma 4.3 allows to estimate the difference between the solutions $\hat{u}$ and $\check{u}$.

$$\begin{aligned}
&\left\|\hat{u}^h(t)-\check{u}^h(t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}\\
&\le e^{(P_1+P_2M)t}\left(\left\|\hat{u}_{o,h}-\check{u}_{o,h}\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}+\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|\hat{\mathcal{U}}^h-\check{\mathcal{U}}^h\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbb{R})}\right) \qquad (54)\\
&\quad+e^{(P_1+P_2M)t}\left(\left\|\hat{\mathcal{Q}}^h-\check{\mathcal{Q}}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}+K\left\|\hat{\mathcal{P}}^h-\check{\mathcal{P}}^h\right\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\right),
\end{aligned}$$

where, by **(Q)** and **(BD)**,

$$\begin{aligned}
K &= \left\|\hat{u}_{o,h}\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}+\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|\check{\mathcal{U}}^h\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbb{R})}+\left\|\check{\mathcal{Q}}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}\\
&\le M+\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}(M+1)\,T+Q_1TM+\|Q_2\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})}TM+Q_3TM^2.
\end{aligned}$$

Using **(BD)**, **(Q)** and **(P)**, we have:

$$\begin{aligned}
\left\|\hat{\mathcal{U}}^h-\check{\mathcal{U}}^h\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbb{R})} &\le \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}\left\|\hat{u}-\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)},\\
\left\|\hat{\mathcal{Q}}^h-\check{\mathcal{Q}}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})} &\le Q_1\left\|\hat{u}^h-\check{u}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}\\
&\quad+Q_3\left(\left\|\hat{u}\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L^1}(\mathcal{X};\mathbb{R}^k))}+\left\|\check{u}\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L^1}(\mathcal{X};\mathbb{R}^k))}\right)\\
&\quad\times\left\|\hat{u}-\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)}\\
&\le Q_1\left\|\hat{u}^h-\check{u}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})}+2MQ_3\left\|\hat{u}-\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)},
\end{aligned}$$

$$\begin{aligned}
\left\|\hat{\mathcal{P}}^h - \check{\mathcal{P}}^h\right\|_{\mathbf{L^1}([0,t];\mathbf{L^\infty}(\mathcal{X};\mathbb{R}))} &\le \int_0^t \left\|\hat{\mathcal{P}}^h(s) - \check{\mathcal{P}}^h(s)\right\|_{\mathbf{L^\infty}(\mathcal{X};\mathbb{R})} \mathrm{d}s \\
&\le \int_0^t \left\| p^h\left(s,\cdot,\hat{u}(s)\right) - p^h\left(s,\cdot,\check{u}(s)\right)\right\|_{\mathbf{L^\infty}(\mathcal{X};\mathbb{R})} \mathrm{d}s \\
&\le P_2 \int_0^t \left\|\hat{u}(s) - \check{u}(s)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \mathrm{d}s \\
&= P_2 \left\|\hat{u} - \check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)}.
\end{aligned}$$

Inserting these estimates into (54) we deduce that

$$\begin{aligned}
&\left\|\hat{u}^h(t) - \check{u}^h(t)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R})} \\
&\le e^{(P_1+P_2M)t}\left\|\hat{u}_o - \check{u}_o\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \\
&\quad + e^{(P_1+P_2M)t}\left(\|v\|_{\mathbf{L^\infty}([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})} + Q_1 + 2MQ_3 + KP_2\right)\left\|\hat{u} - \check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)}.
\end{aligned}$$

Sum over $h = 1, \dots, k$ and use Gronwall Lemma to prove **(WP.3)**, completing the proof. □

Proof of Corollary 2.3. For every $w \in X$, with X as in (44), define $u = \mathcal{T} w$ as the image of w through the operator $\mathcal{T}$, defined in (46). By (24), we deduce that $u^h(t,x) \ge 0$ for a.e. $x \in \mathcal{X}$. This implies that also the unique fixed point of the operator $\mathcal{T}$ has the same property, thus (4) holds. □

Proof of Corollary 2.4. By Theorem 2.2, we know that there exists a solution $u \in \mathbf{C^0}([0,T]; \mathbf{L^1}(\mathcal{X};\mathbb{R}^k))$ and that this solution can be uniquely extended beyond time T as long as $\|u(T)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}$ is bounded. By Corollary 2.3, $\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} = \sum_{h=1}^k \int_{\mathcal{X}} u^h(t,x)\mathrm{d}x$. Using (2), the Divergence Theorem and **(BD)**, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{h=1}^k \int_{\mathcal{X}} u^h(t,x)\mathrm{d}x \\
&= \sum_{h=1}^k \int_{\mathcal{X}} \left(p^h\left(t,x,u(t)\right)u(t) + q^h\left(t,x,u(t,x),u(t)\right)\right)\mathrm{d}x + \sum_{h=1}^k \int_{\partial\mathcal{X}} u_b^h\left(t,\xi,u(t)\right)\mathrm{d}\xi \\
&\le \int_{\mathcal{X}} \left(C_1(t,x) + C_2(t)\sum_{h=1}^k u^h(t,x)\right)\mathrm{d}x + \int_{\partial\mathcal{X}} B(\xi)\left(k + \|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}\right)\mathrm{d}\xi \\
&= \left(\|C_1\|_{\mathbf{L^\infty}([0,t];\mathbf{L^1}(\mathcal{X};\mathbb{R}))} + k\,\|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}\right) + \left(\|C_2\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} + \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})}\right)\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}
\end{aligned}$$

and usual ODE estimates ensure that $\|u(t)\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}$ is bounded on bounded intervals. □

Proof of Theorem 2.5. We divide the proof in several steps. *Theorem 2.2 applies.* We first check that the assumptions of Theorem 2.2 hold.

(P) holds. Fix $h \in \{1, \dots, k\}$, $t \in I$ and $x \in \mathcal{X}$. If $w \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$, then

$$\begin{aligned}\left|p^h(t,x,w)\right| &\le \bar{P}_1 + \bar{P}_2 \left\| \int_{\mathcal{X}} \mathcal{K}_p^h\left(t,x,x'\right) w(x')\mathrm{d}x' \right\| \\ &\le \bar{P}_1 + \bar{P}_2 \left\| \mathcal{K}_p^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kpk})} \|w\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}.\end{aligned}$$

If $w_1, w_2 \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$, then

$$\begin{aligned}\left|p^h(t,x,w_1) - p^h(t,x,w_2)\right| &\le \bar{P}_2 \left\| \int_{\mathcal{X}} \left|\mathcal{K}_p^h\left(t,x,x'\right)\right| \left|w_1(x') - w_2(x')\right| \mathrm{d}x' \right\| \\ &\le \bar{P}_2 \left\| \mathcal{K}_p^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kpk})} \|w_1 - w_2\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}.\end{aligned}$$

Therefore **(P)** holds with $P_1 = \bar{P}_1$ and $P_2 = \bar{P}_2 \left\| \mathcal{K}_p^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kpk})}$.

(Q) holds. Fix $h \in \{1, \dots, k\}$, $t \in I$ and $x \in \mathcal{X}$. If $u \in \mathbb{R}^k$ and $w \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$, then

$$\begin{aligned}\left|q^h(t,x,u,w)\right| &= \left| Q^h\left(t,x,u, \int_{\mathcal{X}} \mathcal{K}_q^h\left(t,x,x'\right) w(x')\mathrm{d}x'\right) \right| \\ &\le \bar{Q}_1 \|u\| + \bar{Q}_2(x) \left\| \int_{\mathcal{X}} \mathcal{K}_q^h\left(t,x,x'\right) w(x')\mathrm{d}x' \right\| + \bar{Q}_3 \|u\| \left\| \int_{\mathcal{X}} \mathcal{K}_q^h\left(t,x,x'\right) w(x')\mathrm{d}x' \right\| \\ &\le \bar{Q}_1 \|u\| + \left(\bar{Q}_2(x) + \bar{Q}_3 \|u\|\right) \left\| \mathcal{K}_q^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kqk})} \|w\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}.\end{aligned}$$

If $u_1, u_2 \in \mathbb{R}^k$ and $w_1, w_2 \in \mathbf{L^1}(\mathcal{X}; \mathbb{R}^k)$, then

$$\begin{aligned}&\left|q^h(t,x,u_1,w_1) - q^h(t,x,u_2,w_2)\right| \\ &\le \bar{Q}_1 \|u_1 - u_2\| + \bar{Q}_3 \left\| \mathcal{K}_u^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kqk})} \|w_1\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \|u_1 - u_2\| \\ &\quad + \bar{Q}_3 \|u_2\| \left\| \mathcal{K}_u^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kqk})} \|w_1 - w_2\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}.\end{aligned}$$

Therefore, condition **(Q)** holds with $Q_1 = \bar{Q}_1$, $Q_2(x) = \bar{Q}_2(x) \left\| \mathcal{K}_q^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kqk})}$, and $Q_3 = \bar{Q}_3(x) \left\| \mathcal{K}_q^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{kqk})}$. **(Q+)** is straightforward.

(BD) holds:

$$\left|u_b^h(t,\xi,w)\right| \le \bar{B}(\xi)\left(1+\left\|\int_{\mathcal{X}} \mathcal{K}_u^h(t,\xi,x')\,w(x')\mathrm{d}x'\right\|\right)$$
$$\le \bar{B}(\xi)\left(1+\left\|\mathcal{K}_u^h\right\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X}\times\mathcal{X};\mathbb{R}^{k_u k})}\|w\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}\right).$$
$$\left|u_b^h(t,\xi,w)-u_b^h(t,\xi,w')\right| \le \bar{B}(\xi)\left\|\mathcal{K}_u^h\right\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X}\times\mathcal{X};\mathbb{R}^{k_u k})}\left\|w-w'\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}$$

so **(BD)** holds with $B(\xi)=\bar{B}(\xi)\left(1+\|\mathcal{K}_u\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X}\times\mathcal{X};\mathbb{R}^{k_u k^2})}\right)$. Clearly, also **(BD+)** holds.

Stability Estimates. We now pass to the stability estimates. In each of the following cases, we keep $t\in I$ fixed and $h\in\{1,\ldots,k\}$. Define

$$\begin{aligned}
&\hat{\mathcal{U}}_b^h(t,\xi)=\hat{u}_b^h\left(t,\xi,\hat{u}(t)\right), \quad \hat{\mathcal{Q}}^h(t,x)=\hat{q}^h\left(t,x,\hat{u}(t,x),\hat{u}(t)\right), \quad \hat{\mathcal{P}}^h(t,x)=\hat{p}^h\left(t,x,\hat{u}(t)\right),\\
&\check{\mathcal{U}}_b^h(t,\xi)=\check{u}_b^h\left(t,\xi,\check{u}(t)\right), \quad \check{\mathcal{Q}}^h(t,x)=\check{q}^h\left(t,x,\check{u}(t,x),\check{u}(t)\right), \quad \check{\mathcal{P}}^h(t,x)=\check{p}^h\left(t,x,\check{u}(t)\right).
\end{aligned} \tag{55}$$

In order to use Proposition 4.3, compute preliminarily

$$\mathcal{P}(t)=\exp\left(t\,\max\left\{\left\|\hat{\mathcal{P}}^h\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})},\left\|\check{\mathcal{P}}^h\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R})}\right\}\right)\le\exp\left(t\left(P_1+P_2M\right)\right),$$

where M is an upper bound for the $\mathbf{L}^\infty$ in time and $\mathbf{L}^1$ in space norms of both solutions. Therefore, Proposition 4.3 implies that

$$\begin{aligned}
&\left\|\hat{u}^h(t)-\check{u}^h(t)\right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R})}\\
\le&\,\mathcal{P}(t)\,\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{n+m})}\left\|\hat{\mathcal{U}}_b^h-\check{\mathcal{U}}_b^h\right\|_{\mathbf{L}^1([0,t]\times\partial\mathcal{X};\mathbb{R})}+\mathcal{P}(t)\left\|\hat{\mathcal{Q}}^h-\check{\mathcal{Q}}^h\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R})}\\
&+\mathcal{P}(t)\left(\|u_o\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}+\left\|\check{\mathcal{Q}}^h\right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R})}\right)\left\|\hat{\mathcal{P}}^h-\check{\mathcal{P}}^h\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}\\
&+\mathcal{P}(t)\,\|v\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X};\mathbb{R}^{k\times(n+m)})}\left\|\check{\mathcal{U}}_b\right\|_{\mathbf{L}^1([0,t]\times\partial\mathcal{X};\mathbb{R}^k)}\left\|\hat{\mathcal{P}}^h-\check{\mathcal{P}}^h\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))}.
\end{aligned} \tag{56}$$

Then, we estimate the terms in (56). Using **(BD)** and (55) we deduce that

$$\begin{aligned}
&\left\|\hat{\mathcal{U}}_b^h-\check{\mathcal{U}}_b^h\right\|_{\mathbf{L}^1([0,t]\times\partial\mathcal{X};\mathbb{R})}\\
=&\int_0^t\int_{\partial X}\left|\hat{u}_b^h\left(\tau,\xi,\hat{u}(\tau)\right)-\check{u}_b^h\left(\tau,\xi,\check{u}(\tau)\right)\right|\mathrm{d}\xi\,\mathrm{d}\tau\\
\le&\int_0^t\int_{\partial X}\left|\hat{u}_b^h\left(\tau,\xi,\hat{u}(\tau)\right)-\hat{u}_b^h\left(\tau,\xi,\check{u}(\tau)\right)\right|\mathrm{d}\xi\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{\partial\mathcal{X}} \left| \hat{u}_b^h\left(\tau,\xi,\check{u}(\tau)\right) - \check{u}_b^h\left(\tau,\xi,\check{u}(\tau)\right)\right| \mathrm{d}\xi\,\mathrm{d}\tau \\
\leq\ & \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})} \left\|\hat{u}-\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)} \\
&+\int_0^t \int_{\partial\mathcal{X}} \left| \hat{U}_b^h\left(\tau,\xi,\int_{\mathcal{X}} \hat{\mathcal{K}}_u^h(\tau,\xi,x')\,\check{u}(\tau,x')\mathrm{d}x'\right) - \hat{U}_b^h\left(\tau,\xi,\int_{\mathcal{X}} \check{\mathcal{K}}_u^h(\tau,\xi,x')\,\check{u}(\tau,x')\mathrm{d}x'\right)\right| \mathrm{d}\xi\,\mathrm{d}\tau \\
&+\int_0^t \int_{\partial\mathcal{X}} \left| \hat{U}_b^h\left(\tau,\xi,\int_{\mathcal{X}} \check{\mathcal{K}}_u^h(\tau,\xi,x')\,\check{u}(\tau,x')\mathrm{d}x'\right) - \check{U}_b^h\left(\tau,\xi,\int_{\mathcal{X}} \check{\mathcal{K}}_u^h(\tau,\xi,x')\,\check{u}(\tau,x')\mathrm{d}x'\right)\right| \mathrm{d}\xi\,\mathrm{d}\tau \\
\leq\ & \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})} \left\|\hat{u}-\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)} \\
&+\int_0^t \int_{\partial\mathcal{X}} \bar{B}(\xi) \left\|\hat{\mathcal{K}}_u^h - \check{\mathcal{K}}_u^h\right\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X}\times\mathcal{X};\mathbb{R}^{k_u k})} \left\|\check{u}(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \mathrm{d}\xi\,\mathrm{d}\tau \\
&+\left\|\hat{U}_b^h - \check{U}_b^h\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbf{L}^\infty(\mathbb{R}^{k_u};\mathbb{R}))} \\
\leq\ & \|B\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})} \left\|\hat{u}-\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)} \\
&+\left\|\bar{B}\right\|_{\mathbf{L^1}(\partial\mathcal{X};\mathbb{R})} \left\|\hat{\mathcal{K}}_u^h - \check{\mathcal{K}}_u^h\right\|_{\mathbf{L}^\infty([0,t]\times\partial\mathcal{X}\times\mathcal{X};\mathbb{R}^{k_u k})} \left\|\check{u}\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R}^k)} \\
&+\left\|\hat{U}_b^h - \check{U}_b^h\right\|_{\mathbf{L^1}([0,t]\times\partial\mathcal{X};\mathbf{L}^\infty(\mathbb{R}^{k_u};\mathbb{R}))}.
\end{aligned}
$$

Using **(Q)** we deduce that

$$
\begin{aligned}
&\left\|\hat{Q}^h - \check{Q}^h\right\|_{\mathbf{L^1}([0,t]\times\mathcal{X};\mathbb{R})} \\
\leq\ & \int_0^t \int_{\mathcal{X}} \left| \hat{q}^h\left(\tau,x,\hat{u}(\tau,x),\hat{u}(\tau)\right) - \hat{q}^h\left(\tau,x,\check{u}(\tau,x),\check{u}(\tau)\right)\right| \mathrm{d}x\,\mathrm{d}\tau \\
&+\int_0^t \int_{\mathcal{X}} \left| \hat{q}^h\left(\tau,x,\check{u}(\tau,x),\check{u}(\tau)\right) - \check{q}^h\left(\tau,x,\check{u}(\tau,x),\check{u}(\tau)\right)\right| \mathrm{d}x\,\mathrm{d}\tau \\
\leq\ & Q_1 \int_0^t \left\|\hat{u}(\tau)-\check{u}(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \mathrm{d}\tau + Q_3 \int_0^t \left\|\hat{u}(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \int_{\mathcal{X}} \left\|\hat{u}(\tau,x)-\check{u}(\tau,x)\right\| \mathrm{d}x\,\mathrm{d}\tau \\
&+Q_3 \int_0^t \left\|\hat{u}(\tau)-\check{u}(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)} \int_{\mathcal{X}} \left\|\check{u}(\tau,x)\right\| \mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathcal{X}} \left| \hat{Q}^h \left(\tau, x, \check{u}(\tau,x), \int_{\mathcal{X}} \hat{\mathcal{K}}_q^h \left(\tau, x, x'\right) \check{u}\left(\tau, x'\right) \mathrm{d}x' \right) \right. \\
&\qquad\qquad \left. -\check{Q}^h \left(\tau, x, \check{u}(\tau,x), \int_{\mathcal{X}} \check{\mathcal{K}}_q^h \left(\tau, x, x'\right) \check{u}\left(\tau, x'\right) \mathrm{d}x' \right) \right| \mathrm{d}x \, \mathrm{d}\tau \\
&\le \left(Q_1 + Q_3 \left(\left\| \hat{u} \right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathcal{X};\mathbb{R}^k))} + \left\| \check{u} \right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathcal{X};\mathbb{R}^k))} \right) \right) \int_0^t \left\| \hat{u}(\tau) - \check{u}(\tau) \right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)} \mathrm{d}\tau \\
&+\int_0^t \int_{\mathcal{X}} \sup_{\eta \in \mathbb{R}^{k_q}} \left| \hat{Q}^h \left(\tau, x, \check{u}(\tau,x), \eta\right) - \check{Q}^h \left(\tau, x, \check{u}(\tau,x), \eta\right) \right| \mathrm{d}x \, \mathrm{d}\tau \\
&+\bar{Q}_3 \int_0^t \int_{\mathcal{X}} \left\| \check{u}(\tau,x) \right\| \left\| \int_{\mathcal{X}} \left(\hat{\mathcal{K}}_q^h(\tau,x,x') - \check{\mathcal{K}}_q^h(\tau,x,x') \right) \check{u}\left(\tau, x'\right) \mathrm{d}x' \right\| \mathrm{d}x \, \mathrm{d}\tau \\
&\le \left(Q_1 + Q_3 \left(\left\| \hat{u} \right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathcal{X};\mathbb{R}^k))} + \left\| \check{u} \right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathcal{X};\mathbb{R}^k))} \right) \right) \left\| \hat{u} - \check{u} \right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbb{R}^k)} \\
&+\left\| \hat{Q}^h - \check{Q}^h \right\|_{\mathbf{L}^1([0,t]\times\mathcal{X};\mathbf{L}^\infty(\mathbb{R}^k\times\mathbb{R}^{k_q};\mathbb{R}))} + \int_0^t \left\| \check{u}(\tau) \right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)}^2 \mathrm{d}\tau \left\| \hat{\mathcal{K}}_q^h - \check{\mathcal{K}}_q^h \right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{X}^2;\mathbb{R}^{k_q})}.
\end{aligned}
$$

Using **(P)**, we have

$$
\begin{aligned}
&\left\| \hat{\mathcal{P}}^h - \check{\mathcal{P}}^h \right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathcal{X};\mathbb{R}))} \\
&\le \int_0^t \sup_{x\in\mathcal{X}} \left| \hat{p}^h \left(\tau, x, \hat{u}(\tau)\right) - \hat{p}^h \left(\tau, x, \check{u}(\tau)\right) \right| \mathrm{d}\tau + \int_0^t \sup_{x\in\mathcal{X}} \left| \hat{p}^h \left(\tau, x, \check{u}(\tau)\right) - \check{p}^h \left(\tau, x, \check{u}(\tau)\right) \right| \mathrm{d}\tau \\
&\le P_2 \int_0^t \left\| \hat{u}(\tau) - \check{u}(\tau) \right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)} \mathrm{d}\tau \\
&+\int_0^t \sup_{x\in\mathcal{X}} \left| \hat{P}^h \left(\tau, x, \int_{\mathcal{X}} \hat{\mathcal{K}}_p^h(\tau,x,x') \check{u}(\tau,x') \mathrm{d}x' \right) - \check{P}^h \left(\tau, x, \int_{\mathcal{X}} \hat{\mathcal{K}}_p^h(\tau,x,x') \check{u}(\tau,x') \mathrm{d}x' \right) \right| \mathrm{d}\tau \\
&+\int_0^t \sup_{x\in\mathcal{X}} \left| \check{P}^h \left(\tau, x, \int_{\mathcal{X}} \hat{\mathcal{K}}_p^h(\tau,x,x') \check{u}(\tau,x') \mathrm{d}x' \right) - \check{P}^h \left(\tau, x, \int_{\mathcal{X}} \check{\mathcal{K}}_p^h(\tau,x,x') \check{u}(\tau,x') \mathrm{d}x' \right) \right| \mathrm{d}\tau \\
&\le P_2 \int_0^t \left\| \hat{u}(\tau) - \check{u}(\tau) \right\|_{\mathbf{L}^1(\mathcal{X};\mathbb{R}^k)} \mathrm{d}\tau + t \left\| \hat{P}^h - \check{P}^h \right\|_{\mathbf{L}^\infty\left([0,t]\times\mathcal{X}\times\mathbb{R}^{k_p};\mathbb{R}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+\bar{P}_2\int_0^t \sup_{x\in\mathcal{X}}\int_{\mathcal{X}}\left|\hat{\mathcal{K}}_p^h(\tau,x,x')-\check{\mathcal{K}}_p^h(\tau,x,x')\right|\left|\check{u}(\tau,x')\right|\mathrm{d}x'\mathrm{d}\tau\\
&\le P_2\int_0^t \left\|\hat{u}(\tau)-\check{u}(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}\mathrm{d}\tau+t\left\|\hat{P}^h-\check{P}^h\right\|_{\mathbf{L}^\infty\left([0,t]\times\mathcal{X}\times\mathbb{R}^{k_p};\mathbb{R}\right)}\\
&+\bar{P}_2\int_0^t \left\|\check{u}(\tau)\right\|_{\mathbf{L^1}(\mathcal{X};\mathbb{R}^k)}\mathrm{d}\tau\left\|\hat{\mathcal{K}}_p^h-\check{\mathcal{K}}_p^h\right\|_{\mathbf{L}^\infty\left([0,t]\times\mathcal{X}^2;\mathbb{R}^{k_p k}\right)}.
\end{aligned}
$$

The above estimate, duly inserted in (56) and followed by a standard application of Gronwall Lemma, completes the proof. □

Proof of Proposition 3.1. Checking **(V)** and **(ID)** is immediate. It is sufficient to verify that the assumptions of Theorem 2.5 hold. It is immediate to check that **($\bar{\mathbf{P}}$)** holds with $\bar{P}_1 = \max\{\|\mu_S\|, \|\mu_I\| + \|\kappa+\theta\|, \|\mu_H\| + \|\eta\|, \|\mu_R\|\}$ (all norms being in $\mathbf{L}^\infty(\mathcal{I}\times\mathbb{R}_+\times\mathbb{R}^2;\mathbb{R})$), $\bar{P}_2 = 1$, thanks to $\rho\in\mathbf{L}^\infty$. Concerning **($\bar{\mathbf{Q}}$)**, choose $\bar{Q}_1 = \max\{\|\kappa\|, \|\eta+\theta\|\}$, $\bar{Q}_2 = 0$, $\bar{Q}_3 = 1$ and use $\rho\in\mathbf{L}^\infty$. Finally, **($\overline{\mathbf{BD}}$)** holds with $\bar{B}(\xi) = \sup_I \|S_b(t)\|_{\mathbf{L}^\infty(\mathcal{X},\mathbb{R})}$.

Positivity is immediate. To apply Corollary 2.4, simply set $C_1\equiv 0$ and $C_2\equiv 0$.

To obtain an $\mathbf{L}^\infty$ bound, note first that since $I\in\mathbf{C^0}\left(\mathcal{I};\mathbf{L^1}(\mathbb{R}_+\times\mathbb{R}^2;\mathbb{R})\right)$, the integral in (12) is bounded on any bounded time interval. Hence, a repeated application of (35) in Lemma 4.2 yields the boundedness of S, I, H and R on any bounded interval. Uniqueness then follows from **(WP.2)**. □

Proof of Proposition 3.2. Assumptions **(V)** and **(ID)** trivially hold. Condition **(P)** holds with $P_1 = \|d\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}/\varepsilon$ and $P_2 = 1/\varepsilon$. Verifying **(Q)** is straightforward. To prove that **(BD)** holds, compute for $y\in\mathbb{R}^n$ with $\|y\| > r$:

$$
\begin{aligned}
|u_b(t,y,w)| &= \left|\frac{1}{A(a=0,y)\,\varepsilon^n}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\mathcal{M}\left(\frac{y'-y}{\varepsilon}\right)b(a',y')\,w(a',y')\mathrm{d}a'\mathrm{d}y'\right|\\
&\le \frac{1}{\varepsilon^n\ \inf A}\left|\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\mathcal{M}\left(\frac{y'-y}{\varepsilon}\right)b(a',y')\,w(a',y')\mathrm{d}a'\mathrm{d}y'\right|\\
&\le \frac{1}{\varepsilon^n\ \inf A}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\left|\mathcal{M}\left(\frac{y'-y}{\varepsilon}\right)\right|\left(\sup_{|y'-y|<r}\left|b(a',y')\right|\right)\left|w(a',y')\right|\mathrm{d}a'\mathrm{d}y'\\
&\le \frac{1}{\varepsilon^n\ \inf A}\|\mathcal{M}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\frac{1}{(1+\|y\|-r)^{n+1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\left|w(a',y')\right|\mathrm{d}a'\mathrm{d}y'
\end{aligned}
$$

proving the first requirement in **(BD)**. Lipschitz continuity is proved by the same procedure.

The assumptions on the signs of data and parameters allow to apply Corollary 2.3 and ensure that (5) holds. □

Data availability

No data was used for the research described in the article.

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