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General stability estimates in nonlocal traffic models for several populations

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Abstract. We prove global existence, uniqueness and $\mathbf{L}^1$ stability of solutions to general systems of nonlocal conservation laws modeling multiclass vehicular traffic. Each class follows its own speed law and has specific effects on the other classes' speeds. Moreover, general explicit dependencies of the speed laws on space and time are allowed. Solutions are proved to depend continuously—in suitable norms—on all terms appearing in the equations, as well as on the initial data. Numerical simulations show the relevance and the effects of the nonlocal terms.

Mathematics Subject Classification. 35L65, 76A30, 65M22.

Keywords. Nonlocal conservation laws, Multipopulation traffic modeling, Nonlocal traffic modeling.

1. Introduction

In this paper we analyze a nonlocal multiclass, or multipopulation, macroscopic traffic flow model. Here, $\rho_1, \ldots, \rho_n$ are the densities of n different populations (or classes) of drivers or vehicles, whose evolution is described by the following system of nonlinear conservation laws:

$$\left\{\begin{array}{ll} \partial_t \rho_i + \partial_x \left(\rho_i \, v_i(t,x,\eta_i * \rho)\right) = 0 & (t,x) \in [t_o, +\infty[\times \mathbb{R} \\ \rho_i(t_o,x) = \rho_i^o(x) & x \in \mathbb{R} \end{array}\right. \qquad i = 1, \ldots, n, \tag{1.1}$$

where $\eta_1, \ldots, \eta_n \colon \mathbb{R} \to \mathbb{R}^n$ are suitable weights, $v_1, \ldots, v_n \colon \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are the time and space dependent speed laws, while $\rho_1^o, \ldots, \rho_n^o$ are the initial densities, and $\rho = (\rho_1, \ldots, \rho_n)$. The present setting allows, in a time dependent framework, to account for space inhomogeneities as well as for the possible different natures of drivers/vehicles and of their interactions. Indeed, the equations in (1.1) are coupled through the nonlocal terms $\eta_i * \rho$ where each of the space convolutions

$$(\eta_i * \rho)_j\,(t,x) := \left(\eta_{ij} * \rho_j(t)\right)(x) = \int_{\mathbb{R}} \eta_{ij}(\xi)\rho_j(t,x-\xi)\,\mathrm{d}\xi \qquad i,j = 1,\ldots,n$$

describes how the i-th population interacts with the j-th one. Note that we consider the possibility of vehicles possessing a non-standard visual horizon, as is the case with autonomous vehicles, which typically have a comprehensive understanding of the traffic environment, both ahead and behind.

Within this general framework, defined only by the two rather simple assumptions **(v)** and (η), defined in Sect. 2, besides proving global in time existence and uniqueness of the solution to (1.1), we show the $\mathbf{L}^1$-Lipschitz continuous dependence of solutions on the initial data, their $\mathbf{L}^1$ local Lipschitz continuity in time and also provide detailed stability estimates on the dependence of solutions on v and on η; see also [15]. Moreover, *a priori* estimates on the total variation in space of the solution are obtained.

The current literature offers several related results, due to the interest in nonlocal traffic models. Well-posedness and stability of multiclass systems with a coupling in the nonlocal terms were considered in [21,25]. In particular, the latter work is set mainly in a Radon measure setting with stability estimates expressed by means of Wasserstein distance, thus requiring initial data to have the same total mass.

Motivated by the effects of moving obstacles on vehicular traffic, [6] considers a coupled ODE-PDE system. Well-posedness of weak solutions is shown for sufficiently small times and, under stronger conditions on the convolution kernels, long time existence is also proved.

The development of *ad hoc* numerical methods for equations of the type (1.1) was considered, for instance, in [4,12,16]. Recall that nonlocal systems fitting in the form (1.1) can be motivated also by entirely different physical settings: see for instance [34] devoted to the dynamics of bolts on a conveyor belt, [22] devoted to the flow of melted metal and [5,24] modeling supply chains. Crowd dynamics is a further widely considered application, see for instance [11,20]. A relaxation representation of nonlocal conservation laws was considered in [10].

Nonlocal conservation laws have been at the center of various papers dealing with the nonlocal to local limit, e.g. [4,17–19,27,29]. In particular, the negative result obtained in [19] is consistent with the stability estimate on the dependence of solution on η obtained hereafter, which requires a rather strong norm.

Particular features of the present analytical setting are: the full nonlinearity of (1.1), possible thanks to the generality of the functions v_i, the explicit dependence of the speed laws on the space variable x and on time t, the variety of the interactions between the different populations allowed by the, possibly different, n^2 functions η_{ij}.

Due to the nonlinearity of (1.1), a natural tool is the Banach Fixed Point Theorem, which we apply in $\mathbf{C}^0\left([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)\right)$ and relies on the representation of solutions to renewal equations by means of characteristics, often referred to also as Lagrangian solutions, as in [25]. With this tool, we prove existence, uniqueness and $\mathbf{L}^1$-Lipschitz continuous dependence of the solution on the initial datum. Careful **BV** bounds allow first to extend the solution globally in time. Then, we also get the local in time $\mathbf{L}^1$-Lipschitz continuous dependence of the solution on t, v and η.

We stress that the stability estimates proved below require that the total variation in space of the initial datum, and hence of the solution, be finite. In the case of the (local) Lipschitz continuity in time this is shown by the following elementary example:

$$\begin{cases} \partial_t \rho + \partial_x \rho = 0 \\ \rho(0,x) = \rho_o^m(x) \end{cases} \qquad \text{where} \qquad \rho_o^m(x) = \sum_{i=1}^{m} \chi_{\left[\frac{2i}{2m}, \frac{2i+1}{2m}\right]}(x)$$

which yields, for $\varepsilon \in \left]0, 1/(2m)\right[$,

$$\left\|\rho(t+\varepsilon) - \rho(t)\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} = \mathrm{TV}(\rho_o^m)\, \varepsilon.$$

In turn, the estimates on the total variation in space depend on the $\mathbf{W}^{\mathbf{2},\infty}$ norm of the convolution kernel. On the one hand, this motivates our assumptions being more restrictive than those in [25], where only continuity in time (not Lipschitz continuity) is proved. On the other hand, this is also consistent with the impossibility, in general, of letting η (formally) converge to a Dirac delta, which is equivalent to pass from a nonlocal to a local problem, see [17,19,29]. We refer for instance to [7] for $\mathbf{L}^p$ continuity estimates, with $p > 1$.

Various other papers use fixed point arguments together with implicit or explicit expressions for the solutions, see [24,25,28,29]. Alternatively, sequences of approximate solutions constructed by means of numerical algorithms can also be used to prove the existence of solutions to systems fitting into (1.1), see for instance [1,2,13,14].

Below, numerical integrations show specific features of the model (1.1), such as the effects of different horizons and overtakes. Moreover, we compare a solution to the nonlocal equation (1.1) with that of the classical Lighthill-Whitham [32] and Richards [33] model in presence of space inhomogeneities (bottleneck).

The paper is organized as follows. In Sect. 2 we present the main analytical results. Numerical integrations of (1.1) are shown in Sect. 3. All proofs are deferred to Sect. 4.

2. Analytical results

Below we denote by ∂_t, ∂_x and ∇_ρ the partial derivatives with respect to, respectively, t, x and $(\rho_1, \ldots, \rho_n)$. Moreover, if $v = v(x, \rho)$ with $x \in \mathbb{R}$ and $\rho \in \mathbb{R}^n$, then the notation $[\partial_x v \quad \nabla_\rho v]$ is the vector $(\partial_x v, \partial_{\rho_1} v, \ldots, \partial_{\rho_n} v)$. Fix $t_o \in \mathbb{R}_+ = [0, +\infty[$. We study (1.1) under the following assumptions:

(v) $v = (v_1, \ldots, v_n) \in \mathbf{C}^0([t_o, +\infty[; \mathcal{V}^n)$, where

$$\mathcal{V} := \Big\{ v \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \colon v(\cdot, 0) \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}), \qquad [\partial_x v \quad \nabla_\rho v] \in \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R} \times \mathbb{R}^n; \mathbb{R} \times \mathbb{R}^n) \Big\}.$$

(η) For $i = 1, \ldots, n$, $\eta_i \in \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)$.

Referring to $\mathcal{V}$ and $\mathcal{V}^n$, we use the norms

$$\begin{aligned} \|v_i\|_{\mathcal{V}} &:= \|v_i(\cdot,0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} + \|[\partial_x v_i \quad \nabla_\rho v_i]\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}\times\mathbb{R}^n)} \\ \|v\|_{\mathcal{V}^n} &:= \sqrt{\sum_{i=1}^n \|v_i\|_{\mathcal{V}}^2} \\ \|v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)} &:= \sup_{\tau\in[t_o,t]} \|v(\tau)\|_{\mathcal{V}^n}. \end{aligned}$$

Remark 2.1. As the proofs below show, the third derivative $\partial^3_{xxx} v$ is not even required to exist. Nevertheless, we state assumption **(v)** and define $\|\cdot\|_{\mathcal{V}}$ as above merely for simplicity.

As a first step, we formalize what we mean by solution to (1.1).

Definition 2.2. Fix a non empty real interval I with $\min I = t_o$. By *solution to* (1.1) *on the time interval* I we mean a map $\rho = (\rho_1, \ldots, \rho_n) \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R};\mathbb{R}^n))$ such that, for $i = 1, \ldots, n$, setting

$$w_i(t,x) := v_i\left(t, x, \left(\eta_{i1} * \rho_1(t)\right)(x), \left(\eta_{i2} * \rho_2(t)\right)(x), \ldots, \left(\eta_{in} * \rho_n(t)\right)(x)\right), \tag{2.1}$$

ρ_i is a solution to

$$\begin{cases} \partial_t \rho_i + \partial_x\left(\rho_i\, w_i(t,x)\right) = 0 & (t,x) \in I \times \mathbb{R} \\ \rho_i(t_o,x) = (\rho_o)_i(x) & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Above, by *solution to* (2.2) we mean a distributional solution [8, Definition 4.2], which is also a weak entropy solution in the sense of [30, Definition 1], see [23, Lemma 5], [25, Theorem 2.7] or [26, Corollary II.1].

We are now ready to state the main analytical result. To this aim, we denote by $C(\cdots)$ a locally bounded quantity dependent on its arguments, whose actual value is specified in the proofs.

Theorem 2.3. *Let **(v)** and (η) hold. Then, problem* (1.1) *generates a unique map*

$$\mathcal{P}\colon [t_o, +\infty[\, \times \mathbb{R}_+ \times (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n) \to (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$$

with the following properties:

($\mathcal{P}$1) *$\mathcal{P}$ is a process: for all $\bar t \geq t_o$ and $t', t'' \in \mathbb{R}_+$, $\mathcal{P}_{\bar t,0}$ is the identity and $\mathcal{P}_{\bar t+t',t''} \circ \mathcal{P}_{\bar t,t'} = \mathcal{P}_{\bar t,t'+t''}$.*

($\mathcal{P}$2) *For any $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$ and any $T > t_o$, the orbit $t \mapsto \mathcal{P}_{t_o,t}\rho_o$ is the unique global solution to* (1.1) *on $[t_o, T]$ in the sense of Definition* 2.2.

($\mathcal{P}$3) *$\mathcal{P}$ is locally Lipschitz continuous in the initial datum: for all $t \geq t_o$ and $\rho_o, \hat\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$*

$$\begin{aligned} &\|\mathcal{P}_{t_o,t}\rho_o - \mathcal{P}_{t_o,t}\hat\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\ &\quad\leq C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})}, \|v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \|\hat\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}, \mathrm{TV}(\hat\rho_o), t - t_o\right) \\ &\qquad\times \|\rho_o - \hat\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}. \end{aligned}$$

($\mathcal{P}$4) *$\mathcal{P}$ is locally Lipschitz continuous in t: for all $T > t_o$, $t', t'' \in [t_o, T]$ and $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^n)$*

$$\begin{aligned}&\|\mathcal{P}_{t_o,t'}\rho_o - \mathcal{P}_{t_o,t''}\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\ &\quad\le C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}, \|v\|_{\mathbf{C}^0([t_o,T];\mathcal{V}^n)}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \mathrm{TV}(\rho_o), T-t_o\right)\left|t'-t''\right|.\end{aligned}$$

($\mathcal{P}$5) *Let $v, \hat v$ satisfy **(v)** and call $\mathcal{P}, \hat{\mathcal{P}}$ the corresponding processes. Then, for all $t \ge t_o$ and $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^n)$*

$$\begin{aligned}&\left\|\mathcal{P}_{t_o,t}\rho_o - \hat{\mathcal{P}}_{t_o,t}\rho_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\ &\quad\le C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}, \|v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \mathrm{TV}(\rho_o), t-t_o\right)\\ &\qquad\times\|v-\hat v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)}\,(t-t_o).\end{aligned}$$

($\mathcal{P}$6) *Let $\eta, \hat\eta$ satisfy (η) and call $\mathcal{P}, \hat{\mathcal{P}}$ the corresponding processes. Then, for all $t \ge t_o$ and $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^n)$*

$$\begin{aligned}&\left\|\mathcal{P}_{t_o,t}\rho_o - \hat{\mathcal{P}}_{t_o,t}\rho_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\ &\quad\le C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}, \|\hat\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}, \|v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \mathrm{TV}(\rho_o), t-t_o\right)\\ &\qquad\times\|\eta-\hat\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}\,(t-t_o).\end{aligned}$$

($\mathcal{P}$7) *For all $t \ge t_o$ and $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^n)$*

$$\begin{aligned}\mathrm{TV}\left(\mathcal{P}_{t_o,t}\rho_o\right) \le{}& \left(\mathrm{TV}(\rho_o) + C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}, \|v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\right)(t-t_o)\right)\\ &\times\exp\left(C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})}, \|v\|_{\mathbf{C}^0([t_o,t];\mathcal{V}^n)}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\right)(t-t_o)\right).\end{aligned}$$

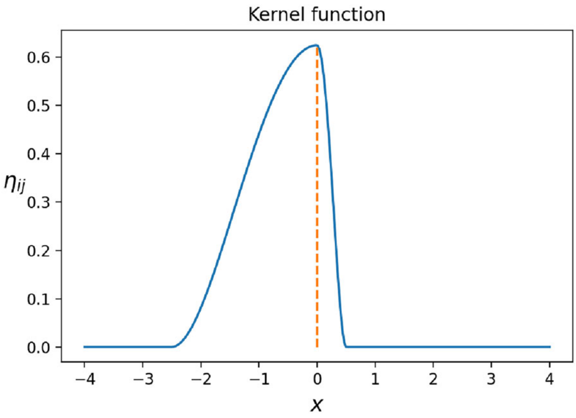
($\mathcal{P}$8) *For every $i = 1, \ldots, n$, if $(\rho_o)_i \ge 0$, then for all $t \ge t_o$, $(\mathcal{P}_{t,t_o}\rho_o)_i \ge 0$.*

The proof is deferred to Sect. 4. Since $\rho \equiv 0$ is a solution, the bounds **($\mathcal{P}$3)** and **($\mathcal{P}$7)** ensure also an $\mathbf{L}^\infty$ bound on ρ.

3. Numerical integrations

In the numerical integrations below, motivated by vehicular traffic, we choose non negative values of each component of the initial density ρ_o. Hence, thanks to **($\mathcal{P}$8)**, it is sufficient that we define for all $i = 1, \ldots, n$ the speed laws v_i only on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n_+$. Then, one can extend them by regularity to all $\mathbb{R} \times \mathbb{R}^n$ in order to formally recover the hypothesis **(v)**.

The kernel functions η_{ij} describe how far the i-th population *"sees"* the j-th population. For simplicity, we standardize their choices as follows:



$$\eta_{ij}(x) := \begin{cases} 0 & x < -f_{ij} \\ A_{ij}\,\left(1-\left(\frac{x}{f_{ij}}\right)^2\right)^2 & x \in [-f_{ij}, 0] \\ A_{ij}\,\left(1-\left(\frac{x}{b_{ij}}\right)^2\right)^2 & x \in [0, b_{ij}] \\ 0 & x > b_{ij} \end{cases} \tag{3.1}$$

where the normalization constants A_{ij} are chosen so that $\int_{\mathbb{R}} \eta_{ij}(x)\mathrm{d}x = 1$. The forward, respectively backward, horizons are f_{ij}, respectively b_{ij}, and are typically chosen so that $f_{ij} \gg b_{ij}$, accounting for the far greater relevance of the forward vision.

Note however that whenever a population of autonomous vehicles (AVs) is present, it is reasonable to assume that they have information about the whole traffic situation, also out of the standard visual horizon, both in front and behind them. This is consistent with our allowing the kernel function η_{ij} to have, possibly, unbounded support. At the same time, the backward horizon for a standard vehicle may well be of the order of a vehicle length, coherently with the little relevance of the behind traffic.

Throughout, we use the Lax-Friedrichs method [31, § 4.6] adapted to deal with nonlocal fluxes. Indeed, fix a spatial mesh size $\Delta x > 0$, a final time $T > 0$, and construct the grid points $x_k := k\,\Delta x$, for $k \in \mathbb{Z}$, and, for $m = 1, \ldots, N_T$, the times $t_m := t_{m-1} + \Delta t_m$, where $t_0 := 0$ and $\Delta t_m > 0$ to be specified below and such that $T = \sum_{m=0}^{N_T} \Delta t_m$. Call, for all $k \in \mathbb{Z}$ and $m = 0, \ldots, N_T - 1$, the cell $C_k := \left[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}\right]$, where $x_{k+\frac{1}{2}} := (x_{k+1} + x_k)/2$, and the time interval $C^m := [t_m, t_{m+1}[$. Then, for all $i = 1, \ldots, n$, the solution ρ_i to (1.1) is approximated by the piecewise constant function $\sum_{k,m} (\rho_i)_k^m\, \chi_{C^m}(t)\, \chi_{C_k}(x)$ where, for $m = 0, \ldots, N_T - 1$ and $k \in \mathbb{Z}$, $(\rho_i)_k^m$ is given by the Lax-Friedrichs scheme

$$\begin{cases} (\rho_i)_k^{m+1} = \frac{1}{2}\left((\rho_i)_{k-1}^m + (\rho_i)_{k+1}^m\right) - \frac{\Delta t_m}{2\Delta x}\left[F\left((\rho_i)_{k+1}^m\right) - F\left((\rho_i)_{k-1}^m\right)\right] \\ (\rho_i)_k^0 = (\rho_o)_i(x_k), \end{cases}$$

with the numerical fluxes $F((\rho_i)_k^m) := (\rho_i)_k^m (v_i)_k^m$. Here, the values $(v_i)_k^m$ are defined by

$$(v_i)_k^m := v_i \left(t_m, x_k, \Delta x \sum_{p\in\mathbb{Z}} \eta_{i1}(x_p)(\rho_1)_{k-p}^m, \ldots, \Delta x \sum_{p\in\mathbb{Z}} \eta_{in}(x_p)(\rho_n)_{k-p}^m \right),$$

which are an approximation of $v_i\left(t_m, x_k, \eta_{i1} * \rho_1(t_m, x_k), \ldots, \eta_{in} * \rho_n(t_m, x_k)\right)$. For each $m = 0, \ldots, N_T - 1$, Δt_m is chosen to satisfy the CFL condition [31, § 4.4]

$$\Delta t_m \leq \frac{\Delta x}{\max\limits_{\substack{i=1,\ldots,n,\\ k\in\mathbb{Z}}} \{(v_i)_k^m\}}. \tag{3.2}$$

Finally, at the numerical level, we employed free flow boundary conditions.

3.1. The role of the horizon

Here we consider $n = 2$ populations of drivers whose unique difference is their visual horizon. Such distinction is carried by the kernel functions η_{ij} in (3.1). More precisely, for $i, j = 1, 2$, fix $b_{ij} = 0.01$, $f_{1j} = 1.5$ and $f_{2j} = 0.3$ so that, forward, the first population is able to see farther than the second one.

We choose the following speed law, satisfying **(v)**:

$$v_i(t,x,q) := \begin{cases} 1 & q < 0 \\ (1-q)^3 & q \in [0,1] \\ 0 & q > 1 \end{cases} \qquad q = \eta_{i1} * \rho_1 + \eta_{i2} * \rho_2, \qquad i = 1, 2. \tag{3.3}$$

System (1.1) is equipped with the initial datum

$$\rho_1^o(x) = \rho_2^o(x) = 0.5\, \chi_{[0,2]}(x),$$

as in Fig. 1, top left. The resulting evolution, approximated by a numerical integration on a mesh of 10000 points on the numerical domain [0, 10], shows the relevance of the forward horizon, see Fig. 1. For the first population, the free space on the road stretch ahead has more relevance in the choice of the speed. As a consequence, the front vehicles in the first population are faster. Moreover, again due to the length of the forward horizon, the front vehicles almost share the same speed, as shown in the graphs at times $t = 0.9$, 3.3, 6.4, where the density of the first population displays a rather steep right front. On the contrary, the slope in the right part of the graph of ρ_2 is significantly lower.

3.2. Different maximal speeds

On the numerical domain [0, 100] we now consider $n = 3$ populations differing only in their maximal speed, i.e., we set for $i = 1, 2, 3$

$$v_i(t,x,q) := \begin{cases} V_i & q < 0 \\ V_i(1-q)^3 & q \in [0,1] \\ 0 & q > 1 \end{cases} \qquad q = \eta_{i1} * \rho_1 + \eta_{i2} * \rho_2 + \eta_{i3} * \rho_3, \qquad \begin{aligned} V_1 &= 1.5 \\ V_2 &= 0.9 \\ V_3 &= 0.5. \end{aligned} \tag{3.4}$$

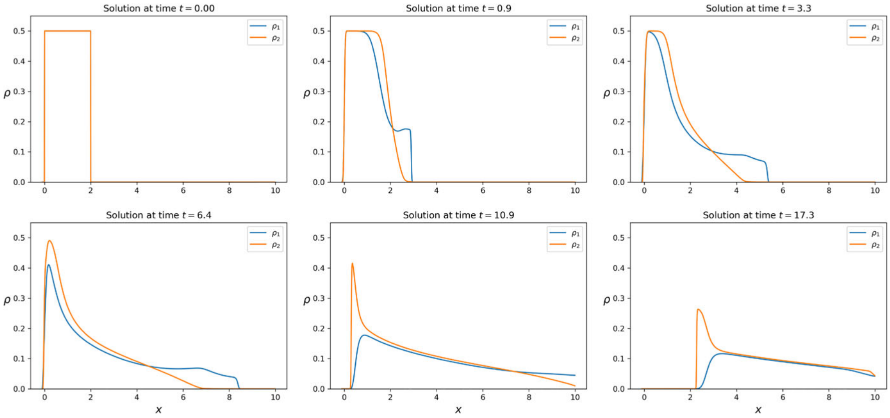


Figure 1. Solutions to (1.1) for $n = 2$ with the parameters as in Sect. 3.1. The two populations have the same speed law, but the forward horizon of the first population is 5 times that of the second one. As a result, the first population moves faster and its front vehicles share almost the same speed. On the contrary, the rightmost part of the graph of ρ_2 is less steep

We choose the initial datum

$$(\rho_o)_1(x) = 0.3\,\chi_{[1,5]}(x), \quad (\rho_o)_2(x) = 0.3\,\chi_{[8,12]}(x), \quad (\rho_o)_3(x) = 0.3\,\chi_{[15,19]}(x) \tag{3.5}$$

and kernel functions (3.1) with forward horizon $f_{ij} = 1.0$ and backward horizon $b_{ij} = 0.01$ for $i, j = 1, 2, 3$.

Figure 2 shows the result of the numerical integration of (1.1)–(3.4)–(3.5) obtained with a uniform mesh of 10000 points.

At the initial time, the slowest population is ahead and the fastest is behind. Model (1.1) allows overtakes and, in fact, at the final time the populations are ordered according to their maximal speeds. It is realistic that during overtakes queues form, see in particular the graphs at times $t = 7.0$, 28.7, 80.9 in Fig. 2. Note that the graph at time $t = 28.7$ clearly shows that the density $\rho_1(t)$ exceeds the maximum of the initial datum $(\rho_o)_1$, coherently with the lack of validity of the maximum principle in the present setting.

3.3. Space inhomogeneity and forward horizon

The generality of system (1.1) allows to evaluate the combined effects of space inhomogeneities and nonlocal terms. The numerical domain [0, 20] is a road stretch with a bottleneck in [5, 10] where the maximal speed smoothly decreases up to 50%. We show below that vehicles with a larger forward horizon better cope with the slowdown and pass through it faster than vehicles with a lower one. More precisely, we confront a solution to (1.1) with that to the standard Lighthill-Whitham [32] and Richards [33] model.

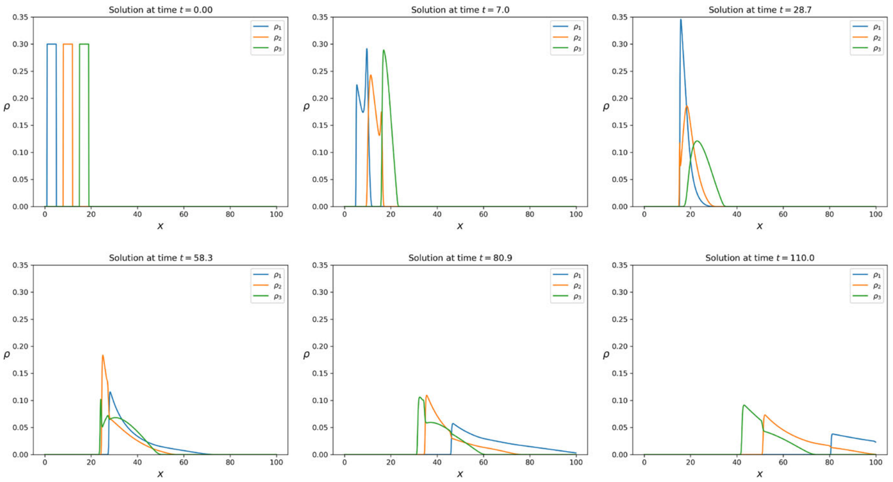


Figure 2. Solutions to (1.1)–(3.4)–(3.5) with $n = 3$ populations on the domain $[0, 100]$ with the parameters given in Sect. 3.2. Note that faster populations overtake slower populations and towards the end of the integration the populations appear ordered according to their maximal speed. Moreover, the graph at $t = 28.7$ shows that a density may reach values higher than those attained at the initial time

First, consider (1.1) with $n = 1$, η as in (3.1) with $f_{11} = 1.0$, $b_{11} = 0.01$ and

$$v(t, x, q) := \begin{cases} V(x) & q < 0 \\ V(x)\,(1-q)^3 & q \in [0,1] \\ 0 & q > 1 \end{cases} \qquad q = \eta * \rho, \tag{3.6}$$

$$V(x) = \begin{cases} 1 - \frac{32}{5^6}\,(x-5)^3\,(10-x)^3 & \text{if} \quad x \in [5,10] \\ 1 & \text{otherwise,} \end{cases} \tag{3.7}$$

$$\rho_o(x) = 0.8\,\chi_{[1,\,3]}(x) \tag{3.8}$$

so that Theorem 2.3 applies. We compare the resulting solution ρ with that, say r, to the LWR model

$$\begin{cases} \partial_t r + \partial_x \left(r\, v(t,x,r)\right) = 0 \\ r(0,x) = \rho_o(x) \end{cases} \tag{3.9}$$

where v is as in (3.6)–(3.7) and ρ_o is as in (3.8). Both solutions are computed on a mesh of 10000 points.

Figure 3 displays the solutions ρ and r to the two problems. It stems out that the vehicles with a positive forward horizon slow down well before the bottleneck, cause less congestion and pass through the bottleneck in less time than the others. Indeed, at time $t = 37.5$ the solution to (1.1) is supported on

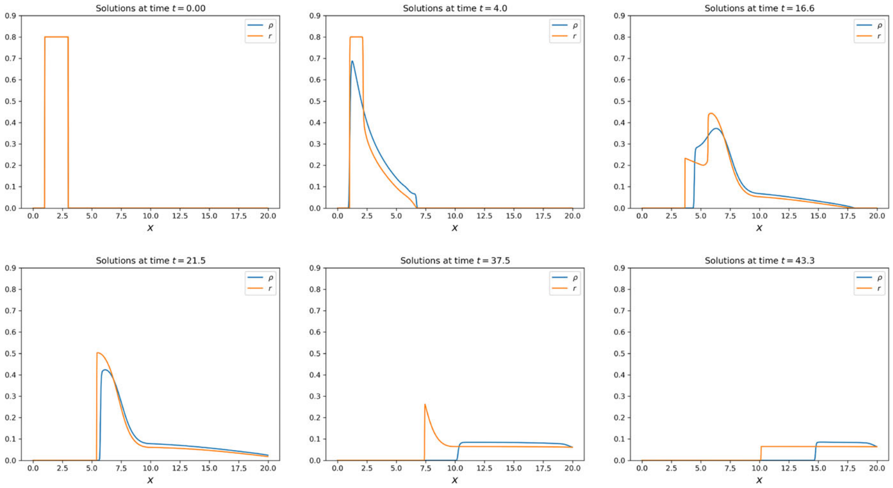


Figure 3. Solution ρ to the nonlocal equation (1.1)–(3.6)–(3.7)–(3.8) compared to r, solving the LWR model (3.9), with the same initial datum (3.8) and speed law (3.6)–(3.7). The effect of the bottleneck in $[5, 10]$ is clearly seen since time $t = 4.0$. Note that already at time $t = 37.5$ the solution to the nonlocal problem is supported on the right of the bottleneck, while the same happens only after $t = 43.3$ for the solution to the LWR model

the right of the bottleneck, while only after $t = 43.3$ the solution to (3.9) has overcome the slowdown.

4. Analytical proofs

Without loss of generality, we assume that $t_o = 0$. Throughout, we use the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$. As usual, we equip the spaces $\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$, $\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}^n)$, and $\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)$ with the norms

$$\begin{aligned}
\|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} &:= \int_{\mathbb{R}} \|\rho(x)\| \mathrm{d}x & \|\rho\|_{\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}^n)} &:= \|\rho\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} + \left\|\frac{\mathrm{d}\rho}{\mathrm{d}x}\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} \\
\|\rho\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} &:= \operatorname{ess\,sup}_{x\in\mathbb{R}} \|\rho(x)\| & \|\rho\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)} &:= \|\rho\|_{\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}^n)} + \left\|\frac{\mathrm{d}^2\rho}{\mathrm{d}x^2}\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}.
\end{aligned}$$

By $\mathrm{TV}(u)$ we denote the total variation of u on its domain, while $\mathrm{TV}(u, I)$ is the total variation of u on the real interval I, see [3, Chapter 3].

The following basic property of **BV** functions is a slight extension of [8, Lemma 2.3].

Lemma 4.1. *For any $u \in \mathbf{BV}(\mathbb{R};\mathbb{R})$ and $h \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$,*

$$\int_{\mathbb{R}} |u\left(x + h(x)\right) - u(x)| dx \leq \left(\max\left\{\operatorname{ess\,sup}_{\mathbb{R}} h, 0\right\} - \min\left\{\operatorname{ess\,inf}_{\mathbb{R}} h, 0\right\}\right) \mathrm{TV}(u).$$

Proof. Let $U(x) := \mathrm{TV}\,(u,]-\infty, x[)$. Then, by [8, Lemma 2.3]

$$\begin{aligned}
&\int_{\mathbb{R}} |u\,(x + h(x)) - u(x)| \mathrm{d}x \\
&\quad\le \int_{\mathbb{R}} \left(U\,(x + \max\{\text{ess sup}_{\mathbb{R}}\, h, 0\}) - U\,(x + \min\{\text{ess inf}_{\mathbb{R}}\, h, 0\})\right) \mathrm{d}x \\
&\quad= \int_{\mathbb{R}} \left(U\,(y + (\max\{\text{ess sup}_{\mathbb{R}}\, h, 0\} - \min\{\text{ess inf}_{\mathbb{R}}\, h, 0\})) - U(y)\right) \mathrm{d}y \\
&\quad\le (\max\{\text{ess sup}_{\mathbb{R}}\, h, 0\} - \min\{\text{ess inf}_{\mathbb{R}}\, h, 0\})\, \mathrm{TV}(U) \\
&\quad= (\max\{\text{ess sup}_{\mathbb{R}}\, h, 0\} - \min\{\text{ess inf}_{\mathbb{R}}\, h, 0\})\, \mathrm{TV}(u),
\end{aligned}$$

completing the proof. □

We now recall a basic stability result on ODE flows.

Proposition 4.2. *Fix $T > 0$ and $w, z \in \mathbf{C}^0\left([0,T]; \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})\right)$. For $t, t_o \in [0,T]$ and $x_o \in \mathbb{R}$, let $t \mapsto X(t; t_o, x_o)$ and $t \mapsto Y(t; t_o, x_o)$ be the solutions to*

$$\begin{cases} \dot{x} = w(t,x) \\ x(t_o) = x_o \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{y} = z(t,y) \\ y(t_o) = x_o. \end{cases} \tag{4.1}$$

Then for all $t \in [0,T]$,

$$\begin{aligned}
|X(t;t_o,x_o) - Y(t;t_o,x_o)| \le{}& \|w - z\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\, |t - t_o| \\
&\times \exp\left(\|\partial_x w\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\, |t - t_o|\right).
\end{aligned}$$

The proof is a standard consequence of Gronwall Lemma and basic ODE theory, see [9, Chapter 3].

Lemma 4.3. *Let (η) hold and $v \in \mathcal{V}^n$. Call*

$$\tilde{X}_M := \left\{\rho \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n) \colon \|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le M\right\}. \tag{4.2}$$

Then, the map

$$\begin{array}{rccl} \tilde{\Pi}_{v,\eta} & : \tilde{X}_M & \to & \mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n) \\ & \rho & \mapsto & v\,(\cdot, (\eta * \rho)(\cdot)), \end{array} \tag{4.3}$$

where we denote

$$v\,(x, (\eta * \rho)(x)) = \left(v_i\,(x, (\eta_{i1} * \rho_1)(x), (\eta_{i2} * \rho_2)(x), \ldots, (\eta_{in} * \rho_n)(x))\right)_{i=1,\ldots,n}, \tag{4.4}$$

is well defined and Lipschitz continuous with

$$\mathbf{Lip}(\tilde{\Pi}_{v,\eta}) := \left(7 + (6 + M)M\|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^{n\times n})}\right) \|v\|_{\mathcal{V}^n}\, \|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^{n\times n})}. \tag{4.5}$$

Moreover, $\tilde{\Pi}_{v,\eta}$ is uniformly bounded by the quantity

$$\begin{aligned}
Q_v := \big(&3 + M\|\eta\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^{n\times n})} + 3M\|\partial_x \eta\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^{n\times n})} \\
&+ M^2\|\partial_x \eta\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^{n\times n})} + M\left\|\partial^2_{xx}\eta\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^{n\times n})}\big)\|v\|_{\mathcal{V}^n}.
\end{aligned} \tag{4.6}$$

Proof. To prove that $\tilde{\Pi}_{v,\eta}$ is well defined, let $\rho \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$, $i = 1, \ldots, n$, and compute:

$$\begin{aligned}
&\left\|(\tilde{\Pi}_{v,\eta}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|v_i\left(\cdot, \eta_i * \rho(\cdot)\right) - v_i(\cdot,0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} + \|v_i(\cdot,0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\ \|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\ \|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} + \|v_i(\cdot,0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \qquad (4.7)\\
&\quad\le \left(1 + M\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\right)\|v_i\|_{\mathcal{V}};\\
&\left\|\partial_x(\tilde{\Pi}_{v,\eta}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|\partial_x v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})} + \|\nabla_\rho v_i\left(\cdot, \eta_i * \rho(\cdot)\right)\ (\partial_x\eta_i * \rho)(\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|\partial_x v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})} + \|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\ \|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\ \|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \qquad (4.8)\\
&\quad\le \left(1 + M\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\right)\|v_i\|_{\mathcal{V}};\\
&\left\|\partial^2_{xx}(\tilde{\Pi}_{v,\eta}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \left\|\partial^2_{xx} v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})} + 2\|\partial_x\nabla_\rho v_i\left(\cdot, \eta_i * \rho(\cdot)\right)\ (\partial_x\eta_i * \rho)(\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\left\|D^2_{\rho\rho} v_i\left(\cdot, \eta_i * \rho(\cdot)\right)\ (\partial_x\eta_i * \rho(\cdot))^2 + \nabla_\rho v_i\left(\cdot, \eta_i * \rho(\cdot)\right)\ (\partial^2_{xx}\eta_i * \rho)(\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \left\|\partial^2_{xx} v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})} + 2\|\partial_x\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\ \|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+\left\|D^2_{\rho\rho} v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\ \|\partial_x\eta_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\ \|\rho\|^2_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+\|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\ \left\|\partial^2_{xx}\eta_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\ \|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\quad\le \left(1 + 2M\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} + M^2\|\partial_x\eta_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} + M\left\|\partial^2_{xx}\eta_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\right)\|v_i\|_{\mathcal{V}};
\end{aligned}\tag{4.9}$$

hence, $\tilde{\Pi}_{v,\eta}$ is well defined.

To prove Lipschitz continuity, let $r, \rho \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$, $i = 1, \ldots, n$ and evaluate

$$\begin{aligned}
&\left\|(\tilde{\Pi}_{v,\eta}r)_i - (\tilde{\Pi}_{v,\eta}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\ \|\eta_i * (r-\rho)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\quad\le \|v_i\|_{\mathcal{V}}\ \|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\ \|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)};\\
&\left\|\partial_x(\tilde{\Pi}_{v,\eta}r)_i - \partial_x(\tilde{\Pi}_{v,\eta}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|\partial_x v_i\left(\cdot, (\eta_i * r)(\cdot)\right) - \partial_x v_i\left(\cdot, (\eta_i * \rho)(\cdot)\right)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\|\nabla_\rho v_i\left(\cdot, (\eta_i * r)(\cdot)\right)\ (\partial_x\eta_i * r)(\cdot) - \nabla_\rho v_i\left(\cdot, (\eta_i * \rho)(\cdot)\right)\ (\partial_x\eta_i * \rho)(\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\le \|\partial_x\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+\|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+M\left\|D^2_{\rho\rho} v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\le \left(2 + M\|\eta_i\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^n)}\right)\|v_i\|_{\mathcal{V}}\|\eta_i\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^n)}\|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)};\\
&\quad\left\|\partial^2_{xx}\left(\tilde{\Pi}_{v,\eta}r\right)_i - \partial^2_{xx}\left(\tilde{\Pi}_{v,\eta}\rho\right)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}
\end{aligned}$$

$$\begin{aligned}
&\leq \left\|\partial^2_{xx} v_i(\cdot,(\eta_i * r)(\cdot)) - \partial^2_{xx} v_i(\cdot,(\eta_i*\rho)(\cdot))\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad +2\|\partial_x \nabla_\rho v_i(\cdot,(\eta_i*r)(\cdot))\left((\partial_x\eta_i*(r-\rho)(\cdot)\right)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad +2\|(\partial_x\nabla_\rho v_i(\cdot,(\eta_i*r)(\cdot)) - \partial_x\nabla_\rho v_i(\cdot,(\eta_i*\rho)(\cdot)))\,(\partial_x\eta_i*\rho)(\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad +\left\|D^2_{\rho\rho}v_i(\cdot,(\eta_i*r)(\cdot))\left((\partial_x\eta_i*r(\cdot))^2-(\partial_x\eta_i*\rho(\cdot))^2\right)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad +\left\|\left(D^2_{\rho\rho}v_i(\cdot,(\eta_i*r)(\cdot)) - D^2_{\rho\rho}v_i(\cdot,(\eta_i*\rho)(\cdot))\right)(\partial_x\eta_i*\rho(\cdot))^2\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad +\left\|\nabla_\rho v_i(\cdot,(\eta_i*r(\cdot)))\left(\partial^2_{xx}\eta_i*(r-\rho)(\cdot)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad +\left\|(\nabla_\rho v_i(\cdot,(\eta_i*r(\cdot))) - \nabla_\rho v_i(\cdot,(\eta_i*\rho(\cdot))))\,(\partial^2_{xx}\eta_i*\rho(\cdot))\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\leq \Big(\left\|\partial^2_{xx}\nabla_\rho v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\quad +2\|\partial_x\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\quad +2\left\|\partial_x D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\quad +\left\|D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\partial_x\eta_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}2M\\
&\quad +\left\|D^3_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\partial_x\eta_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M^2\\
&\quad +\|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\left\|\partial^2_{xx}\eta_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\quad +\left\|D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\left\|\partial^2_{xx}\eta_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\Big)\\
&\quad \times\|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\leq \left(4+5M\|\eta_i\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}+M^2\|\eta_i\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}\right)\\
&\qquad \times\|v_i\|_{\mathcal{V}}\|\eta_i\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}\|r-\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}.
\end{aligned}$$

Thus, $\tilde{\Pi}_{v,\eta}$ is Lipschitz continuous with a Lipschitz constant depending on v, M and η.

The uniform bound of $\tilde{\Pi}_{v,\eta}$ directly follows from (4.8), (4.9) and (4.9). □

We now investigate the dependence of the map $\tilde{\Pi}_{v,\eta}$ on v and η.

Proposition 4.4. *Fix $\rho \in \tilde{X}_M$ as defined in (4.2). With the notation (4.3)–(4.4), define*

$$\begin{array}{lcl} \mathcal{V}^n \times \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n}) & \to & \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)\\ w \ \ , \ \ \xi & \mapsto & \tilde{\Pi}_{w,\xi}\rho. \end{array} \tag{4.10}$$

Then,

(1) *For all $\xi \in \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)$, the map $v \mapsto \tilde{\Pi}_{v,\xi}\rho$ is Lipschitz continuous: $\forall v, w \in \mathcal{V}^n$*

$$\left\|\tilde{\Pi}_{v,\xi}\rho - \tilde{\Pi}_{w,\xi}\rho\right\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)} \leq \left(3+5M\|\xi\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}+M^2\|\xi\|^2_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}\right)\|v-w\|_{\mathcal{V}^n}.$$

(2) *For all $v \in \mathcal{V}^n$, the map $\xi \mapsto \tilde{\Pi}_{v,\xi}\rho$ is locally Lipschitz continuous: $\forall \xi, \eta \in \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)$*

$$\left\|\tilde{\Pi}_{v,\xi}\rho - \tilde{\Pi}_{v,\eta}\rho\right\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}$$

$$
\begin{aligned}
&\leq \Big(7 + 4M\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^n)} + 3M\|\xi\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^n)} + (M+M^2)\|\xi\|^2_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^n)}\Big)\\
&\quad \times M\|v\|_{\mathcal{V}^n}\|\xi-\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^n)}.
\end{aligned}
$$

Proof. Fix $\xi \in \mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})$, $\rho \in \tilde{X}_M$ and $v, w \in \mathcal{V}$. For $i = 1, \ldots, n$, one obtains

$$
\begin{aligned}
&\Big\|(\tilde{\Pi}_{v,\xi}\rho)_i - (\tilde{\Pi}_{w,\xi}\rho)_i\Big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad \leq \|v_i(\cdot,0) - w_i(\cdot,0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} + \|\nabla_\rho v_i - \nabla_\rho w_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\\
&\quad \leq \Big(1 + M\,\|\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\Big)\,\|v_i - w_i\|_{\mathcal{V}}\,.
\end{aligned}
$$

Moreover, for $i = 1, \ldots, n$,

$$
\begin{aligned}
&\Big\|\partial_x(\tilde{\Pi}_{v,\xi}\rho)_i - \partial_x(\tilde{\Pi}_{w,\xi}\rho)_i\Big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad \leq \|\partial_x v_i\left(\cdot,\xi_i*\rho(\cdot)\right) - \partial_x w_i\left(\cdot,\xi_i*\rho(\cdot)\right)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad + \|\left(\nabla_\rho v_i\left(\cdot,\xi_i*\rho(\cdot)\right) - \nabla_\rho w_i\left(\cdot,\xi_i*\rho(\cdot)\right)\right)\left(\partial_x\xi_i*\rho(\cdot)\right)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad \leq \|\partial_x v_i - \partial_x w_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})} + \|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\nabla_\rho v_i\|\\
&\qquad - \nabla_\rho w_{i\,\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)}\\
&\quad \leq \Big(1 + M\,\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\Big)\,\|v_i - w_i\|_{\mathcal{V}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|\partial^2_{xx}(\tilde{\Pi}_{v,\xi}\rho)_i - \partial^2_{xx}(\tilde{\Pi}_{w,\xi}\rho)_i\Big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad = \big\|\partial^2_{xx}v_i(\cdot,(\xi_i*\rho)(\cdot)) - \partial^2_{xx}w_i(\cdot,(\xi_i*\rho)(\cdot))\big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad + \|(\partial_x\nabla_\rho v_i(\cdot,(\xi_i*\rho)(\cdot)) - \partial_x\nabla_\rho w_i(\cdot,(\xi_i*\rho)(\cdot)))\,2(\partial_x\xi_i*\rho)(\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad + \big\|\left(\nabla^2_{\rho\rho}v_i(\cdot,(\xi_i*\rho)(\cdot)) - \nabla^2_{\rho\rho}w_i(\cdot,(\xi_i*\rho)(\cdot))\right)(\partial_x\xi_i*\rho)^2(\cdot)\big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad + \big\|(\nabla_\rho v_i(\cdot,(\xi_i*\rho)(\cdot)) - \nabla_\rho w_i(\cdot,(\xi_i*\rho)(\cdot)))\,(\partial^2_{xx}\xi_i*\rho)(\cdot)\big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad \leq \big\|\partial^2_{xx}v_i - \partial^2_{xx}w_i\big\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})}\\
&\qquad + \|\partial_x\nabla_\rho v_i - \partial_x\nabla_\rho w_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}2\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\qquad + \big\|\nabla^2_{\rho\rho}v_i - \nabla^2_{\rho\rho}w_i\big\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\partial_x\xi_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M^2\\
&\qquad + \|\nabla_\rho v_i - \nabla_\rho w_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\big\|\partial^2_{xx}\xi_i\big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\quad \leq \Big(1 + 2M\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} + M^2\|\partial_x\xi_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)} + M\big\|\partial^2_{xx}\xi_i\big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\Big)\\
&\qquad \times\|v_i - w_i\|_{\mathcal{V}},
\end{aligned}
$$

completing the proof of **(1)**.

Now, we prove that the function $\xi \mapsto \tilde{\Pi}_{v,\xi}\rho$ is locally Lipschitz continuous. Fix $v \in \mathcal{V}^n$, $\rho \in \tilde{X}_M$ and the functions $\eta, \xi \in \mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^{n\times n})$. It is immediate to verify that for $i = 1, \ldots, n$,

$$
\Big\|(\tilde{\Pi}_{v,\eta}\rho)_i - (\tilde{\Pi}_{v,\xi}\rho)_i\Big\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} = \|v_i(\cdot,\eta_i*\rho(\cdot)) - v_i(\cdot,\xi_i*\rho(\cdot))\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}
$$

$$\leq M\|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}$$

and

$$\begin{aligned}
&\left\|\partial_x(\tilde{\Pi}_{v,\eta}\rho)_i-\partial_x(\tilde{\Pi}_{v,\xi}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\leq \|\partial_x v_i(\cdot,\eta_i*\rho(\cdot))-\partial_x v_i(\cdot,\xi_i*\rho(\cdot))\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\|(\nabla_\rho v_i(\cdot,\eta_i*\rho(\cdot))-\nabla_\rho v_i(\cdot,\xi_i*\rho(\cdot)))\,(\partial_x\eta_i*\rho(\cdot))\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\|\nabla_\rho v_i(\cdot,\xi_i*\rho(\cdot))\,((\partial_x\eta_i-\partial_x\xi_i)*\rho(\cdot))\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\leq \|\partial_x\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\qquad+\left\|D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}M^2\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R},\mathbb{R}^n)}\\
&\qquad+\|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\partial_x\eta_i-\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\quad\leq M\|v_i\|_{\mathcal{V}}\left(2+M\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\right)\|\eta_i-\xi_i\|_{\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R};\mathbb{R}^n)}.
\end{aligned}$$

Moreover, for $i=1,\ldots,n$,

$$\begin{aligned}
&\left\|\partial^2_{xx}(\tilde{\Pi}_{v,\eta}\rho)_i-\partial^2_{xx}(\tilde{\Pi}_{v,\xi}\rho)_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\leq \left\|\partial^2_{xx}v_i(\cdot,\eta_i*\rho(\cdot))-\partial^2_{xx}v_i(\cdot,\xi_i*\rho(\cdot))\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\|2\partial_x\nabla_\rho v_i(\cdot,\eta_i*\rho(\cdot))\,((\partial_x\eta_i-\partial_x\xi_i)*\rho(\cdot))\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\|2\,(\partial_x\nabla_\rho v_i(\cdot,\eta_i*\rho(\cdot))-\partial_x\nabla_\rho v_i(\cdot,\xi_i*\rho(\cdot)))\,(\partial_x\xi_i*\rho(\cdot))\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\left\|\nabla^2_{\rho\rho}v_i(\cdot,\eta_i*\rho(\cdot))\left((\partial_x\eta_i*\rho(\cdot))^2-(\partial_x\xi_i*\rho(\cdot))^2\right)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\left\|\left(\nabla^2_{\rho\rho}v_i(\cdot,\eta_i*\rho(\cdot))-\nabla^2_{\rho\rho}v_i(\cdot,\xi_i*\rho(\cdot))\right)(\partial_x\xi_i*\rho(\cdot))^2\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\left\|\nabla_\rho v_i(\cdot,\eta_i*\rho(\cdot))\left((\partial^2_{xx}\eta_i-\partial^2_{xx}\xi_i)*\rho(\cdot)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\qquad+\left\|(\nabla_\rho v_i(\cdot,\eta_i*\rho(\cdot))-\nabla_\rho v_i(\cdot,\xi_i*\rho(\cdot)))\left(\partial^2_{xx}\xi_i*\rho(\cdot)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad\leq \left\|\partial^2_{xx}\nabla_\rho v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\qquad+2\|\partial_x\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\|\partial_x\eta_i-\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\qquad+2\left\|\partial_x D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M^2\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+\left\|D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\partial_x\eta_i-\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M^2\Big(\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\Big)+\left\|D^3_{\rho\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\\
&\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\|\partial_x\xi_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M^3\\
&\qquad+\|\nabla_\rho v_i\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)}\left\|\partial^2_{xx}\eta_i-\partial^2_{xx}\xi_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M\\
&\qquad+\left\|D^2_{\rho\rho}v_i\right\|_{\mathbf{L}^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\eta_i-\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\left\|\partial^2_{xx}\xi_i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}M^2\\
&\quad\leq \Big(4+3M\left(\|\partial_x\eta_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}+\|\partial_x\xi_i\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\right)+M^2\|\partial_x\xi_i\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\\
&\qquad+M\left\|\partial^2_{xx}\xi_i\right\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\Big)\times M\|v_i\|_{\mathcal{V}}\|\eta_i-\xi_i\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)},
\end{aligned}$$

proving **(2)**. □

In the next step we introduce the time dependent version of the map $\tilde{\Pi}_{v,\eta}$ defined by (4.3)–(4.4) on the time interval $[0,T]$, for a fixed $T>0$. To this aim, recall $\tilde{X}_M$ as in (4.2) and introduce

$$X_M := \mathbf{C}^0([0,T];\tilde{X}_M) \quad \text{and} \quad \|\rho\|_{X_M} := \sup_{t\in[0,T]} \|\rho(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}. \tag{4.11}$$

Lemma 4.5. *Let $v \in \mathbf{C}^0([0,T];\mathcal{V}^n)$ and (η) hold. Then, the map*

$$\begin{aligned} \Pi_{v,\eta} : X_M &\to \mathbf{C}^0\left([0,T];\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)\right) \\ \rho &\mapsto \left[t \mapsto \tilde{\Pi}_{v(t),\eta}\left(\rho(t)\right)\right] \text{ i.e. } \left((\Pi_{v,\eta}\rho)(t)\right)(x) = v\left(t,x,(\eta*\rho(t))(x)\right) \end{aligned}$$

is well defined, bounded and satisfies for $r,\rho\in X_M$ the Lipschitz estimates

$$\|\Pi_{v,\eta}r - \Pi_{v,\eta}\rho\|_{\mathbf{C}^0([0,T];\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n))} \le \mathbf{Lip}(\Pi_{v,\eta})\,\|r-\rho\|_{X_M} \tag{4.12}$$

$$\|\Pi_{v,\eta}r - \Pi_{v,\eta}\rho\|_{\mathbf{L}^1([0,T];\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n))} \le \mathbf{Lip}(\Pi_{v,\eta})\,\|r-\rho\|_{\mathbf{L}^1([0,T]\times\mathbb{R};\mathbb{R}^n)} \tag{4.13}$$

where $\mathbf{Lip}(\Pi_{v,\eta}) = \sup_{t\in[0,T]} \mathbf{Lip}\left(\tilde{\Pi}_{v(t),\eta}\right)$.

Proof. Fix $\rho\in X_M$, $\bar{t}\in[0,T]$ and consider a sequence $\{t_n\}_n \subset [0,T]$ converging to $\bar{t}$. Taking advantage of the Lipschitz regularity of $\tilde{\Pi}_{v,\eta}$, by Lemma 4.3 and **(1)** in Proposition 4.4, one can get

$$\begin{aligned} &\left\|\tilde{\Pi}_{v(t_n),\eta}(\rho(t_n)) - \tilde{\Pi}_{v(\bar{t}),\eta}(\rho(\bar{t}))\right\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)} \\ &\quad\le \left\|\tilde{\Pi}_{v(t_n),\eta}(\rho(t_n)) - \tilde{\Pi}_{v(\bar{t}),\eta}(\rho(t_n))\right\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)} \\ &\qquad+ \left\|\tilde{\Pi}_{v(\bar{t}),\eta}(\rho(t_n)) - \tilde{\Pi}_{v(\bar{t}),\eta}(\rho(\bar{t}))\right\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)} \\ &\quad\le C(M,\|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^{n\times n})})\,\|v(t_n)-v(\bar{t})\|_{\mathcal{V}^n} + \mathbf{Lip}\,\tilde{\Pi}_{v(\bar{t}),\eta}\,\|\rho(t_n)-\rho(\bar{t})\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \end{aligned}$$

which tends to zero as $n\to+\infty$, due to the continuity in time of v and ρ. Hence, $\Pi_{v,\eta}$ is well defined.

The boundedness of $\Pi_{v,\eta}$ follows from the same property of v by **(v)** and of $\tilde{\Pi}_{v(t),\eta}$ as proved in Lemma 4.3 for all $t\in[0,T]$.

To prove the Lipschitz estimates on $\Pi_{v,\eta}$, let $r,\rho\in X_M$ and evaluate

$$\begin{aligned} \|\Pi_{v,\eta}r - \Pi_{v,\eta}\rho\|_{\mathbf{C}^0([0,T];\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n))} &= \sup_{t\in[0,T]} \left\|\tilde{\Pi}_{v(t),\eta}\left(r(t)\right) - \tilde{\Pi}_{v(t),\eta}\left(\rho(t)\right)\right\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^n)} \\ &\le \sup_{t\in[0,T]} \mathbf{Lip}(\tilde{\Pi}_{v(t),\eta})\|r(t)-\rho(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\ &\le \mathbf{Lip}(\Pi_{v,\eta})\,\|r-\rho\|_{X_M}, \end{aligned}$$

where $\mathbf{Lip}(\Pi_{v,\eta}) := \sup_{t\in[0,T]} \mathbf{Lip}(\tilde{\Pi}_{v(t),\eta})$ is finite by (4.5) and $v\in\mathbf{C}^0([0,T];\mathcal{V}^n)$, proving (4.12). An analogous procedure gives the other estimate (4.13).
□

For later use, recall that, given $w \in \mathbf{C}^0([0,T];\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R};\mathbb{R}^n))$ and $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$, the Cauchy problems

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i\, w_i) = 0 \\ \rho_i(0,x) = (\rho_o)_i(x) \end{cases} \qquad i = 1, \ldots, n, \tag{4.14}$$

admit the (Lagrangian, see [25, § 2.5]) solutions $\rho_1, \ldots, \rho_n$ such that

$$\rho_i(t,x) = (\rho_o)_i(X_i(0;t,x)) \; \exp\left(- \int_0^t \partial_x w_i(s, X_i(s;t,x))\, \mathrm{d}s \right) \tag{4.15}$$

where the characteristic $t \mapsto X_i(t;t_o,x_o)$ is the solution to the Cauchy problem

$$\begin{cases} \dot{x} = w_i(t,x) & t \in \mathbb{R}_+ \\ x(t_o) = x_o. \end{cases} \tag{4.16}$$

The maps $\rho_1, \ldots, \rho_n$ are also Kružkov solutions to (4.14) in the sense of [30, Definition 1]. Indeed, several results in the literature, see for instance [23, Lemma 5] or [26, Corollary II.1], ensure that for a Cauchy problem of the type (4.14), the concepts of weak and entropy (or Kružkov) solutions coincide.

Useful relations (see [9, Chapter 3] or [28, Lemma 2.6]) which we exploit below are:

$$\begin{aligned} \partial_t X_i(t;t_o,x_o) &= w_i\left(t, X_i(t;t_o,x_o)\right) \\ \partial_{t_o} X_i(t;t_o,x_o) &= -w_i(t_o,x_o)\; \exp\left(\int_0^t \partial_x w_i(s, X_i(s;t_o,x_o))\, \mathrm{d}s \right) \\ \partial_{x_o} X_i(t;t_o,x_o) &= \exp\left(\int_{t_o}^t \partial_x w_i\left(s, X_i(s;t_o,x_o)\right) \mathrm{d}s \right) \end{aligned} \tag{4.17}$$

where $i \in \{1, \ldots, n\}$.

Lemma 4.6. *Let $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$, fix $\tilde{Q} > 0$ and call*

$$W_{\tilde{Q}} := \left\{ w \in \mathbf{C}^0\left([0,T];\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)\right) : \|w\|_{\mathbf{C}^0([0,T];\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n))} \le \tilde{Q} \right\}. \tag{4.18}$$

Define the map Σ_{ρ_o} as

$$\begin{array}{rccc} \Sigma_{\rho_o} : & W_{\tilde{Q}} & \to & \mathbf{C}^0\left([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^N)\right) \\ & w & \mapsto & \rho, \end{array} \tag{4.19}$$

where ρ is the solution to (4.14). Then,

- **(Σ1)** *The map Σ_{ρ_o} is well defined and Lipschitz continuous.*
- **(Σ2)** *For all $w \in W_{\tilde{Q}}$, $t \mapsto (\Sigma_{\rho_o} w)(t)$ is locally $\mathbf{L}^1$-Lipschitz continuous in time.*
- **(Σ3)** *For all $w \in W_{\tilde{Q}}$, $t \in [0,T]$ and $i = 1, \ldots, n$,*

$$\|(\Sigma_{\rho_o} w)_i(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} = \|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \quad \forall t \in [0,T], \quad \forall i = 1, \ldots, n; \tag{4.20}$$

$$\mathrm{TV}\left((\Sigma_{\rho_o} w)_i(t)\right) \le \left(\mathrm{TV}\left((\rho_o)_i\right) + \tilde{Q}\, t\, \|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \right) e^{\tilde{Q}\, t}. \tag{4.21}$$

(Σ4) *For all* $i = 1, \ldots, n$ *and for all* $w \in W_{\tilde{Q}}$*, if* $(\rho_o)_i \geq 0$*, then for all* $t \in [0, T]$*,* $\rho_i(t) \geq 0$*.*

Proof. Fix throughout the index i. We distinguish several steps.

$\mathbf{L}^1$**-norm of** $\Sigma_\rho w$. With the change of variable $\xi = X_i(0; t, x)$, for any $t \in [0, T]$,

$$\begin{aligned}
&\|(\Sigma_{\rho_o} w)_i(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\
&\quad = \int_{\mathbb{R}} \left|(\rho_o)_i(X_i(0;t,x)) \exp\left(-\int_0^t \partial_x w_i(s, X_i(s;t,x))\, \mathrm{d}s\right)\right| \mathrm{d}x \\
&\quad = \int_{\mathbb{R}} |(\rho_o)_i(\xi)| \exp\left(-\int_0^t \partial_x w_i\left(s, X_i(s;t,x)\right)\, \mathrm{d}s\right) \\
&\qquad \times \exp\left(\int_0^t \partial_x w_i\left(s, X_i(s;0,\xi)\right) \mathrm{d}s\right) \mathrm{d}\xi \\
&\quad = \int_{\mathbb{R}} |(\rho_o(\xi))_i|\, \mathrm{d}\xi\,,
\end{aligned}$$

proving (4.20), thanks to (4.15) and (4.17).

TV **estimate.** If $\rho_o \in \mathbf{W}^{\mathbf{1,1}}(\mathbb{R}; \mathbb{R}^n)$ then, for all $t \in [0, T]$, differentiating (4.15) and using the change of variable $\xi = X_i(\tau; t, x)$,

$$\begin{aligned}
&\|\partial_x (\Sigma_{\rho_o} w)_i(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \\
&= \int_{\mathbb{R}} \left|(\rho_o)_i'\left(X_i(0;t,x)\right) \partial_x X_i(0;t,x) \exp\left(-\int_0^t \partial_x w_i\left(\tau, X_i(\tau;t,x)\right) \mathrm{d}\tau\right)\right| \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \Bigg[\left|(\rho_o)_i\left(X_i(0;t,x)\right)\right| \exp\left(-\int_0^t \partial_x w_i\left(\tau, X_i(\tau;t,x)\right) \mathrm{d}\tau\right) \\
&\quad \times \int_0^t \left|\partial^2_{xx} w_i\left(\tau, X_i(\tau;t,x)\right) \partial_x X_i(\tau;t,x)\right| \mathrm{d}\tau\,\Bigg] \mathrm{d}x \\
&\leq \exp\left(\|\partial_x w_i\|_{\mathbf{C}^0([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\, t\right) \mathrm{TV}\left((\rho_o)_i\right) \\
&\quad + \left\|\partial^2_{xx} w_i\right\|_{\mathbf{C}^0([0,t];\mathbf{L}^\infty(\mathbb{R}))} \exp\left(\|\partial_x w_i\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} t\right) \|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} t,
\end{aligned}$$

proving (4.21) when $\rho_o \in \mathbf{W}^{\mathbf{1,1}}(\mathbb{R}; \mathbb{R}^n)$.

We now proceed to consider the case $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^n)$. Thanks to [3, Theorem 3.9], there exists a sequence $\{\rho_o^j\}_j \in \mathbf{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ such that $\rho_o^j \to \rho_o$ in $\mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)$ and $\lim_{j\to\infty} \left\|\partial_x \rho_o^j\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} = \mathrm{TV}(\rho_o)$.

Since, for $i = 1, \ldots, n$, (4.20) ensures that

$$\left\|(\Sigma_{\rho_o^j} w)_i(t) - (\Sigma_{\rho_o} w)_i(t)\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} = \left\|(\rho_o^j)_i - (\rho_o)_i\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})},$$

the lower semicontinuity of the total variation, see [3, Remark 3.5], ensures that

$$\begin{aligned}
\mathrm{TV}\left((\Sigma_{\rho_o} w)_i(t)\right) &\leq \liminf_{j\to\infty} \mathrm{TV}\left((\Sigma_{\rho_o^j} w)_i(t)\right) \\
&\leq \exp\left(\|\partial_x w_i\|_{\mathbf{C}^0([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\, t\right) \mathrm{TV}\left((\rho_o)_i\right)
\end{aligned}$$

$$+\left\|\partial^2_{xx} w_i\right\|_{\mathbf{C}^0([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} \exp\left(\left\|\partial_x w_i\right\|_{\mathbf{C}^0([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\, t\right)$$
$$\left\|(\rho_o)_i\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} t,$$

completing the proof of (4.21).

Σ_{ρ_o} **is well defined.** Given $w \in W_{\tilde{Q}}$ our aim is to prove the continuity in time of the function $\Sigma_{\rho_o} w$. So, fix $0 \le t_1 < t_2 \le T$, by (4.15) evaluate

$$\begin{aligned}
&\|(\Sigma_{\rho_o} w)_i(t_1) - (\Sigma_{\rho_o} w)_i(t_2)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\\
&\quad = \int_{\mathbb{R}} \left|(\Sigma_{\rho_o} w)_i(t_1,x) - (\Sigma_{\rho_o} w)_i(t_1, X_i(t_1;t_2,x)) \exp\left(-\int_{t_1}^{t_2} \partial_x w_i(s; X_i(s;t_2,x))\,\mathrm{d}s\right)\right| \mathrm{d}x\\
&\quad \le \int_{\mathbb{R}} \left|(\Sigma_{\rho_o} w)_i(t_1,x) - (\Sigma_{\rho_o} w)_i(t_1, X_i(t_1;t_2,x))\right| \mathrm{d}x\\
&\qquad + \int_{\mathbb{R}} \left|(\Sigma_{\rho_o} w)_i(t_1, X_i(t_1;t_2,x)) \left(1 - \exp\left(-\int_{t_1}^{t_2} \partial_x w_i(s; X_i(s;t_2,x))\,\mathrm{d}s\right)\right)\right| \mathrm{d}x
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\quad \le \mathrm{TV}((\Sigma_{\rho_o} w)_i(t_1)) \Bigg(\max\left\{\operatorname*{ess\,sup}_{x\in\mathbb{R}}(x - X_i(t_1;t_2,x)), 0\right\}\\
&\qquad - \min\left\{\operatorname*{ess\,sup}_{x\in\mathbb{R}}(x - X_i(t_1;t_2,x)), 0\right\}\Bigg)\\
&\qquad + \int_{\mathbb{R}} \left|(\Sigma_{\rho_o} w)_i(t_1; X_i(t_1;t_2,x)) \int_{t_1}^{t_2} \partial_x w_i(s; X_i(s;t_2,x))\,\mathrm{d}s\right| \mathrm{d}x\,,
\end{aligned} \tag{4.23}$$

where we used Lemma 4.1.

Consider (4.22). Using the estimate (4.21) and the boundedness of w_i we obtain

$$\mathrm{TV}((\Sigma_{\rho_o} w)_i(t_1)) \le e^{\tilde{Q} t_1}\, \mathrm{TV}((\rho_o)_i) + \tilde{Q} t_1 e^{\tilde{Q} t_1} \|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$$

and

$$\begin{aligned}
&\max\left\{\mathrm{ess\,sup}_{x\in\mathbb{R}}(x - X_i(t_1;t_2,x)), 0\right\} - \min\left\{\mathrm{ess\,sup}_{x\in\mathbb{R}}(x - X_i(t_1;t_2,x)), 0\right\}\\
&\quad \le 2 \operatorname*{ess\,sup}_{x\in\mathbb{R}} |x - X_i(t_1;t_2,x)|\\
&\quad \le 2 \|w_i\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} (t_2 - t_1)\\
&\quad \le 2\tilde{Q}(t_2 - t_1)
\end{aligned}$$

so that

$$[(4.22)] \le \left(e^{\tilde{Q} t_1}\, \mathrm{TV}((\rho_o)_i) + \tilde{Q} t_1 e^{\tilde{Q} t_1} \|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right) 2\tilde{Q}\,(t_2 - t_1).$$

Consider now (4.23). By the change of variable $\xi = X_i(t_1;t_2,x)$, (4.20) and (4.18)

$$\begin{aligned}
[(4.23)] &\le \tilde{Q}(t_2 - t_1) \int_{\mathbb{R}} \left|(\Sigma_{\rho_o} w)_i(t_1; X_i(t_1;t_2,x))\right| \mathrm{d}x\\
&\le \tilde{Q}(t_2 - t_1) \int_{\mathbb{R}} \left|\Sigma_{\rho_o} w(t_1,\xi)_i\right| \exp\left(\int_{t_1}^{t_2} \partial_x w_i(s, X_i(s;t_1,\xi))\,\mathrm{d}s\right) \mathrm{d}\xi\\
&\le \tilde{Q}(t_2 - t_1) e^{\tilde{Q}(t_2 - t_1)} \|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}.
\end{aligned}$$

Collecting together the estimates above leads to the following bound:

$$\begin{aligned}&\left\|(\Sigma_{\rho_o}w)_i(t_1)-(\Sigma_{\rho_o}w)_i(t_2)\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\\&\quad\le \tilde Q\,\left[2\left(\mathrm{TV}((\rho_o)_i)+\tilde Q T\|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right)+\|(\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right]e^{\tilde Q T}(t_2-t_1)\end{aligned}\tag{4.24}$$

concluding the proofs of (**Σ2**) and of the well-posedness of Σ_{ρ_o}.

The map Σ_{ρ_o} in (4.19) is Lipschitz continuous. Consider $t\in[0,T]$, $w,z\in W_{\tilde Q}$ and using (4.15), estimate

$$\begin{aligned}\left\|(\Sigma_{\rho_o}w)_i(t)-(\Sigma_{\rho_o}z)_i(t)\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}=\int_{\mathbb{R}}\Big|&\rho_o(X_i(0;t,x))_i\exp\left(-\int_0^t\partial_x w_i(\tau,X_i(\tau;t,x))\,\mathrm{d}\tau\right)\\&-\rho_o(Y_i(0;t,x))_i\exp\left(-\int_0^t\partial_x z_i(\tau,Y_i(\tau;t,x))\,\mathrm{d}\tau\right)\Big|\,\mathrm{d}x\end{aligned}\tag{4.25}$$

where $X_i(t;t_o,x_o)$ and $Y_i(t;t_o,x_o)$ are the solutions to

$$\begin{cases}\dot X_i=w_i(t,x)\\X_i(t_o)=x_o\end{cases}\qquad\text{and}\qquad\begin{cases}\dot Y_i=z_i(\tau,y)\\Y_i(t_o)=x_o.\end{cases}\tag{4.26}$$

Adding and subtracting the term $\rho_o(X_i(0;t,x))_i\exp\left(-\int_0^t\partial_x z_i(\tau,Y_i(\tau;t,x))\right.$ $\left.\mathrm{d}\tau\right)$ to (4.25) and thanks to (4.18), Proposition 4.2 and Lemma 4.1, one obtains

$$\begin{aligned}&\left\|(\Sigma_{\rho_o}w)_i(t)-(\Sigma_{\rho_o}z)_i(t)\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\\&\quad\le\int_{\mathbb{R}}\left|\rho_o(X_i(0;t,x))_i-\rho_o(Y_i(0;t,x))_i\right|\mathrm{d}x\,e^{\tilde Q\,t}\\&\qquad+\int_{\mathbb{R}}\left|\rho_o(X_i(0;t,x))_i\right|\left|\int_0^t\partial_x w_i(\tau,X_i(\tau;t,x))-\partial_x z_i(\tau,Y_i(\tau;t,x))\,\mathrm{d}\tau\right|\mathrm{d}x\,e^{\tilde Q\,t}\\&\quad\le 2\ \mathrm{TV}((\rho_o)_i)\left\|X_i(0;t,\cdot)-Y_i(0;t,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}e^{\tilde Q\,t}\\&\qquad+\int_{\mathbb{R}}\left|\rho_o(X_i(0;t,x))_i\right|\Bigg(\tilde Q\int_0^t\left|X_i(\tau;t,x)-Y_i(\tau;t,x)\right|\mathrm{d}\tau\\&\qquad+\left\|\partial_x w_i-\partial_x z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}t\Bigg)\mathrm{d}x e^{\tilde Q\,t}\\&\quad\le 2\ \mathrm{TV}((\rho_o)_i)\left\|w_i-z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\,t\,e^{2\,\tilde Q\,t}\\&\qquad+\int_{\mathbb{R}}\left|\rho_o(X_i(0;t,x))_i\right|\left[\tilde Q\int_0^t\left\|w_i-z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\left|t-\tau\right|e^{\tilde Q\,(t-\tau)}\ \mathrm{d}\tau\right]\mathrm{d}x\ e^{\tilde Q\,t}\\&\qquad+\int_{\mathbb{R}}\left|\rho_o(X_i(0;t,x))_i\right|\left[t\left\|\partial_x w_i-\partial_x z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\right]\mathrm{d}x\ e^{\tilde Q\,t}\\&\quad\le 2\ \mathrm{TV}((\rho_o)_i)\left\|w_i-z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\,t\,e^{2\,\tilde Q\,t}\\&\qquad+\tilde Q\left\|(\rho_o)_i\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\left\|w_i-z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\frac{t^2}{2}\,e^{2\,\tilde Q\,t}\\&\qquad+\left\|(\rho_o)_i\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\left\|\partial_x w_i-\partial_x z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\,t\,e^{2\,\tilde Q\,t}\\&\quad\le\left(2\,\mathrm{TV}((\rho_o)_i)+\frac{1}{2}\,\tilde Q\,t\left\|(\rho_o)_i\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}+\left\|(\rho_o)_i\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right)\\&\qquad\left\|w_i-z_i\right\|_{\mathbf{C}^0([0,T];\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}))}\,t\,e^{2\,\tilde Q\,t}.\end{aligned}$$

Passing to the supremum over $t\in[0,T]$,

$$\left\|\Sigma_{\rho_o}w-\Sigma_{\rho_o}z\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n))}\le\mathbf{Lip}(\Sigma_{\rho_o})\left\|w-z\right\|_{\mathbf{C}^0([0,T];\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}^n))}$$

where

$$\mathbf{Lip}(\Sigma_{\rho_o}) := \left(2\,\mathrm{TV}(\rho_o) + \frac{1}{2}\,\tilde{Q}\,T\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} + \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right) T\, \exp(2\,\tilde{Q}\,T). \tag{4.27}$$

The proof of (**Σ1**) is completed.

Finally, the positivity (**Σ4**) immediately follows from (4.15). □

Proposition 4.7. *Let $\tilde{Q} > 0$. For a fixed $w \in W_{\tilde{Q}}$ as defined in (4.18), the map*

$$\begin{array}{ccc} (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n) & \to & \mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)) \\ \rho_o & \mapsto & \Sigma_{\rho_o} w \end{array} \tag{4.28}$$

with Σ as in (4.19), is Lipschitz continuous.

Proof. Choose ρ_o and σ_o in $(\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$. Fix $w \in W_{\tilde{Q}}$, $t \in [0,T]$ and $i = 1,\ldots,n$. Then the result is a direct consequence of the linearity of (2.1) and of the equality (4.20):

$$\|\Sigma_{\rho_o} w(t) - \Sigma_{\sigma_o} w(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} = \|\Sigma_{\rho_o - \sigma_o} w(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} = \|\rho_o - \sigma_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}.$$

□

Lemma 4.8. *Let* **(v)** *and* (η) *hold. Fix $M > 0$ and $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$ such that*

$$M = \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}. \tag{4.29}$$

Define $Q := \sup_{t\in[0,T]} Q_{v(t)}$ as in (4.6) and X_M as in (4.11)–(4.2). Then, the map

$$\begin{array}{rcl} \mathcal{T}_{v,\eta,\rho_o} : X_M & \to & X_M \\ \rho & \mapsto & \left(\Sigma_{\rho_o} \circ \Pi_{v,\eta}\right)(\rho) \end{array} \tag{4.30}$$

is well defined and Lipschitz continuous. For each $\rho \in X_M$ the function $\mathcal{T}_{v,\eta,\rho_o}\rho$ is locally Lipschitz continuous in time and with total variation in space bounded by

$$\mathrm{TV}\left(\mathcal{T}_{v,\eta,\rho_o}\rho(t)\right) \le \exp(Qt)\left(\mathrm{TV}(\rho_o) + Q\,\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\,t\right) \quad \forall\, t \in [0,T]. \tag{4.31}$$

Furthermore, if T is sufficiently small, then $\mathcal{T}_{v,\eta,\rho_o}$ is also a contraction.

Proof. Throughout this proof, we keep v, η and ρ_o fixed, hence we omit them.

The map $\mathcal{T}$ is well defined and Lipschitz continuous. Thanks to the property of the maps Π and Σ, it is immediate to verify that if $\rho \in X$ then $\mathcal{T}\rho \in \mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n))$. Furthermore, owing to (4.20), $\|\mathcal{T}\rho\|_{\mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n))} = \|\rho_o\|_{\mathbf{L}^1(\mathbb{R},\mathbb{R}^n)} = M$.

The Lipschitz regularity follows, since $\mathcal{T}$ is the composition of Lipschitz continuous maps, by Lemmas 4.6 and 4.5.

Properties of the function $\mathcal{T}\rho$. Given $\rho \in X_M$, the local $\mathbf{L}^1$-Lipschitz continuity in time of $\mathcal{T}\rho$ directly descends from ($\mathbf{\Sigma 2}$) in Lemma 4.6. Furthermore, for each $t \in [0,T]$, thanks to 4.21, one obtains that

$$\mathrm{TV}(\mathcal{T}\rho(t)) \leq \exp(Q\,t)\left(\mathrm{TV}(\rho_o) + Q\,\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\,t\right)$$

proving (4.31).

The map $\mathcal{T}$ is a contraction for sufficiently small times. Since we have $\mathbf{Lip}(\mathcal{T}) = \mathbf{Lip}(\Sigma)\,\mathbf{Lip}(\Pi)$, we are lead to prove that $\mathbf{Lip}(\Sigma)\,\mathbf{Lip}(\Pi) < 1$.

Choose $T < 2$, which implies that

$$Q = \sup_{t\in[0,T]} Q_{v(t)} \leq C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times m})}, M, \sup_{t\in[0,2]}\|v(t)\|_{\mathcal{V}^n}\right) =: \bar{Q}.$$

Moreover, by (4.5),

$$\mathbf{Lip}(\Pi) \leq C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times m})}, M, \sup_{t\in[0,2]}\|v(t)\|_{\mathcal{V}^n}\right) =: \overline{\mathbf{Lip}\,\Pi}.$$

Hence, by (4.27) the additional conditions

$$\begin{aligned} T &< 1/(2\,\bar{Q}) \\ T &< \frac{1}{2}\left(e\left(2\ \mathrm{TV}((\rho_o)_i) + (\bar{Q}+1)\,M\right)\overline{\mathbf{Lip}(\Pi)}\right)^{-1} \end{aligned}$$

ensure that $\mathbf{Lip}(\mathcal{T}) \leq 1/2$. □

Proposition 4.9. *Fix $\rho \in X_M$ as defined in (4.11). Define the map*

$$\begin{array}{ccc} \mathbf{C}^0([0,T];\mathcal{V}^n) \times \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n}) \times (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n) & \rightarrow & \mathbf{C}^0\left([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)\right) \\ v, \eta, \rho_o & \mapsto & \mathcal{T}_{v,\eta,\rho_o}\rho. \end{array} \tag{4.32}$$

Then

- ***(1)*** *For all $\eta \in \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})$, $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$, the map $v \mapsto \mathcal{T}_{v,\eta,\rho_o}\rho$ is Lipschitz continuous.*
- ***(2)*** *For all $v \in \mathbf{C}^0([0,T];\mathcal{V}^n)$, $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^n)$, the map $\eta \mapsto \mathcal{T}_{v,\eta,\rho_o}\rho$ is locally Lipschitz continuous.*
- ***(3)*** *For all $v \in \mathbf{C}^0([0,T];\mathcal{V}^n)$, $\eta \in \mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})$ the map $\rho_o \mapsto \mathcal{T}_{v,\eta,\rho_o}\rho$ is Lipschitz continuous.*

The proof follows directly from Propositions 4.4 and 4.7.

Proof of Theorem 2.3. **Proof of ($\boldsymbol{\mathcal{P}}\mathbf{2}$)**: Note that Definition 2.2 implies that solving (1.1) is equivalent to finding a fixed point of the map $\mathcal{T}_{v,\eta,\rho_o}$ defined in 4.30. Lemma 4.8 ensures that for a $T^* > 0$ the map $\mathcal{T}_{v,\eta,\rho_o}$ admits a unique fixed point.

Fix an arbitrary $T > 0$. Recall the quantities M defined in (4.29), $Q := \sup_{t\in[0,T]} Q_{v(t)}$ as in (4.6) and by (4.5)

$$\mathbf{Lip}(\Pi_{v,\eta}) \leq C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times m})}, M, \sup_{t\in[0,T]}\|v(t)\|_{\mathcal{V}^n}\right) =: \overline{\mathbf{Lip}(\Pi_{v,\eta})}.$$

Introduce

$$K := \exp(Q\,T)\,(\mathrm{TV}(\rho_o) + Q\,M\,T)$$
$$\Delta T := \min\left\{T, \frac{1}{2\,e\,(2\,K + Q\,M + M)\,\overline{\mathbf{Lip}(\Pi_{v,\eta})}}\right\}.$$

Then, Lemma 4.8 ensures that there exists $\rho \in \mathbf{C}^0([0,\Delta T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n))$ solving (1.1) on $[0,\Delta T]$. Observe that $\|\rho(\Delta T)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} = M$ by (4.20) and $\mathrm{TV}\,(\rho(\Delta T)) \leq K$ by (4.31). We can iterate the application of Lemma 4.8 obtaining a solution to (1.1) on $[0,T]$, since $\|\rho(k\,\Delta T)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} = M$ and $\mathrm{TV}\,(\rho(k\,\Delta T)) \leq K$ for $k = 1, 2, \ldots$. By the arbitrariness of T, **($\mathcal{P}$2)** follows.

Define $\mathcal{P}_{0,t}\rho_o = \rho(t)$.

Proof of ($\mathcal{P}$1): This property directly follows from the construction, for instance from (4.15).

Proof of ($\mathcal{P}$3): Call $M := \max\{\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \|\hat\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\}$. For $t \in \mathbb{R}_+$ denote $\rho'(t) = \mathcal{P}_{0,t}\rho_o$, $\rho''(t) = \mathcal{P}_{0,t}\hat\rho_o$ and introduce the corresponding characteristics X_i' and X_i'' as in (4.16). Then, by (4.30) $\rho' = \mathcal{T}_{v,\eta,\rho_o}\rho'$ and $\rho'' = \mathcal{T}_{v,\eta,\hat\rho_o}\rho''$, so that for $i = 1, \ldots, n$

$$\begin{aligned}
&\|(\rho')_i(t) - (\rho'')_i(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\\
&\leq \int_{\mathbb{R}} \left|(\rho_o)_i(X_i'(0;t,x)) - (\hat\rho_o)_i(X_i'(0;t,x))\right| \exp\left(-\int_0^t \partial_x(\Pi_{v,\eta}\rho')_i(s,X_i'(s;t,x))\,\mathrm{d}s\right)\mathrm{d}x\\
&\quad + \int_{\mathbb{R}} \left|(\hat\rho_o)_i(X_i'(0;t,x)) - (\hat\rho_o)_i(X_i''(0;t,x))\right| \exp\left(-\int_0^t \partial_x(\Pi_{v,\eta}\rho')_i(s,X_i'(s;t,x))\,\mathrm{d}s\right)\mathrm{d}x\\
&\quad + e^{Qt}\int_{\mathbb{R}} \left|(\hat\rho_o)_i(X_i''(0;t,x))\right| \left|\int_0^t \partial_x(\Pi_{v,\eta}\rho')_i(s,X_i'(s;t,x))\right.\\
&\quad \left. - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i''(s;t,x))\,\mathrm{d}s\right|\mathrm{d}x\\
&\leq \|(\rho_o)_i - (\hat\rho_o)_i\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} + A_1 + A_2 + A_3\,,
\end{aligned}$$

where $Q = C(\|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^{n\times n})}, \tilde M, \|v\|_{\mathbf{C}^0([0,t];\mathcal{V}^n)})$ as in Lemma 4.8, and

$$\begin{aligned}
A_1 &= 2\,e^{Qt}\,\mathrm{TV}\left((\hat\rho_o)_i\right)\,\|X_i'(0;t,\cdot) - X_i''(0;t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
A_2 &= e^{Qt}\int_{\mathbb{R}} \left|(\hat\rho_o)_i(X_i''(0;t,x))\right| \left|\int_0^t \partial_x(\Pi_{v,\eta}\rho')_i(s,X_i'(s;t,x))\right.\\
&\qquad \left. - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i'(s;t,x))\,\mathrm{d}s\right|\mathrm{d}x\\
A_3 &= e^{Qt}\int_{\mathbb{R}} \left|(\hat\rho_o)_i(X_i''(0;t,x))\right| \left|\int_0^t \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i'(s;t,x))\right.\\
&\qquad \left. - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i''(s;t,x))\,\mathrm{d}s\right|\mathrm{d}x\,.
\end{aligned}$$

We estimate the latter terms separately:

$$\begin{aligned}
&\|X_i'(0;t,\cdot) - X_i''(0;t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
&\quad \leq \left\|\partial_x\left(\Pi_{v,\eta}\rho'\right)_i - \partial_x\left(\Pi_{v,\eta}\rho''\right)_i\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} && [\text{ By } (2.1)-(4.16)]\\
&\quad \leq \mathbf{Lip}(\Pi_{v,\eta})\,\|\rho' - \rho''\|_{\mathbf{L}^1([0,t]\times\mathbb{R};\mathbb{R}^n)} && [\text{By Lemma } 4.5]
\end{aligned}$$

so that

$$A_1 \leq 2\,\mathbf{Lip}(\Pi_{v,\eta})\,\mathrm{TV}\left((\hat\rho_o)_i\right)\,e^{Q\,t}\,\|\rho' - \rho''\|_{\mathbf{L}^1([0,t]\times\mathbb{R};\mathbb{R}^n)}.$$

Moreover,

$$\begin{aligned}
&\left|\int_0^t \partial_x(\Pi_{v,\eta}\rho')_i(s,X_i'(s;t,x)) - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i'(s;t,x))\,\mathrm{d}s\right| \\
&\quad\le \int_0^t \left\|\partial_x(\Pi_{v,\eta}\rho')_i(s,X_i'(s;t,x)) - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i'(s;t,x))\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}\mathrm{d}s \\
&\quad\le \int_0^t \mathbf{Lip}(\Pi_{v,\eta})\,\|\rho'(s)-\rho''(s)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\,\mathrm{d}s \qquad \text{[By Lemma 4.5]}
\end{aligned}$$

implying that

$$A_2 \le \mathbf{Lip}(\Pi_{v,\eta})\,M\,e^{2\,Q\,t}\,\|\rho'-\rho''\|_{\mathbf{L}^1([0,t]\times\mathbb{R};\mathbb{R}^n)}.$$

Finally,

$$\begin{aligned}
&\left|\int_0^t \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i'(s;t,x)) - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i''(s;t,x))\,\mathrm{d}s\right| \\
&\quad\le \int_0^t \left|\partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i'(s;t,x)) - \partial_x(\Pi_{v,\eta}\rho'')_i(s,X_i''(s;t,x))\right|\mathrm{d}s \\
&\quad\le \int_0^t \|\Pi_{v,\eta}\rho''\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^n)}\left|X_i'(s;t,x)-X_i''(s;t,x)\right|\mathrm{d}s \\
&\quad\le Q\;\mathbf{Lip}(\Pi_{v,\eta})\,t\,\|\rho'-\rho''\|_{\mathbf{L}^1([0,t]\times\mathbb{R};\mathbb{R}^n)}
\end{aligned}$$

so that

$$A_3 \le Q\,M\;\mathbf{Lip}(\Pi_{v,\eta})\,t\,e^{2\,Q\,t}\,\|\rho'-\rho''\|_{\mathbf{L}^1([0,t]\times\mathbb{R};\mathbb{R}^n)}.$$

We thus obtain

$$\begin{aligned}
&\|\rho'(t)-\rho''(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\
&\quad\le \|\rho_o-\hat\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\
&\qquad + \left(2\ \mathrm{TV}\,(\hat\rho_o)+M+Q\,M\,t\right)\,\mathbf{Lip}(\Pi_{v,\eta})\,e^{2\,Q\,t}\|\rho'-\rho''\|_{\mathbf{L}^1([0,t]\times\mathbb{R};\mathbb{R}^n)}.
\end{aligned}$$

An application of Gronwall Lemma completes the proof of **($\boldsymbol{\mathcal{P}}$3)**.

Proof of ($\boldsymbol{\mathcal{P}}$4): Choose $t', t'' \in [0,T]$. Then, calling $w(t,x) = \Pi_{v,\eta}\rho(t,x)$,

$$\begin{aligned}
&\|\rho(t'')-\rho(t')\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\
&\quad= \|(\Sigma_{\rho_o}w)(t'')-(\Sigma_{\rho_o}w)(t')\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \qquad [\text{ Since } \mathcal{T}_{v_\eta,\rho_o}\rho=\rho] \\
&\quad\le C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})},\|v\|_{\mathbf{C}^0([0,T];\mathcal{V}^n)},\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)},TV(\rho_o),T\right)\,|t''-t'| \\
&\qquad \text{[By (4.24) in Lemma 4.6]}
\end{aligned}$$

proving **($\boldsymbol{\mathcal{P}}$4)**.

Proof of ($\boldsymbol{\mathcal{P}}$5): For any $t \in \mathbb{R}_+$, by [8, Theorem 2.9], we have:

$$\left\|\mathcal{P}_{0,t}\rho_o-\hat{\mathcal{P}}_{0,t}\rho_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le \mathbf{Lip}(\mathcal{P})\int_0^t \liminf_{h\to0^+}\frac{\left\|\mathcal{P}_{\tau,h}\hat{\mathcal{P}}_{0,\tau}\rho_o-\hat{\mathcal{P}}_{\tau,h}\hat{\mathcal{P}}_{0,\tau}\rho_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}}{h}\,\mathrm{d}\tau$$

where

$$\mathbf{Lip}(\mathcal{P}) := C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})},\|v\|_{\mathbf{C}^0([0,t];\mathcal{V}^n)},\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)},\mathrm{TV}(\rho_o),t\right)$$

is as in **$(\mathcal{P}3)$**. Call $\rho = \hat{\mathcal{P}}_{0,\tau}\rho_o$. Recall that for h small, owing to Proposition 4.9,

$$\left\|\mathcal{P}_{\tau,h}\rho - \hat{\mathcal{P}}_{\tau,h}\rho\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \leq \frac{B_1 h}{1 - B_2 h} \left\|v - \hat{v}\right\|_{\mathbf{C}^0([0,h];\mathcal{V}^n)}$$

with

$$\begin{aligned} B_1 &:= \left(2\,\mathrm{TV}(\rho_o) + \left(\frac{1}{2}Qh + 1\right)\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\right) C\left(\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})}\right) \\ B_2 &:= \left(2\,\mathrm{TV}(\rho_o) + \left(\frac{1}{2}Qh + 1\right)\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\right) \mathbf{Lip}(\Pi_{\hat{v},\eta}) \end{aligned}$$

and the constant $C(\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})})$ is given by **(1)** in Proposition 4.4. So,

$$\begin{aligned} &\left\|\mathcal{P}_{0,t}\rho_o - \hat{\mathcal{P}}_{0,t}\rho_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\ &\quad\leq \left(2\,\mathrm{TV}(\rho_o) + \|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}\right) C\left(\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}, \|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})}\right) \\ &\qquad \mathbf{Lip}(\mathcal{P})\, t\, \|v - \hat{v}\|_{\mathbf{C}^0([0,t];\mathcal{V}^n)}. \end{aligned}$$

completing the proof of **$(\mathcal{P}5)$**.

Proof of $(\mathcal{P}6)$: For any $t \in \mathbb{R}_+$, using the same technique as in the proof of **$(\mathcal{P}5)$**, one gets

$$\begin{aligned} &\left\|\mathcal{P}_{0,t}\rho_o - \hat{\mathcal{P}}_{0,t}\rho_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\ &\quad\leq C\left(\|\eta\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})}, \|\hat{\eta}\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})}, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R},\mathbb{R}^n)}\right) \\ &\qquad\times \left(2\,\mathrm{TV}(\rho_o) + \|\rho_o\|_{\mathbf{L}^1(\mathbb{R},\mathbb{R}^n)}\right) \|\rho_o\|_{\mathbf{L}^1(\mathbb{R},\mathbb{R}^n)} \|v\|_{\mathbf{C}^0([0,t];\mathcal{V}^n)} \\ &\quad \mathbf{Lip}(\mathcal{P})\, t\, \|\eta - \hat{\eta}\|_{\mathbf{W}^{\mathbf{2},\infty}(\mathbb{R};\mathbb{R}^{n\times n})} \end{aligned}$$

thanks to Proposition 4.9.

The total variation estimate **$(\mathcal{P}7)$** directly follows from (4.31) while the positivity **$(\mathcal{P}8)$** is immediate. Hence, the proof is completed.

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Declarations

Conflict of interest The authors declare no Conflict of interest in this paper.

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