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# Localized inverse design in conservation laws and Hamilton-Jacobi equations

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**Abstract.** Consider the *inverse design* problem for a scalar conservation law, i.e., the problem of finding initial data evolving into a given profile at a given time. The solution we present below takes into account *localizations* both in the final interval where the profile is assigned and in the initial interval where the datum is sought, as well as additional *a priori constraints* on the datum's range provided by the model. These results are motivated and can be applied to data assimilation procedures in traffic modeling and accidents localization.

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**Keywords.** Traffic dynamics, Inverse problems in conservation laws, Hyperbolic partial differential equations.

## 1. Introduction

We deal with the — constrained and localized — inverse design related to the Cauchy problems for the scalar one dimensional conservation law and for the Hamilton-Jacobi equation. These equations read

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_o(x) \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t U + f(\partial_x U) = 0 \\ U(0,x) = U_o(x) \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}_+$ is time and $x \in \mathbb{R}$ is the space coordinate. In (1.1), on the left, $u = u(t,x)$ is the unknown density of a conserved variable and $f$ is the flux, while on the right $U = U(t,x)$ is the unknown value function and $f$ is the Hamiltonian. The Cauchy problems (1.1) generate the semigroups

$$S^{CL} \colon \mathbb{R}_+ \times \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \qquad \text{and} \qquad S^{HJ} \colon \mathbb{R}_+ \times \mathbf{Lip}\,(\mathbb{R};\mathbb{R}) \to \mathbf{Lip}\,(\mathbb{R};\mathbb{R}) \tag{1.2}$$

in the sense that the orbits $t \to S^{CL}_t u_o$ and $t \to S^{HJ}_t U_o$ solve (1.1), respectively in the entropy or viscosity sense. For a given real interval $J$ — the *constraint* — we provide a full characterization of the *constrained inverse designs* for the two equations in (1.1), namely

$$\begin{aligned} I^{CL}_T(u_T;J) &:= \{u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \colon u_o(x) \subseteq J \text{ for a.e. } x \in \mathbb{R} \text{ and } S^{CL}_T u_o = u_T\} \\ I^{HJ}_T(U_T;J) &:= \{U_o \in \mathbf{Lip}\,(\mathbb{R};\mathbb{R}) \colon U_o'(x) \subseteq J \text{ for a.e. } x \in \mathbb{R} \text{ and } S^{HJ}_T U_o = U_T\} \end{aligned} \tag{1.3}$$

for given functions $u_T \in \mathbf{L}^\infty(\mathbb{R};J)$, $U_T \in \mathbf{Lip}\,(\mathbb{R};\mathbb{R})$ with $U_T'(\mathbb{R}) \subseteq J$ and for a fixed $T > 0$.

Then, we extend this characterization to comprehend cases where the profile $u_T$ is known only on a given real interval, say $K_T$:

$$I^{CL}_T(u_T;J)_{|K_o} := \left\{ \tilde u_o \in \mathbf{L}^\infty(K_o;J) \colon \exists\, u_o \in \mathbf{L}^\infty(\mathbb{R};J) \text{ with } \begin{array}{l} {u_o}_{|K_o} = \tilde u_o \text{ and} \\ S^{CL}_T {u_o}_{|K_T} = u_T \end{array} \right\}.$$

and the meaningful interval $K_o$ is singled out below by means of $K_T$ and $u_T$.

Our motivation is a typical situation in traffic management: there, traffic flow measurements are available at a given location, say $x = L$, during a given time interval. Out of these data, one seeks to reconstruct the flow along road segments before and after the measuring site. Thus, one is led to solve

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an inverse design problem, with constraints on the unknown function (traffic density varies in the fixed bounded interval $J$) and localized in space time, say at $\{L\} \times K_T$. Refer to Sect. 3 for the detailed discussion.

Data assimilation and flow reconstruction are a common problem in various disciplines: the monograph [22] deals with the case of weather forecasts, a field classically related to these problems. Applications to sedimentation are considered in [8], to oil reservoirs in [30] while the special issue [32] is devoted to general fluid dynamics and [20] provide a more analytical approach devoted to Navier-Stokes equation. We refer to [25] for further examples. Deeply related to the present result are [21] and [5], here kinetic techniques are employed.

The current literature provides several results about the inverse design for conservation laws. The case of the (inviscid) Burgers' equation is thoroughly considered in [19,26,27], the general homogeneous case is solved in [10] while the smooth $x$ dependent case is tackled in [12] and the piecewise constant case is in [3], see also [17] for an alternative approach and [16] for results on the Hamilton - Jacobi equation in several space dimensions. A specific system is considered in [9]. Further results specific to reachability are in [1,2,4].

The above applications and, in particular, our motivation are based on conservation laws. However, we extensively make use of techniques typical of Hamilton-Jacobi equation and we extensively exploit the deep connection between the two classes of equations, summarized in (2.1) below. This allows, in particular, to obtain Theorem 2.1, which provides a new independent --- and simpler --- proof of the characterization of the reachability of a profile also in the case of data constrained to attain values in $J$. This result improves that in [12] relaxing the regularity assumptions on the target profile.

As the statements and proofs below show, the roles of the constraint $J$ and of the localization to $K_T$ are entirely different. The former one appears as somewhat marginal, see Remark 2.5. This is due to our requiring that the flux $f$ in (1.1) is convex and independent of $x$. Indeed, the example in [12,13] shows that, without these conditions, the range of the solution to a conservation law may well significantly grow.

When dealing with real measurements, it is necessary to further localize the available information since, in general, only discrete samples of data can be collected. Moreover, measurement errors affect these values. A refinement of the analytical techniques below might tackle these practical issues.

The next section presents our results, first only in the constrained case, then also under a localization condition. Section 3 is devoted to the implications of the general results to our original motivation rooted in vehicular traffic modeling. Finally, all proofs are deferred to Sect. 4.

## 2. Analytic results

Throughout, concerning conservation laws, we refer to entropy solutions as classically defined by Kružkov [24, Definition 1], see also [10, Definition 2.1]. In the case of the Hamilton-Jacobi equation we refer to viscosity solutions [14, Definition I.1], see also [10, Definition 2.2]. The connection between the Cauchy problems in (1.1) is summarized by the following commuting diagrams:

$$
\begin{array}{ccc}
\multicolumn{3}{c}{[23,\ \text{Theorem 1.1}]} \\
U_o & \longrightarrow & S_t^{HJ} U_o \\
\partial_x \downarrow & & \downarrow \partial_x \\
u_o & \longrightarrow & S_t^{CL} u_o
\end{array}
\qquad\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{[10,\ \text{Proposition 2.3}]} \\
U_o & \longrightarrow & S_t^{HJ} U_o \\
\int^x \uparrow & & \uparrow\ [10,\ \text{Formula (2.2)}] \\
u_o & \longrightarrow & S_t^{CL} u_o
\end{array}
\tag{2.1}
$$

see also [11,23].

On the flux/Hamiltonian $f$ we require the following condition:

(f) $f \in \mathbf{C}^2(\mathbb{R};\mathbb{R})$ is strongly convex, in the sense that $f''(x) > 0$ for all $x \in \mathbb{R}$, $\lim_{x\to-\infty} f'(x) = -\infty$ and $\lim_{x\to+\infty} f'(x) = +\infty$.

Throughout, for the basic results about Hamilton-Jacobi equation we refer to [18, Theorem 4 and Theorem 5, Section 3, Chapter 3]. In particular, the solution to (1.1), right, can be written as

$$\forall T \in \mathbb{R}_+ \setminus \{0\} \quad \forall x \in \mathbb{R} \qquad (S_T^{HJ} U_o)(x) = \inf_{\xi \in \mathbb{R}} \left[ U_o(\xi) + T\, f^* \left( \frac{x-\xi}{T} \right) \right] \tag{2.2}$$

and the Legendre transform $f^*$ of $f$ is recalled in Definition 4.2.

For fixed $u \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ and $T > 0$, from the theory of conservation laws we define (a.e.)

$$\begin{aligned} \pi_u \; : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x - T\, f'(u(x)) \,. \end{aligned} \tag{2.3}$$

As soon as $u$ is the solution to the conservation law (1.1) at time $T$, $\pi_u(x)$ is the intersection between the axis $t = 0$ and the minimal — for $u$ left continuous at $x$ — backward characteristic from $(T, x)$, see [15, § 10.3 and § 11.1].

Fix $T > 0$. We say that a map $u \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ satisfies Oleinik estimate [29] at $T$, whenever the following condition holds:

(o) For a.e. $x \in \mathbb{R}$ and $\Delta x \in \mathbb{R}_+ \setminus \{0\}$, $\dfrac{f'(u(x+\Delta x)) - f'(u(x))}{\Delta x} \leq \dfrac{1}{T}$.

As is well known, a solution $u$ to (1.1), left, satisfies condition (o) at $T$ if and only if the map $\pi_u$ is (a.e.) non decreasing. Hence, since $u(x) = (f')^{-1}\left(\frac{x-\pi_u(x)}{T}\right)$, by (f) $u$ admits a representative which is left continuous and thus Condition (o) is satisfied for *every* $x \in \mathbb{R}$ and $\Delta x \in \mathbb{R}_+ \setminus \{0\}$. Below, when (o) applies, by $u$ or $u_T$ we understand their left continuous representatives.

### 2.1. Constrained inverse design

For a non empty closed real interval $J$ acting as *constraint* we now head towards detailed descriptions of the sets (1.3) of profiles $u_T$ and $U_T$ that can be attained as solutions to (1.1) for suitable initial data $u_o$ and $U_o$. The case $J = \mathbb{R}$ is not excluded. We stress that the results below depend on and take care of the constraint $J$. By (2.1), the descriptions of the two sets are one consequence of the other.

Proceeding towards a full characterization of $I_T^{HJ}(U_T; J)$, we start collecting information on a *particular* element $U_o^\flat$.

Proofs to statements in this paragraph are deferred to § 4.1.

**Theorem 2.1.** *Let $f$ satisfy (f), $J$ be a non empty closed interval and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R}; J)$ satisfying Condition (o) at $T$. Define, for a fixed $\breve{x} \in \mathbb{R}$, for all $x \in \mathbb{R}$*

$$U_T(x) := \int_{\breve{x}}^{x} u_T(\xi)\, \mathrm{d}\xi \,,\; U_o^\flat(x) := \sup_{\xi \in \mathbb{R}} \left[ U_T(\xi) - T\, f^* \left( \frac{\xi - x}{T} \right) \right],\; u_o^\flat(x) := \frac{\mathrm{d}}{\mathrm{d}x} U_o^\flat(x). \tag{2.4}$$

*Then, $U_o^\flat \in I_T^{HJ}(U_T; J)$ and $u_o^\flat \in I_T^{CL}(u_T; J)$.*

Above, $f^*$ is, as usual, the Legendre transform of $f$, see Definition 4.2.

**Corollary 2.2.** *Let $f$ satisfy (f), $J$ be a non empty closed interval and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R}; J)$. Then, $u_T$ satisfies Condition (o) at $T$ if and only if $I_T^{CL}(u_T; J) \neq \emptyset$.*

The proof of Corollary 2.2 is as follows: on the one hand, Oleinik's result [29] ensures that Condition (o) holds for any reachable profile. On the other hand, Theorem 2.1 shows that Condition (o) implies that $I_T^{CL}(u_T; J)$ is non empty.

Corollary 2.2 was originally stated as [10, Corollary 3.2] in the case $J = \mathbb{R}$, and proved relying on conservation laws techniques, while the proof of Theorem 2.1 is based on the Hamilton-Jacobi equation (1.1).

An explicit construction of $u_o^\flat$ is provided in the following extension of [10, Theorem 3.1] which, besides comprising the constraint $J$, does not require **SBV** regularity.

**Proposition 2.3.** *Let $f$ satisfy (f) and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. Define $U_T$, $U_o^\flat$ and $u_o^\flat$ as in (2.4). Call $v$ and $V$ the entropy and viscosity solutions to*

$$\begin{cases} \partial_t v + \partial_x f(-v) = 0 \\ v(0,x) = -u_T(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t V + f(-\partial_x V) = 0 \\ V(0,x) = -U_T(x). \end{cases} \tag{2.5}$$

*Then, for a.e. $t \in [0,T]$ and $x \in \mathbb{R}$,*

$$\begin{array}{rl} v(T,x) &= -u_o^\flat(x); \\ V(T,x) &= -U_o^\flat(x); \end{array} \qquad v(t,x) = \partial_x V(t,x) \qquad \text{and} \qquad \mathrm{TV}\,(u_T) \geq \mathrm{TV}\,(u_o^\flat).$$

When $f$ depends explicitly on $x$, the bending of characteristic lines significantly complicates a result like Proposition 2.3, see [12].

The function $u_o^\flat$ defined in (2.4) has the minimal range among those profiles that evolve into $u_T$, as proved by the following result.

**Corollary 2.4.** *Let $f$ satisfy (f) and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ satisfying Condition (o) at $T$. Let $u_o^\flat$ be as defined in (2.4). Then,*

$$\mathrm{TV}\,(u_T) = \mathrm{TV}\,(u_o^\flat) \qquad \text{and} \qquad \overline{co}\; u_T(\mathbb{R}) = \overline{co}\; u_o^\flat(\mathbb{R}).$$

Above, for $A \subseteq \mathbb{R}$, $\overline{\mathrm{co}}\, A$ stands for the closed convex hull of $A$.

**Remark 2.5.** Corollary 2.4 underlines that the role of $J$ in reachability is rather marginal, as soon as it is sufficiently large, i.e., as soon as $J \supseteq \overline{\mathrm{co}}\; u_T(\mathbb{R})$.

**Proposition 2.6.** *([15, Theorem 11.4.3]) Let $f$ satisfy (f) and $T$ be positive. For all $U_o \in \mathbf{Lip}\,(\mathbb{R};\mathbb{R})$, define $u_o = U_o'$ and set $u_T = S_T^{CL} u_o$, according to (1.2). Using the notation (2.3), for all $x \in \mathbb{R}$*

$$S_T^{HJ} U_o(x) = U_o\left(\pi_{u_T}(x)\right) + T\, f^*\left(\frac{x - \pi_{u_T}(x)}{T}\right). \tag{2.6}$$

Note that, when $f$ also depends explicitly on $x$, [12, Theorem 3.1] reformulates Proposition 2.6 in terms of the connection between generalized characteristics and minima of the integral functional connected to the Hamilton-Jacobi equation.

**Proposition 2.7.** *Let $f$ satisfy (f), $J$ be a non empty closed interval and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R};J)$ satisfying Condition (o) at $T$. Define $U_T$ and $U_o^\flat$ as in (2.4). Then, for all $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, using the notation (2.3) and setting, for any fixed $\breve{x} \in \mathbb{R}$,*

$$\forall\, t \in \mathbb{R} \qquad U_o(x) := \int_{\pi_{u_T}(\breve{x})}^{x} u_o(\xi)\,\mathrm{d}\xi - T\, f^*\left(-\frac{\pi_{u_T}(\breve{x})}{T}\right) \tag{2.7}$$

*we have the equivalences*

$$u_o \in I_T^{CL}(u_T;J) \iff U_o \in I_T^{HJ}(U_T;J) \iff \begin{cases} (i) \quad U_o \geq U_o^\flat; \\ (ii) \;\; U_o = U_o^\flat \text{ on } \overline{\pi_{U_T'}(\mathbb{R})}; \\ (iii)\; U_o'(\mathbb{R}) \subseteq J. \end{cases} \tag{2.8}$$

The above proposition is strictly related to [12, Theorem 3.3] and [16, Theorem 2.6]: both these results do not consider the constraint $J$ but the former one applies to general $x$ dependent Hamiltonian functions while the latter applies to several space dimensions.

When the constraint $J$ is compact, the introduction of the maps $U_o^\sharp$ and $u_o^\sharp$ in the following Proposition allows the precise description of the inverse design sets provided by Theorem 2.9.

**Proposition 2.8.** *Let $f$ satisfy (f), $J$ be a non empty compact interval and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R}; J)$ satisfying (o) at $T$. Define $U_T$ as in (2.4) and*

$$\forall\, x \in \mathbb{R} \qquad U_o^\sharp(x) := \sup\left\{U_o(x) \colon U_o \in I_T^{HJ}(U_T; J)\right\}, \qquad u_o^\sharp(x) := \frac{\mathrm{d}}{\mathrm{d}x} U_o^\sharp(x). \tag{2.9}$$

*Then, $U_o^\sharp \in I_T^{HJ}(U_T; J)$ and $u_o^\sharp \in I_T^{CL}(u_T; J)$.*

Note that while $U_T$ and $U_o^\sharp$ depend on $\breve{x}$ in (2.4), the map $u_o^\sharp$ is actually independent of it.

We are now ready for a further characterization of the set $I_T^{HJ}(U_T; J)$.

**Theorem 2.9.** *Let $f$ satisfy (f), $J$ be a non empty compact interval and $T$ be positive. Fix $u_T \in \mathbf{L}^\infty(\mathbb{R}; J)$ satisfying (o) at $T$. Then, with the notation (2.4), (2.9),*

$$I_T^{HJ}(U_T; J) = \left\{U_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R}) \colon \begin{array}{l} U_o'(x) \in J \\ U_o(x) \in [U_o^\flat(x), U_o^\sharp(x)] \end{array} \text{ for a.e. } x \in \mathbb{R}\right\}. \tag{2.10}$$

*In particular, $I_T^{HJ}(U_T; J)$ is convex and compact with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}$. Moreover, setting $\breve{y} := \pi_{u_T}(\breve{x})$, for any fixed $\breve{x} \in \mathbb{R}$, we have the characterization*

$$I_T^{CL}(u_T; J) = \left\{u_o \in \mathbf{L}^\infty(\mathbb{R}; J) \colon \int_{\breve{y}}^{y} u_o \mathrm{d}x \in \left[\int_{\breve{y}}^{y} u_o^\flat \mathrm{d}x, \int_{\breve{y}}^{y} u_o^\sharp \mathrm{d}x\right] \text{ for all } y \in \mathbb{R}\right\} \tag{2.11}$$

*so that $I_T^{CL}(u_T)$ is convex and sequentially compact with respect to the weak-$*$ $\mathbf{L}^\infty$ topology.*

Again, note that the arbitrariness of $\breve{x}$ does affect $U_o^\flat$ and $U_o^\sharp$ but has no relevance on $u_o^\flat$, $u_o^\sharp$.

### 2.2. Inverse design localized in space

This section is devoted to the localization of the previous results on two space intervals: the former one, $K_T$, is to be considered at time $t = T$ and the latter one, $K_o$ at time $t = 0$.

Hereafter, we consider only the conservation law in (1.1), the case of the Hamilton-Jacobi equation being entirely analogous.

The proofs related to statements in this section, where necessary, are deferred to § 4.2.

**Definition 2.10.** Let $J$ be a non trivial closed real interval and $T > 0$. Fix a second non trivial closed real interval $K_T$. A profile $u_T \in \mathbf{L}^\infty(K_T; J)$ is *reachable at $t = T$ on $K_T$* if there exists a $u_o \in \mathbf{L}^\infty(\mathbb{R}; J)$ such that the corresponding solution $u$ to the conservation law in (1.1) satisfies ${S_T^{CL} u_o}_{|K_T} = u_T$.

If $K_o$ is another non trivial closed real interval, denote

$$I_T^{CL}(u_T; J)_{|K_o} := \left\{\tilde{u}_o \in \mathbf{L}^\infty(K_o; J) \colon \exists\, u_o \in \mathbf{L}^\infty(\mathbb{R}; J) \text{ with } \begin{array}{ll} {S_T^{CL} u_o}_{|K_T} & = u_T, \text{ and} \\ {u_o}_{|K_o} & = \tilde{u}_o \end{array}\right\}. \tag{2.12}$$

We regret that in the above notation $I_T^{CL}(u_T; J)_{|K_o}$, the set $K_T$ is omitted for simplicity, in spite of its relevance.

We now provide a simple specific extension of any map $\widehat{u}_T \in \mathbf{BV}(K_T; J)$ to $u_T^* \in \mathbf{BV}(\mathbb{R}; J)$ so that the inverse design restricted to $K_o$ remains unaltered, provided $K_o$ is the domain of dependency of $K_T$. More precisely:

**Theorem 2.11.** *Let $f$ satisfy (f). Let $J$ be a non trivial closed real interval and $T$ be positive. Fix $x_1, x_2 \in \mathbb{R}$ with $x_2 > x_1$ and choose a map $\widehat{u}_T \in \mathbf{BV}([x_1, x_2]; J)$. Define*

$$u_T^*(x) = \begin{cases} \widehat{u}_T(x_1+) & x < x_1 \\ \widehat{u}_T(x) & x \in [x_1, x_2] \\ \widehat{u}_T(x_2-) & x > x_2 \end{cases} \tag{2.13}$$

*Then,*

$$I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]} = I_T^{CL}(u_T^*;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}.$$

In other words, setting in Theorem 2.11 $K_T=[x_1,x_2]$ and $K_o=\overline{\mathrm{co}}\,\pi_{\widehat{u}_T}(\mathring{K}_T)$, we have

$$u^*_{T|K_T}=\widehat{u}_T \qquad \text{and} \qquad I_T^{CL}(\widehat{u}_T;J)_{|K_o}=I_T^{CL}(u_T^*;J)_{|K_o}.$$

In spite of its simplicity, the extension $u_T^*$ provided in (2.13) of Theorem 2.11 is sufficient to recover the whole inverse design. The next result shows that any other extension either gives the same result or gives the empty set.

**Theorem 2.12.** *Let $f$ satisfy (f). Let $J$ be a non trivial closed real interval and $T$ be positive. Fix $x_1,x_2\in\mathbb{R}$ with $x_2>x_1$ and choose a map $\widehat{u}_T\in\mathbf{L}^\infty([x_1,x_2];J)$ reachable at $t=T$ on $[x_1,x_2]$. Let $u_T\in\mathbf{L}^\infty(\mathbb{R};J)$ be such that $u_{T|[x_1,x_2]}=\widehat{u}_T$. Then,*

*either:* $I_T^{CL}(u_T;J)=\emptyset$,

*or:* $I_T^{CL}(u_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}=I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}$.

Note that by Corollary 2.2 the assumption that $\widehat{u}_T$ be reachable ensures that $\widehat{u}_T$ has locally bounded variation by (o), hence its traces at $x_1$ and $x_2$ are well defined.

**Corollary 2.13.** *Let $f$ satisfy (f). Let $T$ be positive, $J$ be a non trivial closed real interval and fix $x_1$, $x_2$ in $\mathbb{R}$ with $x_1<x_2$. Let $u_1,u_2\in\mathbf{L}^\infty(\mathbb{R};J)$ be such that $u_{1|[x_1,x_2]}=u_{2|[x_1,x_2]}$, $I_T^{CL}(u_1,J)\neq\emptyset$ and $I_T^{CL}(u_2,J)\neq\emptyset$. Then, with the notation (2.4) and (2.9)*

$$I_T^{CL}(u_1;J)_{|[\pi_{u_1}(x_1+),\pi_{u_1}(x_2-)]}=I_T^{CL}(u_2;J)_{|[\pi_{u_2}(x_1+),\pi_{u_2}(x_2-)]}, \tag{2.14}$$

$$u^\flat_{1|[\pi_{u_1}(x_1+),\pi_{u_1}(x_2-)]}=u^\flat_{2|[\pi_{u_2}(x_1+),\pi_{u_2}(x_2-)]}, \tag{2.15}$$

$$u^\sharp_{1|[\pi_{u_1}(x_1+),\pi_{u_1}(x_2-)]}=u^\sharp_{2|[\pi_{u_2}(x_1+),\pi_{u_2}(x_2-)]}. \tag{2.16}$$

Equality (2.14) directly follows from Theorem 2.12. Then, (2.15) and (2.16) follow combining (2.14) with Theorem 2.9 setting for simplicity $\breve{x}=x_1$.

## 3. Application to traffic

A macroscopic description of the flow of traffic along a highway tract can be based on the well known Lighthill-Whitham and Richards model [28,31], leading to the evolution equation

$$\partial_t\rho+\partial_x\left(\rho\, v(\rho)\right)=0 \qquad (t,x)\in\mathbb{R}_+\times[0,L], \tag{3.1}$$

where $t$ is time, $x$ is the coordinate along the road, $\rho=\rho(t,x)$ roughly measures the amount of vehicles per unit length and $v=v(\rho)$ is the (mean) traffic speed corresponding to the density $\rho$. As usual, we call $q=\rho\, v(\rho)$ the vehicular flux. The space coordinate varies along the interval $[0,L]$, with $L>0$. Note that (3.1) is neither a Cauchy problem nor a standard initial - boundary value problem: nevertheless it is a classical setup in traffic management.

At location $x=L$, the outflow $q_{out}=q_{out}(t)$ is measured. The results in Sect. 2 allow to exhibit conditions implying that traffic underwent some critical event (possibly an accident) and estimate where it happened. Moreover, they also characterize vehicular traffic, providing properties and constraints that any flow reconstruction in a data assimilation procedure must enjoy or fulfill to be coherent with (3.1). The localization results in § 2.2 allow us to provide statements that are intrinsic to any (bounded) time interval and that hold on the natural domain of dependency of the measured data.

The speed law $v = v(\rho)$ plays in traffic modeling a role analogous to that played in thermodynamics by the equation of state. However, while equation of states can be rigorously justified on the basis of physical assumptions, speed laws are typically accepted or rejected on the basis of qualitative considerations. The assumption we pose on $v$ is

(v) $v \in \mathbf{C}^2([0, R]; \mathbb{R}_+)$ is such that $v(R) = 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}(\rho\, v(\rho)) < 0$, for a fixed $R > 0$.

This condition is slightly weaker than the more usual one $v' < 0$ and $v'' < 0$.

A common problem in traffic modeling is the following: given the traffic outflow measured at the position $x = L$, namely $q_L(t) = \rho(t, L)\, v(\rho(t, L))$, reconstruct the function $\rho = \rho(t, x)$ for $x \in [0, L]$ assuming that the outflow $q_L$ results from the — unknown — inflow at position $x = 0$.

We are thus led to exchange the roles of time $t$ and space $x$ in (3.1), using as dependent variable the flow $q(t, x) = \rho(t, x)\, v(\rho(t, x))$ and refer to the (backward) Cauchy problem

$$\begin{cases} \partial_x q + \partial_t f(q) = 0 \\ q(t, L) = q_L(t) \end{cases} \qquad (t, x) \in \mathbb{R} \times [0, L]. \tag{3.2}$$

With reference to (3.1), $f$ is (related to) the inverse of the map $\rho \mapsto \rho\, v(\rho)$ on the interval where this map is strictly increasing which, under assumption (v), is the interval $[0, \widehat{q}]$ where $\widehat{q} = \max_{\rho\in[0,R]} \rho\, v(\rho)$. Note that the choice of the congested interval where $\frac{\mathrm{d}q}{\mathrm{d}\rho} < 0$ is not consistent with (3.2).

**Proposition 3.1.** *Let $v$ satisfy (v) and let $\widehat{\rho}$ be such that $\widehat{\rho}\, v(\widehat{\rho}) = \max_{[0,R]} \rho\, v(\rho)$. Define, for a fixed $\bar{\rho} \in \,]0, \widehat{\rho}[$ and for all $\rho \in [0, \bar{\rho}]$,*

$$f(q) := \rho \iff q = \rho\, v(\rho),$$

*Then:*

1. *If, with reference to (3.1), $E \in \mathbf{C}^2([0, \bar{\rho}]; \mathbb{R})$ is a convex entropy and $F$ a corresponding flux, then, $F \circ f$ is a convex entropy and $E \circ f$ is a corresponding flux for (3.2).*
2. *If $\rho \in \mathbf{L}^\infty(\mathbb{R}_+ \times [0, L]; [0, \bar{\rho}])$ is a weak solution to $\partial_t \rho + \partial_x(\rho\, v(\rho)) = 0$, then, the map $(t, x) \mapsto q(t, x) = \rho(t, x)\, v(\rho(t, x))$ is a weak solution to $\partial_x q + \partial_t f(q) = 0$*
3. *If in distributional sense*
$$\partial_t E(\rho) + \partial_x F(\rho) \le 0, \tag{3.3}$$
*then, in distributional sense*
$$\partial_x (F \circ f)(q) + \partial_t (E \circ f)(q) \le 0.$$
4. *If (3.3) holds for any convex entropy, then, the trace $q_L$ defined by*
$$\lim_{\delta\to 0+} \int_{-T}^{T} |q_L(t) - q(t, L - \delta)|\, \mathrm{d}t = 0$$
*for all $T > 0$, is well defined and $q$ is a weak solution to (3.2).*

The proofs of (1), (2) and (3) are straightforward calculations, while (4) follows from the regularity and convexity of $f$, thanks to [33].

Condition (o) can then be interpreted as a minimal, necessary but not sufficient, requirement for $q_L$ to be compatible with a regular flow of traffic.

**Proposition 3.2.** *Under the assumptions of Proposition 3.1, call $\bar{q} = \bar{\rho}\, v(\bar{\rho})$. Fix a non trivial compact time interval $[T_1, T_2]$ and a measured traffic flow $q_{out} \in \mathbf{L}^\infty([T_1, T_2]; [0, \bar{q}])$. Then, $q_{out}$ is reachable at $x = L$, for some $L > 0$, on $[T_1, T_2]$ in the sense of Definition 2.10 with respect to the problem*

$$\begin{cases} \partial_x q + \partial_t f(q) = 0 & (t, x) \in \mathbb{R} \times [0, L] \\ q(t, 0) = q_{in}(t) & t \in \mathbb{R} \end{cases}$$

*if and only if*

$$\frac{1}{L} \geq \operatorname*{ess\,sup}_{T_1 \leq t_1 < t_2 \leq T_2} \frac{f'\left(q_{out}(t_2)\right) - f'\left(q_{out}(t_1)\right)}{t_2 - t_1}. \tag{3.4}$$

*Moreover, for such an $L$, define*

$$\tau_1 = T_1 - L\, f'\left(q_{out}(T_1)\right) \quad \text{and} \quad \tau_2 = T_2 - L\, f'\left(q_{out}(T_2)\right).$$

*There exist $q^\flat, q^\sharp \in \mathbf{L}^\infty([\tau_1, \tau_2]; [0, \bar q]$ such that if $q_{in} \in \mathbf{L}^\infty(\mathbb{R}; [0, \bar q])$, the following statements are equivalent:*

*(1) There exists an entropy solution $q \in \mathbf{L}^\infty(\mathbb{R} \times [0, L]; [0, \bar q])$ to*

$$\partial_x q + \partial_t f(q) = 0 \quad \text{such that} \quad \begin{array}{l} q(t,0) = q_{in}(t) \ \text{ for } \ t \in [\tau_1, \tau_2]; \\ q(t,L) = q_{out}(t) \ \text{for } \ t \in [T_1, T_2]. \end{array} \tag{3.5}$$

*(2) For any $\tau \in [\tau_1, \tau_2]$*

$$\int_{\tau_1}^{\tau} q_{in}(t)\, \mathrm{d}t \in \left[\int_{\tau_1}^{\tau} q^\flat(t)\, \mathrm{d}t\,, \int_{\tau_1}^{\tau} q^\sharp(t)\, \mathrm{d}t\right]. \tag{3.6}$$

The characterization (3.4) is obtained applying Corollary 2.2. Then, Theorem 2.11, Corollary 2.13 and Theorem 2.9 allow to prove the equivalence. Remark that Proposition 3.2 is *intrinsic* to the time interval $[T_1, T_2]$, thanks in particular to Corollary 2.13.

**Remark 3.3.** Localizing (2.4) in Theorem 2.1, the function $q^\flat$, through one of its primitives $Q^\flat$, can be computed from the measured data $q_{out}$ for any $\tau \in [\tau_1, \tau_2]$

$$Q^\flat(\tau) := \sup_{t \in [T_1, T_2]} \left[\int_{T_1}^{t} q_{out}(s)\, \mathrm{d}s - L\, f^*\left(\frac{t - \tau}{L}\right)\right].$$

From the traffic management point of view, as soon as condition (3.4) is violated, one can infer that the standard flow of traffic was altered — possibly by an accident — at a distance $L$ from the measuring site. For such $L$, a time $\tau$ at which traffic resumes after the road was blocked, is a time where $Q^\flat$ is not differentiable.

## 4. Technical details

Recall first the following elementary definitions and properties.

**Lemma 4.1.** *Let $f: \mathbb{R} \to \mathbb{R}$ be strongly convex. Then, for all $A \in \mathbb{R}$, there exists $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) \geq \alpha + A\,|x|$.*

**Definition 4.2.** Let $f: \mathbb{R} \to \mathbb{R}$ be convex. Its *Legendre Transform* is the map $f^*: \mathbb{R} \to \mathbb{R}$ defined by $f^*(y) := \sup_{x \in \mathbb{R}} \left(y\, x - f(x)\right)$. for all $y \in \mathbb{R}$.

Note that for any compact interval $K$, the values of $f^*$ on $f'(K)$ depend exclusively on the restriction of $f$ to $K$.

**Lemma 4.3.** *([18, Theorem 3, § 3.3.2]) If $f: \mathbb{R} \to \mathbb{R}$ is strongly convex and $f^*$ is its Legendre transform, then*

*(L1) for all $y \in \mathbb{R}$, $f^*(y) = y\,(f')^{-1}(y) - f\left((f')^{-1}(y)\right)$;*
*(L2) for all $y \in \mathbb{R}$, $(f^*)'(y) = (f')^{-1}(y)$;*
*(L3) $f^*$ is strongly convex in the sense it satisfies (f).*

### 4.1. Proofs related to Sect. 2.1

*Proof of Theorem 2.1.* The proof is divided into several short steps. Using the notation (2.4), define the set valued map

$$\begin{aligned} \mathcal{M} : \mathbb{R} \to & \quad \mathcal{P}(\mathbb{R}) \\ x \mapsto & \left\{\xi \in \mathbb{R} \colon U_T(\xi) = U_o^\flat(x) + T\, f^*\left(\frac{\xi - x}{T}\right)\right\} \end{aligned} \tag{4.1}$$

**1.** $U_o^\flat \in \mathbf{Lip}\,(\mathbb{R};\mathbb{R})$ **and for a.e.** $x \in \mathbb{R}$, $\frac{\mathrm{d}}{\mathrm{d}x} U_o^\flat(x) \in J$.

A direct consequence of the assumptions on $u_T$ is that, for $h$ positive, by (2.4)

$$U_T(x) - h\, \operatorname{ess\,inf}_{\mathbb{R}} u_T \geq U_T(x-h) \geq U_T(x) - h\, \operatorname{ess\,sup}_{\mathbb{R}} u_T. \tag{4.2}$$

Moreover, for any $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, setting $\xi = x - (x_2 - x_1)$,

$$U_o^\flat(x_1) = \sup_{\xi \in \mathbb{R}} \left[U_T(\xi) - T\, f^*\left(\frac{\xi - x_1}{T}\right)\right] = \sup_{x \in \mathbb{R}} \left[U_T\left(x - (x_2 - x_1)\right) - T\, f^*\left(\frac{x - x_2}{T}\right)\right]$$

so that by (4.2) with $h = x_2 - x_1$, we have the two estimates

$$\begin{aligned} U_o^\flat(x_1) &\leq \sup_{x \in \mathbb{R}} \left(U_T(x) - T\, f^*\left(\frac{x - x_2}{T}\right)\right) - (x_2 - x_1)\, \operatorname{ess\,inf}_{\mathbb{R}} u_T \\ &= U_o^\flat(x_2) - (x_2 - x_1)\, \operatorname{ess\,inf}_{\mathbb{R}} u_T; \\ U_o^\flat(x_1) &\geq \sup_{x \in \mathbb{R}} \left(U_T(x) - T\, f^*\left(\frac{x - x_2}{T}\right)\right) - (x_2 - x_1)\, \operatorname{ess\,sup}_{\mathbb{R}} u_T \\ &= U_o^\flat(x_2) - (x_2 - x_1)\, \operatorname{ess\,sup}_{\mathbb{R}} u_T, \end{aligned}$$

proving the Lipschitz continuity of $U_o^\flat$ and, since $[\operatorname{ess\,inf}_{\mathbb{R}} u_T, \operatorname{ess\,sup}_{\mathbb{R}} u_T] \subseteq J$, completing the proof of the claim, thanks to Rademacher's Theorem [18, Theorem 6, § 5.8.3]. ✓

**2. For all** $x$, **the set** $\mathcal{M}(x)$ **is not empty.** (In other words, the sup in (2.4) is a maximum). By (f) and Lemma 4.3, the map $f^*$ is strongly convex, the map $U_T$ is globally Lipschitz continuous, and so sublinear at $\pm\infty$, hence by Lemma 4.1 $\lim_{\xi \to -\infty} \left(U_T(\xi) - T\, f^*\left(\frac{\xi - x}{T}\right)\right) = -\infty$ and $\lim_{\xi \to +\infty} \left(U_T(\xi) - T\, f^*\left(\frac{\xi - x}{T}\right)\right) = -\infty$ for all $x \in \mathbb{R}$. An application of Weierstrass Theorem completes the proof of the claim. ✓

**3. There exists** $R > 0$ **such that if** $x \in \mathbb{R}$ **and** $\xi \in \mathcal{M}(x)$, **then** $|x - \xi| \leq R$. Let $\kappa$ be a Lipschitz constant for $U_T$. By (f) and Lemma 4.3, $f^*$ is strongly convex, so that we can apply Lemma 4.1 to $f^*$ with $A = \kappa + 1$. Hence,

$$\begin{aligned} U_T(\xi) - T\, f^*\left(\frac{\xi - x}{T}\right) &\leq U_T(x) + \kappa\, |\xi - x| - T\, \alpha - (\kappa + 1)\, |\xi - x| \\ &\leq U_T(x) - T f^*(0) + T\left(f^*(0) - \alpha\right) - |\xi - x| \end{aligned}$$

and the choice of $\xi$ as a maximizer for the left hand side above ensures that $|x - \xi| \leq T\left(f^*(0) - \alpha\right)$. ✓

**4.** $\mathcal{M}$ **is monotone increasing.** By this, we mean that if $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$, then for all $\xi_1 \in \mathcal{M}(x_1)$ and for all $\xi_2 \in \mathcal{M}(x_2)$, it holds that $\xi_1 \leq \xi_2$.

Proceed by contradiction and assume that $x_1 < x_2$ but $\xi_1 > \xi_2$. Define

$$A = \frac{\xi_1 - x_1}{T}, \quad B = \frac{\xi_2 - x_2}{T}, \quad C = \frac{\xi_2 - x_1}{T} \quad \text{and} \quad D = \frac{\xi_1 - x_2}{T}.$$

Clearly, $A + B = C + D$ and $A > C > B$, $A > D > B$. By construction, there exists a (unique) $\theta \in\, ]0,1[$ such that

$$C = \theta\, A + (1 - \theta) B \quad \text{and} \quad D = (1 - \theta) A + \theta\, B.$$

The strong convexity of $f^*$ then ensures that

$$f^*(C) + f^*(D) < f^*(A) + f^*(B).$$

The above choices of $x_1, x_2, \xi_1, \xi_2$ imply that

$$U_o^\flat(x_1) = U_T(\xi_1) - T\, f^*(A) \quad \text{and} \quad U_o^\flat(x_2) = U_T(\xi_2) - T\, f^*(B),$$

so that

$$U_o^\flat(x_1) + U_o^\flat(x_2) < U_T(\xi_1) + U_T(\xi_2) - T\, f^*(C) - T\, f^*(D)$$

which in turn implies that at least one of the following inequalities hold:

$$U_o^\flat(x_1) < U_T(\xi_2) - T\, f^*(C) \quad \text{or} \quad U_o^\flat(x_2) < U_T(\xi_1) - T\, f^*(D).$$

Both inequalities above contradict the definition (2.4) of $U_o^\flat$. ✓

**5. For all $x$, the set $\mathcal{M}(x)$ is a compact interval.**

By Claim 3. above, for all $x \in \mathbb{R}$, the set $\mathcal{M}(x)$ is bounded since $\mathcal{M}(x) \subseteq [x - R, x + R]$. The definition (4.1) of $\mathcal{M}$ shows that for every fixed $x$, the set $\mathcal{M}(x)$ is closed, thanks to the regularity of $U_o^\flat$ proved in Claim 1 and that of $U_T$ and $f^*$.

Fix $x \in \mathbb{R}$. To prove that $\mathcal{M}(x)$ is an interval, choose $\xi_1, \xi_2 \in \mathcal{M}(x)$ with $\xi_1 < \xi_2$ and introduce the Lipschitz continuous map $\phi\colon [\xi_1, \xi_2] \to \mathbb{R}$ by $\phi(\xi) := U_T(\xi) - T f^*\left((\xi - x)/T\right)$. Proceed by contradiction and. by (2.4) and (4.1), assume that there exists a $\xi \in \, ]\xi_1, \xi_2[$ such that $\phi(\xi) < \phi(\xi_1) = \phi(\xi_2) = \max \phi$. Then, the Lipschitz continuity of $\phi$ ensures that $\phi$ is differentiable a.e. and

$$\begin{array}{l|l}
\quad 0 > \phi(\xi) - \phi(\xi_1) = \int_{\xi_1}^{\xi} \phi'(y)\, \mathrm{d}y & \quad 0 > \phi(\xi_2) - \phi(\xi) = \int_{\xi}^{\xi_2} \phi'(y)\, \mathrm{d}y \\
\Rightarrow \exists\, y_1 \in \, ]\xi_1, \xi[ \, : \begin{cases} \phi \text{ is differentiable at } y_1 \\ \text{and } \phi'(y_1) < 0 \end{cases} & \Rightarrow \exists\, y_2 \in \, ]\xi, \xi_2[ \, : \begin{cases} \phi \text{ is differentiable at } y_2 \\ \text{and } \phi'(y_2) > 0 \end{cases} \\
\Rightarrow U_T'(y_1) < (f^*)'\left((y_1 - x)/T\right) & \Rightarrow U_T'(y_2) > (f^*)'\left((y_2 - x)/T\right) \\
\Rightarrow f'\left(U_T'(y_1)\right) < (y_1 - x)/T & \Rightarrow f'\left(U_T'(y_2)\right) > (y_2 - x)/T
\end{array}$$

so that

$$f'\left(U_T'(y_2)\right) - f'\left(U_T'(y_1)\right) > \frac{y_2 - y_1}{T}$$

which contradicts that $u_T$ satisfies Condition (o) at $T$ since $y_1 < \xi < y_2$. ✓

**6. $\mathcal{M}$ is surjective, in the sense that $\bigcup_{x \in \mathbb{R}} \mathcal{M}(x) = \mathbb{R}$.** Fix $\xi$ in $\mathbb{R}$. We seek an $x \in \mathbb{R}$ such that $\xi \in \mathcal{M}(x)$. Thanks to Claim 5., it is sufficient to prove that $\min \mathcal{M}(x) \leq \xi \leq \max \mathcal{M}(x)$.

To this aim, define $x_* = \sup\left\{x \in \mathbb{R}\colon \min \mathcal{M}(x) < \xi\right\}$ and note that $\min \mathcal{M}(x_* - \frac{1}{n}) < \xi$. The sequence $\xi_n$ defined by $\xi_n = \min \mathcal{M}(x_* - \frac{1}{n})$ is increasing by Claim 4. above and it is bounded by the above choices. Thus, there exits a real $\xi_*$ such that $\lim_{n \to +\infty} \xi_n = \xi_*$. Definition (4.1) and Claim 1. then ensure that $\xi_* \in \mathcal{M}(x_*)$ and $\xi_* \leq \xi$. As a consequence, $\min \mathcal{M}(x_*) \leq \xi$.

To prove the other bound $\max \mathcal{M}(x_*) \geq \xi$, proceed by contradiction. Assume that $\max \mathcal{M}(x_*) < \xi$ and consider the sequence $y_n = \max \mathcal{M}(x_* + \frac{1}{n})$, which is decreasing and bounded below, so that it admits a limit $y_*$, with $y_* \in \mathcal{M}(x_*)$. Thus, $y_* < \xi$ and for at least one index $n$, we have $y_n < \xi$, so that $\min \mathcal{M}(x_* + \frac{1}{n}) \leq y_n < \xi$ which contradicts the inequality $x_* + \frac{1}{n} > x_*$. ✓

**7. Conclusion.** By (2.4), for all $x, \xi \in \mathbb{R}$, we have that $U_T(\xi) \leq U_o^\flat(x) + T\, f^*\left(\frac{\xi - x}{T}\right)$. By Claim 6., for all $\xi \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $U_T(\xi) = U_o^\flat(x) + T\, f^*\left(\frac{\xi - x}{T}\right)$. Therefore,

$$U_T(\xi) = \inf_{x \in \mathbb{R}} U_o^\flat(x) + T\, f^*\left(\frac{\xi - x}{T}\right)$$

proving, by Hopf-Lax Formula[18, Theorem 6, Section 3, Chapter 3], that $U_o^\flat \in I_T^{HJ}(U_T; J)$ and, hence, that $u_o^\flat \in I_T^{CL}(u_T; J)$ by (2.1). □

*Proof of Proposition 2.3.* Call $g(u):=f(-u)$. By Definition 4.2, $g^*(p) = f^*(-p)$ for all $p \in \mathbb{R}$. Definition (2.4) of $U_o^\flat$ implies that for all $x \in \mathbb{R}$

$$-U_o^\flat(x) = \inf_{\xi\in\mathbb{R}} \left(-U_T(\xi) + T\, f^*\left(-\frac{x-\xi}{T}\right)\right) = \inf_{\xi\in\mathbb{R}} \left(-U_T(\xi) + T\, g^*\left(\frac{x-\xi}{T}\right)\right)$$

By the Hopf-Lax Formula[18, Theorem 6, Section 3, Chapter 3], we have $V(T,x) = -U_o^\flat(x)$. An application of [23, Theorem 1.1] as in (2.1) ensures that $v(T,x) = -u_o^\flat(x)$ and that $v(t,x) = \partial_x V(t,x)$. Finally, the desired bound on the total variation follows from [6, Theorem 6.1]. □

*Proof of Corollary 2.4.* The equality of the total variations follows from Proposition 2.3 and [6, Theorem 6.1]. Then, left continuous representatives can be exploited. Finally, the equality $S_T^{CL} u_o^\flat = u_T$ implies that $\overline{\mathrm{co}}\; u_T(\mathbb{R}) \subseteq \overline{\mathrm{co}}\; u_o^\flat(\mathbb{R})$. The other inclusion follows from Theorem 2.1 choosing $J:=\overline{\mathrm{co}}\; u_T(\mathbb{R})$. □

*Proof of Proposition 2.7.* The leftmost equivalence in (2.8) follows directly from (2.1).

Assume now that $U_o \in I_T^{HJ}(U_T;J)$. Then, to prove $(i)$, start from (2.2):

$$\begin{aligned}
&\forall\, x \in \mathbb{R} \quad\; U_T(x) = \inf\nolimits_{\xi\in\mathbb{R}} U_o(\xi) + T\, f^*\left(\tfrac{x-\xi}{T}\right)\\
\Longrightarrow\; &\forall\, x,\xi \in \mathbb{R}\; U_T(x) \le U_o(\xi) + T\, f^*\left(\tfrac{x-\xi}{T}\right)\\
\Longrightarrow\; &\forall\, x,\xi \in \mathbb{R}\;\; U_o(\xi) \ge U_T(x) - T\, f^*\left(\tfrac{x-\xi}{T}\right)\\
\Longrightarrow\; &\forall\, \xi \in \mathbb{R} \quad\; U_o(\xi) \ge \sup\nolimits_{x\in\mathbb{R}} U_T(x) - T\, f^*\left(\tfrac{x-\xi}{T}\right) = U_o^\flat(\xi)
\end{aligned}$$

by (2.4). Condition $(ii)$ follows from Proposition 2.6 and $(iii)$ is a consequence of (1.3).

Finally, assume that $(i)$, $(ii)$ and $(iii)$ hold. Define

$$\forall\, x \in \mathbb{R} \qquad \widehat{U}_T(x) := \inf_{\xi\in\mathbb{R}} U_o(\xi) + T\, f^*\left(\frac{x-\xi}{T}\right) \tag{4.3}$$

so that $\widehat{U}_T \ge U_T$ by $(i)$ and since $U_o^\flat \in I_T^{HJ}(U_T;J)$. On the other hand, for any $x \in \mathbb{R}$,

$$\begin{aligned}
U_T(x) &= U_o^\flat\left(\pi_{u_T}(x)\right) + T\, f^*\left(\frac{x-\pi_{u_T}(x)}{T}\right) && \text{[By Proposition 2.6]}\\
&= U_o\left(\pi_{u_T}(x)\right) + T\, f^*\left(\frac{x-\pi_{u_T}(x)}{T}\right) && \text{[By } (ii)\text{]}\\
&\ge \widehat{U}_T(x) && \text{[By (4.3)]}
\end{aligned}$$

completing the proof, thanks to (2.2), $(iii)$ and (1.3). □

*Proof of Proposition 2.8.* Note first that $U_o^\sharp$ is well defined. Indeed, fix any $\bar{x} \in \overline{\pi_{U_T'}(\mathbb{R})}$. Then, for all $U_o \in I_T^{HJ}(U_T;J)$, we have $U_o(\bar{x}) = U_o^\flat(\bar{x})$, by $(i)$ in Proposition 2.7. Then, calling $M:=\max\{|w| \colon w \in J\}$, for all $x \in \mathbb{R}$, we have that for all $U_o \in I_T^{HJ}(U_T;J)$, $|U_o(x)| \le \left|U_o^\flat(\bar{x})\right| + M|x-\bar{x}|$, showing that the sup in (2.9) is in $\mathbb{R}$.

We now prove that $U_o^\sharp$ satisfies $(iii)$. Indeed, fix $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Then, for any $U_o \in I_T^{HJ}(U_T;J)$

$$\min J \le \frac{U_o(x_2) - U_o(x_1)}{x_2 - x_1} \le \max J. \tag{4.4}$$

So that

$$\left.\begin{array}{l}
U_o(x_2) \le U_o(x_1) + (x_2-x_1)\max J\\
U_o(x_2) \le U_o^\sharp(x_1) + (x_2-x_1)\max J\\
U_o^\sharp(x_2) \le U_o^\sharp(x_1) + (x_2-x_1)\max J
\end{array}\;\right|\;
\begin{array}{l}
U_o(x_1) \le U_o(x_2) - (x_2-x_1)\min J\\
U_o(x_1) \le U_o^\sharp(x_2) - (x_2-x_1)\min J\\
U_o^\sharp(x_1) \le U_o^\sharp(x_2) - (x_2-x_1)\min J
\end{array}$$

proving $(iii)$.

To complete the proof, simply observe that $U_o^\sharp$ satisfies $(i)$ and $(ii)$ by construction. □

*Proof of Theorem 2.9.* To prove the equality in (2.10), note that the inclusion $\subseteq$ follows from (*i*)–(*iii*) in (2.8) and from the definition of $U_o^\sharp$ in (2.9). On the other hand, (*i*) and (*iii*) are obvious, while Proposition 2.8 ensures (*ii*) in (2.8), which then implies the other inclusion $\supseteq$. Convexity is now straightforward. Compactness in the said topology follows from (2.10) and Ascoli-Arzelà Theorem [18, § C.7], which can be applied thanks to the compactness of $J$.

The correspondences described in (2.1) now ensure the equality

$$I_T^{CL}(u_T;J) = \{u_o \in \mathbf{L}^\infty(\mathbb{R};J) \colon \exists\, U_o \in I_T^{HJ}(U_T;J) \text{ such that } u_o = U_o'\}$$

and therefore the convexity of $I_T^{CL}(u_T;J)$. To prove (2.11), use (2.10) and (2.7) and recall that we set $\breve{y} = \pi_{u_T}(\breve{x})$. Sequential compactness follows from (2.11) and from the boundedness of $I_T^{CL}(u_T;J)$ in the weak-$*$ topology, see [7, (ii) in § 4.3.C]. $\square$

### 4.2. Proofs related to Sect. 2.2

**Lemma 4.4.** *Let $f$ satisfy (f). Fix $u_o, v_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ and call $u$, respectively, $v$ the weak entropy solution to*

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_o(x), \end{cases} \qquad \textit{respectively} \qquad \begin{cases} \partial_t v + \partial_x f(v) = 0 \\ v(0,x) = v_o(x). \end{cases}$$

*Assume there exist $T > 0$ and $\bar{x} \in \mathbb{R}$ such that*

$$u(T, \bar{x}-) = v(T, \bar{x}-). \tag{4.5}$$

*Then, the map*

$$w(t,x) := \begin{cases} u(t,x) & x < \pi(t) \\ v(t,x) & x \geq \pi(t) \end{cases} \qquad \textit{where} \qquad \pi(t) := \bar{x} + (t-T)\, f'\left(u(T,\bar{x}-)\right)$$

*is a weak entropy solution to*

$$\begin{cases} \partial_t w + \partial_x f(w) = 0 \\ w(0,x) = w_o(x) \end{cases} \qquad \textit{where} \qquad w_o(x) := \begin{cases} u_o(x) & x < \pi(0) \\ v_o(x) & x \geq \pi(0). \end{cases}$$

*An entirely similar statement holds replacing the left trace with the right trace in (4.5) and in the definition of $\pi$.*

Remark that, since $T > 0$, the existence of the traces in (4.5) is ensured by the strict convexity of $f$ by [15, Theorem 11.2.2]. Moreover, by (4.5), the line $x = \pi(t)$ is a minimal backward characteristics [15, Theorem 11.1.3] common to both $u$ and $v$.

*Proof of Lemma 4.4.* Define

$$\begin{array}{lll} U_o(x) := \int_{\pi(0)}^x u_o(\xi)\,\mathrm{d}\xi\,; & U(t,x) := \int_{\pi(t)}^x u(t,\xi)\,\mathrm{d}\xi\,; & p_u(t,x) := x - t\, f'\left(u(t,x-)\right); \\ V_o(x) := \int_{\pi(0)}^x v_o(\xi)\,\mathrm{d}\xi\,; & V(t,x) := \int_{\pi(t)}^x v(t,\xi)\,\mathrm{d}\xi\,; & p_v(t,x) := x - t\, f'\left(v(t,x-)\right); \\ W_o(x) := \int_{\pi(0)}^x w_o(\xi)\,\mathrm{d}\xi; & W(t,x) := \int_{\pi(t)}^x w(t,\xi)\,\mathrm{d}\xi; & p_w(t,x) := x - t\, f'\left(w(t,x-)\right). \end{array}$$

By (4.5), $\pi$ is a genuine generalized characteristic of both $u$ and $v$, and since genuine backward characteristics do not cross [15, Corollary 11.1.2], we have

$$\forall\,(t,x) \in [0,T] \times \mathbb{R} \qquad \begin{array}{l} x < \pi(t)\ \ p_u(t,x) = p_w(t,x) \\ x \geq \pi(t)\ \ p_v(t,x) = p_w(t,x) \end{array}$$

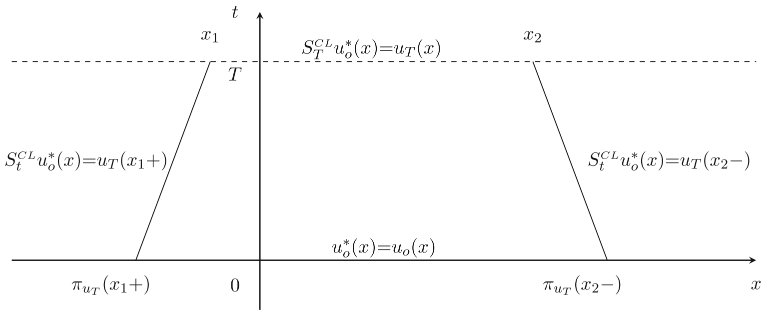


Fig. 1. Notations used in Lemma 4.5

so that, by [15, Theorem 11.4.3],

$$\begin{array}{l|l}
\text{if } x<\pi(t): & \text{if } x>\pi(t): \\
W(t,x) = U(t,x) & W(t,x) = V(t,x) \\
\quad = U_o\left(p_u(t,x)\right) + t\, f^*\left(\frac{x-p_u(t,x)}{t}\right) & \quad = V_o\left(p_v(t,x)\right) + t\, f^*\left(\frac{x-p_v(t,x)}{t}\right) \\
\quad = W_o\left(p_w(t,x)\right) + t\, f^*\left(\frac{x-p_w(t,x)}{t}\right); & \quad = W_o\left(p_w(t,x)\right) + t\, f^*\left(\frac{x-p_w(t,x)}{t}\right).
\end{array}$$

Hence, by [18, Theorem 4, Section 3, Chapter 3] and [15, Theorem 11.4.3], $W$ is the viscosity solution to

$$\begin{cases} \partial_t W + f(\partial_x W) = 0 \\ W(0,x) = W_o(x) \end{cases}$$

which yields the proof by the correspondence in [23, Theorem 1.1], see also (2.1). $\square$

**Lemma 4.5.** *Let $f$ satisfy (f). Let $J$ be a non trivial closed real interval and $T$ be positive. Fix $x_1, x_2 \in \mathbb{R}$ with $x_2 > x_1$ and $u_o, u_T \in \mathbf{L}^\infty(\mathbb{R}; J)$ such that*

$$u_T = S^{CL}_T u_o, \tag{4.6}$$

*Define, as shown in Fig. 1,*

$$u^*_o(x) := \begin{cases} u_o\left(\pi_{u_T}(x_1+)+\right) & x < \pi_{u_T}(x_1+) \\ u_o(x) & x \in [\pi_{u_T}(x_1+), \pi_{u_T}(x_2-)] \\ u_o\left(\pi_{u_T}(x_2-)-\right) & x > \pi_{u_T}(x_2-) \end{cases}$$

$$u^*_T(x) := \begin{cases} u_T(x_1+) & x < x_1 \\ u_T(x) & x \in [x_1, x_2] \\ u_T(x_2-) & x > x_2. \end{cases}$$

*Then,*

$$u^*_T = S^{CL}_T u^*_o.$$

*Proof of Lemma 4.5.* Clearly, both constant maps $(t,x) \mapsto u_T(t_1+)$ and $(t,x) \mapsto u_T(t_2-)$ are entropy solutions to $\partial_t u + \partial_x f(u) = 0$. Thus, the proof follows by an application of Lemma 4.4. $\square$

*Proof of Theorem 2.11.* Definition (2.12), implies the inclusion

$$\underbrace{I^{CL}_T(\widehat{u}_T; J)_{|[\pi_{\widehat{u}_T}(x_1+), \pi_{\widehat{u}_T}(x_2-)]}}_{\text{here } \mathcal{K}_T=[x_1,x_2]} \supseteq \underbrace{I^{CL}_T(u^*_T; J)_{|[\pi_{\widehat{u}_T}(x_1+), \pi_{\widehat{u}_T}(x_2-)]}}_{\text{here } \mathcal{K}_T=\mathbb{R}}.$$

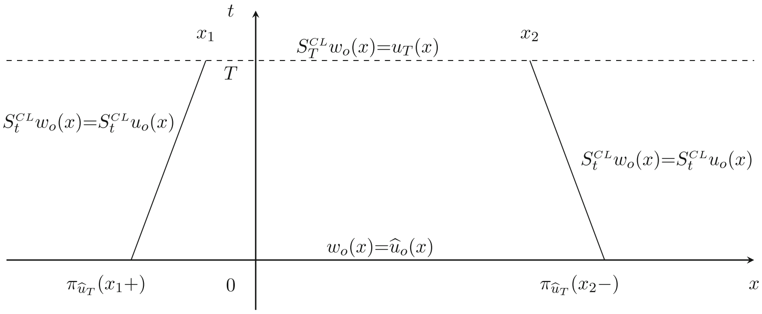


FIG. 2. Notation used in (4.7)

If the set in the left hand side above, namely $I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}$ is empty, then the proof trivially follows. Otherwise, if $I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]} \neq \emptyset$, choose any $\widetilde{u}_o \in I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}$. Then, with reference to (2.12), there exists a $u_o \in \mathbf{L}^\infty(\mathbb{R};J)$ such that $u_{o|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]} = \widetilde{u}_o$ and $S_T^{CL}u_{o|[x_1,x_2]} = \widehat{u}_T$. An application of Lemma 4.5 completes the proof. □

*Proof of Theorem 2.12.* Assume that $I_T^{CL}(u_T;J) \neq \emptyset$. Definition (2.12), ensures the inclusion

$$\underbrace{I_T^{CL}(u_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}}_{\text{here } K_T=\mathbb{R}} \subseteq \underbrace{I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}}_{\text{here } K_T=[x_1,x_2]}.$$

Let $u_o \in \mathbf{L}^\infty(\mathbb{R};J)$ be such that $S_T^{CL}u_o = u_T$. Since also $I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]} \neq \emptyset$, choose any $\widehat{u}_o \in I_T^{CL}(\widehat{u}_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}$. With reference to (2.12), there exists a $\overline{u}_o \in \mathbf{L}^\infty(\mathbb{R};J)$ such that $\overline{u}_{o|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]} = \widehat{u}_o$ and $S_T^{CL}\overline{u}_{o|[x_1,x_2]} = \widehat{u}_T$. Define, see Fig. 2,

$$w_o(x) := \begin{cases} u_o(x) & x < \pi_{\widehat{u}_T(x_1+)} \\ \widehat{u}_o(x) & x \in [\pi_{\widehat{u}_T}(x_1+), \pi_{\widehat{u}_T}(x_2-)] \\ u_o(x) & x > \pi_{\widehat{u}_T(x_2-)} \end{cases} \tag{4.7}$$

and apply twice Lemma 4.4 using the equalities

$$\begin{aligned} S_T^{CL}(u_o)(x_1+) &= \widehat{u}_T(x_1+) = S_T^{CL}(\overline{u}_o)(x_1+) \\ S_T^{CL}(u_o)(x_2-) &= \widehat{u}_T(x_2-) = S_T^{CL}(\overline{u}_o)(x_2-) \end{aligned}$$

in (4.5). One thus obtains $S_T^{CL}w_o = u_T$, showing that $\widehat{u}_o \in I_T^{CL}(u_T;J)_{|[\pi_{\widehat{u}_T}(x_1+),\pi_{\widehat{u}_T}(x_2-)]}$. □

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## Declarations

**Ethics and Consent to Participate** not applicable.



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