Journal of Magnetism and Magnetic Materials 626 (2025) 173038

Contents lists available at ScienceDirect

Journal of Magnetism and Magnetic Materials

journal homepage: www.elsevier.com/locate/jmmm

Research article

On the modeling of magneto-mechanical effects in solids

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ARTICLE INFO

MSC:
74A15
74F15

Keywords:
Magnetostriction
Non-monotonic effects
Lagrangian fields
Thermodynamic consistency

ABSTRACT

The paper develops a thermodynamically-consistent approach to magnetostriction. This is performed by following two different approaches depending on whether a three-dimensional or a one-dimensional setting is considered. In the three-dimensional case the symmetry condition required by the balance of angular momentum results in the need of appropriate variables in the constitutive equations. These variables prove to be Euclidean invariant and comprise the so-called Lagrangian fields usually adopted in the literature. The consequences of the second law of thermodynamics are then determined for a solid described by the temperature, the deformation gradient, and the magnetic field. With this background the magnetostriction is modeled for linear or nonlinear magnetic laws. Next a one-dimensional setting is addressed mainly in connection with available experimental data. The symmetry condition becomes ineffective and hence the classical Eulerian fields are used. Based on the relations established through the thermodynamic consistency a detailed set of constitutive equations, for magnetization and strain, is established. These equations are set up so as to fit the experimental data from a one-dimensional sample under tensile stresses and magnetic fields.

1. Introduction

There are materials that can deform under the action of external stimuli thus allowing the realization of elastomers. Among these materials, magneto-elastic solids have been widely investigated in the literature (see, e.g. [1–4] and refs therein). The appearance of a mechanical deformation, induced by an applied magnetic field, is usually referred to as forced magnetostriction. The modeling of magnetostriction requires a systematic scheme allowing for a magneto-mechanical coupling. This coupling is realized through various approaches and, e.g., Refs. [5–13] give an exhaustive picture of the methods applied in the literature.

A typical magnetostrictive material consists of tiny ferromagnets. These ferromagnets, usually iron, nickel or cobalt, have small magnetic moments as a result of their 3d shells that are not completely filled with electrons. Essentially, the ferromagnets act like tiny permanent bar magnets. When a magnetic field is applied to the material, the randomly located magnets realign themselves with the fields axis. This new ordered structure causes the solid to either stretch or shrink. The strains are usually small, typically 10^{-5} (see, e.g., [1,9,10]). However nanomaterials seem to allow large magnetostrictive effects and this offers a further motivation for the investigation of magnetostriction [11].

Besides depending on the properties of the material, magnetostriction shows different features depending on the temperature. If the absolute temperature θ is above the Curie temperature θ_C then the material is paramagnetic and the law $\mathsf{M}(\mathsf{H})$ of the magnetization M in terms of the magnetic field H is almost linear. If instead $\theta < \theta_C$ the material becomes ferromagnetic in that it breaks up into domains and the molecular field tend to align the spins of the domains; denote by M_s the spin of the domains per unit volume. This partial alignment results in a so-called spontaneous magnetostriction. If a magnetic field H is applied then the magnetization M increases according to a proper law of the material up to the value M_s when the spins are fully aligned; in this process the corresponding magnetostriction is referred to as ordinary (see, e.g., [14]). If the field H is further increased, the structure of the domains is weakly affected thus producing an increase in M and magnetostriction (called saturation magnetostriction). On the whole the curve $\mathsf{M}(\mathsf{H})$ shows the typical nonlinear ferromagnetic dependence. Anyway we look for the description of magnetostriction depending on the form of $\mathsf{M}(\mathsf{H})$. It is a reference property of magnetostriction that, at relatively small values of H, the magnetostriction shows a square law dependence on the magnetic field. Incidentally, an analogous property occurs with electrostriction [1,15–18]. The purpose of this paper is to develop a thermodynamically-consistent approach to electromagnetic deformable solids; for future convenience some results are determined also for the electric field. This is performed by following two different approaches depending on whether a three-dimensional or a one-dimensional setting is considered.

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https://doi.org/10.1016/j.jmmm.2025.173038
Received 11 June 2024; Received in revised form 13 January 2025; Accepted 5 April 2025
Available online 18 April 2025

In the three-dimensional setting the use of Lagrangian fields is within the lines of some works of Ogden and co-workers [4,8,19]. Yet the originality of the approach is related to a restriction on the appropriate electromagnetic fields. The restriction is indicated by the symmetry condition arising from the balance of angular momentum. The technical advantage of the Lagrangian fields is the Euclidean invariance of the time derivatives. Upon the non-unique selection of Lagrangian fields the thermodynamic consistency is developed. The dependence of constitutive functions on the Lagrangian fields proves to be consistent also with the symmetry condition.

In the one-dimensional setting the symmetry condition becomes ineffective and hence the classical Eulerian fields are used. Based on the relations established through the thermodynamic consistency a detailed set of constitutive equations, for magnetization and strain, is established. These equations are specified so as to fit the experimental data from a one-dimensional sample under tensile or compressive stresses and magnetic fields. It is of interest that non-monotonic dependences occur and a simple model is found to fit data with a product of functions of the stress and functions of the magnetic field.

Notation

The motion of a body occupying a time-dependent region $\Omega \subset \mathscr{E}^3$ is described by means of the function $\hat{\mathbf{x}}(\mathbf{X},t)$, providing the position vector $\mathbf{x} \in \Omega = \hat{\mathbf{x}}(\mathrm{R},t)$. The symbols ∇ and ∇_R denote the gradient operator with respect to $\mathbf{x} \in \Omega$ and $\mathbf{X} \in \mathrm{R}$ while $\nabla\cdot$ and $\nabla_R\cdot$ are the corresponding divergences. The function $\hat{\mathbf{x}}$ is assumed to be differentiable and this allows the definition of deformation gradient as $\mathbf{F} = \nabla_R \hat{\mathbf{x}}$ or, in suffix notation, $F_{iK} = \partial_{X_K}\hat{x}_i$. The invertibility of $\mathbf{X} \to \mathbf{x} = \hat{\mathbf{x}}(\mathbf{X},t)$ is guaranteed by letting $J := \det \mathbf{F} > 0$. The symbol $\mathbf{1}$ is the second-order unit tensor,

$\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green deformation tensor and $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})$ is the mechanical Green–Lagrange strain tensor.

Throughout $(\mathbf{x},t) \in \Omega\times\mathbb{R}^+$. We let $\mathbf{v}(\mathbf{x},t)$ be the velocity field and $\mathbf{L}$, $L_{ij} = \partial_{x_j} v_i$, be the velocity gradient. For any function $f(\mathbf{x},t)$ we let $\dot{f}$ be the total time derivative, $\dot{f} = \partial_t f + (\mathbf{v}\cdot\nabla)f$. For any tensor $\mathbf{A}$ we define the magnitude $|\mathbf{A}|$ as $(\mathbf{A}\cdot\mathbf{A})^{1/2}$; the symbols $\mathrm{sym}\mathbf{A}$ and $\mathrm{skw}\mathbf{A}$ denote the symmetric part and the skew part of $\mathbf{A}$ while Sym stands for the space of symmetric tensors. In particular $\mathbf{D} = \mathrm{sym}\,\mathbf{L}, \mathbf{W} = \mathrm{skw}\,\mathbf{L}$. The MKS units are used so that $\mathsf{B} = \mu_0(\mathsf{H}+\mathsf{M})$, where H is the magnetic intensity, M the magnetization, and B the magnetic induction. Throughout the electromagnetic fields are relative to the reference at rest with the point under consideration.

2. Balance equations and selection of variables

We consider an elastic solid that exhibits nonlinear magneto-mechanical couplings. For formal simplicity we let the solid be electrically neutral with zero electric current and electric polarization.

The balance equations of electromagnetism are developed in known books and papers (e.g., [20,21]); here we summarize the main points for the next derivations. The mass density ρ satisfies the continuity equation $\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0$ while the equation of motion has the form

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{T} + \mathbf{b} + \mu_0(\mathsf{M}\cdot\nabla)\mathsf{H},$$

where $\mathbf{T}$ is the mechanical Cauchy stress, $\mathbf{b}$ is the mechanical body force density, and $\mu_0(\mathsf{M}\cdot\nabla)\mathsf{H}$ is the force density due to the interaction between the magnetization M and the magnetic field H. The form $\mu_0(\mathsf{M}\cdot\nabla)\mathsf{H}$ can be justified by starting from the Lorentz force (see, e.g. [22, §5.7]) or using the view of Gilbertian magnetic dipole (see [21] and [23, §2.16.1]). The balance of the angular momentum results in

$$\mathrm{skw}(\mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M}) = \mathbf{0}. \tag{1}$$

The balance of energy is taken in the form

$$\rho\,\dot{\varepsilon} = \rho\mu_0\mathsf{H}\cdot\dot{\mathsf{m}} + \mathbf{T}\cdot\mathbf{L} - \nabla\cdot\mathbf{q} + \rho r,$$

where ε is the internal energy density, $\mathsf{m} = \mathsf{M}/\rho$ is the specific magnetization, $\mathbf{q}$ is the heat flux, and r is the energy supply.

For the purposes of this paper there is no loss of generality by letting $\mathbf{q}/\theta$ be the entropy flux where θ is the absolute temperature. Let η be the entropy density and

$$\phi = \varepsilon - \theta\eta - \mu_0\mathsf{H}\cdot\mathsf{m}$$

be the magnetic free energy density. By the Clausius–Duhem (CD for short) inequality [24,25]

$$\rho\dot{\eta} + \nabla\cdot(\mathbf{q}/\theta) - \rho r/\theta = \rho\gamma \geq 0,$$

where γ is the (rate of) entropy production, we can write

$$-\rho(\dot{\phi} + \eta\dot{\theta}) - \mu_0\mathsf{M}\cdot\dot{\mathsf{H}} + \mathbf{T}\cdot\mathbf{L} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta = \rho\theta\gamma \geq 0. \tag{2}$$

We now investigate the restrictions placed by the CD inequality (2) and check the consistency with the symmetry condition (1).

Let the constitutive functions of $\phi, \eta, \mathbf{T}, \mathbf{q}, \gamma$ depend on the set of variables

$$\Gamma = (\theta, \mathbf{F}, \mathsf{H}, \nabla\theta, \mathbf{D}).$$

while are given by constitutive functions of Γ. Compute $\dot{\phi}$ and substitute in the CD inequality (2) to obtain

$$-\rho(\partial_\theta\phi + \eta)\dot{\theta} + (\mathbf{T} - \rho\partial_{\mathbf{F}}\phi\mathbf{F}^T)\cdot(\mathbf{D} + \mathbf{W}) - (\mu_0\mathsf{M} + \rho\partial_{\mathsf{H}}\phi)\cdot\dot{\mathsf{H}}$$
$$-\rho\partial_{\nabla\theta}\phi\cdot(\nabla\theta)^{\cdot} - \rho\partial_{\mathbf{D}}\phi\cdot\dot{\mathbf{D}} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta = \rho\theta\gamma \geq 0.$$

As to the possible arbitrariness of $\dot{\mathsf{H}}$ we observe that $\dot{\mathsf{H}} = \partial_t\mathsf{H} + \mathbf{v}\cdot\nabla\mathsf{H}$ and $\partial_t\mathsf{H}$ enters Maxwell's equation

$$\nabla\times\mathsf{E} = -\mu_0(\partial_t\mathsf{M} + \partial_t\mathsf{H}),$$

where E is the electric field. An appropriate value of $\nabla\times\mathsf{E}$ allows for any value of $\partial_t\mathsf{H}$ without affecting the constitutive functions. Hence the vector $\dot{\mathsf{H}}$ in the CD inequality (2) can take arbitrary vector values. The linearity and arbitrariness of $(\nabla\theta)^{\cdot}, \dot{\theta}, \dot{\mathbf{D}}$, and $\mathbf{W}, \dot{\mathsf{H}}$ imply that

$$\partial_{\nabla\theta}\phi = \mathbf{0}, \quad \eta = -\partial_\theta\phi, \quad \partial_{\mathbf{D}}\phi = \mathbf{0},$$

and

$$\mathbf{T} - \rho\partial_{\mathbf{F}}\phi\mathbf{F}^T \in \mathrm{Sym}, \quad \mu_0\mathsf{M} = -\rho\partial_{\mathsf{H}}\phi, \tag{3}$$

while the CD inequality reduces to

$$(\mathbf{T} - \rho\partial_{\mathbf{F}}\phi\mathbf{F}^T)\cdot\mathbf{D} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta = \rho\theta\gamma \geq 0. \tag{4}$$

Letting

$$\Xi := \mathbf{T} - \rho\partial_{\mathbf{F}}\phi\mathbf{F}^T$$

we observe that Ξ is a function of $(\theta, \mathbf{F}, \mathsf{H}, \nabla\theta, \mathbf{D})$ such that, by (4),

$$\Xi\cdot\mathbf{D} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta = \rho\theta\gamma \geq 0.$$

The functions Ξ and $\mathbf{q}$ are then required to determine a non-negative entropy production γ.

Comparing (1) and (3) we have

$$\partial_{\mathbf{F}}\phi\mathbf{F}^T - \mathsf{H}\otimes\partial_{\mathsf{H}}\phi \in \mathrm{Sym}. \tag{5}$$

Since

$$\mathsf{H}\otimes\partial_{\mathsf{H}}\phi = -\partial_{\mathsf{H}}\phi\otimes\mathsf{H} - (\mathsf{H}\otimes\partial_{\mathsf{H}}\phi - \partial_{\mathsf{H}}\phi\otimes\mathsf{H})$$

then (5) is equivalent to

$$\partial_{\mathbf{F}}\phi\mathbf{F}^T + \partial_{\mathsf{H}}\phi\otimes\mathsf{H} \in \mathrm{Sym}. \tag{6}$$

The requirements (5) and (6) denote that ϕ cannot depend arbitrarily on $\mathbf{F}$ and H. Now (5) and (6) imply that

$$\partial_{\mathbf{F}}\phi\mathbf{F}^T = \mathsf{H}\otimes\partial_{\mathsf{H}}\phi \quad \text{and} \quad \partial_{\mathbf{F}}\phi\mathbf{F}^T = -\partial_{\mathsf{H}}\phi\otimes\mathsf{H} \tag{7}$$

to within symmetric tensors. The solution to (7) is non-unique. By a direct check we find that

$$\mathfrak{H} = \mathbf{F}^T\mathsf{H}, \qquad \hat{\mathfrak{H}} = \mathbf{F}^{-1}\mathsf{H}$$

are solutions to (7) as well as $f(J)\mathfrak{H}$ and $g(J)\hat{\mathfrak{H}}$, for arbitrary functions f, g of $J = \det \mathbf{F}$; both $f(J)\mathfrak{H}$ and $g(J)\hat{\mathfrak{H}}$ are objective (invariant) vectors (see Appendix A). Furthermore $\mathfrak{H}$ is the Lagrangian magnetic field (see, e.g., [4–6,8]). The dependencies on $\mathfrak{H}$ and $\hat{\mathfrak{H}}$ are consistent with (1). Consistent with the widely-adopted selection in the literature we let $\phi = \phi_0(\theta, \mathfrak{H})$ so that

$$\partial_{\mathbf{F}}\phi_0\mathbf{F}^T - \mathsf{H}\otimes\partial_{\mathsf{H}}\phi_0 = \mathbf{0}.$$

Now, the elastic properties occurring when $\mathsf{H} = \mathbf{0}$ are modeled by an additional (partial) dependence on $\mathbf{F}$. If the dependence on $\mathbf{F}$ is through the invariant strain $\mathbf{E}$,

$$\phi = \Phi(\theta, \mathbf{E}, \mathfrak{H}) \tag{8}$$

then

$$\partial_{\mathbf{F}}\phi\mathbf{F}^T - \mathsf{H}\otimes\partial_{\mathsf{H}}\phi = \mathbf{F}\partial_{\mathbf{E}}\Phi\mathbf{F}^T + \partial_{\mathfrak{H}}\Phi\partial_{\mathbf{F}}\mathfrak{H}\mathbf{F}^T - \mathsf{H}\otimes\partial_{\mathsf{H}}\Phi = \mathbf{F}\partial_{\mathbf{E}}\Phi\mathbf{F}^T \in \mathrm{Sym},$$

and hence Eq. (5) holds identically.

Though with different arguments, an assumption analogous to (8) is considered e.g. in [4] where the free energy is represented in the form $\Omega(\mathbf{F}, \mathbf{B}_L)$, with $\mathbf{B}_L = J\mathbf{F}^{-1}\mathsf{B}$ being the Lagrangian induction field.

Surface forces develop at the boundary as a consequence of large magnetization gradients. The corresponding effects can be modeled by accounting for the gradient $\nabla\mathsf{M}$ among the variables (see, e.g., [26] and also [23, § 12.4] for dielectrics with polarization gradient).

3. Thermodynamic restrictions

We now consider the CD inequality (2) and derive the thermodynamic restrictions associated with the chosen set of variables. Observe that

$$\dot{\mathsf{H}} = \mathbf{F}^{-T}\dot{\mathfrak{H}} - \mathbf{L}^T\mathsf{H},$$

and then

$$\mathsf{M}\cdot\dot{\mathsf{H}} = (\mathbf{F}^{-1}\mathsf{M})\cdot\dot{\mathfrak{H}} - (\mathsf{H}\otimes\mathsf{M})\cdot\mathbf{L},$$

Due to the representation of the mechanical power (see Appendix B) we find

$$\mathbf{T}\cdot\mathbf{L} - \mu_0\mathsf{M}\cdot\dot{\mathsf{H}} = -\mu_0(\mathbf{F}^{-1}\mathsf{M})\cdot\dot{\mathfrak{H}} + (\mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M})\cdot(\mathbf{D} + \mathbf{W}).$$

Furthermore, since $\mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}$ then

$$(\mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M})\cdot\mathbf{D} = (\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} + \mu_0\mathbf{C}^{-1}\mathfrak{H}\otimes\mathbf{F}^{-1}\mathsf{M})\cdot\dot{\mathbf{E}}.$$

Substitute in (2), multiply by J and observe that $\rho J = \rho_R$ is a constant to obtain

$$\begin{aligned} -\rho_R(\dot{\phi} + \eta\dot{\theta}) - \mu_0(J\mathbf{F}^{-1}\mathsf{M})\cdot\dot{\mathfrak{H}} + J(\mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M})\cdot\mathbf{W} \\ + [J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} + \mu_0\mathbf{C}^{-1}\mathfrak{H}\otimes(J\mathbf{F}^{-1}\mathsf{M})]\cdot\dot{\mathbf{E}} - \frac{J}{\theta}\mathbf{q}\cdot\nabla\theta = \rho_R\theta\gamma \geq 0. \end{aligned} \tag{9}$$

This equation shows the occurrence of $J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$, which is just the second Piola stress,

$$\mathbf{T}_{RR} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}.$$

Now the field $J\mathbf{F}^{-1}\mathsf{M}$ occurs as the conjugate magnetization field associated with $\mathfrak{H}$. Hence we let

$$\mathfrak{M} = J\mathbf{F}^{-1}\mathsf{M}$$

and write (9) in the form

$$\begin{aligned} &-(\dot{\phi}_R + \eta_R\dot{\theta}) - \mu_0\mathfrak{M}\cdot\dot{\mathfrak{H}} + (\mathbf{T}_{RR} + \mu_0\mathbf{C}^{-1}\mathfrak{H}\otimes\mathfrak{M})\cdot\dot{\mathbf{E}} \\ &+ J(\mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M})\cdot\mathbf{W} - \frac{J}{\theta}\mathbf{q}\cdot\nabla\theta = \rho_R\theta\gamma \geq 0. \end{aligned} \tag{10}$$

where $\phi_R = \rho_R\phi, \eta_R = \rho_R\eta$. For formal convenience we also let

$$\mathcal{T} := \mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M}, \quad \mathcal{T}_{RR} := \mathbf{T}_{RR} + \mu_0\mathbf{C}^{-1}\mathfrak{H}\otimes\mathfrak{M} = J\mathbf{F}^{-1}\mathcal{T}\mathbf{F}^{-T}. \tag{11}$$

Furthermore we notice that the referential gradient $\nabla_R\theta$ is related to the spatial gradient $\nabla\theta$ in the form $\nabla_R\theta = \mathbf{F}^T\nabla\theta$. Hence it follows that

$$J\mathbf{q}\cdot\nabla\theta = \mathbf{q}_R\cdot\nabla_R\theta,$$

where $\mathbf{q}_R = J\mathbf{q}\mathbf{F}^{-T}$ is the referential heat flux. Hence we can write (10) in the simplified form

$$-(\dot{\phi}_R + \eta_R\dot{\theta}) - \mu_0\mathfrak{M}\cdot\dot{\mathfrak{H}} + \mathcal{T}_{RR}\cdot\dot{\mathbf{E}} + J\mathcal{T}\cdot\mathbf{W} - \frac{1}{\theta}\mathbf{q}_R\cdot\nabla_R\theta = \rho_R\theta\gamma \geq 0. \tag{12}$$

Eq. (12) is merely an expression of the Clausius–Duhem inequality. To proceed with the thermodynamic setting we now need the constitutive assumptions. Various models of materials follow depending on the possible arguments of the constitutive equations.

Let

$$\Gamma = (\theta, \mathbf{E}, \mathfrak{H}, \dot{\mathbf{E}}, \nabla_R\theta)$$

be the set of variables and $\phi_R, \eta_R, \mathbf{T}_{RR}, \mathfrak{M}$ the constitutive functions. To save writing we notice that the time derivative of ϕ_R comprises

$$\partial_{\dot{\mathbf{E}}}\phi_R\cdot\ddot{\mathbf{E}} + \partial_{\nabla_R\theta}\phi_R\cdot\nabla_R\dot{\theta}.$$

The linearity and arbitrariness of $\ddot{\mathbf{E}}$ and $\nabla_R\dot{\theta}$ imply that (12) holds only if

$$\partial_{\dot{\mathbf{E}}}\phi_R = \mathbf{0}, \qquad \partial_{\nabla_R\theta}\phi_R = \mathbf{0}; \qquad \phi_R = \phi_R(\theta, \mathbf{E}, \mathfrak{H}).$$

Hence upon computation of $\dot{\phi}_R$ and substitution in (12) we find

$$\begin{aligned} -(\partial_\theta\phi_R + \eta_R)\dot{\theta} - (\partial_{\mathfrak{H}}\phi_R + \mu_0\mathfrak{M})\cdot\dot{\mathfrak{H}} + (\mathcal{T}_{RR} - \partial_{\mathbf{E}}\phi_R)\cdot\dot{\mathbf{E}} \\ + J\mathcal{T}\cdot\mathbf{W} - \frac{1}{\theta}\mathbf{q}_R\cdot\nabla_R\theta = \rho_R\theta\gamma \geq 0. \end{aligned}$$

First the arbitrariness of $\mathbf{W}$ implies that

$$\mathcal{T} \in \mathrm{Sym},$$

which is just the condition (1). Next, the linearity and arbitrariness of $\dot{\theta}, \dot{\mathfrak{H}}$ imply

$$\eta_R = -\partial_\theta\phi_R, \qquad \mu_0\mathfrak{M} = -\partial_{\mathfrak{H}}\phi_R. \tag{13}$$

Let

$$\Xi := \mathcal{T}_{RR} - \partial_{\mathbf{E}}\phi_R \in \mathrm{Sym}.$$

Hence we have the reduced inequality

$$\Xi\cdot\dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_R\cdot\nabla_R\theta = \rho_R\theta\gamma \geq 0.$$

For example, this inequality holds if

$$\Xi = \Lambda\dot{\mathbf{E}}, \qquad \mathbf{q}_R = -\mathbf{K}\nabla_R\theta,$$

where Λ and $\mathbf{K}$ are positive definite (fourth-order and second-order) tensors. More involved cross-coupling terms and non-linear representations can be considered.

From the referential relation

$$\mathcal{T}_{RR} = \partial_{\mathbf{E}}\phi_R + \Xi \tag{14}$$

we conclude that

$$\mathcal{T} = \mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M} = \frac{1}{J}\mathbf{F}\partial_{\mathbf{E}}\phi_R\mathbf{F}^T + \frac{1}{J}\mathbf{F}\Xi\mathbf{F}^T; \tag{15}$$

the (symmetric) generalized stress $\mathbf{T} + \mu_0\mathsf{H}\otimes\mathsf{M}$ equals the sum of an elastic term $J^{-1}\mathbf{F}\partial_{\mathbf{E}}\phi_R\mathbf{F}^T$ and a dissipative term $J^{-1}\mathbf{F}\Xi\mathbf{F}^T$.

There are cases where it is convenient to consider the stress $\mathcal{T}_{RR}$, instead of $\mathbf{E}$, as a variable. Hence we consider the (magnetic) Gibbs free energy

$$G_R = \phi_R - \mathcal{T}_{RR}\cdot\mathbf{E}, \qquad \mathcal{T}_{RR} \in \mathrm{Sym},$$

and write the CD inequality (12) in the form

$$-(\dot{G}_R + \eta_R \dot{\theta}) - \mu_0 \mathfrak{M} \cdot \dot{\mathfrak{H}} - \mathbf{E} \cdot \dot{\mathcal{T}}_{RR} - \frac{1}{\theta} \mathbf{q}_R \cdot \nabla_R \theta = \rho_R \theta \gamma \ge 0. \tag{16}$$

We then let

$$\hat{\Gamma} = (\theta, \mathcal{T}_{RR}, \mathfrak{H})$$

be the set of variables. Upon computation of the time derivative of the Gibbs free energy $G_R(\theta, \mathcal{T}_{RR}, \mathfrak{H})$ and substitution we have

$$-(\partial_\theta G_R + \eta_R)\dot{\theta} - (\partial_{\mathfrak{H}} G_R + \mu_0 \mathfrak{M}) \cdot \dot{\mathfrak{H}} - (\partial_{\mathfrak{T}_{RR}} G_R + \mathbf{E})\dot{\mathcal{T}}_{RR} - \frac{1}{\theta} \mathbf{q}_R \cdot \nabla_R \theta = \rho_R \theta \gamma \ge 0.$$

The linearity and arbitrariness of $\dot{\theta}, \dot{\mathfrak{H}}, \dot{\mathcal{T}}_{RR}$ imply that

$$\eta_R = -\partial_\theta G_R, \quad \mu_0 \mathfrak{M} = -\partial_{\mathfrak{H}} G_R, \quad \mathbf{E} = -\partial_{\mathfrak{T}_{RR}} G_R. \tag{17}$$

Remark. Magnetostriction is also affected by the magnetic domains. A continuum, systematic framework for the occurrence of magnetic domains might involve the magnetization $\mathfrak{M}$ along with the magnetization gradient $\nabla_R \mathfrak{M}$ as variables. An energy proportional to $|\nabla \mathsf{M}|^2$ is used in the literature to represent the magnetic exchange energy (see, e.g., [27]). Otherwise magnetic domains might be described within a multiscale modeling [3].

4. Magnetostriction in linear magnetoelastic materials

Magnetostriction is the property that causes a deformation of a body in response to a magnetic field. To describe this property we let $\Gamma = (\theta, \mathbf{E}, \mathfrak{H})$ be the set of variables and $\phi_R, \eta_R, \mathbf{T}_{RR}, \mathfrak{M}$ the constitutive functions. Indeed we take ϕ_R in the form

$$\phi_R = \tfrac{1}{2} \mathbf{E} \cdot \mathbb{C}\mathbf{E} - \tfrac{1}{2} \mu_0 \mathfrak{H} \cdot \chi_R \mathfrak{H} \ ; \tag{18}$$

the symmetric second-order tensor χ_R can be viewed as the (magnetic) susceptibility tensor in the reference configuration

and the symmetric fourth-order tensor $\mathbb{C}$ stands for the elasticity tensor. As we will show below, a quadratic energy like (18), though it does not contain mixed magneto-mechanical terms, allows the description of spontaneous magnetostriction. However, to account for the dependence of the magnetostriction on the stress we might consider additional terms for ϕ_R involving both mechanical and magnetic variables as in [3,28].

By (13) and (14), with $\Xi = \mathbf{0}$, it follows

$$\mathfrak{M} = \chi_R \mathfrak{H}, \quad \mathcal{T}_{RR} = \mathbb{C}\mathbf{E} \tag{19}$$

and then from (11)

$$\mathbf{T}_{RR} = \mathbb{C}\mathbf{E} - \mu_0 (\mathbf{C}^{-1} \mathfrak{H}) \otimes (\chi_R \mathfrak{H}).$$

Correspondingly, the Cauchy stress is given by

$$\mathbf{T} = J^{-1} \mathbf{F} (\mathbb{C}\mathbf{E}) \mathbf{F}^T - \mu_0 \mathsf{H} \otimes (\chi \mathsf{H}), \tag{20}$$

where

$$\chi = J^{-1} \mathbf{F} \chi_R \mathbf{F}^T.$$

Hence, owing to the assumption (18), as a thermodynamic restriction it follows that the stress is the sum of an elastic part, $J^{-1}\mathbf{F}(\mathbb{C}\mathbf{E})\mathbf{F}^T$, and a magnetic part, $-\mu_0 \mathsf{H} \otimes (\chi \mathsf{H})$.

According to (20), $\mathbf{T}$ might be viewed as the Maxwell stress tensor [23, § 2.16.3]. However, $\mathbf{T}$ is the mechanical stress and the structure (20) of $\mathbf{T}$ follows directly by the occurrence of $\mathcal{T}$, and not merely $\mathbf{T}$, in the resulting stress power in the entropy inequality. As remarked in [4], there is no need to adopt any form of Maxwell stress within the material.

The dependence of ϕ on $\mathfrak{H}$ yields $\mathbf{T} + \mu_0 \mathsf{H} \otimes \mathsf{M} \in \mathrm{Sym}$, which makes the balance of angular momentum to be satisfied. If, further, $\mathbf{T}$, and hence $\mathbf{T}_{RR}$, vanishes then apart from body force terms the body is at equilibrium. Consequently, the condition $\mathbf{T} = \mathbf{0}$, yields the relation between deformation and magnetic field at equilibrium.

When the (mechanical) stress $\mathbf{T}$ vanishes, the value of the Green–Lagrange strain $\mathbf{E}$ reduces to the magnetostrictive strain, say $\mathbf{E}_m$. Hence, by (20) we find

$$(2\mathbf{E}_m + \mathbf{1})(\mathbb{C}\mathbf{E}_m) = \mu_0 \mathfrak{H} \otimes \chi_R \mathfrak{H}. \tag{21}$$

Our purpose is to determine $\mathbf{E}_m$. Now, a quadratic function $\mathbb{F}(\mathbf{E}_m) = (2\mathbf{E}_m + \mathbf{1})\mathbb{C}\mathbf{E}_m : \mathrm{Sym} \to \mathrm{Sym}$ cannot be inverted in general. Nevertheless we show that a definite result can be determined in the approximation of small deformations.

For each point $\mathbf{X}$ let $\mathbf{u} = \mathbf{x} - \mathbf{X}$ be the displacement. Hence $\boldsymbol{\varepsilon} = \mathrm{sym}\nabla\mathbf{u}$ is the infinitesimal strain tensor. We characterize small deformations by assuming that $|\varepsilon_{ij}| \ll 1$ for any ij-component. To the leading order in $\boldsymbol{\varepsilon}$ we have

$$\mathbf{E}_m \simeq \boldsymbol{\varepsilon}_m, \quad \mathbf{C}_m = 2\mathbf{E}_m + \mathbf{1} \simeq \mathbf{1}, \quad \mathfrak{H} \simeq \mathsf{H}, \quad \mathfrak{M} \simeq \mathsf{M}, \quad \chi_R \simeq \chi.$$

Hence (21) becomes

$$\mathbb{C}\boldsymbol{\varepsilon}_m = \mu_0 \mathsf{H} \otimes \chi \mathsf{H},$$

or, in terms of M, $\mathbb{C}\boldsymbol{\varepsilon}_m = \mu_0 \chi^{-1} \mathsf{M} \otimes \mathsf{M}$. Since the fourth-order tensor $\mathbb{C}$ is assumed to be invertible we apply $\mathbb{C}^{-1}$ to get

$$\boldsymbol{\varepsilon}_m = \mathbb{M}[\mathsf{H} \otimes \mathsf{H}], \qquad \mathbb{M}_{ijhk} := \mu_0 \mathbb{C}^{-1}_{ijhm} \chi_{mk} \tag{22}$$

$$\boldsymbol{\varepsilon}_m = \mathbb{Q}[\mathsf{M} \otimes \mathsf{M}], \qquad \mathbb{Q}_{ijhk} := \mu_0 \mathbb{C}^{-1}_{ijmk} \chi^{-1}_{mh}. \tag{23}$$

Eqs. (22)–(23) describe the magnetostriction effect in the linear approximation. Consequently, at a given stress the resulting strain can be represented formally, in the spatial description, as a quadratic function of the magnetic field (or of the magnetization field).

4.1. Isotropic linear magnetoelastic materials

For definiteness we now restrict attention to mechanically and magnetically isotropic materials within the linear approximation. Let

$$\mathbb{C} = 2\mu_L \mathbf{1} \odot \mathbf{1} + \lambda_L \mathbf{1} \otimes \mathbf{1}, \qquad \chi_R = \chi_R \mathbf{1}, \quad \chi_R > 0,$$

where μ_L, λ_L are the Lamé moduli. Consequently

$$\phi_R(\theta, \mathbf{E}, \mathfrak{H}) = \tfrac{1}{2} \lambda_L (\mathrm{tr}\, \mathbf{E})^2 + \mu_L |\mathbf{E}|^2 - \tfrac{1}{2} \mu_0 \chi_R |\mathfrak{H}|^2, \tag{24}$$

where λ_L, μ_L, χ_R are functions of θ. In general the collinearity of $\mathfrak{H}$ and $\mathfrak{M}$ does not imply the collinearity of H and M. Now, by (19) we have

$$\mathsf{M} = \chi \mathsf{H}, \qquad \chi = J^{-1} \chi_R \mathbf{F}\mathbf{F}^T.$$

Hence collinearity occurs if $\mathbf{F}\mathbf{F}^T = \alpha \mathbf{1}$, $\alpha \in \mathbb{R}$. Since $\mathfrak{H} = \mathbf{F}^T \mathsf{H}$, $\mathfrak{M} = J\mathbf{F}^{-1}\mathsf{M}$ then the collinearity is preserved, i.e.

$$\mathsf{M} \parallel \mathsf{H} \iff \mathfrak{M} \parallel \mathfrak{H},$$

if $\mathbf{F}$ is diagonal relative to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ associated with H, e.g. $\mathbf{e}_1 = \mathsf{H}/|\mathsf{H}|$.

By (24) and (14) we find that

$$\mathbf{T}_{RR} = \lambda_L (\mathrm{tr}\, \mathbf{E})\mathbf{1} + 2\mu_L \mathbf{E} - \mu_0 \chi_R \mathbf{C}^{-1} \mathfrak{H} \otimes \mathfrak{H}. \tag{25}$$

In the spatial description the Cauchy stress $\mathbf{T}$ has the form

$$\mathbf{T} = J^{-1} \mathbf{F}[\lambda_L (\mathrm{tr}\, \mathbf{E})\mathbf{1} + 2\mu_L \mathbf{E}]\mathbf{F}^T - \mu_0 J^{-1} \chi_R \mathsf{H} \otimes (\mathbf{F}\mathbf{F}^T \mathsf{H}).$$

The magnetostriction effect is described by the strain $\mathbf{E}_m$ induced by the magnetic field while the (mechanical) stress vanishes. In view of (25) we have

$$\lambda_L (\mathrm{tr}\, \mathbf{E}_m)\mathbf{1} + 2\mu_L \mathbf{E}_m = \mu_0 \chi_R (\mathbf{C}_m^{-1} \mathfrak{H}) \otimes \mathfrak{H}, \tag{26}$$

the subscript m being a reminder that the stress $\mathbf{T}$ is zero.

We now apply (26) to determine the magnetostrictive effect in a one-dimensional setting. It is understood that the deformation is due to the application of a magnetic field, which is often referred to as *field induced magnetostriction*. Hence, for formal convenience, hereafter the subscript m is omitted.

4.2. One-dimensional restriction of spatial magnetostriction

Let $\mathbf{e}_1$ be a fixed unit vector and let H be applied in the $\mathbf{e}_1$ direction, $\mathsf{H} = \mathsf{H}\mathbf{e}_1$. Hence we look for the deformation gradient in the form

$$\mathbf{F} = \mathrm{diag}[1+\xi, 1-\delta, 1-\delta];$$

since $J = (1+\xi)(1-\delta)^2$ the constraint $J > 0$ requires that

$$\xi > -1, \qquad \delta \neq 1.$$

Hence we have

$$\mathfrak{H} = \mathbf{F}^T\mathsf{H} = (1+\xi)\mathsf{H}\mathbf{e}_1,$$

$$\mathbf{E} = \tfrac{1}{2}\mathrm{diag}[(1+\xi)^2 - 1, (1-\delta)^2 - 1, (1-\delta)^2 - 1],$$

$$\mathbf{C}^{-1} = \mathrm{diag}[(1+\xi)^{-2}, (1-\delta)^{-2}, (1-\delta)^{-2}].$$

By (26) we find the system of equations

$$\lambda_L(\mathrm{tr}\,\mathbf{E}) + 2\mu_L E_{11} = \mu_0\chi_R\mathsf{H}^2, \;\; \lambda_L(\mathrm{tr}\,\mathbf{E}) + 2\mu_L E_{kk} = 0, \;\; k = 2, 3. \tag{27}$$

Hence it follows that

$$\mathrm{tr}\,\mathbf{E} = \frac{\mu_0\chi_R}{3\lambda_L + 2\mu_L}\mathsf{H}^2.$$

Upon substitution of $\mathrm{tr}\,\mathbf{E}$ in the system (27) we obtain

$$E_{11} = \frac{\mu_0\chi_R}{Y}\mathsf{H}^2, \quad E_{22} = E_{33} = -\nu E_{11}, \tag{28}$$

where

$$Y = \frac{\mu_L(3\lambda_L + 2\mu_L)}{\lambda_L + \mu_L}, \quad \nu = \frac{\lambda_L}{2(\lambda_L + \mu_L)};$$

the material parameters Y and ν are the standard Young's modulus and Poisson's ratio.

Eq. (28) describe the one-dimensional magnetostriction and show the quadratic dependence of the diagonal entries E_{11}, E_{22}, E_{33} on the applied magnetic field H. Letting $Q = (\mu_0\chi_R/Y)^{1/2}$ we can write

$$E_{11} = (Q\mathsf{H})^2.$$

This relation is represented by the solid line in Fig. 1 (left). The material parameter Q is a measure of the influence of the magnetic field on the strain and depends on both mechanical and magnetic properties. Since $\mu_0 = 1.256 \cdot 10^{-6}$ N A^{-2}, when Terfenol-D is considered, then [29]

$$Y = 50 \text{ GPa} = 5 \cdot 10^{10} \text{ N/m}^2, \qquad \chi_R = 5,$$

and then

$$Q = \sqrt{\mu_0\chi_R/Y} = 1.12 \cdot 10^{-8} \text{ (m/A)}.$$

Notice that $Q\mathsf{H}$ is a dimensionless parameter because the unit of H is A/m.

Since $E_{11} = [(1+\xi)^2 - 1]/2$ then it follows from (28) that

$$\xi = -1 + (1 + 2Q^2\mathsf{H}^2)^{1/2}, \tag{29}$$

the negative solution being omitted in view of the restriction $\xi > -1$. For small values of H, say $Q|\mathsf{H}| \ll 1$, we have

$$\xi \simeq Q^2\mathsf{H}^2.$$

Instead, for large values of H, namely $Q|\mathsf{H}| \gg 1$, we have

$$\xi \simeq -1 + \sqrt{2}\,Q|\mathsf{H}|.$$

This illustrates why the relation between deformation and magnetic field is almost quadratic for small fields and almost linear for large fields (see the dashed line in Fig. 1, left). As to the example of Terfenol D, a magnetic field with an intensity of at least $3.77 \cdot 10^5$ Oe $= 3 \cdot 10^7$ A/m ($Q|\mathsf{H}| \simeq 0.3$) would be necessary to give evidence of the linear dependence.

Incidentally, let l_0, l be the length in the $\mathbf{e}_1$ direction before and after the deformation. If the expansion or the contraction occurs uniformly then

$$\frac{l - l_0}{l_0} = \varepsilon_{11} = \xi,$$

where ξ, usually denoted by λ, is given by the quadratic–linear property of (29).

Alternatively, the magnetostriction can be described in terms of the magnetization $\mathfrak{M}$. In place of (26) we consider

$$\lambda_L(\mathrm{tr}\,\mathbf{E}_m)\mathbf{1} + 2\mu_L\mathbf{E}_m = \mu_0\chi_R^{-1}(\mathbf{C}_m^{-1}\,\mathfrak{M}) \otimes \mathfrak{M}, \tag{30}$$

and keep assuming that the fields are directed along $\mathbf{e}_1$. Since $\mathfrak{M} = J\mathbf{F}^{-1}\mathsf{M}$ then the significant components of $\mathfrak{M}$ and H are related by

$$\mathfrak{M} = (1-\delta)^2\mathsf{M}.$$

Hence Eq. (30) results into the system

$$E_{11}(1 + 2E_{11}) = (1-\delta)^4\Lambda^2\mathsf{M}^2, \quad E_{22} = E_{33} = -\nu E_{11},$$

where $\Lambda = \mu_0/Y\chi_R$. Now,

$$(1-\delta)^2 = 1 - 2\nu E_{11}$$

and hence we can write

$$E_{11}(1 + 2E_{11}) = (1 - 2\nu E_{11})^2\Lambda^2\mathsf{M}^2.$$

To the linear approximation in E_{11} we find

$$E_{11} = \frac{\Lambda^2\mathsf{M}^2}{1 + 4\nu\Lambda^2\mathsf{M}^2}.$$

Consequently it follows

$$\xi = (1 + 2\frac{\Lambda^2\mathsf{M}^2}{1 + 4\nu\Lambda^2\mathsf{M}^2})^{1/2} - 1.$$

The parabolic dependence of ξ on M, around $\mathsf{M} = 0$, is consistent with the literature. Indeed, the magnetostriction (here ξ) is mostly determined by a parabola-shaped curve with respect to the magnetization M; see Fig. 1 (right), where $\nu = 0.17$, the Poisson ratio value of Terfenol-D. Experimental data provided, e.g., in [30] show the upward parabola-shaped curve for the longitudinal magnetostrictive strain (here $[(1+\xi)^2 - 1]/2$) and the downward transverse magnetostrictive strain $[(1-\delta)^2 - 1]/2$, with $\xi, \delta > 0$.

It is worth looking at numerical aspects connected with magnetostriction. For Terfenol-D samples

$$\Lambda^2 = \mu_0/Y\chi_R = 5 \cdot 10^{-18} \text{ (m/A)}^2$$

and then $\Lambda = 2.24 \cdot 10^{-9}$ m/A. If $\mathsf{M} \simeq 10^7$ A/m then

$$\Lambda\mathsf{M} \simeq 2.24 \cdot 10^{-2}.$$

Consistent with Fig. 1 (right), in this case the longitudinal magnetostriction is approximately quadratic, i.e. $\xi \simeq 0.5 \cdot 10^{-3}$, in agreement with [29].

5. Magnetostriction in non-linear materials

Consider Eq. (12) and for simplicity let $\mathbf{q}_R = \mathbf{0}$ while the arbitrariness of $\mathbf{W}$ implies the symmetry of $\boldsymbol{\mathcal{T}}$. We then have

$$-(\dot{\phi}_R + \eta_R\dot{\theta}) - \mu_0\mathfrak{M} \cdot \dot{\mathfrak{H}} + \boldsymbol{\mathcal{T}}_{RR} \cdot \dot{\mathbf{E}} \geq 0. \tag{31}$$

The constitutive functions $\phi_R, \eta_R, \mathfrak{M}, \boldsymbol{\mathcal{T}}_{RR}$ are assumed to be functions of

$$\boldsymbol{\Lambda} = (\theta, \mathfrak{H}, \mathbf{E}).$$

Hence we have

$$-(\partial_\theta\phi_R + \eta_R)\dot{\theta} - (\partial_{\mathfrak{H}}\phi_R + \mu_0\mathfrak{M}) \cdot \dot{\mathfrak{H}} + (\boldsymbol{\mathcal{T}}_{RR} - \partial_{\mathbf{E}}\phi_R) \cdot \dot{\mathbf{E}} \geq 0.$$

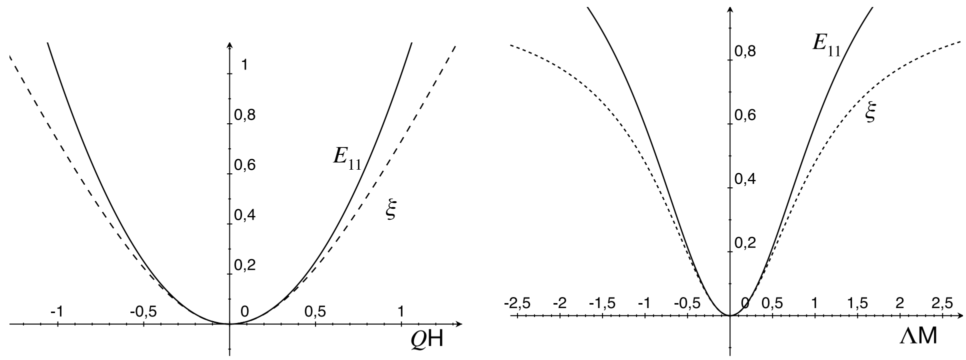

Fig. 1. Plot of E_{11} (solid) and ξ (dashed) as functions of QH (left) and ΛM (right).

The linearity and arbitrariness of $\dot\theta, \dot{\mathfrak{H}}, \dot{\mathbf{E}}$ imply that

$$\eta_R = -\partial_\theta \phi_R, \quad \mu_0 \mathfrak{M} = -\partial_{\mathfrak{H}} \phi_R, \quad \boldsymbol{\mathcal{T}}_{RR} = \partial_{\mathbf{E}} \phi_R. \tag{32}$$

Consequently Eq. (31) holds as an equality and then, by (32), in any evolution process $\boldsymbol{\Lambda}(t) = (\theta(t), \mathfrak{H}(t), \mathbf{E}(t))$, $t \geq t_0$, we have

$$\dot\phi_R = -\eta_R \dot\theta - \mu_0 \mathfrak{M} \cdot \dot{\mathfrak{H}} + \boldsymbol{\mathcal{T}}_{RR} \cdot \dot{\mathbf{E}}. \tag{33}$$

Hence in a process $\tilde{\boldsymbol{\Lambda}}(\tau), \tau \in [t_0, t]$ from $\boldsymbol{\Lambda}_0$ to $\boldsymbol{\Lambda}(t)$ we have

$$\phi_R(\boldsymbol{\Lambda}(t)) - \phi_R(\boldsymbol{\Lambda}_0) = \int_{t_0}^t [-\eta_R \dot\theta - \mu_0 \mathfrak{M} \cdot \dot{\mathfrak{H}} + \boldsymbol{\mathcal{T}}_{RR} \cdot \dot{\mathbf{E}}] d\tau.$$

A particular, simple case occurs when

$$\mu_0 \mathfrak{M} = \chi_R \mathfrak{H}, \qquad \boldsymbol{\mathcal{T}}_{RR} = \mathbb{C}\mathbf{E},$$

where χ_R and $\mathbb{C}$ are second- and fourth-order symmetric constant tensors. In this case it follows that

$$-\mu_0 \mathfrak{M} \cdot \dot{\mathfrak{H}} = -\tfrac{1}{2}(\mathfrak{H} \cdot \chi_R \mathfrak{H})^{\cdot}, \qquad \boldsymbol{\mathcal{T}}_{RR} \cdot \dot{\mathbf{E}} = \tfrac{1}{2}(\mathbf{E} \cdot \mathbb{C}\mathbf{E})^{\cdot}.$$

Hence, to within an additive constant, we have

$$\phi_R(\boldsymbol{\Lambda}(t)) = -\int_{t_0}^t \eta_R \dot\theta d\tau - \tfrac{1}{2}(\mathfrak{H} \cdot \chi_R \mathfrak{H})(t) + \tfrac{1}{2}(\mathbf{E} \cdot \mathbb{C}\mathbf{E})(t).$$

Since $\eta_R = -\partial_\theta \phi_R$, if η_R depends only on θ then

$$\partial_\theta \phi_R \, \dot\theta = [\Phi(\theta)]^{\cdot},$$

Φ being a primitive of $-\eta_R$. Consequently we find

$$\phi_R(\boldsymbol{\Lambda}) = \Phi(\theta) - \tfrac{1}{2}(\mathfrak{H} \cdot \chi_R \mathfrak{H}) + \tfrac{1}{2}(\mathbf{E} \cdot \mathbb{C}\mathbf{E}).$$

Non-linear models are obtained by letting

$$\mu_0 \mathfrak{M} = f(h) \chi_R \mathfrak{H}, \qquad \boldsymbol{\mathcal{T}}_{RR} = g(e)\mathbb{C}\mathbf{E},$$

where

$$h = \mathfrak{H} \cdot \chi_R \mathfrak{H}, \qquad e = \mathbf{E} \cdot \mathbb{C}\mathbf{E}.$$

Hence Eq. (33) holds while ϕ_R is given by

$$\phi_R = \Phi(\theta) - F(h) + G(e),$$

where F is the primitive of $f/2$ and G is the primitive of $g/2$.

6. Nonlinear constitutive relations for one-dimensional settings

A well-known non-linear model in the literature is given by the standard square constitutive equations for strain and induction in terms of stress and magnetic field. These equations though are inadequate at high fields and hence some more involved models are in order [31]. Furthermore, detailed data, by now available in the literature (see, e.g. [3,32]), show deep cross-coupling effects between mechanical and magnetic fields.

To obtain a more flexible family of constitutive equations we consider one-dimensional settings thus introducing reasonable approximations from the start.

Let $\mathbf{e}_1$ be the longitudinal direction of the one-dimensional domain and let the pertinent fields $\mathbf{H}, \mathbf{M}$ be parallel to $\mathbf{e}_1$. Consistently we let $T_{11} = \sigma$ be the only non-zero stress component. With this strict one-dimensional geometry, the symmetry condition vanishes in that $\mathbf{T}, \mathbf{H}$, and $\mathbf{M}$ are now scalars. Furthermore, any projection is Euclidean invariant; letting $\mathsf{H} = \mathbf{H} \cdot \mathbf{e}_1$ and $\mathbf{Q}$ be a time dependent orthogonal tensor (see Appendix A) we have

$$H^* = \mathbf{H}^* \cdot \mathbf{e}_1^* = \mathbf{Q}\mathbf{H} \cdot \mathbf{Q}\mathbf{e}_1 = \mathbf{H} \cdot \mathbf{Q}^T \mathbf{Q}\mathbf{e}_1 = \mathbf{H} \cdot \mathbf{e}_1 = \mathsf{H}$$

and the same is true for M and σ.

As to the mechanical power, observe that $\mathbf{T} \cdot \mathbf{L} = \sigma L_{11} = \sigma F^{-1} \dot F$, F being the longitudinal strain, $F = F_{11} > 0$. For formal simplicity we neglect heat conduction and write the CD inequality (2) in the form

$$-\rho(\dot\phi + \eta\dot\theta) - \mu_0 \mathsf{M} \, \dot{\mathsf{H}} + \sigma F^{-1} \dot F = \rho\theta\gamma \geq 0.$$

Observe that

$$F^{-1} \dot F = (\ln F)^{\cdot}.$$

Hence we write the CD inequality in the form

$$-\rho(\dot\phi + \eta\dot\theta) - \mathcal{M} \, \dot{\mathsf{H}} + \sigma (\ln F)^{\cdot} = \rho\theta\gamma \geq 0, \tag{34}$$

where $\mathcal{M} = \mu_0 \mathsf{M}$.

Let $\rho\zeta = \rho\phi - \sigma \ln F$ denote the (magnetic) Gibbs free energy and observe that

$$-\rho\dot\phi + \sigma(\ln F)^{\cdot} = -\rho\dot\zeta - (\ln F)\dot\sigma.$$

Hence we write (34) in the form

$$-\rho(\dot\zeta + \eta\dot\theta) - \mathcal{M}\dot{\mathsf{H}} - \ln F \, \dot\sigma = \rho\theta\gamma \geq 0.$$

Letting

$$\zeta = \zeta(\theta, \mathsf{H}, \sigma)$$

we find that

$$\eta = -\partial_\theta \zeta, \qquad \mathcal{M} = -\rho \partial_{\mathsf{H}} \zeta, \qquad \ln F = -\rho \, \partial_\sigma \zeta. \tag{35}$$

If $F = 1 + \xi$ and $|\xi| \ll 1$ then $\ln(1 + \xi) \simeq \xi$ and we can write the approximate relation

$$\xi \simeq -\rho \, \partial_\sigma \zeta.$$

The results (35) hold for both compressive and tensile stresses. For compressive stresses we might merely let $\sigma = -\tau, \tau > 0$. Accordingly, using the variable τ we obtain the relations

$$\rho\zeta = \rho\phi + \tau \ln F, \qquad \zeta = \hat\zeta(\theta, \mathsf{H}, \tau),$$

$$\eta = -\partial_\theta \hat\zeta, \qquad \mathcal{M} = -\rho \partial_{\mathsf{H}} \hat\zeta, \qquad \ln F = \rho \, \partial_\tau \hat\zeta.$$

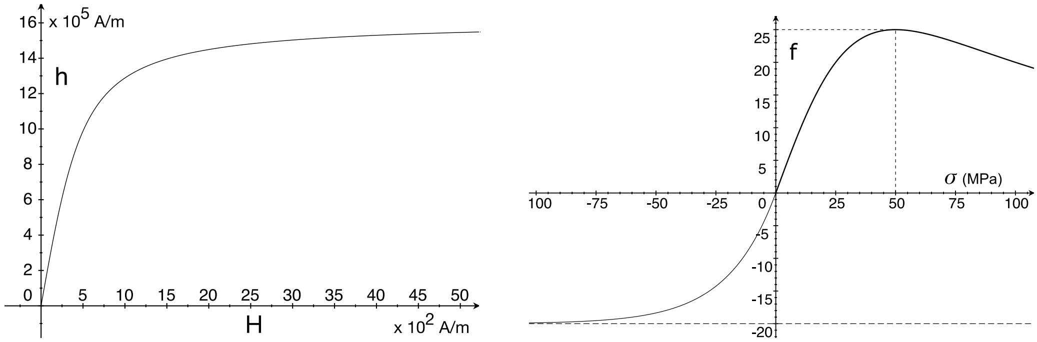

Fig. 2. The function $\mathsf{h}(\mathsf{H})$ with $\mathsf{h}_0 = 16.1 \cdot 10^5$A/m and $\mathsf{H}_S = 2 \cdot 10^2$ A/m (left). The function $\mathsf{f}(\sigma)$ with $\sigma_0 = 1$ MPa, $\sigma_+ = 50$ MPa and $\sigma_- = 20$ MPa (right).

6.1. Comparison with experimental data

To make these equations operative we now establish a detailed model for some materials described by experimental data on joint dependence on the magnetic field H and the stress σ. Models are available (see, e.g., [33]) in terms of Taylor approximations, with respect to M, and based on experimental results (e.g. in [34]). Here, instead, we look for a model in terms of elementary functions so that the understanding of the constitutive properties prove more immediate.

For definiteness we consider the experimental results in [3,32] and start with addressing attention to the magnetostriction through the dependence of the magnetization M on the (normal) stress σ. The data in [32] are relative to both tensile and compressive stresses, $\sigma \in [-100, 100]$ MPa, and show some remarkable properties. First the magnetic properties are not symmetric with respect to σ; they show a different behavior under the inversion $\sigma \to -\sigma$. Secondly, the σ dependence shows a homogeneous change via a factor dependent on H. Consequently we look for the joint dependence on σ and H by assuming the separation of the variables. Thirdly, as already remarked, the data show a non-monotonic dependence on σ. The data are relative to thick strips of NO Fe Si.

The magnetization M shows a saturation property with respect to H for every applied stress σ and a non-monotonic dependence on the stress for every magnetic field. To describe the non-monotonic dependence $\mathcal{M} - \sigma$ we let

$$\mathsf{M} = \mathsf{h}(\mathsf{H}) + \mathsf{f}(\sigma)\mathsf{k}(\mathsf{H}). \tag{36}$$

where f is non-dimensional. Since $\mu_0 \mathsf{M} = -\rho\partial_{\mathsf{H}}\zeta$ then

$$\rho\zeta(\theta, \mathsf{H}, \sigma) = \rho\check{\zeta}(\theta) - \mu_0\mathcal{H}(\mathsf{H}) - \mu_0\mathsf{f}(\sigma)\mathcal{K}(\mathsf{H}) - \Sigma(\sigma),$$

where $\mathcal{H}$ and $\mathcal{K}$ are primitives of h and k. Hence we have

$$\xi \simeq \ln F = \mu_0 \mathsf{f}'(\sigma)\mathcal{K}(\mathsf{H}) + \Sigma'(\sigma). \tag{37}$$

We now determine the functions $\mathsf{h}, \mathsf{f}, \mathcal{K}$. The function f is required to vanish at $\sigma = 0$ and hence, by (36),

$$\mathsf{h}(\mathsf{H}) = \mathsf{M}(\mathsf{H}, 0).$$

A good fit of data is found by letting $\mathsf{h}(\mathsf{H})$ be expressed by the Langevin function, which is a widely used function in the description of the magnetization curve [35].

$$\mathsf{h}(\mathsf{H}) = \mathsf{h}_0[\coth(\mathsf{H}/\mathsf{H}_S) - \mathsf{H}_S/\mathsf{H}],$$

where

$$\mathsf{h}_0 = 16.1 \cdot 10^5 \text{A/m}, \qquad \mathsf{H}_S = 2 \cdot 10^2 \text{ A/m}.$$

The corresponding plot of $\mathsf{h}(\mathsf{H})$ is represented in Fig. 2 (left).

As to $\mathsf{f}(\sigma)$ we observe that the M–σ curve has a maximum as $\sigma > 0$ and though the curve changes significantly under the inversion $\sigma \to -\sigma$, the data indicate a common value of f' as $\sigma \to 0_-$ and $\sigma \to 0_+$. We then let

$$\mathsf{f}(\sigma) = \begin{cases} (\sigma/\sigma_0)/[1 + (\sigma/\sigma_+)^2], & \text{if } \sigma \ge 0, \\ (\sigma_-/\sigma_0)[\exp(\sigma/\sigma_-) - 1], & \text{if } \sigma < 0, \end{cases}$$

where $\sigma_0, \sigma_+, \sigma_-$ are positive parameters. The function f and f' are continuous and $\mathsf{f}(0) = 0$, $\mathsf{f}'(0) = 1/\sigma_0$;

$$\mathsf{f}'(\sigma) = \begin{cases} [1 - (\sigma/\sigma_+)^2]/\sigma_0[1 + (\sigma/\sigma_+)^2]^2, & \text{if } \sigma \ge 0, \\ \exp(\sigma/\sigma_-)/\sigma_0, & \text{if } \sigma < 0. \end{cases}$$

Now, f has a maximum at $\sigma = \sigma_+$, with value $\mathsf{f}(\sigma_+) = \sigma_+/(2\sigma_0)$, while $\lim_{\sigma\to+\infty} \mathsf{f}(\sigma) = 0$, and $\lim_{\sigma\to-\infty} \mathsf{f}(\sigma) = -\sigma_-/\sigma_0$ (see Fig. 2, right).

As is shown below, for constant magnetic fields the magnetic (relative) susceptibility χ takes the form

$$\chi = a + b\,\mathsf{f}(\sigma),$$

where a, b depend on H. The data in Fig. 3 (left) are taken from Fig. 7 of [32] and show the experimental dependence of the magnetic susceptibility on σ parameterized by H. It follows that $\sigma_+ = 50$ MPa, $\mathsf{f}_{max} = \mathsf{f}(\sigma_+) = 25$ and $\sigma_- = 20$ MPa, as it is plotted in Fig. 2 (right).

The function $\mathsf{k}(\mathsf{H})$ is determined by the observation that a mechanical stress leads to a significant increase in the magnetization up to a suitable magnetic field. Furthermore, comparing (28) and (37) we conclude that $\mathcal{K}(\mathsf{H}) \simeq \mathsf{H}^2$ around $\mathsf{H} = 0$. This conclusion is consistent with experimental data; see Fig. 4 of [3] as $\sigma = 0$. Hence we let

$$\mathcal{K}(\mathsf{H}) = \mathsf{k}_0\mathsf{H}^2\exp(-\mathsf{H}^2/\mathsf{H}_0^2), \tag{38}$$

where k_0 and H_0 are positive parameters. It follows that

$$\mathcal{K}'(\mathsf{H}) = k(\mathsf{H}) = 2k_0\mathsf{H}(1 - \mathsf{H}^2/\mathsf{H}_0^2)\exp(-\mathsf{H}^2/\mathsf{H}_0^2),$$

$$\mathcal{K}''(\mathsf{H}) = k'(\mathsf{H}) = 2k_0(1 - 5\mathsf{H}^2/\mathsf{H}_0^2 + 2\mathsf{H}^4/\mathsf{H}_0^4)\exp(-\mathsf{H}^2/\mathsf{H}_0^2);$$

notice that $\mathcal{K}$ and $\mathsf{k}' = \mathcal{K}''$ are even functions of H while $\mathcal{K}' = \mathsf{k}$ is odd. We observe that k vanishes (and $\mathcal{K}$ has a maximum) at H_0, while k has a maximum (and $\mathcal{K}$ has an inflection point) at H_1 given by

$$\mathsf{H}_1 = [(5 - \sqrt{17})/4]^{1/2}\,\mathsf{H}_0 \simeq 0.468\,\mathsf{H}_0.$$

In view of Fig. 3 we let

$$\mathsf{k}_0 = 30, \qquad \mathsf{H}_0 = 6 \cdot 10^3 \text{ A/m}, \qquad \mathsf{H}_1 = 2.82 \cdot 10^3 \text{A/m}.$$

Fig. 4 shows the plots of $\mathcal{K}, \mathsf{k}, \mathsf{k}'$.

We can then write the function $\mathsf{M}(\mathsf{H}, \sigma)$ in the form

$$\mathsf{M}(\mathsf{H}, \sigma) = \mathsf{M}_V(\mathsf{H}, \sigma) + \mathsf{h}_0[\coth(\mathsf{H}/\mathsf{H}_S) - \mathsf{H}_S/\mathsf{H}],$$

where

$$\mathsf{M}_V(\mathsf{H}, \sigma) = 2k_0\mathsf{H}(1 - \mathsf{H}^2/\mathsf{H}_0^2)\exp(-\mathsf{H}^2/\mathsf{H}_0^2)\begin{cases} \dfrac{\sigma/\sigma_0}{1 + (\sigma/\sigma_+)^2} & \text{if } \sigma \ge 0, \\ \dfrac{\sigma_-}{\sigma_0}[\exp(\sigma/\sigma_-) - 1] & \text{if } \sigma < 0. \end{cases} \tag{39}$$

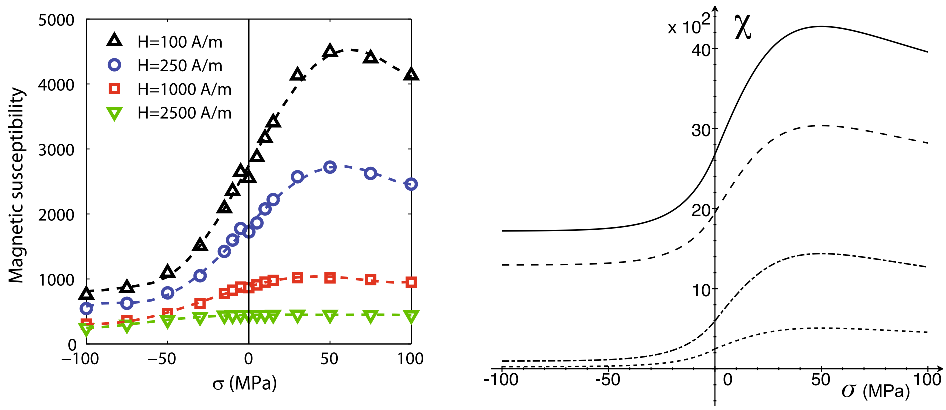

Fig. 3. Experimental data on the magnetic susceptibility χ versus the stress σ from Fig. 7 of [32] (left). Plot of χ after Eq. (40) at different values of the magnetic field (right); H = 100 A/m (solid line), H = 250 A/m (dashed line), H = 1000 A/m (dash-dotted line), H = 2500 A/m (short dashed line).

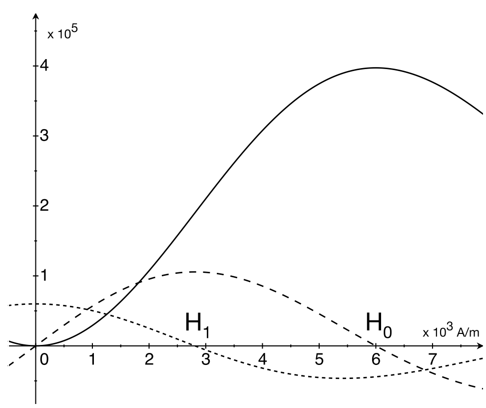

Fig. 4. The functions $\mathcal{K}$ (solid line, unit $\mathrm{A}^2/\mathrm{m}^2$), $\mathrm{k} = \mathcal{K}'$ (dashed line, unit A/m), and $\mathrm{k}' = \mathcal{K}''$ (short dashed line, non-dimensional).

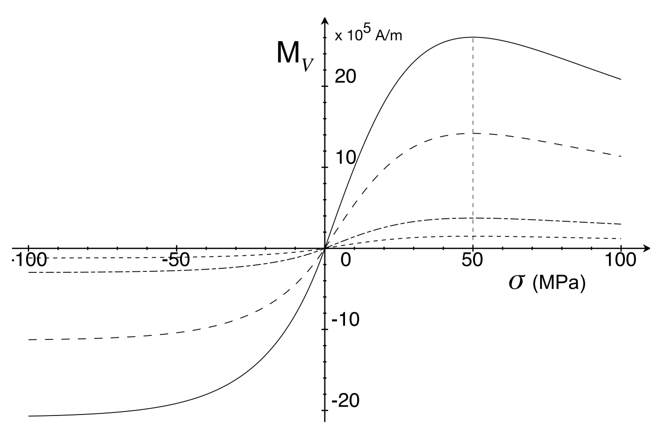

Fig. 5. Plot of M_V as a function of σ at different values of the magnetic field; H = 100 A/m (solid line), H = 250 A/m (dashed line), H = 1000 A/m (dash-dotted line), H = 2500 A/m (short dashed line).

We notice that $\mathrm{M}_V(\mathrm{H},0) = 0$, and $\lim_{\sigma\to+\infty} \mathrm{M}_V = 0$. M_V describes the effect of the stress on M. The function (39) shows that, for any magnetic field H the effect of σ has a maximum at $\sigma = \sigma_+$ (see Fig. 2, right). Hence the parameter σ_+ represents the so-called Villari point or Villari reversal [36,37]. Consistent with [3, Fig. 2b], Fig. 5 shows the dependence of the magnetization M_V on the stress σ for the same values of the magnetic field H considered in Fig. 3 and the same values of the parameters k_0, H_0, σ_0, σ_+, σ_-.

We now examine the magnetic susceptibility $\chi = \partial_\mathrm{H}\mathrm{M}(\mathrm{H},\sigma)$ and the magnetostrictive strain ξ_m. The magnetic susceptibility consists of two additive terms,

$$\partial_\mathrm{H}\mathrm{M}(\mathrm{H},\sigma) = \partial_\mathrm{H}\mathrm{M}_V(\mathrm{H},\sigma) + \mathrm{h}'(\mathrm{H}).$$

Let χ_σ denote the additive part of χ induced by the stress σ. We have

$$\chi_\sigma = \partial_\mathrm{H}\mathrm{M}_V(\mathrm{H},\sigma) = 2\mathrm{k}_0(1 - 5\mathrm{H}^2/\mathrm{H}_0^2 + 2\mathrm{H}^4/\mathrm{H}_0^4)\exp(-\mathrm{H}^2/\mathrm{H}_0^2)\mathrm{f}(\sigma). \tag{40}$$

As a comment, owing to the separation of variables, at constant H the functions $\mathrm{M}(\mathrm{H},\sigma)$ and $\partial_\mathrm{H}\mathrm{M}_V(\mathrm{H},\sigma)$ have a proportional dependence on σ and hence the maximum at the same value of σ. This is the case as is shown by Figures 2b and 3b of [3].

Fig. 3 (right) shows the properties of the magnetic susceptibility

$$\chi = \chi_\sigma(\mathrm{H},\sigma) + \mathrm{h}'(\mathrm{H}), \tag{41}$$

where χ_σ is given by (40) while k_0 has been determined by comparison with the experimental data in [32, Fig. 7] (on the left).

6.2. Longitudinal and transverse magnetostriction

Consider tensile stresses, $\sigma > 0$, and hence $\mathrm{f}(\sigma) > 0$. At constant σ the susceptibility $\chi_\sigma = \partial_\mathrm{H}\mathrm{M}_V$ decreases as $|\mathrm{H}|$ increases. If H is constant then $\partial_\mathrm{H}\mathrm{M}_V$ increases up to the maximum (determined by f) at $\sigma = \sigma_+$. For compressive stresses, $\sigma < 0$, we have $\mathrm{f} < 0$ and $\partial_\mathrm{H}\mathrm{M}_V < 0$. Furthermore, if $|\mathrm{H}|$ increases then $|\partial_\mathrm{H}\mathrm{M}_V|$ decreases and this is consistent with Fig. 3.

As to the strain, by Eq. (37) we let $\xi_\sigma = \Sigma'(\sigma)$ be the elastic strain and

$$\xi_m := \xi - \xi_\sigma = \mu_0\mathcal{K}(\mathrm{H})\mathrm{f}'(\sigma)$$

the magnetostrictive strain. Substitution of $\mathcal{K}$ and f' yields

$$\xi_m(\mathrm{H},\sigma) = \mu_0\mathrm{k}_0\mathrm{H}^2\exp(-\mathrm{H}^2/\mathrm{H}_0^2)\frac{1}{\sigma_0}\begin{cases} \dfrac{1-(\sigma/\sigma_+)^2}{[1+(\sigma/\sigma_+)^2]^2}, & \text{if } \sigma \geq 0,\\ \exp(\sigma/\sigma_-), & \text{if } \sigma < 0; \end{cases} \tag{42}$$

Apparently $\xi_m(\mathrm{H}) = \xi_m(-\mathrm{H})$ and its maximum value occurs at $|\mathrm{H}| = \mathrm{H}_0 = 6\cdot 10^3$ A/m. Fig. 6 shows ξ_m as a function of $\mathrm{H} > 0$ at different values of the stress σ. Consistent with (42), ξ_m decreases at increasing stress level and changes sign across $\sigma = \sigma_+$.

Let $0 < \sigma < \sigma_+$. Hence $\mathrm{f}'(\sigma) > 0$ and, for small values of H,

$$\xi_m \propto \mathrm{H}^2.$$

As σ increases then ξ_m decreases and is negative when $\sigma > \sigma_+$. The change of sign of ξ_m around σ_+ (50 MPa in Fig. 6) shows that the magnetostrictive strain changes sign at the Villari point. If, instead, $\sigma < 0$ (compressive stresses) then ξ_m is positive for any σ and decreases as $|\sigma|$ increases. This is the expected (and also real) behavior of longitudinal

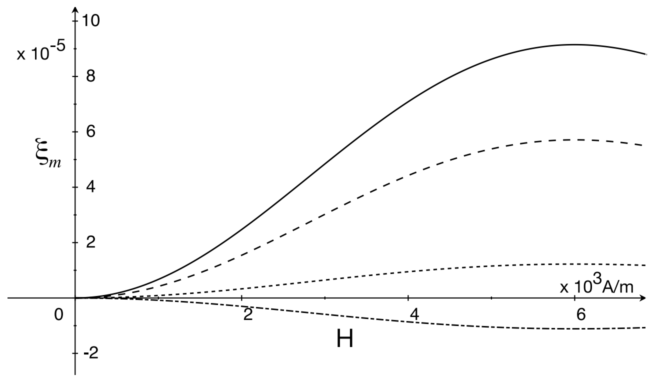

Fig. 6. Plot of ξ_m as a function of H at $\sigma = 0$ (solid line), $\sigma = 20$ MPa (dashed line), $\sigma = 40$ MPa (short dashed line), $\sigma = 78$ MPa (dash-dotted line).

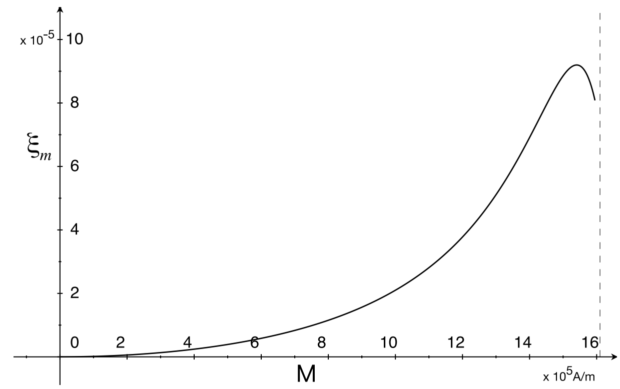

Fig. 7. Plot of ξ_m as a function of M at $\sigma = 0$.

magnetostriction, namely when ξ_m is in the direction of H and along the axis (segment) of the sample.

To find the transverse magnetostriction we notice that, by the sample geometry and the experimental set (see, e.g., [38, Fig. 1]) we can only assume the value of J while $F_{22} = F_{33}$. To fix ideas we let $J = F_{11} F_{22}^2 = 1$ whence

$$F_{22} = F_{33} = 1/\sqrt{F_{11}}.$$

Let

$$F_{11} = 1 + \xi_\sigma + \xi_m, \qquad F_{22} = 1 - \delta_\sigma - \delta_m.$$

Hence

$$1 - \delta_\sigma - \delta_m = \frac{1}{\sqrt{1 + \xi_\sigma + \xi_m}}, \qquad \delta_m \simeq \tfrac{1}{2}\xi_m, \tag{43}$$

whence the transverse magnetostriction strain is expected to be half the longitudinal magnetostriction strain. While the condition $F_{22} = F_{33}$ is due to the symmetry of the setting, relation (43) holds true for the data in [39, Fig. 6] where the transverse strain δ_m turns out to be $-1/2$ times the longitudinal strain ξ_m. This proves that $J = 1$ is a physically sound assumption.

6.3. Strain as a function of the magnetization

Most often in the literature the magnetostrictive strain ξ_m is plotted as a function of M rather than H (see, e.g., [3, Fig. 4] and [32, Fig. 8a]). The plots are quite different and we point out that the difference is related to the saturation property and hence to the nonlinear dependence of M on H. Restrict attention to the un-loaded condition and observe that $\mathsf{M}(\mathsf{H}, 0) = \mathsf{h}(\mathsf{H})$. Hence we let

$$\mathsf{M}_0 = \mathsf{h}(\mathsf{H}_0) = 15.46 \cdot 10^5 \text{ A/m}, \qquad \mathsf{M}_1 = \mathsf{h}(\mathsf{H}_1) = 14.72 \cdot 10^5 \text{ A/m}.$$

We then look for $\mathcal{K}^*(\mathsf{M}) = \mathcal{K}(\mathsf{h}^{-1}(\mathsf{M}))$. To determine h^{-1} we observe that

$$\mathsf{H} = \mathsf{H}_S \mathcal{L}^{-1}(\mathsf{H}/\mathsf{h}_0), \qquad \mathcal{K}^*(\mathsf{M}) = \mathcal{K}(\mathsf{H}_S \mathcal{L}^{-1}(\mathsf{M}/h_0)).$$

As the last step we consider $\mathcal{L}^{-1}$ in the approximate form of [40]. Fig. 7 shows the dependence of ξ_m on M in the un-loaded condition, which shows the standard behavior of experimental data (see, e.g., [3, Fig. 4] and [32, Fig. 5]). The function $\xi_m(\mathsf{M}, 0)$ is symmetric and its maximum value occurs at $|\mathsf{M}| = \mathsf{M}_0 = 15.46 \cdot 10^5$ A/m.

7. Conclusions

The subject of this paper is a thermodynamically-consistent scheme of magnetomechanical coupling in solids. The purpose was to establish a mathematical model of magnetostriction where deformation and magnetization depend non-linearly on the magnetic field and the stress.

In the first part the paper develops a scheme for a three-dimensional body and is based on the general aspects of continuum mechanics. First a theory has to be consistent with the objectivity principle whereby the constitutive equations are required to be invariant under Euclidean transformations. Though this is not crucial for the present purposes, we find that

$$\mathbf{F}^T \mathbf{H}, \quad \mathbf{F}^{-1}\mathbf{M}$$

are invariant and so are the vectors obtained by multiplying with any power of the Jacobian J (see the Appendix 8.1). As a second requirement, the theory has to comply with the balance equations. The balance of angular momentum leads to

$$\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} \in \mathrm{Sym}.$$

This condition is satisfied identically by considering the dependence on

$$\mathfrak{H} = \mathbf{F}^T \mathbf{H}, \quad \mathfrak{M} = J \mathbf{F}^{-1} \mathbf{M}$$

instead of $\mathbf{H}, \mathbf{M}$. These fields have already appeared in the literature and are denoted as Lagrangian fields, namely fields relative to the reference configuration. Next we examine the consequences of the second law of thermodynamics and we find that

$$\mu_0 \mathfrak{M} = -\partial_{\mathfrak{H}} \phi_R, \quad \mathcal{T}_{RR} = \partial_{\mathbf{E}} \phi_R,$$

where $\mathcal{T}_{RR} = J\mathbf{F}^{-1}(\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M})\mathbf{F}^{-T}$ and $\mathbf{E}$ is the Green–Lagrange strain tensor. Hence the magnetostriction is described for linear magnetoelastic materials by starting with a free energy ϕ_R of the form (18).

For a non-linear model, and motivated by some experimental works described in the literature, we then restrict attention to one dimensional settings. The symmetry condition on $\mathbf{T}, \mathbf{H}, \mathbf{M}$ is no longer a restriction and then we can use the Eulerian, though one-dimensional, fields H, M. A family of non-linear models has been established with these fields on the basis of the thermodynamic relations

$$\mu_0 \mathsf{M} = -\rho \partial_{\mathsf{H}} \zeta, \quad \ln F = -\rho \partial_\sigma \zeta.$$

Besides the conceptual interest for nonlinear models of magneto-mechanical couplings, the need of revisiting the modeling of magnetostriction in continuum physics is emphasized by experimental evidence of non-monotonic dependences in the dependence of strain and magnetization in terms of stress and magnetic field. Section 6 shows how the modeling is derived on the basis of the thermodynamic consistency and the requirement of a good fit of the experimental data available for NO Fe Si.

The results established in Section 6 show a threefold interest. First, they constitute an explicit mathematical model of magnetostriction in terms of elementary functions. Secondly, from the conceptual side, for a spatial description the dependence on the magnetic field $\mathbf{H}$ and the deformation $\mathbf{F}$ has to be in a joint form as with $\mathfrak{H} = \mathbf{F}^T\mathsf{H}$. Thirdly, in a one-dimensional setting, the constitutive Eq. (36) is thermodynamically consistent thus allowing modeling of the magnetization through the product $\mathcal{K}(\mathsf{H})\mathsf{f}(\sigma)$ and thus taking into account for the non-monotonic behavior. Remarkably, the non-monotonic behavior expressed by data show that the cross-coupling occurs through the product of functions, e.g. $\mathsf{f}(\sigma)\mathsf{k}(\mathsf{H})$, and not merely through the product of variables, e.g. $\sigma\mathsf{H}$.

The nonlinear magnetostriction model thus established is overall characterized by the functions $\mathsf{f}(\sigma), \mathcal{K}(\mathsf{H}), \mathsf{h}(\mathsf{H})$ while $\Sigma(\sigma)$ describes the purely elastic response. These functions are parameterized by a minimal set of positive parameters, namely $\sigma_+, \sigma_-, \mathsf{k}_0, \mathsf{H}_0, \mathsf{H}_S$, and hence differentiations and integrations can be performed by hand. Increasing the number of parameters would allow a better fit of material properties though at the expense of simplicity.

The expression (36) of M is strictly related to the one-dimensional model. Analogous properties in three dimensions might perhaps involve invariants of the pertinent fields (see, e.g., [28]).

CRediT authorship contribution statement

C. Giorgi: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **A. Morro:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Informed consent statement

Not applicable.

Funding

This research received no external funding.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The research leading to this work has been developed under the auspices of INDAM-CNR. Open access funding is provided by CRUI Consortium agreement.

Appendix A. SO(3) invariant fields

The selection of variables is required to satisfy the objectivity principle ([41, § 17], [42, § 4.1.2]): the constitutive functions must be form invariant under the group of Euclidean transformations SO(3). In particular this means that invariant quantities $(\phi, \eta, \varepsilon)$ can depend only on invariants.

A change of frame $\mathscr{F} \to \mathscr{F}^*$ given by a Euclidean transformation, such that $\mathbf{x} \mapsto \mathbf{x}^*$, is expressed by

$$\mathbf{x}^* = \mathbf{c} + \mathbf{Q}\mathbf{x}, \qquad \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1}, \quad \det\mathbf{Q} = 1. \tag{A.1}$$

Under the transformation (A.1), the deformation gradient $\mathbf{F}$ and the magnetic field H change as vectors,

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \qquad \mathsf{H}^* = \mathbf{Q}\mathsf{H}.$$

Hence it follows that the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and the Green–Lagrange strain tensor $\mathbf{E} = (\mathbf{C} - \mathbf{1})/2$ are invariant. Both $\mathbf{F}^T\mathsf{H}$ and $\mathbf{F}^{-1}\mathsf{H}$ are invariant,

$$(\mathbf{F}^T\mathsf{H})^* = (\mathbf{Q}\mathbf{F})^T(\mathbf{Q}\mathsf{H}) = \mathbf{F}^T\mathbf{Q}^T\mathbf{Q}\mathsf{H} = \mathbf{F}^T\mathsf{H},$$

$$(\mathbf{F}^{-1}\mathsf{H})^* = (\mathbf{Q}\mathbf{F})^{-1}(\mathbf{Q}\mathsf{H}) = \mathbf{F}^T\mathbf{Q}^T\mathbf{Q}\mathsf{H} = \mathbf{F}^{-1}\mathsf{H}.$$

Since $J = \det\mathbf{F}$ is invariant, in that

$$J^* = \det(\mathbf{Q}\mathbf{F}) = \det\mathbf{Q}\det\mathbf{F} = \det\mathbf{F} = J,$$

then $f(J)\mathbf{F}^T\mathsf{H}$ and $g(J)\mathbf{F}^{-1}\mathsf{H}$ are invariant too for any functions f, g.

Appendix B. Representation of the mechanical power

If $\mathbf{T}$ is non-symmetric then the mechanical power is $\mathbf{T}\cdot\mathbf{L}$. Now, the velocity gradient $\mathbf{L}$ is decomposed in the form $\mathbf{L} = \mathbf{D} + \mathbf{W}$, where $\mathbf{D}$ is the stretching tensor and $\mathbf{W}$ is the spin. Under SO(3), $\mathbf{D}$ is a tensor or frame-indifferent ([43], § 20.3) while $\mathbf{W}$ is not frame-indifferent,

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \qquad \mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T.$$

Since $\mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot\mathbf{D} + \mathbf{T}\cdot\mathbf{W}$ and $\mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}$ then

$$\mathbf{T}\cdot\mathbf{D} = (\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T})\cdot\dot{\mathbf{E}} = J^{-1}\mathbf{T}_{RR}\cdot\dot{\mathbf{E}},$$

where $\mathbf{T}_{RR} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ is the second Piola stress. Hence

$$\mathbf{T}\cdot\mathbf{L} = J^{-1}\mathbf{T}_{RR}\cdot\dot{\mathbf{E}} + \mathbf{T}\cdot\mathbf{W};$$

the power $\mathbf{T}\cdot\mathbf{L}$ is the sum of an invariant term $J^{-1}\mathbf{T}_{RR}\cdot\dot{\mathbf{E}}$ and a frame-dependent term $\mathbf{T}\cdot\mathbf{W}$.

Data availability

All data generated or analyzed during this study are included in this published article.

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