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Integral Representations and Zeros of the Lommel Function and the Hypergeometric $_1F_2$ Function

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Abstract. We give different integral representations of the Lommel function $s_{\mu,\nu}(z)$ involving trigonometric and hypergeometric $_2F_1$ functions. By using classical results of Pólya, we give the distribution of the zeros of $s_{\mu,\nu}(z)$ for certain regions in the plane (μ, ν). Further, thanks to a well known relation between the functions $s_{\mu,\nu}(z)$ and the hypergeometric $_1F_2$ function, we describe the distribution of the zeros of $_1F_2$ for specific values of its parameters.

Keywords. Lommel functions, hypergeometric functions, integral representations, distribution of zeros.

1. Introduction

The Lommel function $s_{\mu,\nu}(z)$ is a particular solution of the inhomogeneous Bessel differential equation:

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = z^{\mu+1}. \tag{1}$$

More precisely, $s_{\mu,\nu}(z)$ is the solution of Eq. (1) satisfying $s_{\mu,\nu}(z) \sim \frac{z^{\mu+1}}{(\mu+1)^2-\nu^2}$ $(1 + O(z^2))$ around $z = 0$ when $\mu \pm \nu$ is not an odd negative integer. The explicit Taylor series of $s_{\mu,\nu}(z)$ around $z = 0$ is given by [24]

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu+1)^2 - \nu^2} \left(1 - \frac{z^2}{(\mu+3)^2 - \nu^2} + \frac{z^4}{((\mu+3)^2 - \nu^2)((\mu+5)^2 - \nu^2)} + \cdots\right) \tag{2}$$

The previous expression can be equivalently written in terms of hypergeometric ${}_1F_2$ function as [24]

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu+1)^2 - \nu^2}\,{}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right) \tag{3}$$

Notice that the function ${}_1F_2$ in (3), or $z^{-\mu-1}s_{\mu,\nu}(z)$, is an entire function of z with order of growth equal to 1. The Lommel function has many applications in mathematical physics and applied sciences: for example, it is fundamental in the description of piezoelectricity phenomena [1], optics [2], radiative transfer processes in the atmosphere [3], mechanics [17], elastic scattering theory [22].

In this paper we will analyze the distribution of the zeros of the function $s_{\mu,\nu}(z)$. The identification of the values (μ, ν) giving a non negative function $s_{\mu,\nu}(z)$ on the real axis and the description of the location of the zeros have been investigated by many authors. In [5] Cooke showed that if $\nu \geq 0$ and $\nu \leq \mu - 1$ then $s_{\mu,\nu}(z) > 0$ for $z > 0$. Equally, if $\nu \geq 1/2$ and $\nu \leq \mu$ $s_{\mu,\nu}(z) > 0$ for $z > 0$, unless $\mu = \nu = 1/2$ when $s_{\mu,\nu} \geq 0$ for $z > 0$. Successively, Steinig [19] showed that $s_{\mu,\nu}(z) > 0$ for $z > 0$ if $|\nu| < \mu$ and $\mu \geq 1/2$ except when $|\nu| = \mu = 1/2$, where one has $s_{\mu,\nu}(z) \geq 0$ for $z > 0$. Also, Steinig established the following properties of $s_{\mu,\nu}(z)$: if $\mu < 1/2$ or if $\mu = 1/2$ and $|\nu| > 1/2$ then $s_{\mu,\nu}(z)$ has an infinite number of zeros for $z \in (0, \infty)$. An extension of the positivity results of Cooke and Steinig has been given more recently by Cho & Chung [4], since they showed that $s_{\mu,\nu} > 0$ also in the region $\mu > 1/2$ and $\nu^2 \leq (\mu+1)^2 - 2$, whereas $s_{\mu,\nu} \geq 0$ for $\mu = |\nu| = 1/2$, giving also interlacing properties among the zeros of $s_{\mu,\nu}$ and those of the Bessel function J_ν for certain values of the parameters (μ, ν). In [12] Koumandos and Lamprecht give estimates about the location of the zeros of $s_{\mu,\nu}(z)$ for $\mu \in (-1/2, 1/2)$ and $\nu = 1/2$, whereas in [11] Koumandos, extends these to $\mu \in (-3/2, 1/2)$, $\mu \neq -1/2$ and $\nu = 1/2$. For more properties of the Lommel functios the reader can look for example at [14], Ch. 11.9 or [24], Ch. 10.7.

One of the tools that we will use is a result due to Pólya [15] about the zeros of entire functions possessing suitable integral representations. The theorem is the following:

Theorem 1.1 [Pólya, 1918 [15]]. *Suppose that the function $f(t)$ is positive and not decreasing in $(0, 1)$. Then the functions of z defined by*

$$V(z) = \int_0^1 \sin(zt)f(t)dt, \quad U(z) = \int_0^1 \cos(zt)f(t)dt \tag{4}$$

possesses only real zeros. Further, if $f(t)$ grows steadily these zeros are simple and the intervals $(k\pi, (k+1)\pi)$, $k = 1, 2, \ldots$ contain the positive zeros of $V(z)$, whereas the intervals $(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2})$, $k = 0, 1, 2, \ldots$ contain the zeros of $U(z)$, each interval containing just one zero in both cases.

By growing steadily it is meant that the function is not piecewise constant with a finite number of rational points of discontinuity in $(0, 1)$.

2. Some integral representations and comments

In this section we will report some known facts about integral representations of the function $s_{\mu,\nu}(z)$. In particular, in 1936 Szymanski [21] got an integral representation for the Lommel function in the case $\nu-\mu$ is an even positive integer. Actually, Szymanski firstly considers Eq. (1) for imaginary argument:

$$z^2\frac{d^2T}{dz^2}+z\frac{dT}{dz}-(z^2+\nu^2)T=z^{\mu+1}. \tag{5}$$

and noticed that the solution of (5) satisfying $T=t_{\mu,\nu}(z)\sim\frac{z^{\mu+1}}{(\mu+1)^2-\nu^2}(1+O(z))$ can be expressed as

$$t_{\mu,\nu}=z^{\mu+1}\int_0^1\cosh(zt)W_{\mu,\nu}(t)dt, \tag{6}$$

where $W_{\mu,\nu}$ is the solution of the following differential equation

$$(1-t^2)\frac{d^2W_{\mu,\nu}}{dt^2}+(2\mu-1)t\frac{dW_{\mu,\nu}}{dt}+(\nu^2-\mu^2)W_{\mu,\nu}=0, \tag{7}$$

with the following boundary conditions

$$W(1)=0, \quad W'(0)=-1. \tag{8}$$

The solution of Eq. (7) with the boundary conditions (8) is written by Szymanski in terms of the Gegenbauer functions C_n^k:

$$W_{\mu,\nu}(t)=-(1-t^2)^{\mu+\frac{1}{2}}\frac{C^{\mu+1}_{\nu-\mu-1}(t)}{\left(C^{\mu+1}_{\nu-\mu-1}(t)\right)'_{t=0}}. \tag{9}$$

where, for integers values of $\nu-\mu-1$ (odd values, for even positive integers the denominator vanishes), the functions C_n^k are defined by the generating function $(1-2xt+x^2)^{-k}$, i.e. $C_n^k(t)$ is the coefficients of x^n in the expansion of $(1-2xt+x^2)$ in powers of x:

$$(1-2xt+x^2)^{-k}=\sum_{n=0}C_n^k(t)x^n. \tag{10}$$

For the solution $s_{\mu,\nu}(z)$ of (1) equally one finds, for $\nu-\mu$ even integers:

$$s_{\mu,\nu}=z^{\mu+1}\int_0^1\cos(zt)W_{\mu,\nu}(t)dt, \tag{11}$$

where $W_{\mu,\nu}(t)$ is again given by the expression (9).

In [25] another integral representation of the function $s_{0,v}(z)$ is given. In particular, by a direct check, it is possible to show that the function

$$s_{0,\nu}(z)=\frac{1}{1+\cos(\pi\nu)}\int_0^\pi\sin(z\sin(t))\cos(\nu t)dt. \tag{12}$$

solves Eq. (1) when $\mu = 0$ with $s_{0,\nu}(z) \sim \frac{z}{1-\nu^2}(1 + O(z))$. Equivalently, the function $s_{0,\nu}(z)$ can be also represented as

$$s_{0,\nu}(z) = \frac{1}{1+\cos(\pi\nu)} \int_0^1 \sin(zt) \frac{\cos(\nu \arcsin(t)) + \cos(\nu\pi - \nu \arcsin(t))}{\sqrt{1-t^2}} dt, \tag{13}$$

where the range of arcsin is $(-\pi/2, \pi/2)$. It is possible to show that the function multiplying $\sin(zt)$ in (13) is positive and increasing for $|\nu| < 1$ and $t \in (0, 1)$. It is then possible to apply the Theorem (1.1) to get the following

Corollary 2.1. *The function $s_{0,\nu}(z)$, for $|\nu| < 1$, possesses only real zeros. The zeros are simple and the intervals $(k\pi, (k+1)\pi)$, $k = 0, 1, 2, \ldots$ contain the non-negative zeros of $s_{0,\nu}(z)$, each interval containing just one zero.*

In the next section we will extend the integral representation (12) and the results described in Corollary (2.1) to other values of μ. Actually, the existence of a formula generalizing (12), given by formula (21) below, has been pointed out to us after the writing of this work: it is a special case of an equality given by Prudnikov et al. [16]. For the sake of completeness, we re-derive such formula in a manner that is closer to what Szymanski did in [21] and to what is reported in formula (13). Some comments on the formula given in [16] will be also given. The application to this formula to get a generalization of (2.1) seems instead to be new to us.

3. Other integral representations and the zeros

In this section we will assume that $\mu \pm \nu$ is not an odd negative integer so that the series (2) is defined. By generalizing the case $\mu = 0$ considered in [25], we make the following ansatz for the function $s_{\mu,\nu}(z)$:

$$s_{\mu,\nu}(z) = z^{\mu} \int_0^1 \sin(zt) f_{\mu,\nu}(t) dt. \tag{14}$$

We assume that the functions $f_{\mu,\nu}(t)$ are finite in $z = 0$ whereas may be unbounded for $t \to 1$. By inserting (14) in (1) we get that the functions $f_{\mu,\nu}(t)$ must be the solutions of the following differential equations

$$(1-t^2)\frac{d^2 f_{\mu,\nu}}{dt^2} + (2\mu - 3)t\frac{df_{\mu,\nu}}{dt} + (\nu^2 - (\mu-1)^2) f_{\mu,\nu} = 0, \tag{15}$$

with the boundary conditions:

$$f_{\mu,\nu}(0) = 1, \quad \lim_{t\to 1^-} (1-t) f_{\mu,\nu}(t) = 0,$$
$$\lim_{t\to 1^-} \left((1-t^2)\frac{df_{\mu,\nu}}{dt} + (2\mu - 1)t f_{\mu,\nu} \right) = 0. \tag{16}$$

The general solution of Eq. (15) can be represented in terms of hypergeometric functions as:

$$A(1-t)^{\mu-\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\mu+\frac{1}{2};\frac{1-t}{2}\right)$$
$$+B\,{}_2F_1\left(1-\nu-\mu,1+\nu-\mu;\frac{3}{2}-\mu;\frac{1-t}{2}\right) \tag{17}$$

where A and B are two arbitrary constants. The condition $\lim_{t\to 1^-}(1-t)f_{\mu,\nu}(t) = 0$ gives

$$\mu+\frac{1}{2}>0, \tag{18}$$

whereas the series around $t=1$ of the function $(1-t^2)\frac{df_{\mu,\nu}}{dt}+(2\mu-1)tf_{\mu,\nu}$ is given by

$$(1-t^2)\frac{df_{\mu,\nu}}{dt}+(2\mu-1)tf_{\mu,\nu}$$
$$\sim\left(A(1-t)^{\mu+\frac{1}{2}}\frac{\nu^2-\mu^2}{2\mu+1}+B(2\mu+1)\right)(1+O(1-t)), \tag{19}$$

and results in $B=0$. Finally, from the boundary condition (16) $f_{\mu,\nu}(0)=1$ we get

$$A\,{}_2F_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\mu+\frac{1}{2};\frac{1}{2}\right)=1. \tag{20}$$

The previous result is summarized in the following.

Proposition 3.1. *For $\mu>-\frac{1}{2}$ the Lommel function $s_{\mu,\nu}(z)$ possess the following integral representation:*

$$s_{\mu,\nu}(z)=z^\mu\int_0^1\sin(zt)f_{\mu,\nu}(t)dt$$
$$=z^\mu\int_0^1\frac{\sin(zt)}{(1-t)^{\frac{1}{2}-\mu}}\frac{{}_2F_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\mu+\frac{1}{2};\frac{1-t}{2}\right)}{{}_2F_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\mu+\frac{1}{2};\frac{1}{2}\right)}dt. \tag{21}$$

Notice that the function ${}_2F_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\mu+\frac{1}{2};\frac{1-t}{2}\right)$ is proportional to the Ferrers functions $P_{\nu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(t)$. Indeed, by definition, one has [16]

$$P_\nu^\mu(t)=\frac{1}{\Gamma(1-\mu)}\left(\frac{1+t}{1-t}\right)^{\frac{\mu}{2}}\,{}_2F_1\left(-\nu,\nu+1,1-\mu,\frac{1-t}{2}\right), \tag{22}$$

giving

$$s_{\mu,\nu}(z)=z^\mu\int_0^1\frac{\sin(zt)}{(1-t^2)^{\frac{1}{4}-\frac{\mu}{2}}}\frac{P_{\nu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(t)}{P_{\nu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(0)}dt. \tag{23}$$

The previous equation can be derived also from a formula given in [16]. Indeed one has (see [16] equation 2.17.9.1)

$$\int_0^1 \frac{t^{\alpha-1}}{(1-t^2)^{\frac{\mu}{2}}} \sin(zt) P_\nu^\mu(t) dt = z\frac{\sqrt{\pi}}{2^{\alpha-\mu+1}} \frac{\Gamma(\alpha+1)}{\Gamma(\frac{2+\alpha-\mu-\nu}{2})\Gamma(\frac{3+\alpha-\mu+\nu}{2})} \\ {}_2F_3\left(\frac{\alpha+1}{2}, \frac{\alpha}{2}+1; \frac{3}{2}, \frac{2+\alpha-\mu-\nu}{2}, \frac{3+\alpha-\mu+\nu}{2}; -\frac{z^2}{4}\right) \tag{24}$$

By setting $\alpha = 1$ and changing $\mu \to -\mu + \frac{1}{2}$ and $\nu \to \nu - \frac{1}{2}$, Eq. (24) gives (23) via (3). In this work, however, since we are going to use an old result of Hurwitz [9] on the zeros of hypergeometric functions, we prefer to keep the integral (21) as it is.

We are interested in the monotonicity of the integrand $f_{\mu,\nu}(t)$ for $t \in (0, 1)$ and $\mu > -1/2$. We notice that the derivative of $f_{\mu,\nu}$ with respect to t is proportional again to a hypergeometric function:

$$\frac{df_{\mu,\nu}}{dt} = \frac{1-2\mu}{2(1-t)^{\frac{3}{2}-\mu}} \frac{{}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; \frac{1-t}{2}\right)}{{}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu+\frac{1}{2}; \frac{1}{2}\right)}. \tag{25}$$

The previous relation can be written in a more compact form as

$$\frac{df_{\mu,\nu}}{dt} + a_{\mu,\nu} f_{\mu-1,\nu} = 0, \tag{26}$$

where we set

$$a_{\mu,\nu} \doteq 2\frac{\Gamma\left(\frac{\mu+1+\nu}{2}\right)\Gamma\left(\frac{\mu+1-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)}. \tag{27}$$

Notice that $a_{\mu,\nu}$ is related to the integral of $f_{\mu,\nu}(t)$ between 0 and 1: indeed, from (26) we get for $\mu + 1/2 > 0$

$$\int_0^1 f_{\mu,\nu}(t) dt = \frac{1}{a_{\mu+1,\nu}} = \frac{\Gamma\left(\frac{\mu+1+\nu}{2}\right)\Gamma\left(\frac{\mu+1-\nu}{2}\right)}{2\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma\left(\frac{\mu-\nu}{2}+1\right)}. \tag{28}$$

The previous quantity is finite under the given assumptions that $\mu \pm \nu$ is not an odd negative integer.

As regards the monotonicity, from (26) we see that the zeros of $\frac{df_{\mu,\nu}}{dt}$ between 0 and 1/2 are the zeros of ${}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; \frac{1-t}{2}\right)$, since the constant coefficient is zero only for $\mu + \frac{1}{2} = -n$, $n = 0, 1, 2, ...$ as can be seen directly from the explicit expression

$${}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu+\frac{1}{2}; \frac{1}{2}\right) = 2^{1/2-\mu}\sqrt{\pi}\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2})} \tag{29}$$

The number of zeros of the hypergeometric functions ${}_2F_1(a, b; c; x)$ for $x \in (0, 1)$ have been analyzed by Klein [10] and Hurwitz [9]. For completeness, we

TABLE 1. The number of zeros N of ${}_2F_1(a, b; c; x)$ for $x \in (0, 1)$, $a > b$, $c \leq a + b$

a	b	c	N
+	+	+	0
+	+	−	$\frac{1+(-1)^{\lfloor -c \rfloor}}{2}$
+	−	+	$1 + \lfloor -b \rfloor$
+	−	−	$\lfloor -b \rfloor - \lfloor -c \rfloor$, for $\lfloor -b \rfloor > \lfloor -c \rfloor$
+	−	−	$\frac{1-(-1)^{\lfloor -b \rfloor + \lfloor -c \rfloor}}{2}$, for $\lfloor -b \rfloor \leq \lfloor -c \rfloor$
−	−	−	$\frac{1+(-1)^{\lfloor -a \rfloor + \lfloor -b \rfloor + \lfloor -c \rfloor}}{2}$

report these results as given by Hurwitz. They are summarized in Table (1) for $a > b$, $c \leq a + b$[1]

We notice however that from Table 1 we can provide only partial results about the region of the parameters (ν, μ) where the function ${}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; \frac{1-t}{2}\right)$ is free from zeros for $t \in (0, 1)$. Indeed, the Table 1 gives this information for $t \in (-1, 1)$. We need to extend the results of Hurwitz and Klein and look at the zeros of ${}_2F_1(a, b; c; x)$ in the region $x \in (0, 1/2)$, i.e. the zeros of ${}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; \frac{1-t}{2}\right)$ for $t \in (0, 1)$. To this aim, the quadratic (in the independent variable) functional identities for the hypergeometric functions are very useful. The identity we need is the following (see [7], formula 41 at page 120):

$$\frac{{}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; x\right)}{(1-2x)(1-x)^{\mu-3/2}} = {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu-\frac{1}{2}; 4x(1-x)\right) \quad (30)$$

where $x \in (0, 1/2)$. Equation (30) provides an explicit relation between the zeros of ${}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu-\frac{1}{2}; x\right)$ and the zeros of ${}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; x\right)$. In particular it follows that if ${}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu-\frac{1}{2}; x\right)$ is free from zeros for $x \in (0, 1)$, ${}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \mu-\frac{1}{2}; x\right)$ is free from zeros for $x \in (0, 1/2)$. By looking at table (1) for ${}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu-\frac{1}{2}; x\right)$ we get the following

Proposition 3.2. *For $\mu > -1/2$ the function $f_{\mu,\nu}(t)$ is monotonic for $t \in (0, 1)$ iff the following conditions on the parameters are satisfied:*

- $\mu > 1/2$ *and* $|\nu| \in (0, \mu)$.
- $\mu < 1/2$ *and* $|\nu| \in (|\mu|, \mu + 2)$

The region in the plane (μ, ν) corresponding to monotonic function $f_{\mu,\nu}(t)$ for $t \in (0, 1)$ is illustrated in figure 1.

[1] These choices are not restrictive, since a and b can be interchanged and, for $c > a + b$ one can use the equivalence ${}_2F_1(a, b; c; x) = (1-x)^{c-a-b}_2 F_1(c-a, c-b; c; x)$.

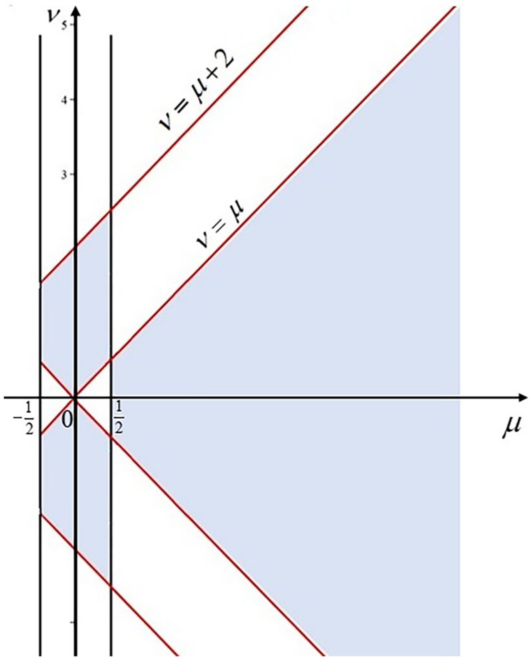

FIGURE 1. The region of monotonicity of $f_{\mu,\nu}(t)$ for $t \in (0,1)$

The function $f_{\mu,\nu}(t)$ can be decreasing or increasing for $t \in (0,1)$. Since $f_{\mu,\nu}(0) = 1$, by looking at what happens in a neighborhood of $t = 1$ it is simple to get the following

Proposition 3.3. *For $\mu > -1/2$ the function $f_{\mu,\nu}(t)$ has the following monotonicity properties going from $t = 0$ to $t = 1$:*

- *For $\mu > 1/2$ and $|\nu| \in (0, \mu)$ the function is decreasing from 1 to 0.*
- *For $\mu < 1/2$ and $|\nu| \in (|\mu|, \mu + 1)$ the function is increasing from 1 to $+\infty$.*
- *For $\mu < 1/2$ and $|\nu| \in (\mu + 1, \mu + 2)$ the function is decreasing from 1 to $-\infty$.*

From Proposition (3.3) and Theorem (1.1) we immediately get the following

Corollary 3.4. *Apart the branch point at $z = 0$, the function $s_{\mu,\nu}(z)$ for $\mu \in (-1/2, 1/2)$ and $|\nu| \in (|\mu|, \mu+1)$ possesses only real zeros. The zeros are simple and the intervals $(k\pi, (k + 1)\pi)$, $k = 1, 2, \ldots$ contain the non-negative zeros, each interval containing just one zero.*

The previous generalizes the Corollary (2.1) given in [25]. Further, since for $\mu > 1/2$ and $|\nu| \in (0, \mu)$ the function $f_{\mu,\nu}(t)$ is positive and decreasing, we get the Corollary:

Corollary 3.5. *The function $s_{\mu,\nu}(z)$ for $\mu > 1/2$ and $|\nu| \in (0, \mu)$ is positive on the positive real axis and possesses only complex zeros.*

Corollary (3.5) is not new, it has been given by [19] and then appears in [6]. The proof given here is however simpler. The zeros of $s_{\mu,\nu}(z)$ can be better characterized by looking at the asymptotic expansion of the hypergeometric ${}_1F_2$ function, related to $s_{\mu,\nu}(z)$ by the formula (3). Indeed, by using the results given in [13] and with the aid of (3) we get, for z real and positive, the asymptotic relation

$$\frac{s_{\mu,\nu}(z)}{z^{\mu+1}} \sim \frac{1}{z^2}\left(1 + O\left(\frac{1}{z}\right)\right) + \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)}{4\sqrt{\pi}}\left(\frac{2}{z}\right)^{\mu+\frac{3}{2}} \cos\left(z - \frac{\pi}{2}\left(\mu + \frac{3}{2}\right)\right)\left(1 + O\left(\frac{1}{z}\right)\right) \tag{31}$$

and if $\mu < 1/2$ we see that the zeros are asymptotically given by $z_k \sim \pi\left(k + \frac{2\mu+5}{4}\right)$, for k suitable large integers.

The behavior of $f_{\mu,\nu}(t)$ for $t \in (0, 1)$, $\mu \in (-1/2, 1/2)$ and $|\nu| \in (\mu+1, \mu+2)$ gives the following result: yhe function $c - f_{\mu,\nu}(t)$ is positive and increasing for $t \in (0, 1)$ for any $c \geq 1$. Indeed, if we introduce the function

$$V_{\mu,\nu}(z) \doteq z^{\mu} \int_0^1 (c - f_{\mu,\nu}(t)) \sin(zt) dt = z^{\mu-1} c(1 - \cos(z)) - s_{\mu,\nu}(z), \tag{32}$$

we can apply the Pólya Theorem (1.1) and get the following

Corollary 3.6. *For any $c \geq 1$, the function $V_{\mu,\nu}(z) = cz^{\mu-1}(1-\cos(z))-s_{\mu,\nu}(z)$ for $\mu \in (-1/2, 1/2)$ and $|\nu| \in (\mu+1, \mu+2)$ possesses only real zeros. The zeros are simple and the intervals $(k\pi, (k+1)\pi)$, $k = 1, 2, \ldots$ contain the non-negative zeros, each interval containing just one zero.*

Actually, (21) is not the only integral representation in terms of a trigonometric kernel. Indeed, it is also possible to give a cosine integral representation. Let us assume $\mu > 1/2$. Then, by integrating by parts Eq. (21) we get

$$s_{\mu,\nu} = z^{\mu}\left(\frac{1}{z} - \frac{a_{\mu,\nu}}{z}\int_0^1 \cos(zt) f_{\mu-1,\nu}(t)\right), \tag{33}$$

where we used Eq. (26). Let us set

$$c_{\mu,\nu} \doteq z^{\mu+1} a_{\mu+1,\nu} \int_0^1 \cos(zt) f_{\mu,\nu}(t). \tag{34}$$

Equation (34) is well defined for $\mu > -1/2$. Again, by integrating by parts (34) and by using the equation $a_{\mu+1,\nu}a_{\mu,\nu} = (\mu^2 - \nu^2)$ we get, for $\mu > +1/2$

$$c_{\mu,\nu} = z^{\mu}\left(\mu^2 - \nu^2\right)\int_0^1 \sin(zt)f_{\mu-1,\nu}(t) = (\mu^2 - \nu^2)zs_{\mu-1,\nu}. \tag{35}$$

From (35) it follows that, for $\mu > -1/2$

$$s_{\mu,\nu} = \frac{1}{z((\mu+1)^2 - \nu^2)}c_{\mu+1,\nu} = \frac{a_{\mu+2,\nu}}{((\mu+1)^2 - \nu^2)}z^{\mu+1}\int_0^1 \cos(zt)f_{\mu+1,\nu}(t). \tag{36}$$

Notice that the previous integral representation is still convergent for $\mu > -3/2$. It is also possible to check directly, by using the series for the hypergeometric function around $t = 1$ that indeed the integral in (36) gives $s_{\mu,\nu}$ also for $\mu \in (-3/2, -1/2)$, so we get the following

Proposition 3.7. *For $\mu > -\frac{3}{2}$ the Lommel function $s_{\mu,\nu}(z)$ possesses the following integral representation:*

$$s_{\mu,\nu}(z) = \frac{a_{\mu+2,\nu}}{((\mu+1)^2 - \nu^2)}z^{\mu+1}\int_0^1 \cos(zt)f_{\mu+1,\nu}(t)$$
$$= \frac{z^{\mu+1}}{a_{\mu+1,\nu}}\int_0^1 \cos(zt)f_{\mu+1,\nu}(t). \tag{37}$$

where $a_{\mu,\nu}$ is given in Eq. (27) and $f_{\mu,\nu}(t)$ by Eq. (21).

It is possible to get the integral representation (37) also by directly making the ansatz $s_{\mu,\nu} = z^{\mu+1}\int_0^1 \cos(zt)f(t)$ for some $f(t)$ and then by looking at the differential equation and boundary conditions that $f(t)$ must obey so that $s_{\mu,\nu}$ satisfies Eq. (1), like we did with the integral representation (21). Since in the integral (37) it appears the function $f_{\mu+1,\nu}(t)$, i.e. the same function appearing in the integral (21) but with $\mu \to \mu + 1$, from Proposition (3.7) we get directly the following

Proposition 3.8. *For $\mu > -3/2$ the function $f_{\mu+1,\nu}(t)$ is monotonic for $t \in (0, 1)$ iff the following conditions on the parameters are satisfied:*

- *$\mu > -1/2$ and $|\nu| \in (0, \mu + 1)$.*
- *$\mu < -1/2$ and $|\nu| \in (|\mu + 1|, \mu + 3)$*

To understand the behavior of $\frac{f_{\mu+1,\nu}(t)}{a_{\mu+1,\nu}}$, we look at the end-points $t = 0$ and $t = 1$. For $t = 0$ we get

$$\frac{f_{\mu+1,\nu}(0)}{a_{\mu+1,\nu}} = \frac{1}{a_{\mu+1,\nu}} = \frac{\Gamma\left(\frac{\mu+1+\nu}{2}\right)\Gamma\left(\frac{\mu+1-\nu}{2}\right)}{2\Gamma\left(1+\frac{\mu+\nu}{2}\right)\Gamma\left(1+\frac{\mu-\nu}{2}\right)} \tag{38}$$

For $\mu > -1/2$ $f_{\mu+1,\nu}(1) = 0$, giving a decreasing function for $|\nu| \in (0, \mu + 1)$ since the values of the function (38) is positive in this interval. For $\mu < -1/2$ $f_{\mu+1,\nu}(t)$ diverges at $t = 1$, the sign being negative. Also, for $|\nu| \in (|\mu+1|, \mu+2)$

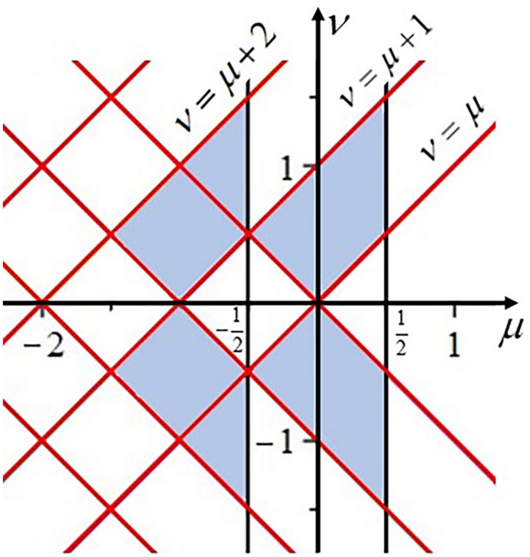

FIGURE 2. The regions described in Corollaries (3.4) and (3.9)

the values of the function (38) are negative, whereas for $|\nu| \in (\mu+2, \mu+3)$ are positive. It follows that the function

$$-\frac{1}{a_{\mu+1,\nu}} f_{\mu+1,\nu}(t) \tag{39}$$

is positive and increasing for $\mu \in (-3/2, -1/2)$ and $|\nu| \in (|\mu+1|, \mu+2)$. The previous results let to expand the region in the (μ, ν) plane for the Corollary (3.4). Indeed we immediately get the following

Corollary 3.9. *Apart the branch point at $z = 0$, the function $s_{\mu,\nu}(z)$ for $\mu \in (-3/2, -1/2)$ and $|\nu| \in (|\mu+1|, \mu+2)$ possesses only real zeros. The zeros are simple and the intervals $(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2})$, $k = 0, 1, 2, \ldots$ contain the non-negative zeros, each interval containing just one zero.*

With the same considerations given after Corollary (3.5) it is possible to show that the zeros in Corollary (3.9) are asymptotically given by $z_k \sim \pi\left(k + \frac{2\mu+5}{4}\right)$ for large k.

As we did with Corollary (3.6), we can introduce the function

$$U_{\mu,\nu}(z) = z^{\mu}\left(c + \frac{1}{a_{\mu+1,\nu}}\right)\sin(z) - s_{\mu,\nu}(z), \tag{40}$$

and, from the Pólya Theorem (1.1) we get the following

Corollary 3.10. *For any $c \geq 0$, the function $U_{\mu,\nu}(z)$ (40) for $\mu \in (-3/2, -1/2)$ and $|\nu| \in (\mu+2, \mu+3)$ or $\mu > -1/2$ and $|\nu| \in (0, \mu+1)$ possesses only real zeros. The zeros are simple and the intervals $(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2})$, $k = 0, 1, 2, \ldots$ contain the non-negative zeros, each interval containing just one zero.*

The Corollaries (3.6) and (3.10) are trivial when the value of c is large and for large real values of z, due to the asymptotic (31). They are less obvious for smaller values of z.

By looking closer at the paper [15], one sees that Theorem (1.1) can be actually extended to linear combinations of the functions $U(z)$ and $V(z)$ as also underline by Pólya himself (see also [20]). Under the same assumptions about $f(t)$ of Theorem (1.1) indeed, one has [20] that the entire function

$$\cos(\theta)U(z) + \sin(\theta)V(z) = \int_0^1 f(t)\cos(zt - \theta)dt, \quad \theta \in (0, \pi) \tag{41}$$

possesses only real simple zeros, each zero belonging to the intervals

$$\left(\left(k - \frac{1}{2}\right)\pi + \theta, \left(k + \frac{1}{2}\right)\pi + \theta\right), \quad k = 0, \pm 1, \pm 2, \ldots \tag{42}$$

Due to the results given in Proposition (3.3), the Corollaries (3.4) and (3.9) can be summarized in the following.

Corollary 3.11. *The function $z^{-\mu}\left(a_{\mu,\nu}\cos(\theta)s_{\mu-1,\nu}(z) + \sin(\theta)s_{\mu,\nu}(z)\right)$ for $\mu \in (-1/2, 1/2)$ and $|\nu| \in (|\mu|, \mu+1)$ possesses only real zeros for any $\theta \in [0, \pi]$. The zeros are simple and each of the intervals (42) contain just one zero.*

Let us finally notice that for particular values of the parameters μ and ν it is possible to get algebraic functions for the kernel of the integral (21). Also, when μ is a positive integer, it is possible to give explicit formulae for the hypergeometric kernel in terms of trigonometric functions. These observations will be further investigated in a separate paper.

4. Some properties of the hypergeometric ${}_1F_2$ function

Pólya [15] and Hille [8] have been two of the few authors to investigate about the distribution of the zeros of the hypergeometric function ${}_1F_2(1; b, c; z)$. For uniformity of notation, we use the same set of parameters for ${}_1F_2$ as in (3). The results of Pólya and Hille can be summarized as follows:

Proposition 4.1. *The function ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ possesses only real zeros for $\nu = 1/2$ and $\mu \in (-5/2, -1/2)$ or $\mu \in (-7/2, -5/2)$. For $\nu = 1/2$ and $\mu > 1/2$ it has only complex zeros. For $\nu = 1/2$ and $\mu < 1/2$ there are infinitely many real zeros.*

Very recently Sokal [18] gives certain conditions ensuring that ${}_pF_q(a_1, ... a_p; b_1, ...b_q; x)$, $p \leq q$ possesses only real, non positive roots. In particular he showed that ${}_pF_q(a_1, ...a_p; b_1, ...b_q; x)$, $p \leq q$ are entire functions belonging to the Laguerre-Pólya class LP^+ (i.e. they can be obtained as a limit, uniformly on compact subsets of $\mathbb{C}$, of a sequence of polynomials with roots on $(-\infty, 0]$) for arbitrarily large $b_{p+1}, ..., b_q$ if and only if, after a possible reordering, the differences $a_i - b_i$, $i = 1, ..., p$, are nonnegative integers. When the differences $a_i - b_i$, $i = 1, ..., p$, are not integers it is still possible for the functions ${}_pF_q$ to belong to the Laguerre-Pólya class LP^+ for some finite values of $b_{p+1}, ...b_q$, as it is shown also by the results of Pólya and Hille stated above.

As far as we know, if one considers the function ${}_1F_2(a_1; b_1, b_2; z)$, there are no results in the literature about two or three dimensional sets of $(a_1, b_1, b_2) \in \mathbb{R}^3$ for which ${}_1F_2(a_1; b_1, b_2; z)$ belong to LP^+ (see also at the end of [18]). However, due to the equivalence (3), any result about the zeros of $s_{\mu,\nu}(z)$ can be directly transferred to ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$. Actually, it would be preferable to investigate the zeros of the function ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ rather then those of $s_{\mu,\nu}(z)$ since ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ is an entire function of z for any choice of the parameters (μ, ν). In this section, for completeness, we report the Corollaries for ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ corresponding to the Corollaries (3.4), (3.5), (3.6), (3.9) and (3.10) for $s_{\mu,\nu}(z)$. Also, for completeness, we will give the corresponding regions in the plane (b_1, b_2) for which ${}_1F_2(a_1; b_1, b_2; z)$ belong to LP^+.

Corollary 4.2. *The function* ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ *for* $\mu \in (-1/2, 1/2)$ *and* $|\nu| \in (|\mu|, \mu + 1)$ *possesses only real negative zeros. The zeros are simple and are contained in the intervals* $(-\frac{k^2\pi^2}{4}, -\frac{(k+1)^2\pi^2}{4})$, $k = 1, 2, \ldots$, *each interval containing just one zero.*

Corollary 4.3. *The function* ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ *for* $\mu \in (-3/2, -1/2)$ *and* $|\nu| \in (|\mu + 1|, \mu + 2)$ *possesses only real negative zeros. The zeros are simple and are contained in the intervals* $(-\frac{(2k+1)^2\pi^2}{16}, -\frac{(2k+3)^2\pi^2}{16})$, $k = 0, 1, 2, \ldots$, *each interval containing just one zero.*

From the previous two Corollaries, by setting $a_1 = 1$, $b_1 = \frac{\mu-nu+3}{2}$, $b_2 = \frac{\mu+\nu+3}{2}$ and by noticing that ${}_1F_2(a_1; b_1, b_2; z)$ is an entire function of fractional order of growth and hence possesses an infinite number of zeros [23], we get the following

Proposition 4.4. *In the regions enclosed by the lines* $b_1 = \frac{1}{2} + \delta$, $b_1 = 1 + \delta$, $b_2 = 1 + \delta$ *and* $b_1 + b_2 = \frac{5}{2} + \delta$, *where* $\delta = 0, \frac{1}{2}$, *and in the regions enclosed by the lines obtained by the exchange* $b_1 \leftrightarrow b_2$, *the function* ${}_1F_2(a_1; b_1, b_2; z)$ *belong to the Laguerre-Pólya class* LP^+.

The regions described in Proposition (4.4) are a rotation and translation of the regions in Fig. 2 and are reported in Fig. 3.

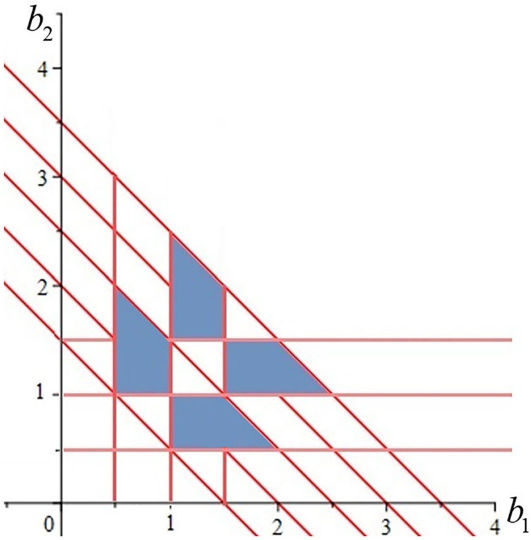

FIGURE 3. The regions (in grey) described in Proposition (4.4)

Finally, we give the following results as consequences of the Corollaries (3.5), (3.6) and (3.10).

Corollary 4.5. *The function* ${}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; z\right)$ *for* $\mu > 1/2$ *and* $|\nu| \in (0, \mu)$ *is positive on the positive real axis and possesses only complex zeros.*

Corollary 4.6. *For any* $c \geq 1$*, the function*

$$c\left((\mu+1)^2 - \nu^2\right) \frac{(1-\cos(z))}{z^2} - {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right) \quad (43)$$

for $\mu \in (-1/2, 1/2)$ *and* $|\nu| \in (\mu+1, \mu+2)$ *possesses only real zeros. The zeros are simple and the intervals* $(k\pi, (k+1)\pi)$*,* $k = 1, 2, \ldots$ *contain the non-negative zeros, each interval containing just one zero.*

Corollary 4.7. *For any* $c \geq 0$*, the function*

$$a_{\mu+2,\nu}(ca_{\mu+1,\nu} + 1)\frac{\sin(z)}{z} - {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right) \quad (44)$$

for $\mu \in (-3/2, -1/2)$ *and* $\nu \in (\mu+2, \mu+3)$ *or* $\mu > -1/2$ *and* $\nu \in (0, \mu+1)$ *possesses only real zeros. The zeros are simple and the intervals* $(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2})$*,* $k = 0, 1, 2, \ldots$ *contain the non-negative zeros, each interval containing just one zero.*

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