Structures 69 (2024) 107403

Contents lists available at ScienceDirect

Structures

journal homepage: www.elsevier.com/locate/structures

# Conservative systems showing instability in tension

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ARTICLE INFO

*Keywords:*
Conservative loads
Instability under tension
Analogies

ABSTRACT

Within the framework of a second-order theory, the stability of a straight beam, hinged at one end and supported at an intermediate point, is studied. Two different tensile loading conditions are applied to the beam, causing bifurcation and instability. A new analogy is shown between one of these stability problems and the elastic circular arch problem on spring soil with negative stiffness. The relationship between the results presented here and those presented in Gajewski and Palej (1974), in Zaccaria et al. (2011), and in Feriani and Carini (2017) is shown.

## 1. Introduction

The phenomenon of instability of beams under tensile loading was discovered, it seems for the first time, by Grammel [1], see also [2]. Grammel's results, together with other similar examples, are reported in Ziegler's book ([3]) and have also recently been reconsidered ([4]). Gajewski and Palej [5] gave other examples of instability of beams under tensile loads. Kelly [6] demonstrated the existence of the phenomenon of tensile instability in multilayer elastomeric bearings with non-negligible shear deformation (Timoshenko beam). Timoshenko beams under tension were later studied by many other authors (see, for instance, [7–9]). Zaccaria et al. [10] found another class of structures that exhibit instability in tension and demonstrated this phenomenon experimentally. In particular, [10] considered two inextensible elastic rods, clamped at one end and connected to each other by a 'slider', which buckled under tensile 'dead' loads.

Caddemi et al. [11] and Caddemi et al. [12] studied the tensile buckling in slender beams in the presence of multiple internal sliders endowed with shear deformation singularities (due to the internal sliders). They stated that this type of beam can be considered as the discrete counterpart of the uniform Timoshenko column and that its accuracy in terms of tensile buckling load depends on the number of sliders considered in the model.

A more complex multimodular beam model incorporating possible imperfections at the discrete level was obtained by Palumbo et al. [13].

In [14] two conservative systems have been presented which show instability due to a tensile loading. However, it seems that their loading cannot be defined as 'dead' loading in the sense of Zaccaria et al. [10].

Simão and da Silva [15,16] studied the buckling and post-buckling behaviour of a repetitive rod system under axial tensile loading. Each part of this repetitive rod system is similar to the structures mentioned by Ziegler and first studied by Grammel, *i.e.* a two-rod system with a rotational joint connected to a compressed rod overlapping the tensioned one.

Experimental measurements of the critical load of structures subjected to tensile loads have been carried out by Efraim and Blostatsky [17].

The motivations that led and still lead to the study of the stability of structures under tensile loads are many. For instance, in the work of Zaccaria et al. [10] it was suggested that the influence of the axial load on the behaviour of columns with internal sliders could both provide application to innovative actuators for mechanical wave control and open up perspectives for understanding of certain failure models in material elements.

Other motivations of particular importance relate to the development of metamaterials. A tensile unstable structural element can indeed be used as the *building block* of a metamaterial. Introductions to the general idea of metamaterials can be found in [18], while in [19–22], an overview of recent developments in this field is provided.

Unstable structures can be used to generate the desired material properties to the macrostructure. For example, the classical bounds (Voigt and Reuss) related to elastic composites, can be exceeded by unstable inclusions and the Poisson coefficient can be practically unlimited in the case of microstructures containing unstable elements ([23–25]). Metamaterials with negative Poisson's coefficient have attracted much interest in recent years for their potential applications in biomedical engineering, aerospace, sensors, etc. Elastic multistable metamaterials are also being developed (see, for example, [26]).

Metamaterials can also be designed for damping and energy absorption applications: indeed, unstable internal structures can be designed

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https://doi.org/10.1016/j.istruc.2024.107403
Received 4 October 2023; Received in revised form 9 September 2024; Accepted 26 September 2024
Available online 8 October 2024

such that the deformation is reversible and the material can absorb multiple impacts without losing its functionality ([27–29]).

The Kirchhoff–Love equations governing the spatial equilibrium of an elastic beam subjected to tension and torsion have been used to study the existence of local instabilities, using the Kirchhoff analogy ([30]).

This paper examines a new system subjected to two different tensile loads: a straight beam with a constant cross-sectional area and stiffness, hinged at one end and simply supported at an intermediate point. Two different tensile load cases are considered on this structure. Firstly, a conservative load acting at the free end consisting of the sum of two non-conservative forces and, secondly, a uniformly distributed conservative load consisting of the sum of two uniformly distributed non-conservative forces. A new analogy is shown between the structure loaded at the free end and the elastic circular arch on spring soil with negative stiffness. A discussion of the results obtained is presented. In particular, it is shown that there is a close relationship between the structures studied by Gajewski and Palej [5], Zaccaria et al. [10] and Feriani and Carini [14].

## 2. Problem position

Consider the structures in Fig. 1. Fig. 1a shows a straight beam of length $L$ with constant cross-section and constant bending stiffness. The beam is hinged at one end and free at the other, where a non-conservative[1] follower force is applied, and is constrained by an intermediate support at a distance $a$ from the hinge ($0 \le a \le L$). For $a \to 0$ the beam reduces to a cantilever beam, also known as the Beck column. In the more general case, that is, with $a \neq 0$, we will refer to the beam as a 'generalized' Beck column. In the case of compressive loads, the stability conditions of this structure have already been extensively studied by Zorii and Chernukha [31] and then by Elishakoff and Hollkamp [32],Elishakoff and Lottati [33],De Rosa and Franciosi [34],Lee [35],Abdullatif and Mukherjee [36]. Fig. 1b shows the same beam as Fig. 1a but with a different non-conservative load. Unlike the previous load, which remains anchored at the free end of the beam and tilts as the tangent to the axis line at the point of application of the load itself, the load in Fig. 1b acts along a fixed straight line. For $a \to 0$ the beam reduces to the well-known Reut column and then we will refer to the beam of Fig. 1b as the 'generalized' Reut column. The structure of Fig. 1c has the same geometrical and material properties as the beams in Fig. 1a and Fig. 1b but with both non-conservative loads applied, and will therefore be refer to as the 'generalized' Beck plus Reut column. In the following, compressive loads $P$ are considered positive.

The 'generalized' Beck (Fig. 1a), Reut (Fig. 1b), and Beck plus Reut (Fig. 1c) columns, in the absence of inertial forces, are governed by the following differential equations with the relevant boundary conditions (second order theory):

$$\text{'generalized' Beck column:}\quad \begin{cases} EJv_i'''' + Pv_i'' = 0 \quad \text{with} \quad i=1 \quad \text{for} \quad 0 \le x \le a \\ \text{and} \quad i=2 \quad \text{for} \quad a \le x \le L \\ v_1(0)=0, \quad v_1''(0)=0, \\ v_2''(L)=0, \quad v_2'''(L)=0 \\ v_1(a)=v_2(a)=0, \quad v_1'(a)=v_2'(a), \quad v_1''(a)=v_2''(a) \end{cases} \tag{1}$$

$$\text{'generalized' Reut column:}\quad \begin{cases} EJv_i'''' + Pv_i'' = 0 \quad \text{with} \quad i=1 \quad \text{for} \quad 0 \le x \le a \\ \text{and} \quad i=2 \quad \text{for} \quad a \le x \le L \\ v_1(0)=0, \quad v_1''(0)=0 \\ v_2''(L)+\frac{P}{EJ}v_2(L)=0, \quad v_2'''(L)+\frac{P}{EJ}v_2'(L)=0 \\ v_1(a)=v_2(a)=0, \quad v_1'(a)=v_2'(a), \quad v_1''(a)=v_2''(a) \end{cases} \tag{2}$$

$$\text{'generalized' Beck plus Reut column:}\quad \begin{cases} EJv_i'''' + 2Pv_i'' = 0 \quad \text{with} \quad i=1 \quad \text{for} \quad 0 \le x \le a \\ \text{and} \quad i=2 \quad \text{for} \quad a \le x \le L \\ v_1(0)=0, \quad v_1''(0)=0 \\ v_2''(L)+\frac{P}{EJ}v_2(L)=0, \quad v_2'''(L)+\frac{P}{EJ}v_2'(L)=0 \\ v_1(a)=v_2(a)=0, \quad v_1'(a)=v_2'(a), \quad v_1''(a)=v_2''(a) \end{cases} \tag{3}$$

where $v(x)$ is the transversal displacement of the centroid of the cross section of the beam, $J$ and $E$ are the moment of inertia of the cross section and the Young modulus, respectively.

For all beams (Beck, Reut and Beck plus Reut), the conditions at $x=0$ establish that both the displacement and the rotation of the beam axis are zero, *i.e.* they establish the presence of a clamped end, while the conditions at $x=a$ establish that the transverse displacement is zero and the continuity of the axis rotation and the bending moment. The conditions at $x=L$ establish that both the bending moment and the shear are zero in the Beck beam, while they define the value of the bending moment equal to $Pv_2(L)$ and the shear equal to $Pv_2'(L)$ in both the Reut beam and the Beck plus Reut beam.

The differential operator, including the boundary conditions, for the Reut rod is the adjoint operator of the Beck rod operator. Therefore, the operator of the Beck plus Reut problem is self-adjoint, *i.e.*

$$\int_0^a \left(EJv_1'''' + 2Pv_1''\right)u_1 \mathrm{d}x + \int_a^L \left(EJv_2'''' + 2Pv_2''\right)u_2 \mathrm{d}x = \int_0^a v_1\left(EJu_1'''' + 2Pu_1''\right)\mathrm{d}x + \int_a^L v_2\left(EJu_2'''' + 2Pu_2''\right)\mathrm{d}x \tag{4}$$

for all $u_1$, $u_2$, $v_1$, $v_2$ satisfying the boundary conditions in Eq. (3).

The self-adjointness of the differential operator is a necessary condition for the operator itself to be the gradient of a potential, *i.e.* for the system to be conservative. This is supported by the analogy shown in the next section. The concepts of differential operator, adjoint operator, self-adjoint operator, potential operator and of conservative system are very well explained, for instance, in [37–39] (see also Appendix A for a very basic exposition of these concepts).

In this paper, the structures in Fig. 2 were also analysed: they have the same geometrical and material properties as the beams in Fig. 1 but with uniformly distributed, rather than concentrated, loads. The stability of the beam of Fig. 2a, in the case $a \to 0$, was analysed by Leipholz [40] (a column clamped at one end with a uniformly distributed, tangential load), and will therefore be referred to as the 'generalized' Leipholz beam. Fig. 2b shows the same beam as in Fig. 2a but with a distributed load that always keeps the line of action coincident with the axis of the undeformed beam.

On the structure in Fig. 2c both the non-conservative loads of structures in Fig. 2a and 2b are applied simultaneously. The beam in Fig. 2c will be referred to as the 'generalized' symmetrized Leipholz column. The 'generalized' symmetrized Leipholz column, in the absence of inertial forces, is governed by the following differential equations with relevant boundary conditions:

$$\text{'generalized' symmetrized Leipholz column:}\quad \begin{cases} EJv_i'''' + 2p(l-x)v_i'' = 0 \quad \text{with} \quad i=1 \quad \text{for} \quad 0 \le x \le a \\ \text{and} \quad i=2 \quad \text{for} \quad a \le x \le L \\ v_1(0)=0, \quad v_1''(0)=0 \\ v_2''(L)=0, \quad v_2'''(L)=0 \\ v_1(a)=v_2(a)=0, \quad v_1'(a)=v_2'(a), \quad v_1''(a)=v_2''(a). \end{cases} \tag{5}$$

This last new elastic system is also conservative.

[1] Let us define a conservative force by the condition that its work in any admissible displacement of the system on which it acts depends solely on the initial and final configuration of the system ([3]).

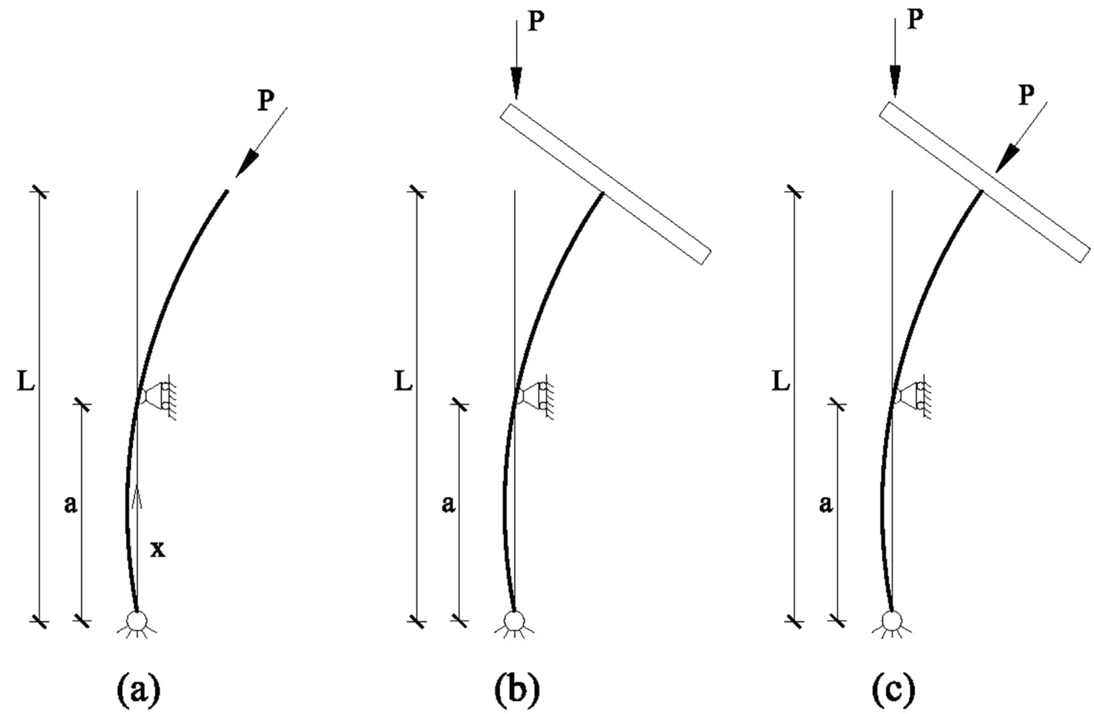



**Fig. 1.** Beam hinged at one end, free at the other end, simply supported at an intermediate point, and subjected to (a) a follower force $P$ at the free end ('generalized' Beck column); (b) a force $P$ which is always directed along the initial undeformed axis, with a fixed direction line of action ('generalized' Reut column); (c) a conservative load as sum of the previous non-conservative loads ('generalized' Beck plus Reut column).

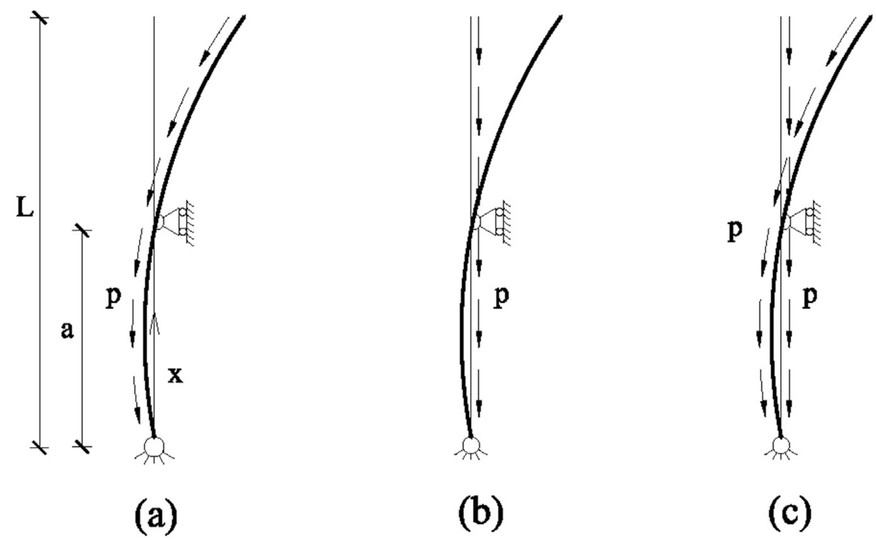



**Fig. 2.** Beam hinged at one end, free at the other end, simply supported at an intermediate point, and subjected to (a) a uniformly distributed follower force $p$; (b) a uniformly distributed force $p$ with a fixed direction; (c) a conservative load as sum of the previous two non-conservative loads.

## 3. An analogy between the 'generalized' Reut plus Beck column and the elastic circular arch

*Analogies are not simply a formal fact but a structural fact. Analogies reveal the existence of an underlying structure, but they do not explain the reason. Our scientific activity is strongly based on the use of analogies. We use them, implicitly or explicitly, because they are like roads already traced on the land we are investigating. Analogy is a fundamental tool for knowledge because it allows us to explore a new field using the established knowledge of another field: we only need to find the homologous entities of the two fields. Analogies are a powerful tool in discovery, learning and teaching, i.e. in the creation and transmission of knowledge. Analogies have another important merit: when dealing with a set of abstract notions, one may find an analogy with a set of concrete notions; in this way the abstract notions can be more easily understood by referring to the concrete ones. The understanding of abstract notions is greatly facilitated by a well chosen analogy!* ([39]).

For the reasons masterfully stated by Tonti, in this paragraph we show an analogy between the stability problem (3) and the statics of a circular arch on spring soil with negative stiffness.

The most famous analogy, developed by Gustav Kirchhoff in 1859 ([41]), is between the dynamics of a rigid body rotating about a fixed point (spinning top) and the deformation of an elastic beam subjected to forces and torques at one end and clamped at the other. The analogy was later extended by Larmor [42] to beams which have curvature and twist in the unstressed state, provided that the components of the initial curvature and initial twist are constants. This is the case, for instance, when the axis of the beam, in the unstressed state, is a circular arch, and the beam is free of twist. In this case, the kinetic analogue is a pendulum with a flywheel symmetrically mounted on its axis.

When a rod which is straight and prismatic in the unstressed state is held bent and twisted by terminal couples, the kinetic analogue is a rigid body moving under no forces. The analogue has been worked out in detail by Hess [43].

Kirchhoff's analogy has recently regained interest in the study of the stability of beams subjected to follower forces (see, for instance, [44]).

Some analogies among simple structural problems (including the Reut column) and the mass–spring system can be useful for studying the stability of rods subjected to non-conservative loads.

Consider, for instance, the set of analogous physical problems governed by the following differential equation with *initial conditions* in the unknown function $v(\xi)$:

$$\frac{d^2 v(\xi)}{d\xi^2} + \alpha^2 v(\xi) = f(\xi) \quad \text{for} \quad 0 \le \xi \le \bar{\xi}$$
$$v(0) = v_0, \quad \left[\frac{dv}{d\xi}\right]_0 = \dot{v}_0 \tag{6}$$

where the parameter $\alpha$ (independent from $\xi$), the function $f(\xi)$, the initial values $v_0$ and $\dot{v}_0$, and the domain boundary $[0, \bar{\xi}]$ of the independent variables $\xi$ are known. Together with the specific values of the relevant physical parameters, in Table 1 some structural problems governed by Eq. (6) are depicted in the case of homogeneous initial conditions. This means that there is an analogy among all the problems listed in Table 1, which implies that any specific property of the solution of each problem can be extended to all the other problems.

For example, the structure number 3 of Table 1 is identified as the Reut column in the case where only the axial load $P$ is present. From the analogy with the mass–spring problem, number 1 in the Table, it immediately follows that problem 3 (second order theory) has only the trivial solution whatever the value of the compressive load $P$. The same can be said for the Beck column, which is identical to the Reut column but with external loads and constraint reactions exchanged roles. The same result can be obtained immediately by analogy with the arch structure (first order theory), number 4 of Table 1. As it is well known, problem 1 has a unique solution even in the case of negative spring stiffness $K$, and, by analogy, problem 3, with tensile load $P$, also has a unique solution. This means that, in the case of the presence of only the tensile load $P$, problem 3 admits only the trivial solution.

Another example of using the analogy between problem 1 (mass–spring) and problem 3 (Reut rod) concerns the variational formulation of the two problems. It is well known that the dynamic mass–spring problem and, more generally, the elastodynamics problem, has a variational formulation of the convolutive type, first introduced by Gurtin ([45]) and then, in simplified form, by Tonti ([46]):

$$\mathcal{F}(u) = \underset{\hat{u}}{\text{stat}}\, \mathcal{F}(\hat{u}) \tag{7}$$

('stat' means 'stationarity') for any $\hat{u}$ satisfying the initial conditions (here assumed to be homogeneous, for simplicity) and with

$$\mathcal{F}(\hat{u}) = \frac{1}{2} m \int_0^T \dot{\hat{u}}(T-t)\dot{\hat{u}}(t)dt + \frac{1}{2} K \int_0^T \hat{u}(T-t)\hat{u}(t)dt - \int_0^T F(T-t)\hat{u}(t)dt. \tag{8}$$

By analogy it is then possible to derive the variational convolutive formulation also for the problem 3 of Table 1 (the Reut rod, in the case $M(x) = 0$):

$$\mathcal{F}(v) = \underset{\hat{v}}{\text{stat}}\, \mathcal{F}(\hat{v}) \tag{9}$$

for any $\hat{v}$ satisfying the (homogeneous) initial conditions and with

$$\mathcal{F}(\hat{v}) = \frac{1}{2} EJ \int_0^L \hat{v}'(L-x)\hat{v}'(x)dx + \frac{1}{2} P \int_0^L \hat{v}(L-x)\hat{v}(x)dx + \int_0^L M(L-x)\hat{v}(x)dx. \tag{10}$$

The analogies shown in Table 1 concern problems governed by the second-order differential equation with initial conditions, Eq. (6). Eq. (6) represents the equation of motion of the mass $m$ of problem 1, while for problems 3 and 4 it represents the compatibility equation and the constitutive law, the equilibrium equation being used to define of the respective known terms (bending moments).

Unlike the Reut rod and the Beck rod problems, the Beck plus Reut rod problem is not analogous to the mass–spring problem through the second-order Eq. (6). However, an analogy can be re-established between the Beck plus Reut problem and the arched beam problem, with suitable modifications, by referring to the respective differential equations of the fourth order (and no longer of the second order), *i.e.* by considering as a known term not the bending moment but the transverse external load.

Indeed, consider the arched beam in Fig. 3b. The governing equations of the arch problem in Fig. 3b in terms of radial displacements $w(s)$ only, with the appropriate boundary conditions, and, for the moment, without the bed of radial springs, are as follows (see Appendix B):

$$\text{circular arch problem:} \quad \begin{cases} EJ w_i'''' + \frac{2}{R^2} w_i'' + \frac{1}{R^4} w = 0 \quad \text{with} \quad i = 1 \quad \text{for} \quad 0 \le s \le a \\ \text{and} \quad i = 2 \quad \text{for} \quad a \le s \le L \\ w_1(0) = 0, \quad w_1''(0) = 0 \\ w_2''(L) + \frac{1}{R^2} w_2(L) = 0, \quad w_2'''(L) + \frac{1}{R^2} w_2'(L) = 0 \\ w_1(a) = w_2(a) = 0, \quad w_1'(a) = w_2'(a), \quad w_1''(a) = w_2''(a). \end{cases} \tag{11}$$

We immediately see that these equations and the relevant boundary conditions are very similar to those of the 'generalized' Beck plus Reut column (Fig. 3a) when $1/R^2$, in Eqs. (11), is replaced by $P/EJ$, except for the presence of the term $w/R^4$. Now, considering the new arched beam of Fig. 3b with a bed of radial springs with negative stiffness $K = -EJ/R^4$, the above term $w/R^4$ disappears from the Eqs. (11) and the set of Eqs. (11) becomes the same set of Eqs. (3) dealing with the 'generalized' Beck plus Reut column problem (Fig. 2a). Then, since the modified arch system is conservative, we can say that also the 'generalized' Beck plus Reut column system is conservative, and its total potential energy is analogous to the following deformation energy of the arch in Fig. 3b:

$$U(w) = \frac{1}{2} \int_0^L EJ \left(w'' + \frac{w}{R^2}\right)^2 ds - \frac{EJ}{2R^4} \int_0^L w^2 ds. \tag{12}$$

After replacing $s$ by $x$, $w$ by $v$ and $1/R^2$ by $P/EJ$, we get the following form of the total potential energy of the 'generalized' Beck plus Reut column by analogy:

$$U(v) = \frac{1}{2} \int_0^L EJ \left(v''\right)^2 dx + P \int_0^L v v'' dx \tag{13}$$

which, after an integration by parts, becomes

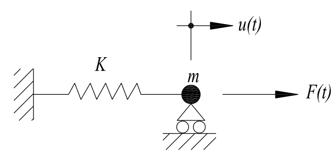
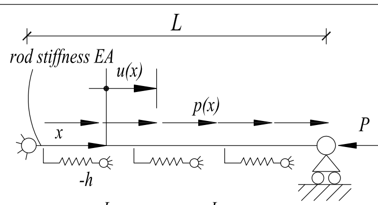
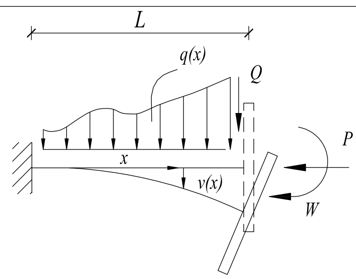
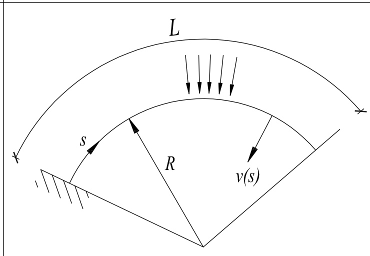
$$U(v) = \frac{1}{2} \int_0^L EJ \left(v''\right)^2 dx + P \int_0^L \left(v'\right)^2 dx + P v(L) v'(L) \tag{14}$$

where we recognize the strain energy of the beam $\frac{1}{2}\int_0^L EJ \left(v''\right)^2 dx$ and the potential energy of the external loads $P \int_0^L \left(v'\right)^2 dx + P v(L) v'(L)$.

## 4. Stability of the new conservative systems

First consider the structure in Fig. 1c. As it can be seen, its load is equal to the sum of the 'generalized' Beck and the 'generalized' Reut loading conditions. The resulting load is conservative; therefore,

**Table 1**
Some examples of structural mechanics problems governed by Eq. (6), in the case of homogeneous initial conditions.

| PROBLEM NUMBER | THE PHYSICAL PROBLEM | $\xi$ | $v$ | $\alpha^2$ | $f$ | $\bar{\xi}$ |
|---|---|---|---|---|---|---|
| 1 |   | TIME $t$ | $u(t)$ | $\frac{K}{m}$ | $\frac{F(t)}{m}$ | FINAL TIME OF THE MOTION $T$ |
| 2 |   $P=\int_0^L p(x)\,dx+\int_0^L hu(x)\,dx$ | SPATIAL COORDINATE $x$ | $u(x)$ | $\frac{h}{EA}$ | $-\frac{p(x)}{EA}$ | LENGTH OF THE ROD $L$ |
| 3 |   | SPATIAL COORDINATE $x$ | $v(x)$ | $\frac{P}{EJ}$ | $-\frac{M(x)}{EJ}$ $M=-\int_x^l (z-x)q(z)dz - Q(L-x) - W$ | LENGTH OF THE BEAM $L$ |
| 4 |   | SPATIAL CURVILINEAL COORDINATE $s$ | $v(s)$ | $\frac{1}{R^2}$ | $-\frac{M(s)}{EJ}$ | ARCH LENGTH $L$ |

the buckling conditions are independent of the mass distribution, and the critical buckling load can be determined using the standard static criterion *i.e.*, looking for equilibrium states other than the trivial one, if they exist. The cases $P > 0$ and $P < 0$ are considered separately.

Case $P > 0$.

As it is well known, the general solution of the differential Eq. (3) is:

$$\begin{cases} v_1(z) = A_1 \cos \alpha z + B_1 \sin \alpha z + C_1 z + D_1 \\ v_2(z) = A_2 \cos \alpha z + B_2 \sin \alpha z + C_2 z + D_2 \end{cases} \tag{15}$$

with $\alpha^2 = P/EJ$. Substituting the Eqs. (15) into the boundary conditions of the problem (3), the following homogeneous system of eight linear algebraic equations, in the unknowns $A_1$, $B_1$, $C_1$, $D_1$, $A_2$, $B_2$, $C_2$, $D_2$, is obtained:

$$\begin{cases} A_1 + D_1 = 0 \\ \alpha^2 A_1 = 0 \\ \alpha^2 \cos(\alpha L) A_2 + \alpha^2 \sin(\alpha L) B_2 - \alpha^2 L C_2 - \alpha^2 D_2 = 0 \\ -\alpha^3 \sin(\alpha L) A_2 + \alpha^3 \cos(\alpha L) B_2 - \alpha^2 C_2 = 0 \\ \cos(\alpha a) A_1 + \sin(\alpha a) B_1 + a C_1 + D_1 = 0 \\ \cos(\alpha a) A_2 + \sin(\alpha a) B_2 + a C_2 + D_2 = 0 \\ -\alpha \sin(\alpha a) A_1 + \alpha \cos(\alpha a) B_1 + C_1 + \alpha \sin(\alpha a) A_2 - \alpha \cos(\alpha a) B_2 - C_2 = 0 \\ -\alpha^2 \cos(\alpha a) A_1 - \alpha^2 \sin(\alpha a) B_1 + \alpha^2 \cos(\alpha a) A_2 + \alpha \sin(\alpha a) B_2 = 0. \end{cases} \tag{16}$$

By equating to zero the determinant of the coefficient matrix of the above linear system, the following equation is obtained:

$$-\ \alpha^2 a(L-a)\cos(\alpha L) + 2\alpha a \sin(\alpha a) + \alpha a \sin(\alpha L)$$
$$+\ \alpha L \sin(\alpha a)\,[\sin(\alpha L)\sin(\alpha a) + \cos(\alpha a)\cos(\alpha L)]$$

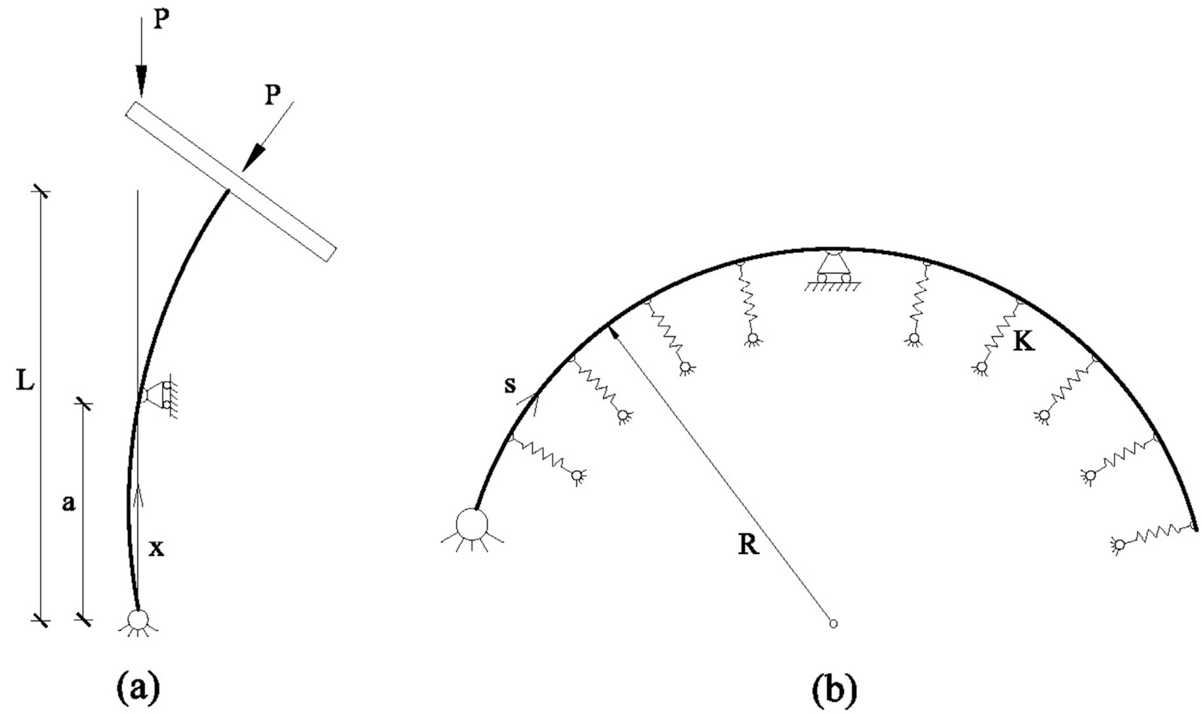



**Fig. 3.** Two analogous systems: (a) the 'generalized' Beck plus Reut column, and (b) the circular arch on a bed of springs with stiffness $K = -EJ/R^4$.

$$+\ \sin(\alpha a)\left[\sin(\alpha a)\cos(\alpha L) - \cos(\alpha a)\sin(\alpha L)\right] = 0 \tag{17}$$

whose roots give the values of the critical loads. Table 2 shows the values of the first ten critical loads for different values of the parameter $c = a/L$. For $c = 1$ the critical loads of the Euler pinned–pinned rod are found. For $c \to 0$ we find the first critical load already identified in [14]. Interestingly, the first critical load is the same for both the pinned–pinned rod ($c = 1$) and the clamped-free rod ($c \to 0$).

Case $P < 0$.

In this case, the general solution of the differential Eqs. (3) is:

$$\begin{cases} v_1(z) = A_1 \sinh\alpha z + B_1 \cosh\alpha z + C_1 z + D_1 \\ v_2(z) = A_2 \sinh\alpha z + B_2 \cosh\alpha z + C_2 z + D_2 \end{cases} \tag{18}$$

where $\alpha^2 = -P/EJ$. Substituting Eq. (18) into the boundary conditions of the problem (3) we obtain:

$$\begin{cases} B_1 + D_1 = 0 \\ \alpha^2 B_1 = 0 \\ \alpha^2 \sinh(\alpha L) A_2 + \alpha^2 \cosh(\alpha L) B_2 - \alpha^2 L C_2 - \alpha^2 D_2 = 0 \\ \alpha^3 \cosh(\alpha L) A_2 + \alpha^3 \sinh(\alpha L) B_2 - \alpha^2 C_2 = 0 \\ \sinh(\alpha a) A_1 + \cosh(\alpha a) B_1 + a C_1 + D_1 = 0 \\ \sinh(\alpha a) A_2 + \cosh(\alpha a) B_2 + a C_2 + D_2 = 0 \\ \alpha \cosh(\alpha a) A_1 + \alpha \sinh(\alpha a) B_1 + C_1 - \alpha \cosh(\alpha a) A_2 - \alpha \sinh(\alpha a) B_2 - C_2 = 0 \\ \alpha^2 \sinh(\alpha a) A_1 + \alpha^2 \cosh(\alpha a) B_1 - \alpha^2 \sinh(\alpha a) A_2 - \alpha \cosh(\alpha a) B_2 = 0. \end{cases} \tag{19}$$

By equating to zero the determinant of the coefficient matrix of the above system, the following equation is obtained:

$$\begin{aligned} &-\ \alpha^2 a(L-a)\cosh(\alpha L) + 2\alpha a \sinh(\alpha a) + \alpha a \sinh(\alpha L) \\ &+\ \alpha L \sinh(\alpha a)\left[\cosh(\alpha L)\cosh(\alpha a) - \sinh(\alpha a)\sinh(\alpha L)\right] \\ &+\ \sinh(\alpha a)\left[\sinh(\alpha a)\cosh(\alpha L) - \cosh(\alpha a)\sinh(\alpha L)\right] = 0 \end{aligned} \tag{20}$$

whose roots give the values of the critical loads. Unlike the case of compressive loading, in this case there is only one root for each value of $c$. Table 3 shows the critical load values for different values of the parameter $c = a/L$.

The same analysis applied to the problem in Fig. 1c can be extended to the problem in Fig. 2c. Unfortunately, the latter problem requires a numerical analysis, typically involving the finite element method. The beam is divided into standard straight Euler–Bernoulli finite elements with two nodes, neglecting the axial strain. Each node has only two degrees of freedom (transversal displacement and rotation), with cubic interpolation functions. As usual, each element has a constant cross-section and homogeneous material. Stiffness matrices are computed dividing the beam into 10 finite elements to ensure accuracy.

Tensile buckling loads are shown in Table 3 for different values of the parameter $c$, namely for different positions of the intermediate support. The tensile buckling is possible for $0 \le c < 1$. As $c$ increases, the critical load also increases. For $c \to 0$, the value of the dimensionless critical load tends to $-21.389$ as in [14]. Table 4 details the dimensionless critical tensile load obtained for $c = 0.5$ as a function of the number $n$ of finite elements used.

## 5. Link with earlier works

In the case of $a \to 0$, *i.e.* a cantilever beam, and in the case of concentrated loads at the free end, the results presented here can be related to the previous results of Gajewski and Palej [5], Zaccaria et al. [10] and Feriani and Carini [14].

Gajewski and Palej [5] have considered a cantilever beam of variable cross-section with yielding constraint described by a rotational spring of stiffness $\psi$ and loaded by a tensile force at the free end of the cantilever. The tensile force is assumed to be inclined with respect to the tangent of the axis line at the end point of the angle $\chi$ and displaced of $e^*$ from the end point of the axis line (see Fig. 4a).

Gajewski and Palej [5] assume that $e^*$ and $\chi$ are functions of both the transverse displacement of the free end $v(L)$ and the rotation $v'(L)$ of the free end and, through the four parameters $\rho$, $\theta$, $\mu$, $\nu$, in the following form:

$$\begin{cases} e = \rho y'(1) + \theta y(1) \\ \chi = \mu y'(1) + \nu y(1) \end{cases} \tag{21}$$

**Table 2**
Values of dimensionless critical compressive loads $\tilde{P}_i = P_{cr_i} L^2/2EJ$ related to the structure in Fig. 1c.

| $c = \frac{a}{L}$ | $\tilde{P}_1$ | $\tilde{P}_2$ | $\tilde{P}_3$ | $\tilde{P}_4$ | $\tilde{P}_5$ | $\tilde{P}_6$ | $\tilde{P}_7$ | $\tilde{P}_8$ | $\tilde{P}_9$ | $\tilde{P}_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| → 0 | 9.87 | 31.33 | 88.84 | 149.90 | 246.77 | 347.34 | 483.74 | 623.73 | 799.54 | 979.08 |
| 0.1 | 11.33 | 36.11 | 101.73 | 172.27 | 281.24 | 396.83 | 546.30 | 704.97 | 888.65 | 1086.06 |
| 0.2 | 13.11 | 42.32 | 114,88 | 194.61 | 293.35 | 409.94 | 535.21 | 706.72 | 885.49 | 1082.90 |
| 0.3 | 15.23 | 50.05 | 117.25 | 194.34 | 283.15 | 414.84 | 546.09 | 695.31 | 888.49 | 1085.91 |
| 0.4 | 17.60 | 57.68 | 104.17 | 197.52 | 296.29 | 399.49 | 552.01 | 708.52 | 880.24 | 1078.33 |
| 0.5 | 19.65 | 59.41 | 105.12 | 187.89 | 296.17 | 414.68 | 540.53 | 700.31 | 888.36 | 1085.80 |
| 0.6 | 20.06 | 58.32 | 178.43 | 282.65 | 414.48 | 551.83 | 870.76 | 1070.94 | 1302.67 | 1538.79 |
| 0.7 | 18.12 | 59.64 | 118.13 | 196.81 | 278.73 | 538.07 | 708.89 | 888.36 | 1085.79 | 1299.84 |
| 0.8 | 15.06 | 56.81 | 118.70 | 197.53 | 296.30 | 414.02 | 540.38 | 678.13 | 848.85 | 1053.78 |
| 0.9 | 12.16 | 48.32 | 107.67 | 189.06 | 291.17 | 412.71 | 552.71 | 711.08 | 888.55 | 1085.96 |
| 1 | 9.87 | 39.48 | 157.91 | 246.74 | 355.31 | 483.61 | 631.65 | 799.44 | 1194.22 | 1667.96 |

**Table 3**
Values of dimensionless critical tensile loads related to the structure in Fig. 1c (on the left) and in Fig. 2c (on the right).

| $c = \frac{a}{L}$ | $\frac{P_{cr} L^2}{2EJ}$ | $c = \frac{a}{L}$ | $\frac{P_{cr} L^3}{2EJ}$ |
|---|---|---|---|
| → 0 | −5.76 | → 0 | −21.39 |
| 0.1 | −6.64 | 0.1 | −26.65 |
| 0.2 | −7.84 | 0.2 | −34.82 |
| 0.3 | −9.55 | 0.3 | −48.45 |
| 0.4 | −12.12 | 0.4 | −73.10 |
| 0.5 | −16.33 | 0.5 | −122.14 |
| 0.6 | −24.05 | 0.6 | −233.34 |
| 0.7 | −40.74 | 0.7 | −544.79 |
| 0.8 | −88.35 | 0.8 | −1827.31 |
| 0.9 | −343.01 | 0.9 | −15134.24 |
| 1 | −∞ | 1 | −∞ |

**Table 4**
Values of dimensionless critical tensile loads related to the structure in Fig. 2c for various discretizations and for $c = 0.5$ ($n$ is the number of finite elements).

| $n$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| $p_{cr} L^3/2EJ$ | −129.2980 | −123.2868 | −122.3634 | −122.1920 | −122.1435 |

where $y = v/L$, $x = \xi/L$, $y' = \frac{dy}{dx}$, $e = e^*/L$, and $v$ is the transverse displacement of the cantilever (here, the notation is that of Gajewski and Palej [5]).

Gajewski and Palej [5] studied the stability of the cantilever under tension in the case of constant cross section, by using the static criterion, and, therefore, the results are valid only for those parameters for which the static stability criterion can be applied. This is obviously always the case for conservative loading behaviour, *i.e.* when $\theta = 1 - \mu$, and in some cases of non-conservative loading. In any case, Gajewski and Palej [5] have found the general conditions that the parameters must satisfy in order for the static stability criterion to be applicable.

Gajewski and Palej [5] studied a number of special cases: 1. Tensile force applied at the end of the beam and with a line of action passing through a fixed point of the undeformed axis of the beam; 2. Force with variable direction and point of application (case coinciding with that studied by Zaccaria et al. [10]); 3. Non-conservative force applied to the end of the beam; 4. Dead conservative force whose direction and point of application remain unchanged, attached to a rigid arm (example reported by Ziegler [3]); 5. Rod loaded by opposite forces attached to a rigid arm (example reported by Ziegler [3]).

It is easy to verify that the structure in Fig. 4a, for some values of the parameters $\rho$, $\theta$, $\mu$, $\nu$, and for $\psi \to \infty$ (infinite spring stiffness), coincides in geometry and loading with half of the structure in Fig. 4b, a structure studied by Zaccaria et al. [10]. Moreover, these structures are related to the structure in Fig. 4c studied by Feriani and Carini [14]. In fact, the internal slider actions in Fig. 4b coincide with the loads in Fig. 4a and differ from the loads in Fig. 4c only by a 'dead' load $P$ (see Fig. 5).

In turn, the results in [14] coincide with those presented here in the case of $a \to 0$.

## 6. Conclusions

A structural system consisting of a straight beam with constant stiffness, hinged at one end and resting at an intermediate point at a given distance from the hinge, subjected to two different loading conditions was studied: a load applied at the free end, consisting of two non-conservative forces, one follower and one always maintained the same direction; another a load uniformly distributed along the beam, that also consisting of two non-conservative distributed loads, of the same nature as the first two concentrated ones. The loads were applied in both compression and tension.

In the case of tensile loads, only one critical buckling load is always obtained for each applied load condition and for each position of the intermediate support. Therefore, there is no intermediate support position that marks the transition between tensile buckling (always of Eulerian type) and no tensile buckling, for any value of the load. Only for the pinned–pinned beam, there is no tensile instability. In the case of the cantilever beam, the results obtained by Gajewski and Palej [5], Zaccaria et al. [10] and Feriani and Carini [14] were compared. For certain values of the parameters, the structures of Zaccaria et al. [10] and Gajewski and Palej [5] coincide while diverging from that of Feriani and Carini [14] for a conservative 'dead' load.

## CRediT authorship contribution statement

**F. Levi:** Writing – original draft, Software, Methodology, Formal analysis, Data curation, Conceptualization. **A. Carini:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## Acknowledgements

This work was conducted as part of a research project funded by the Italian Ministry of University, and Research (MUR). The authors wish to thank Professor Francesco Genna and Dr. Francesco Ancellotti for their helpful suggestions.

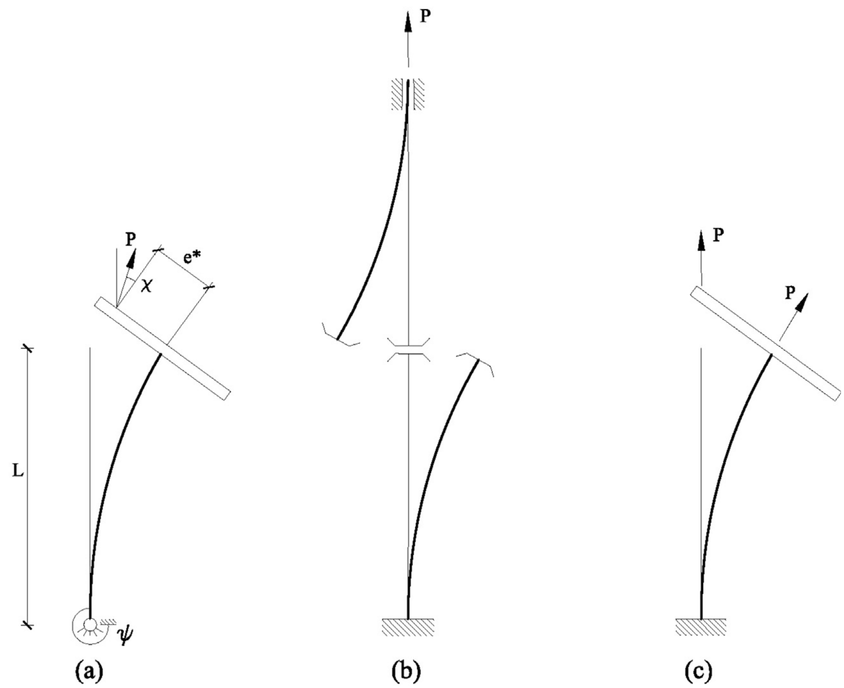



**Fig. 4.** (a) Gajewski and Palej [5] beam; (b) System under tensile 'dead' loading composed by two inextensible elastic rods clamped at one end and jointed through a 'slider' (Zaccaria et al. [10]); (c) Beck plus Reut beam studied by Feriani and Carini [14].

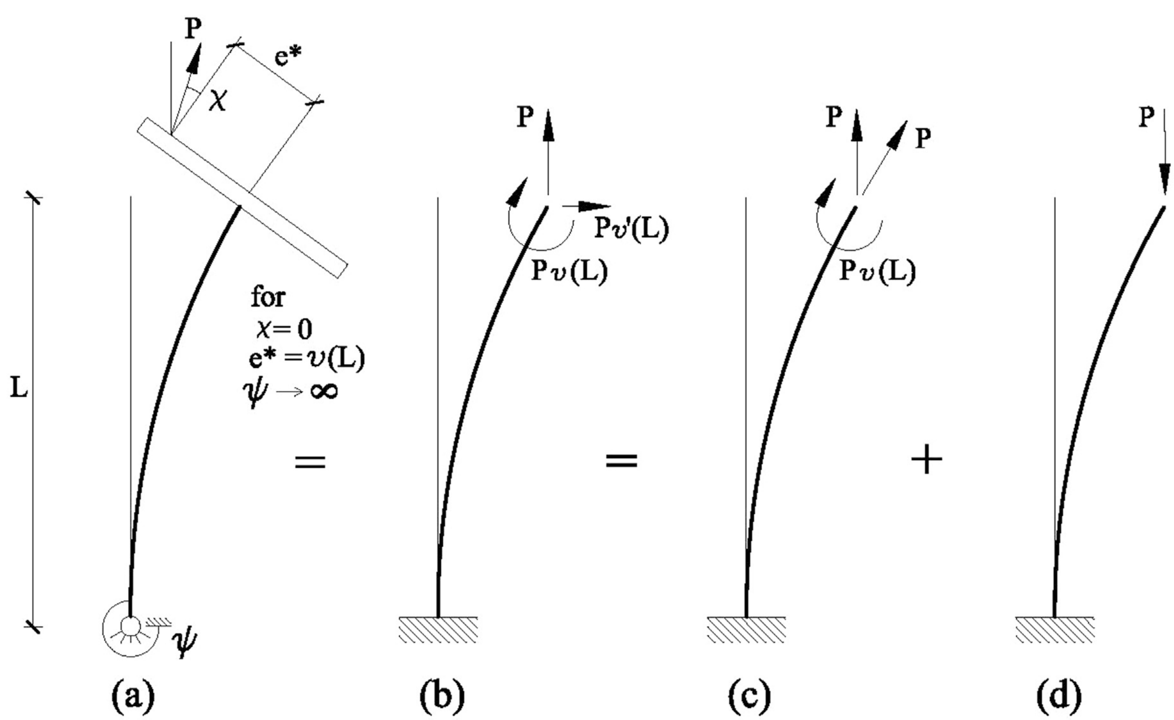



**Fig. 5.** (a) Gajewski and Palej [5] beam; (b) 'Slider' internal actions of the structure of Fig. 4b ([10]); (c) Beck plus Reut beams ([14]); (d) Beam subjected to a 'dead' conservative load.

## Appendix A

(Differential operator, adjoint operator, self-adjoint operator, potential operator, potential energy)
*Scalar product of two functions*.

The *scalar product* of the two functions $u(P)$ and $v(P)$ defined over a given domain $\Omega$ (namely $P \in \Omega$) is the integral of the product of these functions taken over the given domain, and is usually denoted by the symbol $(u, v)$ so that

$$(u, v) = \int_{\Omega} u(P)v(P)\mathrm{d}\Omega. \tag{A.1}$$

The scalar produc has the following properties

$$(u, v) = (v, u); \quad (u, u) > 0 \quad \text{for all} \quad u \neq 0. \tag{A.2}$$

*Differential operator*.

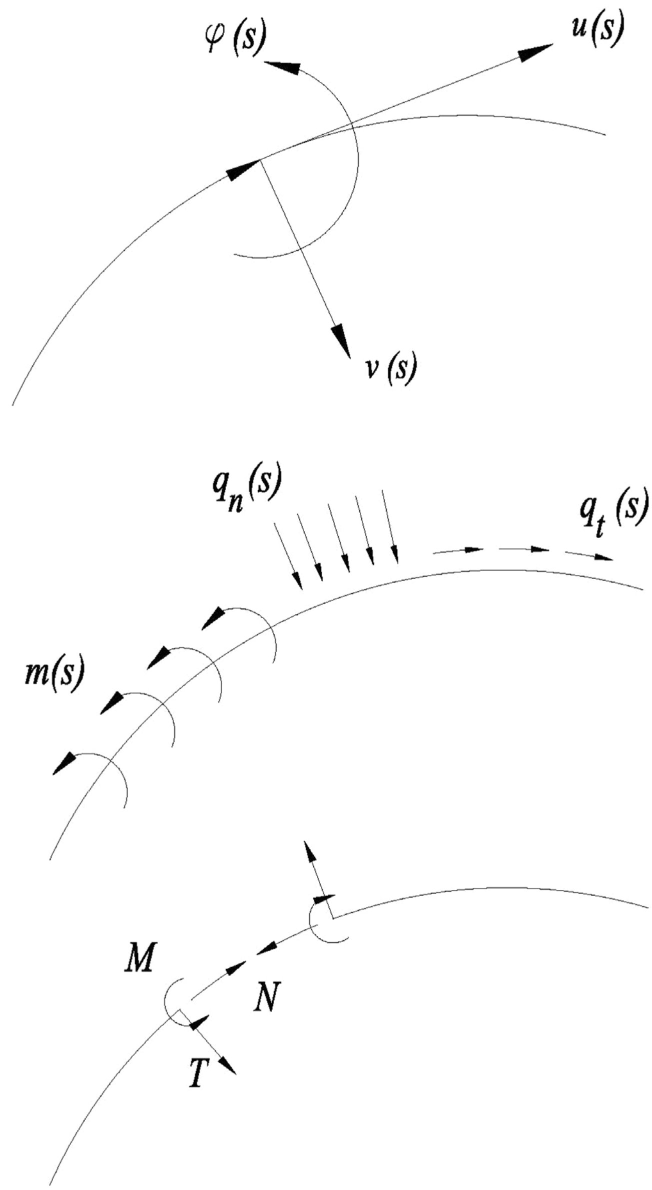



**Fig. 6.** Adopted notation for the governing equation of the elastic circular arch.

An *operator* (or transformation or mapping) $A$ is a correspondence which assigns to each function of a set of functions a unique function of a new set. The set of elements $u$ which satisfies the given initial or boundary conditions and the given functional class is called the *domain* of the operator, denoted as $D(A)$. The set of elements $v = A(u)$ constitutes the *range* of the operator, denoted as $R(A)$. An operator containing the operation of differentiation is called a *differential operator*.

A set of functions is called linear if it has the following property: if the functions $u$ and $v$ belong to the given set, then the functions $u + v$ and $\alpha u$, (where $\alpha$ is an arbitrary constant), also belong to it.

*Linear operator and linear problem*.

The operator $L$ is called linear if its field of definition $D(L)$ is a linear set and if $L(\alpha u_1 + \beta u_2) = \alpha L u_1 + \beta L u_2$ for all scalars $\alpha$ and $\beta$ and for all $u_1$ and $u_2 \in D(L)$.

A linear problem can be written in the following form:

$$Lu = f \tag{A.3}$$

where $L$ is a *linear operator*, $u$ is the unknown function and $f$ is the data, *i.e.* a known function.

*Adjoint operator*.

The linear operator $L^*$ is said to be the *adjoint* of the linear operator $L$ if

$$(Lu, v) = (u, L^* v) \tag{A.4}$$

for all $u$, $v$ belonging to the domain $D(L)$ of the operator $L$. For example, the adjoint operator of the following linear differential operator $L_1$:

$$L_1 = \left\{ \frac{\mathrm{d}}{\mathrm{d}x}; \quad x \in [a, b]; \quad u(a) = 0 \right\} \tag{A.5}$$

can be obtained by integration by parts as follows:

$$\int_a^b \frac{\mathrm{d}u(x)}{\mathrm{d}x} v(x)\mathrm{d}x = [u(x)v(x)]_a^b + \int_a^b -\frac{\mathrm{d}v(x)}{\mathrm{d}x} u(x)\mathrm{d}x. \tag{A.6}$$

Since $u(a) = 0$, in order to eliminate the boundary term in Eq. (A.6), we obtain the adjoint operator $L_1^*$:

$$L_1^* = \left\{ -\frac{\mathrm{d}}{\mathrm{d}x}; \quad x \in [a, b]; \quad v(b) = 0 \right\}. \tag{A.7}$$

*Self-adjoint operator*.

A linear operator $L$ is said to be *self-adjoint*, if for any two functions $u$ and $v$ from its domain, the following identity holds

$$(Lu, v) = (u, Lv). \tag{A.8}$$

As an example, we consider the following linear differential operator $L_2$:

$$L_2 = \left\{ \frac{d^2}{dx^2}; \quad x \in [a, b]; \quad u(a) = 0; \quad u(b) = 0 \right\}. \tag{A.9}$$

By doing a double integration by parts, we get:

$$\int_a^b \frac{d^2u(x)}{dx^2} v(x)dx = \left[\frac{du(x)}{dx} v(x)\right]_a^b - \left[u(x)\frac{dv(x)}{dx}\right]_a^b + \int_a^b u(x)\frac{d^2v(x)}{dx^2}dx. \tag{A.10}$$

Since $u(a) = 0$ and $u(b) = 0$, in order to eliminate the boundary terms, we obtain the adjoint operator $L_2^*$:

$$L_2^* = \left\{ -\frac{d^2}{dx^2}; \quad x \in [a, b]; \quad v(a) = 0; \quad v(b) = 0 \right\}. \tag{A.11}$$

Since $L_2^* = L_2$, $L_2$ is a self-adjoint operator.

*Potential operators.*

By a *line* in the space of functions $u \in D(L)$ we mean the one-parameter family of functions $\eta(P, \lambda)$ where $\lambda$ is a real parameter.

The *circulation* between two elements $u_0$ and $u_1$ of the domain $D(L)$ of an operator $L(u)$ along the elements of a line connecting $u_0$ and $u_1$, refers to the quantity

$$\Gamma = \int_{u_0}^{u_1} \int_{\Omega} L(\eta(P, \lambda)) \frac{\partial \eta(P, \lambda)}{\partial \lambda} d\Omega d\lambda \tag{A.12}$$

the integral on $\Omega$ simply expresses a scalar product of two functions, while the integral on $\lambda$ is the analogue of the integral of the circulation in an ordinary vector field.

Suppose the domain $D(L)$ is convex. If the circulation (A.12) does not depend on the line connecting two points, we say that $L$ is a *potential operator*.

The condition for potentialness is (Volterra symmetry condition)

$$(Lu, v) = (u, Lv) \tag{A.13}$$

namely, for it to be a potential operator, $L$ must be a self-adjoint operator .

*Variational formulation for linear problems.*

If $S$ is a linear self-adjoint operator, the solution of the linear problem $Su = f$ makes the following quadratic functional stationary:

$$F(u) = \frac{1}{2}(Su, u) - (f, u). \tag{A.14}$$

The condition of stationarity

$$\delta F(u) = 0 \tag{A.15}$$

is equivalent to the problem $Su = f$. In this case we say that the problem $Su = f$ admits a variational formulation; the equation $Su = f$ is called the Euler–Lagrange equation of the functional $F(u)$ which, in turn, is called the *potential* of the operator $S$.

*Conservative system.*

We say *material system*, or simply *system*, a discrete or continuous set of material points. On a material system can act: forces external to the system and forces internal to the system, active forces and reactions. If all the forces acting on the system are conservative, the system is referred to as a conservative system. For example, in an Euler stability problem all reactions are nonworking, and both the external dead load $P$ and the elastic stresses (the internal loads) are conservative. If the system is conservative, the total work of the internal and external forces can be represented by a potential energy.

## Appendix B

(The governing equations of elastic circular arches. First order theory.)

Under the hypotheses of small strains, displacements and curvature, the governing equations of elastic circular arches with radius $R$ are (see Fig. 6):

*Compatibility*

$$\frac{du}{ds} - \frac{v}{R} = \epsilon \tag{B.1}$$

$$\frac{dv}{ds} + \frac{u}{R} = -\phi + t \tag{B.2}$$

$$\frac{d\phi}{ds} = \chi \tag{B.3}$$

where $u(s)$ and $v(s)$ are the tangential and radial components of the displacement of the cross section at $s$, respectively, $\phi(s)$ is the absolute rotation of the cross section, $\epsilon(s)$, $t(s)$ are the axial and shear deformation, $\chi(s)$ is the curvature;

*Equilibrium*

$$\frac{dN}{ds} - \frac{T}{R} + q_t = 0 \tag{B.4}$$

$$\frac{dT}{ds} + \frac{N}{R} + q_n = 0 \tag{B.5}$$

$$\frac{dM}{ds} - T + m = 0 \tag{B.6}$$

where $N$, $T$ are the axial and shear actions at $s$, while $M$ is the internal bending moment, $q_t$ and $q_n$ are the axial and radial components of the distributed external loads, while $m$ is the external moment per unit length along $s$;

*Constitutive laws*

$$\epsilon = \frac{N}{EA} \tag{B.7}$$

$$t = \frac{T}{GA_*} \tag{B.8}$$

$$\chi = \frac{M}{EJ} \tag{B.9}$$

where $E$ is the Young modulus, $G$ is the shear elastic modulus, while $A$ is the cross section area, $J$ is the cross section inertia moment, $A_*$ is the reduced shear area.

When the assumptions of vanishing shear deformation and circular arches are made, *i.e.*:

$$t \simeq 0 \tag{B.10}$$

$$R = const \tag{B.11}$$

the above governing equations become:

*Compatibility*

$$\frac{d^2u}{ds^2} + \frac{u}{R^2} = \frac{d\epsilon}{ds} - \frac{\phi}{R} \tag{B.12}$$

$$\frac{d^2v}{ds^2} + \frac{v}{R^2} = -\chi - \frac{\epsilon}{R} \tag{B.13}$$

*Equilibrium*

$$\frac{d^2N}{ds^2} + \frac{N}{R^2} = -\frac{dq_t}{ds} - \frac{q_n}{R} \tag{B.14}$$

$$\frac{d^2T}{ds^2} + \frac{T}{R^2} = \frac{q_t}{R} - \frac{dq_n}{ds} \tag{B.15}$$

$$\frac{d^3M}{ds^3} + \frac{1}{R^2}\frac{dM}{ds} = -\frac{m}{R^2} + \frac{q_t}{R} - \frac{dq_n}{ds} - \frac{d^2m}{ds^2} \tag{B.16}$$

*Constitutive law*

$$\epsilon = \frac{N}{EA} \tag{B.17}$$

$$\chi = \frac{M}{EJ}. \tag{B.18}$$

Finally, when the inextensibility assumption is made (*i.e.* $\epsilon = 0$), the whole set of the governing equation in terms of $u$ and $v$ is given by:

$$\frac{d^2 v}{ds^2} + \frac{v}{R^2} = -\frac{M}{EJ} \tag{B.19}$$

$$\frac{d^3 u}{ds^3} + \frac{1}{R^2}\frac{du}{ds} = -\frac{1}{R}\frac{M}{EJ} \tag{B.20}$$

$$\frac{du}{ds} - \frac{v}{R} = 0. \tag{B.21}$$

The same set, written in terms of the radial displacement $v$ only, reduces (when $\epsilon = 0$, $t = 0$, $R = const$ and $EJ = const$) to:

$$\frac{d^5 v}{ds^5} + \frac{2}{R^2}\frac{d^3 v}{ds^3} + \frac{1}{R^4}\frac{dv}{ds} = \frac{1}{EJ}\left(\frac{m}{R^2} - \frac{q_t}{R} + \frac{dq_n}{ds} + \frac{d^2 m}{ds^2}\right) \tag{B.22}$$

which reduces, when $m = 0$ and $q_t = 0$, to:

$$\frac{d^4 v}{ds^4} + \frac{2}{R^2}\frac{d^2 v}{ds^2} + \frac{1}{R^4}v = \frac{1}{EJ}q_n. \tag{B.23}$$

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