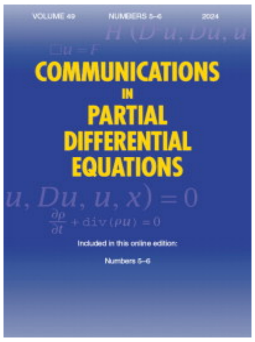

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Initial data identification in space dependent conservation laws and Hamilton-Jacobi equations

Rinaldo M. Colombo[a], Vincent Perrollaz[b], and Abraham Sylla[c]

[a]INdAM Unit & Department of Information Engineering, University of Brescia, Brescia, Italy; [b]Institute Denis Poisson, University of Tours, CNRS UMR 7013, University of Orléans, Orléans, France; [c]Department of Mathematics and Applications, University of Milano – Bicocca, Milan, Italy

ABSTRACT
Consider a Conservation Law and a Hamilton-Jacobi equation with a flux/Hamiltonian depending also on the space variable. We characterize first the attainable set of the two equations and, second, the set of initial data evolving at a prescribed time into a prescribed profile. An explicit example then shows the deep differences between the cases of x-independent and x-dependent fluxes/Hamiltonians.

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1. Introduction

This paper presents a full characterization of the attainable set for conservation laws and for Hamilton-Jacobi equations with space dependent flux / Hamiltonian (see Theorem 3.2). Then, again for both equations, we provide necessary and sufficient conditions for a function to evolve into a given profile at a given time (see Theorem 3.3). These results are obtained under exactly the same assumptions on the flux / Hamiltonian. Finally, we construct an explicit example to highlight the consequences of the explicit space dependence (Theorem 4.1).

As is well known, both Conservation Laws and Hamilton-Jacobi equations generate Lipschitz continuous semigroups whose orbits are solutions, either in the entropy sense or in the viscosity sense. However, the insurgence of singularities implies that these evolutions may not be time reversible, in general. As a result, inverse designs, when non empty, may well display interesting—infinite dimensional—geometric or topological properties.

From a control theoretic point of view, the characterization of inverse designs solves the most elementary controllability problem, thus playing a key role in subsequent developments. Indeed, the first step in the study of inverse designs consists in a full characterization of the attainable sets, i.e., of the profiles leading to non empty inverse designs. In this connection, the current literature offers a few results, typically limited to the x-independent case. We refer the reader to [1] for a characterization of the attainable set for a conservation law (here, with

CONTACT Rinaldo M. Colombo rinaldo.colombo@unibs.it INdAM Unit & Department of Information Engineering, University of Brescia, Brescia, Italy.

boundary); to [2] for a result on the attainable set for Hamilton-Jacobi equations in several space dimensions and to [3] for the case of an x-dependent source term. A triangular system of conservation laws is considered in [4].

Below, we proceed beyond reachable sets and fully characterize inverse designs. More precisely, we consider the conservation law

$$\begin{cases} \partial_t u + \partial_x \left(H(x,u)\right) = 0 & (t,x) \in \,]0,+\infty[\, \times \mathbb{R} \\ u(0,x) = u_o(x) & x \in \mathbb{R} \end{cases} \tag{CL}$$

and the Hamilton-Jacobi equation

$$\begin{cases} \partial_t U + H(x, \partial_x U) = 0 & (t,x) \in \,]0,+\infty[\, \times \mathbb{R} \\ U(0,x) = U_o(x) & x \in \mathbb{R} \end{cases} \tag{HJ}$$

both in the scalar, one dimensional, *non homogeneous*, i.e., x-dependent, case. Denote by

$$\begin{array}{rcccl} S^{CL}: \mathbb{R}_+ \times \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) & \to & \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \\ t \quad , \quad u_o & \mapsto & S^{CL}_t u_o \end{array} \quad \text{and} \quad \begin{array}{rcl} S^{HJ}: \mathbb{R}_+ \times \mathbf{Lip}(\mathbb{R};\mathbb{R}) & \to & \mathbf{Lip}(\mathbb{R};\mathbb{R}) \\ t \quad , \quad u_o & \mapsto & S^{HJ}_t u_o \,, \end{array} \tag{1.1}$$

respectively, the semigroups whose orbits are entropy solutions to (**CL**) and viscosity solutions to (**HJ**), see [5, Section 2.5]. For any positive T and for any assigned profiles $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$, the inverse designs are

$$\begin{array}{rcl} I^{CL}_T(w) & := & \left\{u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \colon S^{CL}_T u_o = w\right\} \quad \text{and} \\ I^{HJ}_T(W) & := & \left\{U_o \in \mathbf{Lip}(\mathbb{R};\mathbb{R}) \colon S^{HJ}_T U_o = W\right\}. \end{array} \tag{1.2}$$

In the homogeneous—x-independent—case, a general characterization of $I^{CL}_T(w)$ and $I^{HJ}_T(W)$ is given in [6]. Other results in this setting are [7], devoted to Burgers' equation also from a numerical point of view; [1], specific to boundary value problems arising in the modeling of vehicular traffic. The multi–dimensional setting is considered in [8], specifically in the case of (**HJ**). A classical reference for analytic techniques used in these papers is [9].

The present non homogeneous case significantly differs from the homogeneous one and significantly less results in the literature are available. The explicit example constructed below shows that when H depends on x (even smoothly), the inverse design $I^{CL}_T(w)$ may have properties substantially different to the general ones that hold in the homogeneous case, see Section 3. In particular, for instance, the results in [6] ensure that in the x-independent case

$$S^{CL}_T \left(\mathbf{L}^\infty(\mathbb{R};\mathbb{R})\right) = \mathbf{cl}_{\mathbf{L}^1} \left(S^{CL}_T \left\{u_o \in \mathbf{C}^1(\mathbb{R};\mathbb{R}) \colon S^{CL}_t u_o \in \mathbf{C}^1(\mathbb{R};\mathbb{R}) \text{ for all } t \in [0,T]\right\}\right),$$

where $\mathbf{cl}_{\mathbf{L}^1}$ stands for the closure in the strong $\mathbf{L}^1(\mathbb{R};\mathbb{R})$ topology, which can be false when H depends on x, as in the case of the example in Section 4. It thus appears that non homogeneous Conservation Laws are, in a sense, *more singular* than homogeneous ones.

Assume $I^{CL}_T(w)$ is non empty. Then, in the x-independent case, the presence of a shock in w (i.e., of a jump discontinuity satisfying Lax entropy conditions, see [10, Section 8.3]) is a necessary and sufficient condition for $I^{CL}_T(w)$ to be infinite or, equivalently, $I^{CL}_T(w)$ is a singleton if and only if w is continuous. More precisely, in the x-independent case, the presence of a shock in w implies that $I^{CL}_T(w)$ is a closed convex cone without extremal faces of finite dimension. On the contrary, in the x-dependent case, we exhibit an example where $I^{CL}_T(w)$ is a singleton although w displays a shock. This is explained in Section 4, where the theory of generalized characteristics, see [11], is deeply exploited. Graphs of the constructed

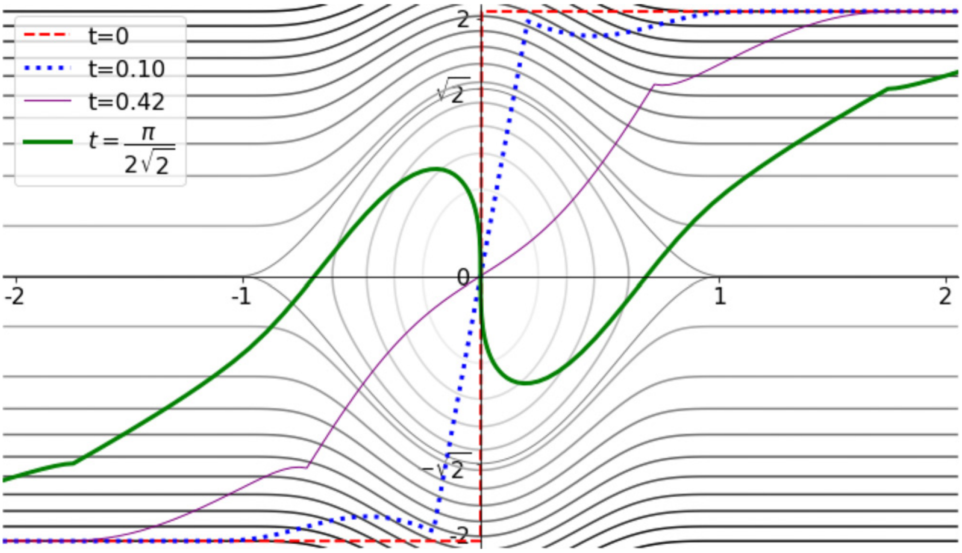

Figure 1. Superposition of a solution to (**CL**) at different times with the orbits of the Hamiltonian system (**HS**). x (or q) is on the horizontal axis and u (or p) on the vertical axis. As proved later in Theorem 4.1, the initial datum (5.28) is the unique one that evolves into the depicted profiles where, at time $T = \pi \Big/ (2\sqrt{2})$, a shock arises.

solution are in Figure 1. The inverse design problem for non convex Hamiltonian is largely open. Interesting techniques referred to (**HJ**) in the x independent case are in [12, 13], that deal also with several space variables.

Further remarks on the consequences of the x dependence in H are deferred to Theorem 4.1 and to the subsequent discussion. Let us also recall the related result [14], limited to the study of the attainable set, where H in (**CL**) consists of a smooth function of u for $x > 0$ and another smooth function of u for $x < 0$, see also the related preprint [15].

While inverse design refers to going back in time, the *dual* approach is connected to the problem of the compactness of the range of the semigroup S_t^{CL}, apparently considered only in the homogeneous case [16], extended in [17] to balance laws, but the case of fluxes depending on the space variable is, to our knowledge, still open.

The next section provides the basic background. Then, on the basis of [5], Section 3 extends to the x-dependent case several classical results, see [6]. On the contrary, the example constructed in Section 4 shows how deep can be the differences between the homogeneous and non homogeneous case. All proofs are deferred to Section 5 and some technical lemmas are collected in the appendices.

2. Notations and definitions

The analytic techniques developed below take advantage of the deep connection between (**CL**) and (**HJ**). We know, on the basis of [5], that both these Cauchy problems are (globally) well posed under the same set of assumptions, namely

$$\textbf{Smoothness: } H \in \mathbf{C}^3(\mathbb{R}^2;\mathbb{R})\,. \qquad \textbf{(C3)}$$

$$\textbf{Compact NonHomogeneity:}\quad \begin{array}{l}\exists X > 0,\ \forall (x,p) \in \mathbb{R}^2,\\ |x| \geq X \implies \partial_x H(x,p) = 0.\end{array} \tag{CNH}$$

$$\textbf{Strong Convexity:}\quad \begin{array}{l}\forall x \in \mathbb{R},\ p \mapsto \partial_p H(x,p) \text{ is an}\\ \text{increasing } \mathbf{C}^1\text{−diffeomorphism}\\ \text{of } \mathbb{R} \text{ onto itself.}\end{array} \tag{CVX}$$

Rather than tackling directly the characterization of the inverse design for (**CL**), we do it for (**HJ**) and use the correspondence to get back to (**CL**).

Assumption (**CVX**) implies that H is strictly convex with respect to the second variable. As is well known, the mappings $x \mapsto -x$ and $H \mapsto -H$ transform the convex case into the concave one, and *vice versa*. Recall that (**CVX**) is a recurrent assumption in the context of (**HJ**) where it allows a connection to optimal control, see [18–20]. On the contrary, the use of Assumption (**CNH**) in conservation laws, to the authors' knowledge, was recently introduced in [5].

It is worth noting that the assumptions (**C3**)–(**CNH**)–(**CVX**) comprise fluxes (Hamiltonians) that do not fit in the classical Kružkov paper [21]. Indeed, following [5, Example 1.1] consider the Hamiltonian

$$H(x,u) := V(x)\, u \left(1 - \frac{u}{R(x)}\right), \tag{2.1}$$

where $V, R \in \mathbf{C}^3(\mathbb{R};\mathbb{R})$ are both strictly positive and with compactly supported derivative. The conservation law (**CL**)–(2.1) describes the time evolution of the density $u = u(t,x)$ of a flow of vehicles along a one-dimensional road that allows a space dependent maximal density $R = R(x)$ and maximal speed $V = V(x)$. It is readily checked that H in (2.1) satisfies (**C3**), and it is strongly concave—analogously to (**CVX**). On the other hand, this H may not meet the assumptions of [21]. In particular, it fails the growth assumption $\sup_{(x,u)\in\mathbb{R}^2} \left(-\partial^2_{xu} H(x,u)\right) < +\infty$, see [21, Formula (4.2)].

Recall the classical definition of entropy solution [21, Definition 1], as tweaked in [5].

Definition 2.1. *Fix $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. A bounded function $u \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R};\mathbb{R})$ is a* solution *to* (**CL**) *if for all test functions $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}_+ \times \mathbb{R};\mathbb{R}_+)$ and for all scalar $k \in \mathbb{R}$:*

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} |u(t,x) - k|\, \partial_t \phi(t,x)\, \mathrm{d}x\, \mathrm{d}t\\
&+ \int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}\left(u(t,x) - k\right)\ \left(H\left(x, u(t,x)\right) - H(x,k)\right)\ \partial_x \phi(t,x)\, \mathrm{d}x\, \mathrm{d}t\\
&- \int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}\left(u(t,x) - k\right)\ \partial_x H(x,k)\, \phi(t,x)\, \mathrm{d}x\, \mathrm{d}t\\
&+ \int_{\mathbb{R}} |u_o(x) - k|\, \phi(0,x)\, \mathrm{d}x \quad \geq \quad 0\,.
\end{aligned}$$

Definition 2.1, taken from by [5, Definition 2.1] is apparently weaker than the classical Kružkov definition since it does not require the *"trace at $t = 0$ condition"* [21, Formula (2.2)]. Nevertheless, under Assumption (**C3**), Definition 2.1 ensures uniqueness and uniform $\mathbf{L}^1_{\mathbf{loc}}$–continuity in time of the solution, as proved in [5, Theorem 2.6].

The following Lemma ensures the existence of left and right traces in the space variable at any point. In the homogeneous—x-independent—case, this is classically obtained through the well known Oleinik estimates [10, Theorem 11.2.1 and Theorem 11.2.2].

Lemma 2.2. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $T > 0$ and $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ so that $I_T^{CL}(w) \neq \emptyset$. Then, for all $x \in \mathbb{R}$, w admits finite left and right traces at x.*

The proof is deferred to Section 5. Once this Lemma is proved, we are able to use Dafermos' techniques based on generalized characteristics from [11], where solutions are however **required** to have traces at each point. Alternatively, another reference is [10, Chapter 10] or [10, Section 11.11] for the inhomogeneous case, but here solutions are **required** to be in **BV**, i.e., to have finite total variation, see [22, Section 2.4].

We now recall the framework of viscosity solutions to (**HJ**), introduced by Crandall–Lions.

Definition 2.3. [23, Definition 5.3] *Let $U \in \mathbf{Lip}([0,T] \times \mathbb{R};\mathbb{R})$ satisfy $U(0) = U_o$.*

(i) U is a subsolution to (**HJ**) *when for all test functions $\phi \in \mathbf{C}^1(]0,T[\times \mathbb{R};\mathbb{R})$ and for all $(t_o,x_o) \in]0,T[\times \mathbb{R}$, if $U - \phi$ has a point of local maximum at the point (t_o,x_o), then $\partial_t\phi(t_o,x_o) + H(x_o, \partial_x\phi(t_o,x_o)) \leq 0$;*
(ii) U is a supersolution to (**HJ**) *when for all test functions $\phi \in \mathbf{C}^1(]0,T[\times \mathbb{R};\mathbb{R})$ and for all $(t_o,x_o) \in]0,T[\times \mathbb{R}$, if $U - \phi$ has a point of local minimum at the point (t_o,x_o), then $\partial_t\phi(t_o,x_o) + H(x_o, \partial_x\phi(t_o,x_o)) \geq 0$.*
(iii) U is a viscosity solution to (**HJ**) *if it is both a supersolution and a subsolution.*

The literature offers a standardized framework for the well posedness of (**CL**), typically referred to the classical paper [21], see also [10]. On the contrary, a wide variety of assumptions are available, where results ensuring the well posedness of (**HJ**) can be proved, see for instance [18–20, 23] or the textbooks [24, Chapter 9], [25, Chapter 10]. Here we recall in particular [26], devoted to the convex case, and [5] where the two equations are considered under the **same** set of assumptions, thus allowing a detailed description of the correspondence between the solutions to the two equations. Indeed, the orbits of the semigroups (1.1) are solution to (**CL**) in the sense of Definition 2.1, respectively (**HJ**) in the sense of Definition 2.3, see [5, Theorems 2.18 and 2.19]. Thanks to their $\mathbf{L}^1_{\mathbf{loc}}$ continuity, both these semigroups a uniquely defined for all $t \in \mathbb{R}_+$.

For any positive T and for any assigned profiles $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$, we first present conditions ensuring that the sets $I_T^{CL}(w)$ and $I_T^{HJ}(W)$ in (1.2) are not empty and then prove geometrical/topological properties. In light of the correspondence $U \to u = \partial_x U$ between S^{HJ} and S^{CL}, see [5, Theorem 2.20] or also [6, 27–29], each of the two characterizations can be deduced from the other one.

As usual, in connection with (**HJ**) and (**CL**), we use of the system of ordinary differential equations

$$\left\{\begin{array}{rcl} \dot{q} &=& \partial_u H(q,p) \\ \dot{p} &=& -\partial_x H(q,p) \end{array}\right. \tag{HS}$$

which we consider equipped with initial or with final conditions. Basic properties of (**HS**) under (**C3**)–(**CNH**)–(**CVX**) are proved in Lemma 5.1 and in the subsequent ones. For a fixed positive T, with reference to (**HS**), we also introduce the set

$$\mathcal{R}_T \coloneqq \left\{q \in \mathbf{C}^1([0,T];\mathbb{R}) \colon \exists\, p \in \mathbf{C}^1([0,T];\mathbb{R}) \text{ such that } (q,p) \text{ solves } (\mathbf{HS})\right\}, \tag{2.2}$$

whose elements we call Hamiltonian rays. For all $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$, so that Lemma 2.2 applies and we can define

$$\begin{array}{rcl} \pi_w : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & q(0), \end{array} \quad \text{where } (q,p) \text{ solves } (\mathbf{HS}) \text{ with datum } \left\{ \begin{array}{l} q(T) = x \\ p(T) = w(x-)\,. \end{array} \right. \tag{2.3}$$

The map π_w assigns to $x \in \mathbb{R}$ the intersection of the minimal backward characteristics emanating from (T,x), see [11, Definition 3.1, Theorems 3.2 and 3.3], with the axis $t = 0$. Lemmas 2.2 and 5.1 ensure that π_w is well defined. Remark that in the x-independent case, all Hamiltonian rays are straight lines, as also any extremal characteristics, a key simplification exploited in [6, Formula (2.3)].

As is well known, thanks to (**CVX**), Hamilton-Jacobi equation (**HJ**) is deeply related and motivated by the search for minima of functionals of the type

$$\begin{array}{rcl} \mathcal{J}_t : \mathbf{W}^{1,1}([0,t];\mathbb{R}) & \longrightarrow & \mathbb{R} \\ y & \longmapsto & \displaystyle\int_0^t L\left(y(s), \dot y(s)\right) \mathrm{d}s + U_o\left(y(0)\right) \end{array} \tag{2.4}$$

where $U_o \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$ and L is the Legendre transform of H in p, i.e.,

$$\begin{array}{rcl} L: \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,v) & \longmapsto & \sup\limits_{p\in\mathbb{R}} \left(p\, v - H(x,p)\right)\,. \end{array} \tag{2.5}$$

As general references for this minimization problem, we refer to [20, Chapter 5], [30, Part III], [25, Chapter 3], see also [31] for a weak—KAM point of view. Below, for detailed proofs about the connection between solutions to (**HJ**) and to minimization problems in our specific functional setting, we refer to the statements in the Appendix B, taken from [26, § 8.3]. Recall, in particular, that U solves (**HJ**) if and only if for all $(T,x) \in [0,+\infty[\times \mathbb{R}$,

$$U(T,x) = \inf_{\substack{\gamma(T)=x \\ \gamma\in\mathcal{R}_T}} \left(\int_0^T L\left(\gamma(s), \dot\gamma(s)\right) \mathrm{d}s + U_o\left(\gamma(0)\right) \right), \tag{2.6}$$

see Lemma B.9. Note moreover that by Lemma B.7

$$U(T,x) = \inf_{\substack{\gamma(T)=x \\ \gamma\in\mathbf{Lip}([0,T];\mathbb{R})}} \left(\int_0^T L\left(\gamma(s), \dot\gamma(s)\right) \mathrm{d}s + U_o\left(\gamma(0)\right) \right).$$

As a first step, we verify that the present assumptions (**C3**)–(**CNH**)–(**CVX**) allow to apply the results in [5], where convexity was relaxed to genuine nonlinearity and uniform coercivity.

Proposition 2.4. *Let H satisfy* (**C3**)–(**CNH**)–(**CVX**)*. Then. the following properties hold:*

$$\textbf{Uniform Coercivity}: \quad \begin{array}{l} \forall\, h \in \mathbb{R} \quad \exists\, \mathcal{U}_h \in \mathbb{R} \colon \forall\, (x,u) \in \mathbb{R}^2 \\ \text{if } |H(x,u)| \le h \text{ then } |u| \le \mathcal{U}_h\,. \end{array} \tag{UC}$$

$$\textbf{Weak Genuine NonLinearity}: \quad \begin{array}{l} \text{for a.e. } x \in \mathbb{R} \text{ the set} \\ \left\{ w \in \mathbb{R} \colon \partial^2_{ww} H(x,w) = 0 \right\} \\ \text{has empty interior.} \end{array} \tag{WGNL}$$

The proof is deferred to Section 5.

3. Extensions from homogeneous to non homogeneous

This section is focused on those properties known to hold in the homogeneous case, see [6], whose statement admits a natural extension to the non homogeneous case. However, the proofs typically require a new approach.

An interesting connection between (**CL**) and (**HJ**) is the following result, which shows that minimal and maximal backward characteristics are minima of the functional (2.4).

Theorem 3.1. *Let H satisfy* (**C3**)–(**CNH**)–(**CVX**). *Let $U_o \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$ and U solve* (**HJ**) *in the sense of Definition 2.3. Fix $(t,x) \in \left]0,+\infty\right[\times \mathbb{R}$ and let $\check\zeta$, respectively $\hat\zeta$, be the minimal, respectively maximal, backwards characteristics, related to $u = \partial_x U$ which solves* (**CL**), *emanating from (t,x), see [11, Definition 3.1]. Then, with reference to the functional* (2.4),

$$U(t,x) = \mathcal{J}_t(\check\zeta) = \mathcal{J}_t(\hat\zeta)\,.$$

The proof is deferred to Section 5.1.

We are now ready to state the conditions ensuring that $I_T^{HJ}(W)$, as defined in (1.2), is not empty. In other words, the next result completely characterizes the reachable set for (**HJ**).

Theorem 3.2. *Let H satisfy* (**C3**)–(**CNH**)–(**CVX**). *Fix $T > 0$, $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$ and define*

$$\begin{array}{rcl} U_o^* : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\sup_{\substack{q(0)=x\\ q\in\mathcal{R}_T}} \left(W\left(q(T)\right) - \int_0^T L\left(q(s),\dot q(s)\right)\mathrm{d}s \right), \end{array} \tag{3.1}$$

where L is as in (2.5) *and $\mathcal{R}_T$ as in* (2.2). *Then, the following conditions are equivalent:*

(1) $U_o^* \in I_T^{HJ}(W)$.
(2) $I_T^{HJ}(W) \neq \emptyset$.
(3) The set

$$\mathcal{G} := \left\{ (x_o, x_T) \in \mathbb{R}^2 : \exists\, q \in \mathcal{R}_T, \begin{array}{ll} (i) & q(0) = x_o\,,\ q(T) = x_T \\ (ii) & U_o^*(x_o) = W(x_T) - \int_0^T L\left(q(s),\dot q(s)\right)\mathrm{d}s \end{array} \right\} \tag{3.2}$$

has the following property:

$$(x_o, x_T'), (x_o, x_T'') \in \mathcal{G} \implies \forall\, x_T \in [x_T', x_T''],\ (x_o, x_T) \in \mathcal{G}\,. \tag{3.3}$$

Moreover, any of the conditions above implies that the map $\pi_{W'}$ defined in (2.3) *is well defined and nondecreasing.*

The proof is deferred to Section 5.2. The set $\mathcal{G}$ is more readily interpreted from the point of view of (**CL**). In particular, (3.3) describes the structure of rarefaction-like waves and, limited to the x-independent case, deriving the condition on $\mathcal{G}$ from the property of $\pi_{W'}$ is straightforward. In this connection, the x-dependent case is significantly more intricate. We signal in Lemma 5.8 additional properties of the set $\mathcal{G}$.

We are now ready to provide a full and general characterization of the inverse designs.

Theorem 3.3. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $T > 0$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$ such that $I_T^{HJ}(W) \neq \emptyset$ and define U_o^* as in* (3.1). *Then, for all $U_o \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$,*

$$U_o \in I_T^{HJ}(W) \iff \begin{cases} (i)\ U_o \geq U_o^* \\ (ii)\ U_o = U_o^* \text{ on } \overline{\pi_{W'}(\mathbb{R})}. \end{cases} \tag{3.4}$$

where $\pi_{W'}$ is defined as in (2.3).

The proof is deferred to Section 5.3.

Corollary 3.4. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $T > 0$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$ such that $I_T^{HJ}(W) \neq \emptyset$. Then, $I_T^{HJ}(W)$ is a closed convex cone with vertex U_o^*, defined in* (3.1) *and moreover $U_o^* = \min I_T^{HJ}(W)$ — meaning that for any $U \in I_T^{HJ}(W)$, it holds that for all $x \in \mathbb{R}$, $U(x) \geq U_o^*(x)$.*

The proof is an immediate consequence of the characterization provided by Theorem 3.3.

Corollary 3.4 admits a clear counterpart related to (**CL**), on the basis of the correspondence between (**CL**) and (**HJ**) proved in [5, Theorem 2.20]. An analogous characterization in the x-independent case is provided by [6, Proposition 5.2, Item (G2)].

Corollary 3.5. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $T > 0$ and $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$. Then, $I_T^{CL}(w)$ is a closed convex cone with vertex u_o^*, defined by $u_o^* = \partial_x U_o^*$ and U_o^* is as in* (3.1).

4. Peculiarities of the x-dependent case

The extension to the x-dependent case can not be merely reduced to the rise of technical difficulties. Indeed, some properties are irremediably lost and new phenomena arise, as shown below.

The most apparent difference between the two situations is described in Figure 2, with reference to extremal backward generalized characteristics, whose behaviors in the two cases are quite different. In the x-independent case, extremal backward characteristics emanating from a shock define a *non uniqueness gap*, see Figure 2. On the contrary, in the x-dependent

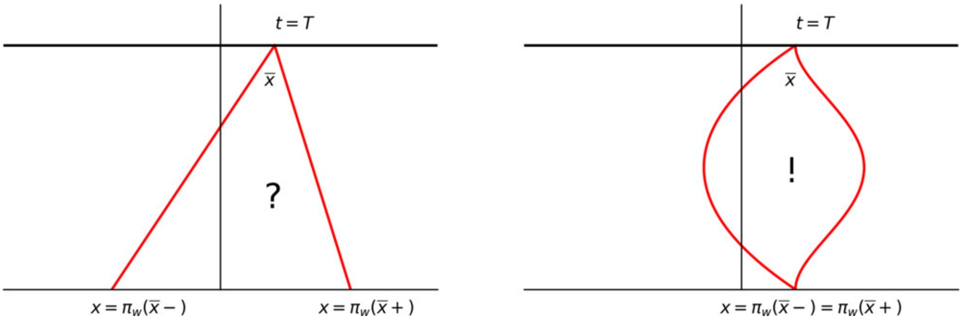

Figure 2. Left, in the x-independent case, extremal characteristics are straight lines and those emanating from the point of jump $\bar{x}$ in w at time T select the *non uniqueness gap* $]\pi_w(x-), \pi_w(x+)[$ along the x axis at time 0 where the initial data has no effect on w. Right, in our x-dependent choice (4.1) of the flow, characteristics could bend and uniquely determine the initial data evolving into w. Note that the solution in the region delimited by the extremal characteristics is unique.

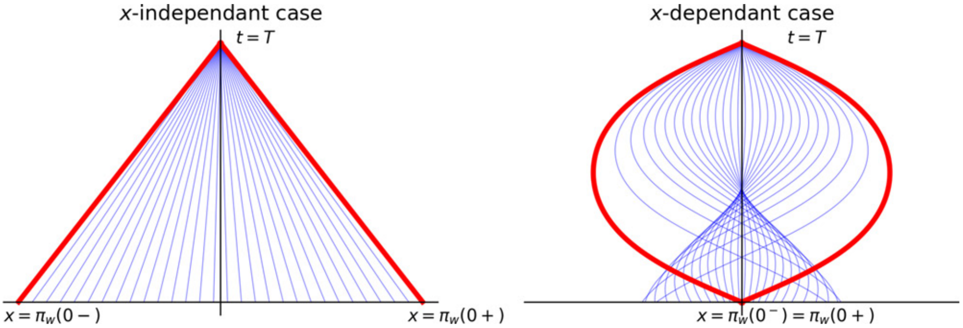

Figure 3. In both graphs, the thicker curves are the extremal backward characteristics while the Hamiltonian rays are thinner. Left, in the x-independent case, the Hamiltonian rays fill the non uniqueness gap described in Figure 2. Right, in the x-dependent case defined by the Hamiltonian (4.1), extremal characteristics still do not intersect, but Hamiltonian rays do and may well exit the non uniqueness gap or also intersect.

case, extremal backward characteristics may well intersect at the initial time, so that the non uniqueness gap disappears.

Furthermore, in the x-independent case, an isentropic solution, see [5, Theorem 3.1], is constructed filling the non uniqueness gap with Hamiltonian rays (2.2) emanating from $q(T) = x$, $p(T) = \theta\, w(x+) + (1-\theta)\, w(x-)$, for $\theta \in [0,1]$. On the contrary, the same idea fails in the x-dependent case. The numerical integrations in Figure 3 referred to (**HS**) with Hamiltonian (4.1), show that extremal backward characteristics still do not intersect in $]0,T[\times\mathbb{R}$, but the intermediate Hamiltonian rays may well cross each other and even exit the region bounded by the extremal characteristics.

When H does not depend on x, the u_o^* defined in Corollary 3.5 is characterized by [6, (G2) in Proposition 5.2]. Then, [6, (R1) in Lemma 7.2] ensures not only that u_o^* is one sided Lipschitz continuous, but also that the solution $\tilde{u}$ to (**CL**) with datum u_o^* evolving into w is Lipschitz continuous on any compact subset of $]0,T[\times\mathbb{R}$. Thus, $\tilde{u}$ satisfies the inequality in Definition 2.1 with an equality, i.e., it is an *isentropic* and also reversible in time solution, see related multi-dimensional results in [9].

This actually is specific to the homogeneous case: there exist an x-dependent Hamiltonian H, a profile w and a time $T>0$ such that $I_T^{CL}(w) \neq \emptyset$ but in any solution evolving from an initial datum in $I_T^{CL}(w)$ shocks arise at a time $t<T$, so that no reversible solution is possible, see Figure 1. In other words, the profile w can be reached exclusively producing a sufficient amount of entropy and no isentropic solution evolves into w. Each of these facts necessarily requires H to depend on x and can not take place in an x-independent setting, as shown in [6]. A consequence is that no direct definition of u_o^* is available, as it was in the x-independent case, and we have to resort to (**HJ**) for its construction.

Theorem 4.1. *Define, see Figure 4,*

$$H(x,u) := \frac{u^2}{2} + g(x) \quad \textit{where} \quad g(x) := \begin{cases} 1-(1-x^2)^4 & \textit{if } |x| \le 1\,, \\ 1 & \textit{otherwise.} \end{cases} \tag{4.1}$$

Then, (**C3**), (**CNH**), *and* (**CVX**) *hold. Moreover, for all* $T > \pi/(2\sqrt{2})$, *there exists* $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ *that contains a discontinuity and such that* $I_T^{CL}(w)$ *is a singleton.*

The proof is deferred to Section 5.4.

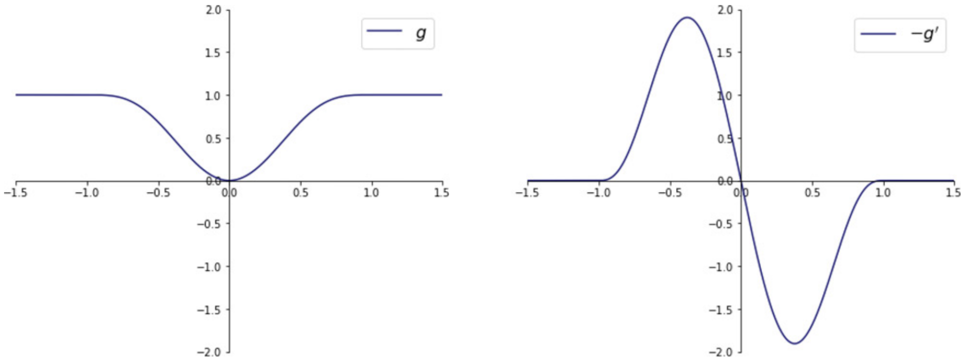

Figure 4. Left, graph of g and, right, the graph of $-g'$, according to (4.1). Clearly, g is $\mathbf{C}^3(\mathbb{R};[0,1])$, even, strictly increasing on $[0,1]$, g' attains values in $[-2,2]$ and H in (4.1) satisfies (**CNH**) with $X=1$.

Remark that if H does not depend on x, as soon as w has a jump, then the contrary to the conclusion of Theorem 4.1 holds true, see [6]. Indeed, $I_T^{CL}(w)$ is either empty or infinite, whenever w has a discontinuity. In particular, [6, (G1) in Proposition 5.2] does not hold.

Recall that [9, Section 5] presents, in the n dimensional case, a backward procedure to construct what corresponds here, in the x-independent case, to U_o^* in (3.1). In [2, 7, 8], a systematic use of such backward—forward construction is developed to deal with the inverse design problem in the x independent case. Formula (3.1) in Theorem 3.2 pushes this techniques at the optimal control level. Indeed, in the x dependent case the forward and backward trajectories may well differ, see [9, Example 6.3].

In Theorem 4.1, which is however restricted to the 1 dimensional case, the function H also satisfies (**CNH**), showing that the behavior for $|x| \to +\infty$ is not relevant in this context. More relevant, Theorem 4.1 shows that there may well be an *intrinsic* minimal entropy production, independently of any constructive procedure. As a matter of fact, the U_o^* in (3.1) corresponds to the construction in [9], although it is built by means of optimal control problems rather than by means of backward Hamilton-Jacobi equations. However, we are here interested in the broader inverse design characterization discussed in Section 3, rather than in time reversibility.

The evolution of the numerical solution computed with a standard finite volume scheme, is represented in Figure 5, see also Figure 1. Remark, and this is intrinsic to the heterogeneous case, that the initial rarefaction profile evolves into a shock wave. The time asymptotic behavior shows further differences with the x-independent case, see [32] for more details.

5. Proofs

Several results of use below can be obtained through rather classical techniques but can hardly be precisely localized in the literature. In these cases, we refer to [26], where all details are provided.

Proof of Lemma 2.2 Let $u_o \in I_T^{CL}(w)$. Call U_o a primitive of u_o, so that $\partial_x S_T^{HJ} U_o = w$ by [5, Theorem 2.20]. Then, by Lemma B.2, $(t,x) \mapsto (S_t^{HJ} U_o)(x)$ is locally semiconcave in the sense of Definition B.1. Thus, $(t,x) \mapsto (S_t^{CL} u_o)(x)$ is locally one sided Lipschitz continuous in the

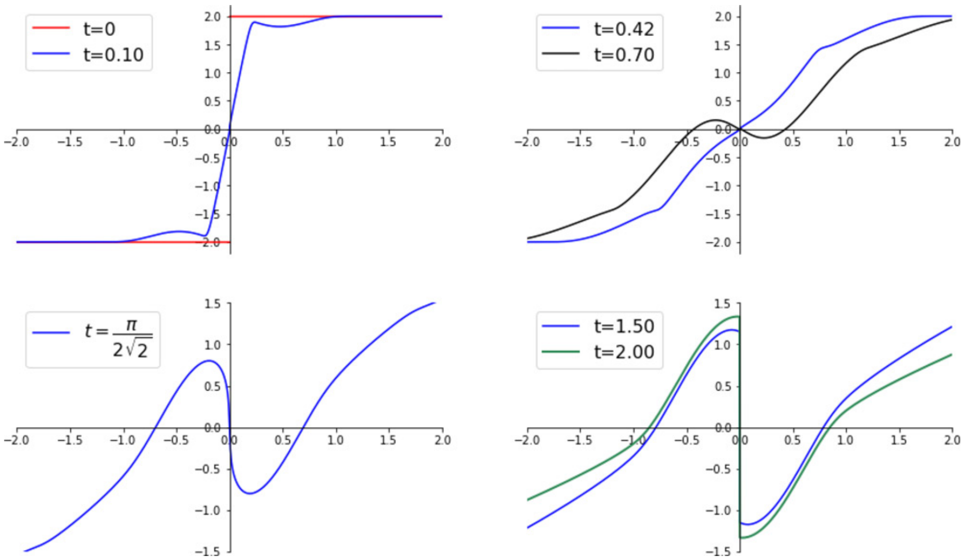

Figure 5. Evolution in time of (a numerical approximation of) the solution u to (**CL**)–(4.1)–(5.28), constructed in Theorem 4.1, as a function of the space variable x, computed at different times, see also Figure 1. Note the initial rarefaction profile turning into a shock at time $T = \pi \big/ (2\sqrt{2})$.

space variable and hence in $\mathbf{BV_{loc}}(\mathbb{R};\mathbb{R})$, for all $t > 0$. As is well known, this ensures the existence of left and right traces at any point of the map $x \mapsto (S_t^{CL} u_o)(x)$, for all $t > 0$. □

Proof of Proposition 2.4 It is immediate to prove that (**CVX**) implies (**WGNL**). Thanks to (**CNH**) and (**CVX**), we can use Lemmas B.3 and B.4 which ensure that there exists a function $\phi \in \mathbf{C}^0(\mathbb{R}_+;\mathbb{R})$ that verifies

$$\forall (x,p) \in \mathbb{R}^2, \quad H(x,p) \geq \phi\left(|p|\right) \quad \text{with} \quad \frac{\phi(r)}{r} \underset{r\to+\infty}{\longrightarrow} +\infty\,. \tag{5.1}$$

Then, (**UC**) readily follows. □

5.1. Proof of Theorem 3.1

Proof of Theorem 3.1 We only prove the result for the maximal backward characteristic $\hat{\zeta}$, which we denote for simplicity ζ. The details of the proof for the minimal characteristic $\check{\zeta}$ are similar.

Fix $\varepsilon > 0$. Apply Lemma B.8 with ζ and $\xi = \zeta - \varepsilon$ on $[0,t]$ and $s = 0$, $\tau = t$. After dividing by ε, we obtain:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\zeta(t)-\varepsilon}^{\zeta(t)} U(t,y)\,\mathrm{d}y - \frac{1}{\varepsilon}\int_{\zeta(0)-\varepsilon}^{\zeta(0)} U_o(y)\,\mathrm{d}y + \frac{1}{\varepsilon}\int_0^t \int_{\zeta(s)-\varepsilon}^{\zeta(s)} H\left(y, \partial_x U(s,y)\right) \mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{\varepsilon}\int_0^t \dot{\zeta}(s)\left(U\left(s,\zeta(s)\right) - U\left(s,\zeta(s)-\varepsilon\right)\right)\mathrm{d}s\,. \end{aligned} \tag{5.2}$$

We want to pass to the limit $\varepsilon \to 0$ in (5.2). To this aim, recall that $U(t,\cdot)$ and U_o are continuous in x by Definition 2.3. Moreover, $\partial_x U$ solves (**CL**) in the sense of Definition 2.1 with initial data U_o', see [5, Theorem 2.20]. For a.e. $s \in [0,t]$, $\partial_x U(s,\cdot)$ has left and right limits at $x = \zeta(s)$

that exist by Lemma 2.2 and coincide, since ζ is genuine [11, Definition 3.2 and Theorem 3.2]. The map U is Lipschitz continuous, hence $U(s,x_2) - U(s,x_1) = \int_{x_1}^{x_2} \partial_x U(s,y)\,\mathrm{d}y$, so that for a.e. $s \in [0,t]$, $U(s,\cdot)$ is differentiable at $x = \zeta(s)$ and $\partial_x U\,(s,\zeta(s)) = \lim_{h\to 0} \partial_x U(s,\zeta(s)+h)$. We thus obtain:

$$U\,(t,\zeta(t)) - U_o\,(\zeta(0)) = \int_0^t \left(\dot\zeta\,(s)\;\partial_x U\,(s,\zeta\,(s)) - H\,(\zeta\,(s),\partial_x U\,(s,\zeta\,(s)))\right) \mathrm{d}s\,. \tag{5.3}$$

Since ζ is genuine [11, Definition 3.2 and Theorem 3.2], there exists a function $\omega \in \mathbf{C}^1(]0,t[;\mathbb{R})$ such that (ζ,ω) is a solution to system (**HS**) with final conditions $\zeta(t) = x$ and $\omega(t) = \partial_x U(t,x+)$, since ζ is a maximal backward characteristics, see [11, Theorem 3.3]. Moreover, for a.e. $s \in [0,t]$, $\omega(s) = \partial_x U\,(s,\zeta\,(s))$. Combining these details with (5.3) and (**CVX**), classical computations lead to:

$$\begin{aligned}
U(t,x) - U_o\,(\zeta\,(0)) &= \int_0^t \left(\dot\zeta\,(s)\;\omega(s) - H\,(\zeta\,(s),\omega(s))\right)\mathrm{d}s\\
&= \int_0^t \left(\partial_p H\,(\xi\,(s),\omega(s))\;\omega(s) - H\,(\zeta\,(s),\omega(s))\right)\mathrm{d}s\\
&= \int_0^t L\left(\zeta\,(s),\partial_p H\,(\xi\,(s),\omega(s))\right)\mathrm{d}s\\
&= \int_0^t L\left(\zeta\,(s),\dot\zeta\,(s)\right)\mathrm{d}s\;,
\end{aligned}$$

concluding the proof. □

5.2. Proof of Theorem 3.2

In the light of the regularity proved in Lemma 2.2, whenever $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ is such that $I_T^{CL}(w) \neq \emptyset$, then by $w(x)$, we mean the left trace $w(x-)$ of w at x, for all $x \in \mathbb{R}$.

Lemma 5.1. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Then, for all $(q_o,p_o) \in \mathbb{R}^2$, the Cauchy problem* (**HS**) *with initial datum (q_o,p_o) at time 0 admits a unique maximal solution (q,p) defined on all $\mathbb{R}$ and satisfying, with the notation* (2.5),

$$\left|p(t)\right| \le \sup_{\substack{|u|\le|p_o|\\ x\in\mathbb{R}}} |H(x,u)| + \sup_{\substack{|v|\le 1\\ x\in\mathbb{R}}} L(x,v)\,. \tag{5.4}$$

Moreover, calling (q,p) the solution to (**HS**) *with datum (q_o,p_o) at time 0, the maps*

$$\begin{array}{rcccrcccrccc}
\mathcal{F}\colon & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 & \mathcal{F}_q\colon & \mathbb{R}^3 & \longrightarrow & \mathbb{R} & \mathcal{F}_p\colon & \mathbb{R}^3 & \longrightarrow & \mathbb{R}\\
& (t,q_o,p_o) & \longmapsto & \left(q(t),p(t)\right) & & (t,q_o,p_o) & \longmapsto & q(t) & & (t,q_o,p_o) & \longmapsto & p(t)
\end{array} \tag{5.5}$$

are of class $\mathbf{C}^2$.

Proof of Lemma 5.1 By (**C3**), the standard Cauchy Lipschitz Theorem ensures local existence and uniqueness of a solution (q,p) to the Cauchy problem for (**HS**) with datum (q_o,p_o). Moreover, since H is conserved along solutions to (**HS**), for all t where (q,p) is defined,

$$\sup_{\substack{|u|\le|p_o|\\ x\in\mathbb{R}}} |H(x,u)| \ge H(q_o,p_o) = H\left(q(t),p(t)\right) \ge \left|p(t)\right| - \sup_{\substack{|v|\le 1\\ x\in\mathbb{R}}} L(x,v)\,,$$

where we used (2.5), see also (B.1) with $\lambda = 1$, proving (5.4). By (**HS**), (**C3**), and (5.4), we also have that the solution (q, p) is bounded and uniformly continuous on bounded intervals. Hence, it is globally defined.

Standard results on ordinary differential equations, see e.g. [24, Theorem 3.9, Theorem 3.10], ensure that the flow $\mathcal{F}$ is as regular as $\partial_q H$, $\partial_x H$ and, by (**C3**), the proof is completed. □

The next three lemmas state simple geometric properties that are consequences of (**CVX**) and (**CNH**) on the graph of H (which looks like a sort of *canyon* along the x direction).

Lemma 5.2. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Then, there exists a unique function*

$$\begin{array}{rcl} z\colon \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & z(x) \end{array} \quad \text{such that} \quad \forall x \in \mathbb{R} \quad \partial_p H\left(x, z(x)\right) = 0\,. \tag{5.6}$$

Moreover, $z \in \mathbf{C}^2(\mathbb{R};\mathbb{R})$, if $|x| \geq X$ then $z'(x) = 0$ and the following quantities are well defined

$$\underline{z} := \min_{x\in\mathbb{R}} z(x)\,, \qquad \overline{z} := \max_{x\in\mathbb{R}} z(x)\,, \qquad \mathbf{K} := \max_{x\in\mathbb{R}} H\left(x, z(x)\right)\,. \tag{5.7}$$

Proof of Lemma 5.2 Existence and uniqueness of z follow from (**CVX**). Together, (**C3**) and (**CVX**) allow to apply the Implicit Function Theorem, proving both the $\mathbf{C}^2$ regularity of z and, by (**CNH**), that $z'(x) = 0$ whenever $|x| \geq X$. The completion of the proof is now immediate. □

Lemma 5.3. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Referring to the function z and to the constant $\mathbf{K}$ defined in Lemma 5.2, there exist functions*

$$\begin{array}{rcl} m\colon \mathbb{R}\times\left]\mathbf{K},+\infty\right[& \longrightarrow & \mathbb{R} \\ (x,c) & \longmapsto & m(x,c) \end{array} \quad \text{and} \quad \begin{array}{rcl} M\colon \mathbb{R}\times\left]\mathbf{K},+\infty\right[& \longrightarrow & \mathbb{R} \\ (x,c) & \longmapsto & M(x,c) \end{array} \tag{5.8}$$

uniquely characterized, for $c > \mathbf{K}$ and $x \in \mathbb{R}$, by

$$\begin{array}{rcl} H\left(x, m(x,c)\right) = c & \text{and} & m(x,c) < z(x) \\ H\left(x, M(x,c)\right) = c & \text{and} & M(x,c) > z(x) \end{array} \tag{5.9}$$

Moreover,

(i) $m, M \in \mathbf{C}^1(\mathbb{R}\times\left]\mathbf{K},+\infty\right[;\mathbb{R})$.
(ii) *m and M have a compact space dependency:*

$$|x| > X \text{ and } c > \mathbf{K} \implies \partial_x m(x,c) = 0 \text{ and } \partial_x M(x,c) = 0\,. \tag{5.10}$$

(iii) For all $x \in \mathbb{R}$, $m(x,\cdot)$ is decreasing while $M(x,\cdot)$ is increasing.
(iv) For all $x \in \mathbb{R}$, $\lim_{c\to+\infty} m(x,c) = -\infty$ and $\lim_{c\to+\infty} M(x,c) = +\infty$.

Proof of Lemma 5.3 We only prove the results for M, the details for m are entirely similar.

Assumption (**CVX**) ensures that condition (5.9) uniquely defines the map M in (5.8). An application of the Implicit Function Theorem shows the regularity, by (**C3**), proving (i).

Again, by the Implicit Function Theorem and the chain rule, we have

$$\partial_x H\left(x, M(x,c)\right) + \partial_x M(x,c)\ \partial_p H\left(x, M(x,c)\right) = 0\,,$$

which implies (5.10) by (**CNH**) and by (5.8) and (5.9), proving (ii).

By the definitions (5.6) of z and (5.8)–(5.9) of M, we have that for all $(x,c) \in \mathbb{R} \times]\mathbf{K}, +\infty[$, $M(x,c) > z(x)$ and hence $\partial_p H(x, M(x,c)) > 0$. Again using the Implicit Function Theorem, $\partial_c M(x,c)\, \partial_p H(x, M(x,c)) = 1$ proving that $\partial_c M(x,c) > 0$, proving (iii).

Since M is increasing, $\lim_{c\to+\infty} M(x,c) = \sup_{c>\mathbf{K}} M(x,c)$. The boundedness of $c \mapsto M(x,c)$ for some x then contradicts the equality $H(x, M(x,c)) = c$, proving (iv). ☐

Lemma 5.4. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Referring to the constant* $\mathbf{K}$ *in Lemma 5.2 and to the functions m, M in Lemma 5.3, define the functions:*

$$v\colon \begin{array}{ccc}]\mathbf{K}, +\infty[& \longrightarrow & \mathbb{R} \\ c & \longmapsto & \sup\limits_{x\in\mathbb{R}} \partial_p H(x, m(x,c)) \end{array} \quad \textit{and} \quad V\colon \begin{array}{ccc}]\mathbf{K}, +\infty[& \longrightarrow & \mathbb{R} \\ c & \longmapsto & \inf\limits_{x\in\mathbb{R}} \partial_p H(x, M(x,c)) \end{array}$$

Then:

(i) v *is nonincreasing and* V *is nondecreasing;*
(ii) $\lim_{c\to+\infty} v(c) = -\infty$ *and* $\lim_{c\to+\infty} V(c) = +\infty$.

Proof of Lemma 5.4 By (**CNH**) and (5.3) in Lemma 5.3, v and V are well-defined. We now prove the statements (ii) and (ii) for V, the case of v being entirely analogous.

From the monotonicity of M and $\partial_p H$ with respect to their second argument:

$$\forall x \in \mathbb{R},\ \forall c_1, c_2 > \mathbf{K}, \ \text{ if } c_1 < c_2 \text{ then } \partial_p H(x, M(x,c_1)) \le \partial_p H(x, M(x,c_2))\,.$$

Taking the infimum over $x \in \mathbb{R}$ we prove (i).

By (i), $\lim_{c\to+\infty} V(c) = \sup_{c>\mathbf{K}} V(c)$. By contradiction, assume that $\overline{V} := \sup_{c>\mathbf{K}} V(c)$ is finite. By the definition of V and (**CNH**),

$$\forall n \in \mathbb{N} \cap]\mathbf{K}, +\infty[\quad \exists x_n \in [-X, X], \qquad \text{such that} \qquad \partial_p H(x_n, M(x_n, n)) \le \overline{V}\,. \tag{5.11}$$

Up to a subsequence, we can assume that $(x_n)_n$ converges to some $\bar{x} \in [-X, X]$. Item (5.3) in Lemma 5.3 and (**CVX**) imply that $\lim_{n\to+\infty} \partial_p H(\bar{x}, M(\bar{x}, n)) = +\infty$. Therefore,

$$\exists N \in \mathbb{N} \cap]\mathbf{K}, +\infty[\quad \text{such that} \quad \forall n \ge N, \quad \partial_p H(\bar{x}, M(\bar{x}, n)) > \overline{V}\,. \tag{5.12}$$

The monotonicity of M and (**CVX**), combined with (5.11), result in:

$$\forall n \ge N, \quad \partial_p H(x_n, M(x_n, N)) \le \overline{V}\,,$$

which contradicts (5.12), proving (ii). ☐

Lemma 5.5. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $q_o \in \mathbb{R}$ and $T \in \mathbb{R}$. Then, the map $p \mapsto \mathcal{F}_q(T, q_o, p)$ defined in* (5.5) *is surjective, in the sense that*

$$\forall q_T \in \mathbb{R}, \quad \exists p_* \in \mathbb{R} \quad \textit{such that} \quad \mathcal{F}_q(T, q_o, p_*) = q_T\,, \tag{5.13}$$

or, with the notation (2.2),

$$\forall (x_o, x_T) \in \mathbb{R}^2, \quad \exists q \in \mathcal{R}_T \quad \textit{such that} \quad q(0) = x_o \textit{ and } q(T) = x_T\,. \tag{5.14}$$

Proof of Lemma 5.5 Recall the map z and the scalar $\mathbf{K}$ defined in Lemma 5.2. Fix $c_o > \mathbf{K}$ and let $p_o = M(q_o, c_o)$. By the conservation of H, we have $H\left(\mathcal{F}_q(t, q_o, p_o), \mathcal{F}_p(t, q_o, p_o)\right) = c_o$ for all $t \in \mathbb{R}$. Hence, by the definition (5.7) of $\mathbf{K}$, for all $t \in \mathbb{R}$, it follows that $\mathcal{F}_p(t, q_o, p_o) \neq z\left(\mathcal{F}_q(t, q_o, p_o)\right)$.

Using the continuity of $\mathcal{F}$, proved in Lemma 5.1, as well as the fact that $p_o > z(q_o)$, we deduce that

$$\forall t \in \mathbb{R}, \quad \mathcal{F}_p(t, q_o, p_o) > z\left(\mathcal{F}_q(t, q_o, p_o)\right) .$$

Therefore, for all $t \in \mathbb{R}$, $\mathcal{F}_p(t, q_o, p_o) = M\left(\mathcal{F}_q(t, q_o, p_o), c_o\right)$, as defined in (5.8)–(5.9). Thus, by (**HS**) and the definition of V in Lemma 5.4, we have:

$$\mathcal{F}_q(T, q_o, p_o) = q_o + \int_0^T \dot{q}(s)\,\mathrm{d}s \geq q_o + V(c_o)\,T \underset{c_o \to +\infty}{\longrightarrow} +\infty .$$

A similar argument, with $p_o = m(q_o, c_o)$, yields $\mathcal{F}_q(T, q_o, p_o) \leq q_o + v(c_o)\,T \underset{c_o \to +\infty}{\longrightarrow} -\infty$. The continuity of $\mathcal{F}_q$ coupled with the Intermediate Value Theorem concludes the proof of Lemma 5.5. □

Lemma 5.6. *Let H satisfy (**C3**), (**CNH**), and (**CVX**). Fix $T > 0$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$ such that $I_T^{HJ}(W) \neq \emptyset$. Then, for all $U_o \in I_T^{HJ}(W)$, with the notation (3.1),*

$$\forall x \in \mathbb{R}, \quad U_o(x) \geq U_o^*(x). \tag{5.15}$$

Proof of Lemma 5.6 Fix $x \in \mathbb{R}$ and $y \in \mathcal{R}_T$ so that $y(0) = x$. Since $U_o \in I_T^{HJ}(W)$, by (2.6) we have:

$$\begin{aligned}
& W\left(y(T)\right) - \int_0^T L\left(y(s), \dot{y}(s)\right)\mathrm{d}s \\
&= \inf_{\substack{\gamma(T) = y(T) \\ \gamma \in \mathcal{R}_T}} \left(\int_0^T L\left(\gamma(s), \dot{\gamma}(s)\right)\mathrm{d}s + U_o\left(\gamma(0)\right)\right) - \int_0^T L\left(y(s), \dot{y}(s)\right)\mathrm{d}s \\
&\leq \int_0^T L\left(y(s), \dot{y}(s)\right)\mathrm{d}s + U_o\left(y(0)\right) - \int_0^T L\left(y(s), \dot{y}(s)\right)\mathrm{d}s \\
&\leq U_o\left(y(0)\right) \\
&= U_o(x) .
\end{aligned}$$

By taking the supremum over $y \in \mathcal{R}_T$, by (3.1) we complete the proof. □

By the second condition in (5.1), for any $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$, there exists $\mathbf{C}_{H,W} > 0$ such that

$$\forall r \in \mathbb{R}_+,\ r \geq \mathbf{C}_{H,W} \implies \frac{\phi(r)}{1+r} > \left(\sup_{\substack{|p| \leq \|W'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \\ q \in \mathbb{R}}} \left|H(q,p)\right| + \sup_{\substack{|v| \leq 1 \\ q \in \mathbb{R}}} \left|L(q,v)\right|\right), \tag{5.16}$$

Lemma 5.7. *Let H satisfy (**C3**), (**CNH**), and (**CVX**). Fix $T > 0$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$. Then, U_o^* defined by (3.1) is Lipschitz continuous and*

$$\left\|(U_o^*)'\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq T\left(\sup_{\substack{|v| \leq \mathbf{C}_{H,W} \\ x \in \mathbb{R}}} \left|\partial_x L(x,v)\right| + \left\|W'\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\right). \tag{5.17}$$

Moreover, in its definition (3.1), the sup is attained and for any Hamiltonian ray q realizing the maximum in (3.1), $\left\|\dot{q}\right\|_{\mathbf{L}^\infty([0,T];\mathbb{R})} \leq \mathbf{C}_{H,W}$.

Proof of Lemma 5.7 First, thanks to (3.1), remark that

$$\forall x \in \mathbb{R}, \quad -U_o^*(x) = \inf_{\substack{q(0)=x \\ q \in \mathcal{R}_T}} \left(\int_0^T L\left(q(s), \dot q(s)\right) \mathrm{d}s - W\left(q(T)\right) \right).$$

Now reverse time applying the change of variable $\tau := T - s$ and introducing $q^r(\tau) := q(T - \tau)$, $L^r(x,v) := L(x,-v)$, $H^r(x,p) := \sup_{v\in\mathbb{R}} \left(p\, v - L^r(x,v)\right)$ so that $H^r(x,p) = H(x,-p)$. Moreover, using $p^r(\tau) = -p(T-\tau)$, $q \in \mathcal{R}_T$ if and only if $q^r \in \mathcal{R}_T^r$, where $\mathcal{R}_T^r$ is the set of Hamiltonian rays (2.2) defined by H^r, with reversed time.

$$\forall x \in \mathbb{R}, \quad -U_o^*(x) = \inf_{\substack{q^r(T)=x \\ q^r \in \mathcal{R}_T^r}} \left(\int_0^T L^r\left(q^r(\tau), \dot q^r(\tau)\right) \mathrm{d}\tau - W\left(q^r(0)\right) \right).$$

Then, in view of Lemma B.9,

$$\forall x \in \mathbb{R}, \quad -U_o^*(x) = \inf_{\substack{q^r(T)=x \\ q^r \in \mathbf{Lip}([0,T];\mathbb{R})}} \left(\int_0^T L^r\left(q^r(\tau), \dot q^r(\tau)\right) \mathrm{d}\tau - W\left(q^r(0)\right) \right). \tag{5.18}$$

So that, by Lemma B.7, $-U_o^*(x) = U^r(T,x)$, with U^r being the viscosity solution to the Hamilton-Jacobi equation

$$\begin{cases} \partial_t U^r + H^r(x, \partial_x U^r) = 0 \\ \tilde U(0,x) = -W(x), \end{cases}$$

proving (5.17).

The result in Lemma B.9 ensures that the supremum in Definition (3.1) is attained as a maximum. We now combine Lemmas B.6 and B.9 to complete the proof. □

Remarkably, the next Lemma does not require W to be reachable.

Lemma 5.8. *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $T > 0$ and $W \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$. Then $\mathcal{G}$, as defined in* (3.2) *has the following properties.*

(i) $\mathcal{G}$ is surjective in the following sense:

$$\forall x_o \in \mathbb{R},\ \exists\, x_T \in \mathbb{R} \text{ s.t. } (x_o, x_T) \in \mathcal{G}. \tag{5.19}$$

(ii) $\mathcal{G}$ is a closed subset of $\mathbb{R}^2$.

(iii) For all $(x_o, x_T) \in \mathcal{G}$, we have

$$|x_o - x_T| \le T\, \mathbf{C}_{H,W}. \tag{5.20}$$

(iv) $\mathcal{G}$ is monotone in the following sense:

$$\forall\, (x_o,x_T), (y_o,y_T) \in \mathcal{G}, \qquad \begin{array}{l} x_o < y_o \implies x_T \le y_T\,; \\ x_T < y_T \implies x_o \le y_o\,. \end{array} \tag{5.21}$$

Note the connection between (iv) and [6, Lemma 8.1], inspired by [33, Lemma 2.2], see also [34, Sections 5 and 6]. However, the x dependence makes the present proof significantly different and more intricate.

Proof of Lemma 5.8 Consider the different items separately.

Property (5.19) comes from the definition of U_o^*, since the sup is actually a max by Lemma 5.7.

Proof of (ii): Let $(x_o^n, x_T^n)_n$ be a sequence taking values in $\mathcal{G}$ which converges to some $(x_o, x_T) \in \mathbb{R}^2$. By definition, for all $n \in \mathbb{N}$, there exists $q^n \in \mathcal{R}_T$ such that

$$x_o^n = q^n(0), \quad x_T^n = q^n(T), \quad U_o^*(x_o^n) = W(x_T^n) - \int_0^T L\left(q^n(s), \dot{q}^n(s)\right) \mathrm{d}s\,. \tag{5.22}$$

For all $n \in \mathbb{N}$, let us denote by $p^n \in \mathbf{C}^1([0,T];\mathbb{R})$ a curve associated with q^n, given by $(x_o^n, x_T^n) \in \mathcal{G}$. Lemma 5.7 ensures that for all $n \in \mathbb{N}$, $\left\|\dot{q}^n\right\|_{\mathbf{L}^\infty(]0,T[;\mathbb{R})} \le \mathbf{C}_{H,W}$. Note that for all $n \in \mathbb{N}$, $q^n \in \mathbf{C}^1([0,T];\mathbb{R})$ by (2.2). Thanks to (**CVX**) and (2.5),

$$\dot{q}^n(t) = \partial_p H\left(q^n(t), p^n(t)\right) \iff p^n(t) = \partial_v L\left(q^n(t), \dot{q}^n(t)\right)\,.$$

This proves the boundedness of $\left(p^n(0)\right)$ and, up to a subsequence, we can assume that $\left(p^n(0), q^n(0)\right)_n$ converges to (p_o, x_o) with $p_o \in \mathbb{R}$. By Lemma 5.1, the flow of the Hamiltonian system is continuous and we establish the existence of $(q,p) \in \mathbf{C}^0([0,T];\mathbb{R}^2)$ such that $(q^n)_n$ and $(p^n)_n$ converge uniformly on $[0,T]$ to q and p, respectively. Using the integral form of (**HS**), we deduce that (q,p) solves (**HS**). Hence, $q \in \mathcal{R}_T$ and $(x_o, x_T) \in \mathcal{G}$.

Proof of (iii): Let $(x_o, x_T) \in \mathcal{G}$ and let $q \in \mathcal{R}_T$ with $q(0) = x_o$ and $q(T) = x_T$. Then, in view of Lemma 5.7 (latter part), we have

$$|x_o - x_T| = \left|q(0) - q(T)\right| \le T \left\|\dot{q}\right\|_{\mathbf{L}^\infty(]0,T[;\mathbb{R})} \le T\,\mathbf{C}_{H,W}\,.$$

Proof of (iv): We only prove the first implication in (5.21), the details of the proof for the second one are similar so we omit them. Let $x, y \in \mathcal{R}_T$ be two maximizers for $U_o^*(x_o)$ and $U_o^*(y_o)$, respectively. By assumption, we have $x(0) < y(0)$ so that we can define

$$\tau = \sup\left\{t \in [0,T] \colon x(s) < y(s) \text{ for all } s \in [0,t]\right\} \tag{5.23}$$

and assume, by contradiction, that $\tau < T$, so that $x(\tau) = y(\tau)$. Define the concatenation

$$\forall t \in [0,T], \quad \xi(t) = \begin{cases} y(t) & \text{if } \ 0 \le t \le \tau\,, \\ x(t) & \text{if } \ \tau < t \le T\,. \end{cases}$$

Clearly, $\xi \in \mathbf{Lip}([0,T];\mathbb{R})$ and $\xi(0) = y_o$.

We now prove that $\dot{x}(\tau) \neq \dot{y}(\tau)$. Denote $p_x, p_y \in \mathbf{C}^1([0,T];\mathbb{R})$ the curves associated with x and y, respectively, given by $(x_o, x_T), (y_o, y_T) \in \mathcal{G}$. Then,

$$\begin{aligned} \dot{x}(\tau) = \dot{y}(\tau) &\iff \partial_p H\left(x(\tau), p_x(\tau)\right) = \partial_p H\left(y(\tau), p_y(\tau)\right) \\ &\iff \partial_p H\left(y(\tau), p_x(\tau)\right) = \partial_p H\left(y(\tau), p_y(\tau)\right) \iff p_x(\tau) = p_y(\tau)\,, \end{aligned}$$

since $p \mapsto \partial_p H\left(y(\tau), p\right)$ is a bijection by (**CVX**). However this contradicts the uniqueness of solutions to (**HS**), see Lemma 5.1.

Hence, ξ is not differentiable at point τ. Moreover, since x and y are maximizers, in light of Lemma B.9 we have:

$$\begin{aligned} & U_o^*(x_o) - W(x_T) \\ &= - \inf_{\substack{q(0)=x_o \\ q \in \mathcal{R}_T}} \int_0^T L\left(q(s), \dot{q}(s)\right) \mathrm{d}s \\ &= - \inf_{\substack{q(0)=x_o \\ q \in \mathbf{Lip}(]0,T[;\mathbb{R})}} \int_0^T L\left(q(s), \dot{q}(s)\right) \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= - \inf_{\substack{q(0)=x_o, q(\tau)=\xi(\tau)\\ q\in \mathbf{Lip}(]0,\tau[;\mathbb{R})}} \int_0^\tau L\left(q(s),\dot q(s)\right) \mathrm{d}s - \inf_{\substack{q(\tau)=\xi(\tau), q(T)=x_T\\ q\in \mathbf{Lip}(]\tau,T[;\mathbb{R})}} \int_\tau^T L\left(q(s),\dot q(s)\right) \mathrm{d}s\\
&= - \int_0^\tau L\left(y(s),\dot y(s)\right) \mathrm{d}s - \int_\tau^T L\left(x(s),\dot x(s)\right) \mathrm{d}s\\
&= - \int_0^T L\left(\xi(s),\dot \xi(s)\right) \mathrm{d}s\,.
\end{aligned}
$$

This ensures that ξ is a Lipschitz maximizer for $U_o^*(x_o)$, therefore, $\xi \in \mathbf{W}^{2,\infty}(]0,T[;\mathbb{R})$ by Lemma B.5. However, this contradicts the fact that ξ is not differentiable at $t=\tau$. We conclude that x and y do not cross in $]0,T[$ implying $x(T)\le y(T)$ and, hence, $x_T \le y_T$. ☐

Proof of Theorem 3.2 The proof of the implication (1) $\implies$ (2) is clear.

Proof of (2) $\implies$ (3). Suppose that $I_T^{HJ}(W)\neq\emptyset$ and set $w=W'$, so that $I_T^{CL}(w)\neq\emptyset$ by the correspondence between (**CL**) and (**HJ**) proved in [5, Theorem 2.20]. We check that $\mathcal{G}$ in (3.2) enjoys the maximal property (3.3). Fix $U_o\in I_T^{HJ}(W)$, $(x_o,x_T'),(x_o,x_T'')\in\mathcal{G}$, with $x_T'<x_T''$, and $x_T\in\,]x_T',x_T''[$. Let $y,z\in\mathcal{R}_T$, as defined in (2.2), be solutions to (**HS**) connecting (x_o,x_T') and (x_o,x_T''), respectively, and let ξ be the minimal backward generalized characteristics emanating from (T,x_T), associated with (**CL**) with initial data U_o'. By [11, Theorem 3.2], ξ is genuine, $\xi\in\mathcal{R}_T$ and by Theorem 3.1,

$$
W(x_T)=\int_0^T L\left(\xi(s),\dot\xi(s)\right)\mathrm{d}s + U_o\left(\xi(0)\right) \ge \int_0^T L\left(\xi(s),\dot\xi(s)\right)\mathrm{d}s + U_o^*\left(\xi(0)\right).
$$

Above, we used the fact that $U_o\ge U_o^*$, see Lemma 5.6. We deduce that

$$
U_o^*\left(\xi(0)\right)\le W(x_T)-\int_0^T L\left(\xi(s),\dot\xi(s)\right)\mathrm{d}s\,.
$$

By definition (3.1) of U_o^*, we have equality above, and therefore ξ is a point of maximum of the functional in (3.1). We deduce that $(\xi(0),x_T)\in\mathcal{G}$. By (5.21) in Lemma 5.8,

$$
x_T'<x_T \implies x_o\le\xi(0) \quad\text{and}\quad x_T''>x_T\implies x_o\ge\xi(0).
$$

We deduce that $\xi(0)=x_o$ and, therefore, $(x_o,x_T)\in\mathcal{G}$.

Proof of (3) $\implies$ (1). We now show that U_o^*, as defined in (3.1), is in $I_T^{HJ}(W)$. We first check that:

$$
\forall\, x_T\in\mathbb{R},\ \exists\, x_o\in\mathbb{R},\quad (x_o,x_T)\in\mathcal{G}. \tag{5.24}
$$

Note that (5.24) differs from (i) in Lemma 5.8, since the roles of the elements in the pair (x_o,x_T) are reversed.

Fix $x_T\in\mathbb{R}$ and introduce the subset:

$$
E=\left\{x\in\mathbb{R}\colon \exists\, y\in\,]-\infty,x_T[\ \text{s.t.}\ (x,y)\in\mathcal{G}\right\}.
$$

Fix $x\in\mathbb{R}$ such that $x<x_T-T\,\mathbf{C}_{H,W}$. As a consequence of (i) in Lemma 5.8, there exists $y\in\mathbb{R}$ such that $(x,y)\in\mathcal{G}$. Now, using (iii) in Lemma 5.8, we can write

$$
y=(y-x)+x<\left|x-y\right|+(x_T-T\,\mathbf{C}_{H,W})\le x_T\,,
$$

which ensures that $]-\infty, x_T - T\,\mathbf{C}_{H,W}[\subset E$ and, therefore, E is non-empty. Moreover, for all $x \in E$, if $y \in \mathbb{R}$ ($y < x_T$) is such that $(x,y) \in \mathcal{G}$, then we have

$$x \leq (x-y) + y < \left|x-y\right| + x_T \leq T\,\mathbf{C}_{H,W} + x_T\,,$$

proving that E is bounded above. Hence, $\overline{x} = \sup E$ is finite. Likewise, the subset

$$F = \left\{x \in \mathbb{R} \colon \exists\, y \in \left]x_T, +\infty\right[\text{ s.t. } (x,y) \in \mathcal{G}\right\}$$

is nonempty and bounded below. Therefore, $\underline{x} = \inf F$ is finite. The monotonicity of $\mathcal{G}$ in (iv) of Lemma 5.8 ensures that $\overline{x} \leq \underline{x}$.

Let $(x_n)_n$ be a sequence of E which converges to $\overline{x}$. For all $n \in \mathbb{N}$, there exists $y_n < x_T$ such that $(x_n, y_n) \in \mathcal{G}$. Since $(x_n)_n$ is bounded, $(y_n)_n$ is bounded as well, as a consequence of (iii) in Lemma 5.8. Up to the extraction of a subsequence, we can assume that $(y_n)_n$ converges to some $\overline{y} \leq x_T$. Since $\mathcal{G}$ is closed, by (i) in Lemma 5.8, $(\overline{x}, \overline{y}) \in \mathcal{G}$. The same way, there exists $\underline{y} \geq x_T$ such that $(\underline{x}, \underline{y}) \in \mathcal{G}$. Let us conclude the proof by a case by case study.

Case 1: $\overline{x} = \underline{x}$. Call x_o this common value. Since $\overline{y} \leq x_T \leq \underline{y}$, we have by (3.3)

$$(x_o, \overline{y}), (x_o, \underline{y}) \in \mathcal{G} \implies (x_o, x_T) \in \mathcal{G}\,.$$

Case 2: $\overline{x} < \underline{x}$. Fix $x_o \in \left]\overline{x}, \underline{x}\right[$. By (i) in Lemma 5.8, there exists $y \in \mathbb{R}$ such that $(x_o, y) \in \mathcal{G}$. However, by the definition of $\overline{x}$, we necessarily have $y \geq x_T$. Similarly, the definition of $\underline{x}$ ensures that $y \leq x_T$. We proved that $y = x_T$ and therefore, $(x_o, x_T) \in \mathcal{G}$ for any $x_o \in \left]\overline{x}, \underline{x}\right[$.

Equality (5.24) rewrites as:

$$\forall\, x \in \mathbb{R},\ \exists\, q \in \mathcal{R}_T \text{ s.t. } q(T) = x \text{ and } W(x) = \int_0^T L\left(q(s), \dot{q}(s)\right) \mathrm{d}s + U_o^*\left(q(0)\right)\,. \tag{5.25}$$

Moreover, by the definition of U_o^*, we also have

$$\forall\, q \in \mathcal{R}_T\,, \qquad \int_0^T L\left(q(s), \dot{q}(s)\right) \mathrm{d}s + U_o^*\left(q(0)\right) \geq W\left(q(T)\right)\,. \tag{5.26}$$

Together (5.25) and (5.26) imply that

$$\begin{aligned} \forall x \in \mathbb{R}, \quad W(x) &= \inf_{\substack{q(T)=x \\ q \in \mathcal{R}_T}} \int_0^T L\left(q(s), \dot{q}(s)\right) \mathrm{d}s + U_o^*\left(q(0)\right) \\ &= \inf_{\substack{q(T)=x \\ q \in \mathbf{Lip}(]0,T[;\mathbb{R})}} \int_0^T L\left(q(s), \dot{q}(s)\right) \mathrm{d}s + U_o^*\left(q(0)\right)\,, \end{aligned}$$

by Lemma B.9. This last equality means that the viscosity solution U to (**HJ**) associated with initial datum U_o^* verifies $U(T) = W$, using the classical correspondence viscosity solution/calculus of variations, see Lemma B.7. We proved that $U_o^* \in I_T^{HJ}(W)$.

Proof of (2) $\Longrightarrow$ $\pi_{W'}$ is well defined and nondecreasing. Suppose that $I_T^{HJ}(W) \neq \emptyset$ and set $w = W'$, so that $I_T^{CL}(w) \neq \emptyset$ by [5, Theorem 2.20]. In the light of both Lemmas 2.2 and 5.1, π_w is well-defined by (2.3).

Fix $x, y \in \mathbb{R}$ with $x < y$. Since $I_T^{CL}(w) \neq \emptyset$, π_w assigns to x, respectively y, the value at time $t = 0$ of the minimal backward generalized characteristics emanating from (T, x), respectively from (T,y), which we denote by ξ_x, respectively ξ_y. By [11, Theorem 3.2], ξ_x and ξ_y are genuine, hence they do not intersect in $]0,T[$, see [11, Corollary 3.2]. This implies in particular that $\xi_x(0) \le \xi_y(0)$, proving that π_w is nondecreasing. ☐

5.3. *Proof of Theorem 3.3*

Proof of Theorem 3.3 We prove the two implications separately.

Claim: If $U_o \in I_T^{HJ}(W)$, then (i) *and* (ii) *hold.* Point *(i)* comes from Lemma 5.6. Let us prove that *(ii)* holds. Fix $x_o \in \pi_{W'}(\mathbb{R})$. By definition, there exists an $x \in \mathbb{R}$ such that $x_o = \pi_{W'}(x)$. This means that x_o is the value at time $t = 0$ of the minimal backward characteristics ξ, see [11, Definition 3.1, Theorems 3.2 and 3.3] emanating from (T,x). Since $U_o \in I_T^{HJ}(W)$, Theorem 3.1 ensures that

$$W(x) = \int_0^T L\left(\xi(s), \dot{\xi}(s)\right) \mathrm{d}s + U_o(x_o) .$$

On the other hand, by (3.1),

$$U_o^*(x_o) = \sup_{\substack{y(0)=x_o \\ y \in \mathcal{R}_T}} \left(W\left(y(T)\right) - \int_0^T L\left(y(s), \dot{y}(s)\right) \right) \mathrm{d}s \ge W(x) - \int_0^T L\left(\xi(s), \dot{\xi}(s)\right) \mathrm{d}s = U_o(x_o).$$

So, $U_o = U_o^*$ on $\pi_{W'}(\mathbb{R})$. These two functions are continuous, hence they coincide on $\overline{\pi_{W'}(\mathbb{R})}$.

Claim: If (i) *and* (ii) *hold, then $U_o \in I_T^{HJ}(W)$.* Fix $x \in \mathbb{R}$. Recall that $I_T^{HJ}(W) \neq \emptyset$ which, by Theorem 3.2, ensures that $U_o^* \in I_T^{HJ}(W)$. This, together with the inequality $U_o \ge U_o^*$, immediately implies:

$$\begin{aligned} W(x) &= \inf_{\substack{y(T)=x \\ y \in \mathcal{R}_T}} \left(\int_0^T L\left(y(s), \dot{y}(s)\right) \mathrm{d}s + U_o^*\left(y(0)\right) \right) \\ &\le \inf_{\substack{y(T)=x \\ y \in \mathcal{R}_T}} \left(\int_0^T L\left(y(s), \dot{y}(s)\right) \mathrm{d}s + U_o\left(y(0)\right) \right) . \end{aligned} \tag{5.27}$$

Denote by ξ the minimal backward characteristics emanating from (T,x). Using both the facts that $U_o^* \in I_T^{HJ}(W)$ and that, by Theorem 3.1, ξ is a minimizer, we have:

$$W(x) = \int_0^T L\left(\xi(s), \dot{\xi}(s)\right) \mathrm{d}s + U_o^*\left(\xi(0)\right) .$$

Clearly, $\xi(0) \in \overline{\pi_{W'}(\mathbb{R})}$ and therefore, by *(ii)*, we can replace $U_o^*\left(\xi(0)\right)$ by $U_o\left(\xi(0)\right)$ in the last equality. This ensures that we have equality in (5.27), which means that $U_o \in I_T^{HJ}(W)$. ☐

5.4. *Proof of Theorem 4.1*

In all proofs in this section, the reader might want to keep Figure 1 in mind for a helpful geometrical visualization.

Long but straightforward computations show that H, as defined in (4.1), satisfies (**C3**), (**CNH**) with $X = 1$, and (**CVX**), see Figure 4. With this flux, the conservation law (**CL**) is also the inviscid Burger equation with source term $-g'$, see Figure 4. We fix the initial datum

$$u_o(x) := \begin{cases} -2 & \text{if } x < 0\,, \\ 2 & \text{if } x > 0\,, \end{cases} \tag{5.28}$$

which would evolve into a rarefaction in the homogeneous case. The proof of Theorem 4.1 is based on the Cauchy problem for (**HS**) which, in this case, reads

$$\left\{ \begin{array}{rcl} \dot{q} & = & p \\ \dot{p} & = & -g'(q) \\ q(0) & = & q_o \\ p(0) & = & p_o \end{array} \right. \qquad \text{with } g \text{ as in (4.1)}, \tag{5.29}$$

and to which Lemma 5.1 applies. The first equation in (5.29) will be tacitly used throughout this section. By the Hamiltonian nature of (5.29), H is conserved along solutions, so that

$$\forall t \in \mathbb{R}, \quad \frac{p(t)^2}{2} + g\big(q(t)\big) = \frac{p_o^2}{2} + g(q_o)\,. \tag{5.30}$$

Lemma 5.9. *Let H be as in (4.1) and u_o be as in (5.28). Fix $q_o \geq 0$. Denote by (q,p) the solution to (5.29) with initial datum $\big(q_o, u_o(q_o+)\big) = (q_o, 2)$. Then, q is increasing on $[0, +\infty[$ and $q(t) \underset{t\to+\infty}{\longrightarrow} +\infty$.*

Proof of Lemma 5.9 Note that $p_o > 0$. By (5.30), for all $t \in \mathbb{R}$

$$p(t)^2 = \underbrace{p(0)^2}_{=4} + 2\underbrace{g(q_o)}_{\geq 0} - 2\underbrace{g\big(q(t)\big)}_{\leq 1} \geq 2\,.$$

Thus, for $t \in \mathbb{R}$, $p(t) \geq \sqrt{2}$. By (**HS**), q is strictly increasing and $q(t) \geq \sqrt{2}\, t$ for $t \in [0, +\infty[$. □

Refer to the lines on the right in Figure 6 for an illustration of the different behaviors of q described in Lemmas 5.9 and 5.10.

Lemma 5.10. *Let H be as in (4.1) and u_o be as in (5.28). Fix $0 \leq q_o < \tilde{q}_o$ and denote by (q,p), respectively $(\tilde{q}, \tilde{p})$, the global solution to (**HS**) with initial datum $\big(q_o, u_o(q_o+)\big) = (q_o, 2)$, respectively $\big(\tilde{q}_o, u_o(\tilde{q}_o)\big) = (\tilde{q}_o, 2)$. Then, $q(t) < \tilde{q}(t)$, for all $t \geq 0$.*

Proof of Lemma 5.10 Set $p_o = u_o(q_o)$ and $\tilde{p}_o = u_o(\tilde{q}_o)$. We proceed by contradiction. Let $\tau > 0$, be the smallest time where $q(\tau) = \tilde{q}(\tau)$. Since $q_o < \tilde{q}_o$, we have that $p(\tau) \geq \tilde{p}(\tau)$. By Lemma 5.9, $p(\tau) \geq \tilde{p}(\tau) \geq 0$. Then,

$$\begin{array}{rll} \left.\begin{array}{l} q_o \in [0, \tilde{q}_o[\\ p_o = \tilde{p}_o \end{array}\right\} \implies & \dfrac{p_o^2}{2} + g(q_o) \leq \dfrac{\tilde{p}_o^{\,2}}{2} + g(\tilde{q}_o) & \text{[By (4.1) and (5.28)]} \\ \implies & \dfrac{p(\tau)^2}{2} + g\big(q(\tau)\big) \leq \dfrac{\tilde{p}(\tau)^2}{2} + g\big(\tilde{q}(\tau)\big) & \text{[By (5.30)]} \\ \implies & p(\tau) \leq \tilde{p}(\tau). & [\text{By } q(\tau) = \tilde{q}(\tau),\ p(\tau) \geq \tilde{p}(\tau) > 0] \end{array}$$

We then deduce that $p(\tau) = \tilde{p}(\tau)$, which contradicts the uniqueness proved by Cauchy Lipschitz theorem. □

Lemma 5.11. *Let H be as in (4.1) and u_o be as in (5.28). Fix $p_o \in [\sqrt{2}, 2[$. Denote by (q,p) the global solution to (5.29) with initial datum $(0, p_o)$. (Refer to Figure 6.)*

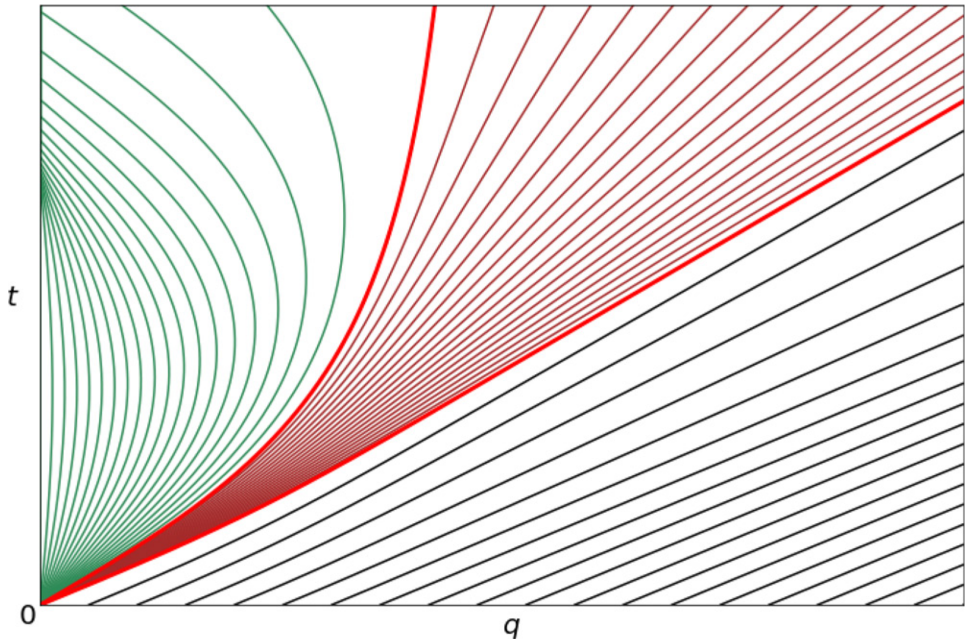

Figure 6. On the horizontal axis, the q component of solutions to (5.29), while t is on the vertical axis. Brown curves are those considered in (5.11) of Lemma 5.11; green curves refer to Lemma 5.12. The 2 red thicker curves depict solutions corresponding to the initial data $(0, \sqrt{2})$ and $(0, 2)$. The black curves are those considered in Lemmas 5.9 and in 5.10.

(i) If $p_o \in \,]\sqrt{2}, 2[$, *then* q *is increasing on* $[0, +\infty[$ *and* $q(t) \xrightarrow[t\to+\infty]{} +\infty$.

(ii) If $p_o = \sqrt{2}$, *then* q *is increasing on* $[0, +\infty[$, $q(t) \xrightarrow[t\to+\infty]{} 1$ *and* q *is concave.*

Refer to the middle curves in Figure 6 for an illustration of the different behaviors of q described in Lemma 5.11.

Proof of Lemma 5.11 The proof of (i) is identical to that of Lemma 5.9, so we omit it.

Concerning (ii), p is positive on $]0, +\infty[$. Indeed, assume by contradiction that there exists a minimal $\tau > 0$ such that $p(\tau) = 0$. By (5.30), we deduce that $q(\tau) = 1$. However, $(q_s, p_s)\colon t \mapsto (1, 0)$ is the unique global solution to (5.29) with datum $(1, 0)$. Hence, p is positive on $]0, +\infty[$.

Thus, by (5.29), q is increasing on $[0, +\infty[$ and positive on $]0, +\infty[$. Once again, (5.30) and the presence of the stationary solution (q_s, p_s) ensure that for all $t \in [0, +\infty[$, $q(t) < 1$. Therefore, as $t \to +\infty$, q admits a finite limit, say q_∞, which is not greater than 1.

Moreover, the positivity of q ensures, by (5.29), that p is nonincreasing on $[0, +\infty[$, so that by (5.29), q is concave. Since p is also bounded, by (5.4) in Lemma 5.1, p admits a finite limit as $t \to +\infty$. Hence, $\dot{q}$ has a finite limit as $t \to +\infty$ and, since we already showed that q converges to q_∞ as $t \to +\infty$, then $\dot{q} \to 0$ and therefore $p \to 0$ as $t \to +\infty$. By (5.30), we get $g(q_\infty) = 1$, and hence $q_\infty \geq 1$, therefore $q_\infty = 1$, completing the proof of (ii). $\square$

Lemma 5.12. *Let H be as in (4.1) and u_o be as in (5.28). Let $p_o \in\,]0,\sqrt{2}[$. Denote by (q,p) the global solution to (5.29) with initial data $(0,p_o)$. Then, q is periodic. Introduce the map*

$$\begin{array}{rcl} \mathcal{T}\colon]0,\sqrt{2}[& \longrightarrow &]0,+\infty[\\ p_o & \longmapsto & \min\left\{\tau > 0\colon \begin{array}{l}(q,p) \text{ solves } (5.29) \text{ with datum } (0,p_o)\\ q \text{ is } \tau\text{-periodic}\end{array}\right\}\end{array} \tag{5.31}$$

(i) q is concave on $[0,\mathcal{T}(p_o)/2]$.
(ii) For all $t \in \left[0,\mathcal{T}(p_o)/2\right]$, $q(t) = q\left(\mathcal{T}(p_o)/2 - t\right)$.
(iii) For all $t \in \left[0,\mathcal{T}(p_o)\right]$, $q(t) = -q\left(\mathcal{T}(p_o) - t\right)$.
(iv) q admits its maximum at $\mathcal{T}(p_o)/4$.

Proof of Lemma 5.12 Note first that by (5.30), q is bounded, since for all $t \in \mathbb{R}$, $g\left(q(t)\right) = \frac{p_o^2}{2} - \frac{(p(t))^2}{2} \le \frac{p_o^2}{2} < 1$ and hence $\left|q(t)\right| < 1$ for all $t \in \mathbb{R}$.

Assume now, by contradiction, that q does not vanish on $]0,+\infty[$. Since $q(0) = 0$ and $\dot q(0) > 0$, we have that for all $t \in\,]0,+\infty[$, $q(t) > 0$. Therefore, p is decreasing on $[0,+\infty[$ by (5.29), bounded by (5.30) and thus admits a finite limit as $t \to +\infty$.

Thus, q is bounded and its derivative $\dot q = p$ has a finite limit as $t \to +\infty$, hence $\lim_{t\to+\infty} \dot q(t) = 0$ and also $\lim_{t\to+\infty} p(t) = 0$. Therefore, by monotonicity, p is nonnegative on all $[0,+\infty[$. Consequently, q is nondecreasing and bounded, therefore it admits a finite limit $q_\infty \ge 0$ as $t \to +\infty$. On the one hand, by taking the limit as $t \to +\infty$ in (5.30), we get:

$$g(q_\infty) = \frac{p_o^2}{2} \in\,]0,1[\implies q_\infty \in\,]0,1[\implies g'(q_\infty) \neq 0.$$

On the other hand, $\dot p(t) = -g'\left(q(t)\right) \to -g'(q_\infty)$. Moreover, p has a finite limit as $t \to +\infty$, $\dot p(t) \to 0$. This provides the needed contradiction.

Thus we proved that there exists $\tau > 0$ such that $q(\tau) = 0$. As a consequence, the number

$$\tau_* := \sup\left\{t \in\,]0,+\infty[\colon \forall\, s \in\,]0,t[,\ q(s) > 0\right\} \tag{5.32}$$

is well-defined and satisfies $q(\tau_*) = 0$. Note that for $t \in\,]0,\tau_*[$, q is positive, p is decreasing and hence q is concave on $[0,\tau_*]$, proving (i) on $[0,\tau_*]$. Furthermore, $p(\tau_*) < p_o$. Apply (5.30) at time $t = \tau_*$ to obtain $p(\tau_*)^2 = p_o^2$, which implies $p(\tau_*) = -p_o$.

Now we verify that q is $2\tau_*$–periodic. To this aim, introduce $\xi(t) = -q(t+\tau_*)$ and $\nu(t) = -p(t+\tau_*)$. Thanks to g being even, it is straightforward to check that both (q,p) and (ξ,ν) solve the same Cauchy problem (5.29). Consequently,

$$\forall\, t \in \mathbb{R} \qquad q(t+2\tau_*) = -q(t+\tau_*) = q(t)\,. \tag{5.33}$$

Hence, q is $2\tau_*$–periodic. By (5.32), $2\tau_*$ is the minimal period, completing the proof of (i).

Finally, define $\widehat q(t) = q(\tau_* - t)$ and $\widehat p(t) = -p(\tau_* - t)$. Note that (q,p) and $(\widehat q,\widehat p)$ both solve (5.29) with datum $(0,p_o)$, since $p(\tau_*) = -p_o$. Hence, for all $t \in [0,\tau_*]$, $q(t) = q(\tau_* - t)$, proving (ii). Combined with the concavity of q, this ensures that $\max_{t\in[0,\tau_*]} q(t) = q(\tau_*/2)$, proving (iv). Finally, (5.33) and (ii) imply (iii). □

Lemma 5.13. *Let H be as in (4.1) and u_o be as in (5.28). Call $\mathcal{T}$ the map defined in (5.31). Then:*

(i) $\mathcal{T}$ is continuous.

(ii) $\mathcal{T}$ *strictly increasing.*
(iii) $\inf_{p_o \in]0,\sqrt{2}[} \mathcal{T}(p_o) = \pi/\sqrt{2}$.
(iv) $\lim_{p_o \to \sqrt{2}} \mathcal{T}(p_o) = +\infty$.

Proof of Lemma 5.13

Proof of (i). Fix $p_o \in]0, \sqrt{2}[$ and let $q_o = 0$. Let (q, p) be the solution to (5.29). Then, by Lemma 5.12, we get $p(t) = \dot{q}(t) > 0$ for all $t \in]0, \mathcal{T}(p_o)/4[$. Hence, using (5.30) we get

$$\forall\, t \in]0, \mathcal{T}(p_o)/4[\qquad p(t) = \sqrt{p_o^2 - 2g\left(q(t)\right)}$$

and by (5.29), we have

$$\frac{\mathcal{T}(p_o)}{4} = \int_0^{\mathcal{T}(p_o)/4} \frac{\dot{q}(t)}{\sqrt{p_o^2 - 2g\left(q(t)\right)}}\, \mathrm{d}t\ . \tag{5.34}$$

Note that the integrand in the right hand side above is singular when $t = \mathcal{T}(p_o)/4$, but it is positive for all t. Use the change of variable $x = q(t)$ to get

$$\frac{\mathcal{T}(p_o)}{4} = \int_0^{g^{-1}(p_o^2/2)} \frac{1}{\sqrt{p_o^2 - 2g(x)}}\, \mathrm{d}x \tag{5.35}$$

where g^{-1} is the inverse of the $\mathbf{C}^1$ diffeomorphism $g_{|]0,1[} \colon]0, 1[\to]0, 1[$.

Define $\mathcal{A} \colon]0, 1[\to \mathbb{R}_+$ by

$$\mathcal{A}(r) := \int_0^1 \frac{r}{\sqrt{g(r) - g(\theta\, r)}}\, \mathrm{d}\theta\ , \tag{5.36}$$

so that the change of variable $x = \theta\, r$ with $r = g^{-1}(p_o^2/2)$ in (5.35) leads to

$$\mathcal{T}(p_o) = 2\sqrt{2}\, \mathcal{A}\left(g^{-1}(p_o^2/2)\right)\ . \tag{5.37}$$

The continuity of $\mathcal{A}$ is proved in Lemma A.2 in Appendix A, completing the proof of (i).

Proof of (ii). By [35, Theorem A], (5.37) and (5.30), the condition

$$\forall x \in]0, 1[, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{g(x)}{g'(x)^2}\right) \geq 0$$

ensures that $E \mapsto 2\sqrt{2}\, \mathcal{A}\left(g^{-1}(E)\right)$ is increasing. Hence, by (5.37), also $\mathcal{T}$ is increasing on $]0, \sqrt{2}[$. By (4.1), for $x \in]0, 1[$ we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{g(x)}{g'(x)^2}\right) = -\frac{3}{32}\,\frac{7x^8 - 32x^6 + 59x^4 - 56x^2 - 6}{(x^2 - 1)^8}\,.$$

We leave to Lemma A.1 in Appendix A the proof that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{g(x)}{g'(x)^2}\right) \geq 0$ for all $x \in]0, 1[$ by means of Sturm theorem, see [36].

Proof of (iii). To prove the lower bound on $\mathcal{T}$, introduce for any $p_o \in]0, \sqrt{2}[$, $\tilde{q} > 0$ so that $g(\tilde{q}) = p_o^2/2$:

$$\inf_{p_o \in]0,\sqrt{2}]} \mathcal{T}(p_o) = \lim_{p_o \to 0+} \mathcal{T}(p_o) \qquad \text{[By the monotonicity of } \mathcal{T}\text{]}$$

$$
\begin{aligned}
&= \lim_{E\to 0} 2\sqrt{2}\,\mathcal{A}\left(g^{-1}(E)\right) && \text{[By (5.37)]}\\
&= \lim_{r\to 0} 2\sqrt{2}\,\mathcal{A}(r) && \text{[By (4.1)]}\\
&= 2\sqrt{2}\lim_{r\to 0}\int_0^1 \frac{r}{\sqrt{g(r)-g(\theta\, r)}}\,\mathrm{d}\theta && \text{[By (5.36)]}\\
&= 4\int_0^1 \frac{\mathrm{d}\theta}{\sqrt{g''(0)\,(1-\theta^2)}}\\
&= \frac{\pi}{\sqrt{2}}\,.
\end{aligned}
$$

This completes the proof of (iii).

Proof of (iv). Similar computations, using now Fatou's lemma, lead to

$$
\lim_{p_o\to\sqrt{2}} \mathcal{T}(p_o) = 2\sqrt{2}\lim_{r\to 1}\int_0^1 \frac{r}{\sqrt{g(r)-g(\theta\, r)}}\,\mathrm{d}\theta \geq 2\sqrt{2}\int_0^1 \frac{1}{(1-\theta^2)^2}\,\mathrm{d}\theta = +\infty\,,
$$

completing the proof of (iv) and of Lemma 5.11. □

Lemma 5.14. *Let H be as in (4.1) and u_o be as in (5.28). Fix $0 < p_o < \tilde{p}_o < 2$ and denote by (q,p), respectively $(\tilde{q},\tilde{p})$, the global solution to (5.29) with initial datum $(0,p_o)$, respectively $(0,\tilde{p}_o)$. Then,*

$$
\begin{array}{llcll}
(i) & p_o\in\,]0,\sqrt{2}[& \Longrightarrow & \forall\, t\in\,]0,\mathcal{T}(p_o)/2] & q(t)<\tilde{q}(t)\,;\\
(ii) & p_o\in[\sqrt{2},2[& \Longrightarrow & \forall\, t\in\,]0,+\infty[& q(t)<\tilde{q}(t)\,.
\end{array}
$$

Refer to the middle and left curves in Figure 6 for an illustration of the different behaviors of q described in Lemma 5.14.

Proof of Lemma 5.14 We split the proof in several steps.

Claim 1: Let $T > 0$ be such that $p(t) > 0$ and $\tilde{p}(t) > 0$ for all $t \in [0,T]$. Then, for all $t \in [0,T]$, $q(t) < \tilde{q}(t)$. By contradiction, since $p_o < \tilde{p}_o$, there exists $s \in\,]0,T]$ such that $q(t) < \tilde{q}(t)$ for $t \in\,]0,s[$, $q(s) = \tilde{q}(s)$ and thus $p(s) \geq \tilde{p}(s)$. Then, by (5.30),

$$
\begin{aligned}
0 < p_o < \tilde{p}_o &\Longrightarrow H(0,p_o) < H(0,\tilde{p}_o) && \text{[By (4.1)]}\\
&\Longrightarrow H\left(q(s),p(s)\right) < H\left(\tilde{q}(s),\tilde{p}(s)\right) && \text{[By (5.30)]}\\
&\Longrightarrow p(s) < \tilde{p}(s) && \text{[Since } q(s)=\tilde{q}(s) \text{ and } p(s)>0,\ \tilde{p}(s)>0]
\end{aligned}
$$

which yields a contradiction, proving Claim 1.

Claim 2: (i) holds for $\tilde{p}_o \in\,]0,\sqrt{2}[$. By Claim 1, for all $t \in\,]0,\mathcal{T}(p_o)/4[$, $q(t) < \tilde{q}(t)$. Indeed, by (5.29) together with Lemmas 5.12 and 5.13, both p and $\tilde{p}$ are positive on $]0,\mathcal{T}(p_o)/4[$ and Claim 1 applies. Hence, by the symmetry in (ii) of Lemma 5.12, we have

$$
\forall\, t\in \left]\frac{\mathcal{T}(\tilde{p}_o)}{2}-\frac{\mathcal{T}(p_o)}{4},\frac{\mathcal{T}(\tilde{p}_o)}{2}\right[\,,\qquad q\left(\frac{\mathcal{T}(\tilde{p}_o)}{2}-t\right)<\tilde{q}(t)\,.
$$

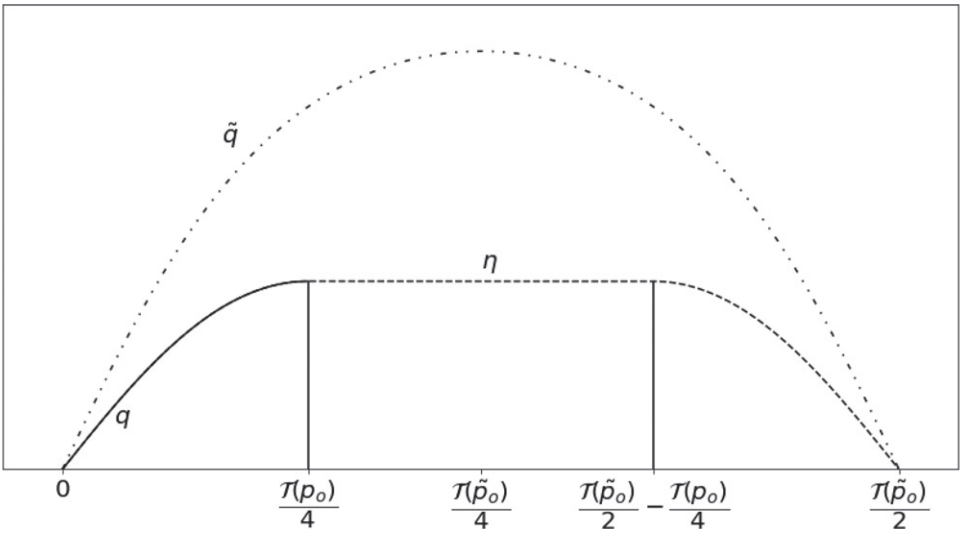

Figure 7. Curves used in Claim 2 in the proof of Lemma 5.14. The dashed curve is the graph of η in (5.38). The continuous curves are the graphs of q restricted to $[0, \mathcal{T}(p_o)/2]$ and of its translate. The dashed–dotted curved is the graph of $\widetilde{q}$.

Introduce the concave function

$$\begin{array}{rcll} \eta : \left[0, \frac{\mathcal{T}(\widetilde{p}_o)}{2}\right] & \longrightarrow & \mathbb{R} & \\ t & \longmapsto & \begin{cases} q(t) & t \in \left[0, \frac{\mathcal{T}(p_o)}{4}\right] \\ q\left(\frac{\mathcal{T}(p_o)}{4}\right) & t \in \left]\frac{\mathcal{T}(p_o)}{4}, \frac{\mathcal{T}(\widetilde{p}_o)}{2} - \frac{\mathcal{T}(p_o)}{4}\right[\\ q\left(\frac{\mathcal{T}(\widetilde{p}_o)}{2} - t\right) & t \in \left[\frac{\mathcal{T}(\widetilde{p}_o)}{2} - \frac{\mathcal{T}(p_o)}{4}, \frac{\mathcal{T}(\widetilde{p}_o)}{2}\right] \end{cases} & \end{array} \tag{5.38}$$

and note that, see Figure 7,

$$\begin{array}{lll} \forall\, t \in \left[0, \frac{\mathcal{T}(p_o)}{4}\right] & q(t) \le \eta(t) & \text{[By (5.38)]} \\ \forall\, t \in \left[0, \frac{\mathcal{T}(\widetilde{p}_o)}{2}\right] & \eta(t) \le \widetilde{q}(t) & \text{[By concavity of } \eta \text{ and } \widetilde{q}\text{]} \\ \forall\, t \in \left[\frac{\mathcal{T}(p_o)}{4}, \frac{\mathcal{T}(p_o)}{2}\right] & q(t) \le \eta(t) & \text{[By symmetry]} \end{array}$$

completing the proof of Claim 2.

***Claim 3:* (i) holds for $\widetilde{p}_o \in [\sqrt{2}, +\infty[$.** By Claim 1, for all $t \in \left]0, \mathcal{T}(p_o)/4\right]$, $q(t) < \widetilde{q}(t)$.

By (iv) in Lemma 5.12, for all $t \in \left]0, \mathcal{T}(p_o)/2\right[$, $q(t) \le q(\mathcal{T}(p_o)/4)$, while by (i) and (ii) in Lemma 5.11, for $\widetilde{p}_o \in [\sqrt{2}, +\infty[$ and for all $t \in [\mathcal{T}(p_o)/4, +\infty[$, $\widetilde{q}(t) > \widetilde{q}(\mathcal{T}(p_o)/4)$. All this ensures that

$$\forall\, t \in \left[\frac{\mathcal{T}(p_o)}{4}, \frac{\mathcal{T}(p_o)}{2}\right], \qquad q(t) \le q\left(\frac{\mathcal{T}(p_o)}{4}\right) < \widetilde{q}\left(\frac{\mathcal{T}(p_o)}{4}\right) \le \widetilde{q}(t),$$

completing the proof of Claim 3.

Claim 4: Proof of **(ii)***.* If $p_o, \tilde{p}_o \in [\sqrt{2}, 2[$, then by (i) and (ii) in Lemma 5.11, q and $\tilde{q}$ are increasing, so that by (5.29) p and $\tilde{p}$ are positive. So, Claim 1 applies, completing the proof. □

We use below the flow $\mathcal{F}$ introduced in (5.5) with reference to (**HS**), which we now particularize to (5.29). By Lemma 5.1, $\mathcal{F}$ is of class $\mathbf{C}^2$. Define

$$(q^\flat, p^\flat)(t) := \mathcal{F}(t, 0, \sqrt{2}) \qquad \text{and} \qquad (q^\sharp, p^\sharp)(t) := \mathcal{F}(t, 0, 2)\,, \tag{5.39}$$

$q^\flat$, respectively $q^\sharp$, being the leftmost, respectively rightmost, red line in Figure 6.

Lemma 5.15. *Let H be as in (4.1) and u_o be as in (5.28). Define the set*

$$D := ([0, +\infty[\times \{2\}) \cup (\{0\} \times]0, 2])\,. \tag{5.40}$$

Then, there exists a unique map

$$\begin{array}{rccc} \Delta : &]0, +\infty[\times]0, +\infty[& \longrightarrow & D \\ & (t, x) & \longmapsto & (q_o, p_o) \end{array} \tag{5.41}$$

such that

$$\mathcal{F}_q(t, q_o, p_o) = x \quad \textit{and} \quad \forall s \in]0, t[,\ \mathcal{F}_q(s, q_o, p_o) > 0\,. \tag{5.42}$$

Moreover,

(1) Δ is continuous.
(2) Δ is monotone, in the sense that setting $\Delta(t_o, x_o) = (0, p_o)$ and $\Delta(t_o, x_o') = (0, p_o')$, if $0 < x_o < x_o' < q^\sharp(t_o)$, then $p_o < p_o'$.
(3) For all $x \in]0, +\infty[$, $\lim_{t \to 0+} \Delta(t, x) = (x, 2)$.

Proof of Lemma 5.15 We split the proof in several steps.

***For all $(t, x) \in]0, +\infty[^2$, there exists $(p_o, q_o) \in D$ satisfying* (5.42).** Fix $(t, x) \in]0, +\infty[\times]0, +\infty[$. If $x = q^\sharp(t)$ as in (5.39), then set $(q_o, p_o) = (0, 2)$. Otherwise, introduce the functions

$$\begin{array}{rccc} h : & [0, +\infty[& \longrightarrow & \mathbb{R} \\ & q_o & \longmapsto & \mathcal{F}_q(t, q_o, 2) - x \end{array} \quad \text{and} \quad \begin{array}{rccc} k : &]0, 2] & \longrightarrow & \mathbb{R} \\ & p_o & \longmapsto & \mathcal{F}_q(t, 0, p_o) - x\,. \end{array}$$

Note that if $x > q^\sharp(t)$ then $h(0) < 0$ and $h(x+1) = \mathcal{F}_q(t, x+1, 2) - x > 0$ by Lemma 5.9. By the Intermediate Value Theorem, there exists a q_o such that $h(q_o) = 0$, hence $\mathcal{F}_q(t, q_o, 2) = x$. By Lemma 5.9, for all $s > 0$, $\mathcal{F}_q(s, q_o, 2) > 0$, proving (5.42) in the case $x > q^\sharp(t)$.

If $x \in [q^\flat(t), q^\sharp(t)]$, then $k(\sqrt{2}) \leq 0 \leq k(2)$ by (5.39). By the Intermediate Value Theorem, there exists p_o such that $k(p_o) = 0$, i.e., $\mathcal{F}_q(t, 0, p_o) = x$. Then, by Lemma 5.14, the right part of (5.42) follows in the case $x \in [q^\flat(t), q^\sharp(t)]$.

Similarly, if $x \in]0, q^\flat(t)[$, then $k(\sqrt{2}) > 0$ and $\lim_{\check{p} \to 0+} k(\check{p}) = -x < 0$. By the Intermediate Value Theorem, we can define

$$p_o := \max \left\{ \pi_o \in [0, \sqrt{2}] : k(\pi_o) = 0 \right\}\,.$$

Hence, for all $\check{p} \in]p_o, \sqrt{2}[$, $k(\check{p}) > 0$. Proceed now by contradiction: assume there exists $s \in]0, t[$ such that $\mathcal{F}_q(s, 0, p_o) < 0$. By Lemma 5.12, we get $t > \mathcal{T}(p_o)$. Using Lemma 5.13

and the Intermediate Value Theorem, it follows that there exists $p'_o \in \,]p_o, \sqrt{2}[$ such that $\mathcal{T}(p'_o) = t$. By Lemma 5.12, this implies that $\mathcal{F}_q(t, 0, p'_o) = 0$ and therefore $k(p'_o) < 0$, which contradicts the choice of p_o.

Δ is uniquely defined. For all $(t, x) \in \,]0, +\infty[\, \times \,]0, +\infty[$ the uniqueness of a (q_o, p_o) satisfying (5.42) follows from the monotonicity properties proved above. Indeed, recalling $q^\sharp$ as defined in (5.39), if $0 < q_o < \tilde{q}_o$ and $p_o = \tilde{p}_o = 2$, then, by Lemma 5.10, for all $s \in [0, +\infty[$, $q^\sharp(s) < \mathcal{F}_q(s, q_o, p_o) < \mathcal{F}_q(s, \tilde{q}_o, \tilde{p}_o)$. On the other hand, if $q_o = \tilde{q}_o = 0$ and $0 < p_o < \tilde{p}_o < 2$, then by Lemmas 5.11, 5.12, and 5.14, for all s such that $\mathcal{F}_q(\tau, q_o, p_o) \geq 0$ for all $\tau \in [0, s]$, we have $\mathcal{F}_q(s, q_o, p_o) < \mathcal{F}_q(s, \tilde{q}_o, \tilde{p}_o) < q^\sharp(s)$. Finally, if $q_o = 0$, $\tilde{q}_o > 0$, $p_o \in \,]0, 2[$ and $\tilde{p}_o = 2$, then Lemmas 5.14 and 5.10 ensure that for all s such that $\mathcal{F}_q(\tau, q_o, p_o) \geq 0$ for all $\tau \in [0, s]$, we have $\mathcal{F}_q(s, q_o, p_o) < q^\sharp(s) < \mathcal{F}_q(s, \tilde{q}_o, \tilde{p}_o)$. The uniqueness of (q_o, p_o) follows.

Δ is continuous. For any $(q_o, p_o) \in D$ and $(t, x) \in \mathbb{R}^2_+$, if $(q_o, p_o) = \Delta(t, x)$ then by (5.30), we have $\left|\mathcal{F}_p(t, q_o, p_o)\right| \leq \sqrt{p_o^2 + 2}$, so that by (5.29), $\left|\mathcal{F}_q(t, q_o, p_o) - q_o\right| \leq t\sqrt{p_o^2 + 2}$. Therefore,

$$\left|q_o\right| \leq x + t\sqrt{p_o^2 + 2}\,. \tag{5.43}$$

Choose now a sequence (t_n, x_n) in $]0, +\infty[\, \times \,]0, +\infty[$ converging to (t, x) also in $]0, +\infty[\, \times \,]0, +\infty[$. Define $(q_o^n, p_o^n) = \Delta(t_n, x_n)$. The sequence p_o^n is in $[0, 2]$ by (5.40) and (5.41). By (5.43), also the sequence q_o^n is bounded, since also (t_n, x_n) is bounded. Call (q_o, p_o) the limit of any convergent subsequence, so that $(q_o, p_o) \in \overline{D}$. By the continuity of $\mathcal{F}$ proved in Lemma 5.1. up to a subsequence we have

$$\mathcal{F}_q(t, q_o, p_o) = \lim_{n \to +\infty} \mathcal{F}_q(t_n, q_o^n, p_o^n) = \lim_{n \to +\infty} x_n = x\,. \tag{5.44}$$

This also shows that $(q_o, p_o) \in D$. Otherwise, if $(q_o, p_o) \in \overline{D} \setminus D$, then $(q_o, p_o) = (0, 0)$ and for all $t \in \mathbb{R}$, $\mathcal{F}_q(t, 0, 0) = 0 \neq x$.

Since $(q_o^n, p_o^n) = \Delta(t_n, x_n)$, then $x_n = \mathcal{F}_q(t_n, q_o^n, p_o^n)$. Thus, $\mathcal{F}_q(s, q_o^n, p_o^n) > 0$ for $s \in \,]0, t_n[$. In the limit $n \to +\infty$, we have $x = \mathcal{F}_q(t, q_o, p_o)$ and if $s \in \,]0, t[$, then $\mathcal{F}_q(s, q_o, p_o) \geq 0$.

The possible behaviors of $s \to \mathcal{F}_q(s, q_o, p_o)$ classified in Lemma 5.9, Lemma 5.11 and in Lemma 5.12 ensure that for all $s \in \,]0, t[$ we have $\mathcal{F}_q(s, q_o, p_o) > 0$ so that also the second condition in (5.42) is met and $\Delta(t, x) = (q_o, p_o)$, the limit (q_o, p_o) being independent of the subsequence. This completes the proof of the continuity of Δ.

Proof of (2) and (3). Fix a positive x. Let t_n be any positive sequence converging to 0. Then, $\mathcal{F}_q\left(t_n, \Delta(t_n, x)\right) = x$. The bound (5.43) ensures that, up to a subsequence, $\lim_{n \to +\infty} \Delta(t_n, x) = \xi$, with $\xi \in \overline{D}$ satisfying $\mathcal{F}_q(0, \xi) = x$. Hence, $\xi = (x, 2)$, proving (3).

The monotonicity of Δ follows from Lemma 5.14, completing the proof of (2). ☐

Proposition 5.16. *Let H be as in (4.1) and u_o be as in (5.28). Recall the notations (5.5) and (5.41). The function*

$$\begin{array}{rcl} u \colon]0, +\infty[\, \times (\mathbb{R} \setminus \{0\}) & \longrightarrow & \mathbb{R} \\ (t, x) & \longmapsto & \left\{ \begin{array}{rll} \mathcal{F}_p\left(t, \Delta(t, x)\right) & \text{if} & x > 0\,, \\ -\mathcal{F}_p\left(t, \Delta(t, -x)\right) & \text{if} & x < 0\,. \end{array} \right. \end{array} \tag{5.45}$$

is in $\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R};\mathbb{R})$*, solves* (CL) *with datum* (5.28) *in the sense of Definition 2.1, it is a classical strong solution outside* $x=0$ *and outside* $|x|=q^\sharp(t)$*, it is continuous along* $|x|=q^\sharp(t)$ *and there is an entropic stationary shock along* $x=0$ *for* $t>\pi/(2\sqrt{2})$.

The lack of differentiability along $|x|=q^\sharp(t)$ is visible in Figure 1.

Proof Proposition 5.16 Call Γ the graph of the map $t\mapsto q^\sharp(t)$ as defined in (5.39) and define $\Omega :=]0,+\infty[^2\setminus\Gamma$. Note that by Lemma 5.15 and (5.45), $u\in\mathbf{C}^0(]0,+\infty[\times(\mathbb{R}\setminus\{0\});\mathbb{R})$.

Claim 1: $u\in\mathbf{C}^1(\boldsymbol{\Omega};\mathbb{R})$ ***and is a classical solution to*** **(CL)–(4.1)–(5.28)** ***in*** $\boldsymbol{\Omega}$**.** This follows from an application of the Implicit Function Theorem. Indeed, let $(t_o,x_o)\in\Omega$. Then, either $\Delta(t_o,x_o)=(q_o,2)$ or $\Delta(t_o,x_o)=(0,p_o)$ for suitable $q_o>0$ or $p_o\in]0,2[$. Thus, $\mathcal{F}_q(t_o,q_o,2)-x_o=0$ or $\mathcal{F}_q(t_o,0,p_o)-x_o=0$. Introduce the functions

$$\begin{array}{rclrcl} F\colon]0,+\infty[^3 & \longrightarrow & \mathbb{R} & G\colon]0,+\infty[^2\times]0,2[& \longrightarrow & \mathbb{R}\\ (t,x,q) & \longmapsto & \mathcal{F}_q(t,q,2)-x & (t,x,p) & \longmapsto & \mathcal{F}_q(t,0,p)-x\,. \end{array}$$

By Lemma 5.1, both F and G are of class $\mathbf{C}^2$, $F(t_o,x_o,q_o)=0$ or $G(t_o,x_o,p_o)=0$.

Moreover, Lemma 5.10 implies that for all $t>0$, $q\mapsto\mathcal{F}_q(t,q,2)$ is increasing and therefore,

$$\forall\,(t,q)\in]0,+\infty[^2,\quad \partial_q\mathcal{F}_q(t,q,2)\geq 0\,.$$

If $\partial_q\mathcal{F}_q(t_o,q_o,2)=0$, then t_o minimizes the map $t\mapsto\partial_q\mathcal{F}_q(t,q_o,2)$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\partial_q\mathcal{F}_q(t,q_o,2)=0$ and $y\colon t\mapsto\partial_q\mathcal{F}_q(t,q_o,2)$ solves the Cauchy problem

$$\begin{cases}\ddot{y}(t)=-g''\left(\mathcal{F}_q(t,q_o,2)\right)y(t)\\ y(t_o)=0\\ \dot{y}(t_o)=0\,.\end{cases}$$

The uniqueness of solutions is ensured by Cauchy Lipschitz theorem, we thus have that $y\equiv 0$. On the other hand, deriving (5.29) with respect to q_o, we see that y also solves

$$\begin{cases}\ddot{y}(t)=-g''\left(\mathcal{F}_q(t,q_o,2)\right)y(t)\\ y(0)=1\\ \dot{y}(0)=0\,,\end{cases}$$

which is a contradiction. Therefore, $\partial_qF(t_o,x_o,q_o)>0$.

Similarly, Lemma 5.14 implies that for all $t>0,p\mapsto\mathcal{F}_q(t,x,p)$ is increasing and therefore

$$\forall\,(t,p)\in]0,+\infty[\times]0,2[,\quad \partial_p\mathcal{F}_q(t,0,p)\geq 0\,.$$

If $\partial_p\mathcal{F}_q(t_o,0,p_o)=0$, then t_o minimizes the map $t\mapsto\partial_p\mathcal{F}_q(t,0,p_o)$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\partial_p\mathcal{F}_q(t,0,p_o)=0$ and $t\mapsto\partial_p\mathcal{F}_q(t,0,p_o)$ solves the Cauchy problem

$$\begin{cases}\ddot{y}(t)=-g''\left(\mathcal{F}_q(t,0,p_o)\right)y(t)\\ y(t_o)=0\\ \dot{y}(t_o)=0\,.\end{cases}$$

The uniqueness of solutions ensured by Cauchy Lipschitz theorem, we thus have that $y \equiv 0$. On the other hand, deriving (5.29) with respect to p_o, we see that y also solves

$$\begin{cases} \ddot{y}(t) = -g''\left(\mathcal{F}_q(t,0,p_o)\right) y(t) \\ y(0) = 0 \\ \dot{y}(0) = 1\,, \end{cases}$$

which is a contradiction. Therefore, $\partial_p G(t_o, x_o, p_o) > 0$.

The Implicit Function Theorem allows us to obtain a locally unique map Q such that $q_o = Q(t_o, x_o)$ from the relation $F(t_o, q_o, x_o) = 0$ and, in the same way, to obtain $p_o = P(t_o, x_o)$ from the relation $G(t_o, x_o, p_o) = 0$, with both functions Q and P of class $\mathbf{C}^1$. Note that by (5.41), by (5.42), by (1) in Lemma 5.15 and by the local uniqueness of Q and P, we get

$$\Delta(t,x) = \left\{ \begin{array}{ll} (Q(t,x), 2) & \text{if } x > q^\sharp(t) \\ (0, P(t,x)) & \text{if } x < q^\sharp(t) \end{array} \right.$$

and, by (5.45), the $\mathbf{C}^1$ regularity of u in Ω is proved.

We now prove that u solves (**CL**) with H as in (4.1) and initial datum (5.28). To this aim, observe that the map $x \mapsto u(t,x)$ is odd, for all $t \in \mathbb{R}_+$.

Assume $x > 0$. Then, by the Implicit Function Theorem and by (5.45), for all $(t,x) \in \Omega$ we have

$$u(t,x) = \left\{ \begin{array}{ll} \mathcal{F}_p\left(t, Q(t,x), 2\right) & \text{if } x > q^\sharp(t) \\ \mathcal{F}_p\left(t, 0, P(t,x)\right) & \text{if } x < q^\sharp(t) \end{array} \right. \quad \text{and} \quad \begin{array}{ll} \partial_t Q = -\frac{\partial_t \mathcal{F}_q}{\partial_q \mathcal{F}_q} = -\frac{\mathcal{F}_p}{\partial_q \mathcal{F}_q} & \partial_x Q = \frac{1}{\partial_q \mathcal{F}_q}\,; \\ \partial_t P = -\frac{\partial_t \mathcal{F}_q}{\partial_p \mathcal{F}_q} = -\frac{\mathcal{F}_p}{\partial_p \mathcal{F}_q} & \partial_x P = \frac{1}{\partial_p \mathcal{F}_q}\,. \end{array}$$

Hence, recalling also (5.29)

$$\partial_t u + u\,\partial_x u = \left\{ \begin{array}{l} \partial_t \mathcal{F}_p + \partial_q \mathcal{F}_p\, \partial_t Q + \mathcal{F}_p\, \partial_q \mathcal{F}_p\, \partial_x Q = -g' \\ \partial_t \mathcal{F}_p + \partial_p \mathcal{F}_p\, \partial_t P + \mathcal{F}_p\, \partial_p \mathcal{F}_p\, \partial_x P = -g' \end{array} \right.$$

ensuring that u solves (**CL**)–(4.1) in the classical sense in Ω. The case $x < 0$ is entirely similar by (5.45), since H is even in x and p.

Claim 2: Conclusion. The monotonicity proved in Lemma 5.15 ensures that Δ, and hence u, admits traces along $x = 0$ for all $t > 0$ and $u(t, 0-) = -u(t, 0+)$, because $u(t)$ is odd. Since $p \mapsto \frac{p^2}{2}$ is an even function, by (4.1) and (5.45) $H\left(0, u(t,0+)\right) = H\left(0, u(t,0-)\right)$. Hence, either $u(t,0+) = u(t,0-)$, or Rankine-Hugoniot conditions hold along the stationary discontinuity along $x = 0$.

Assume that $u(t,0+) \neq u(t,0-)$. Then, $u(t,0+) \neq 0$. Moreover, by (5.45), for a positive sequence x_n converging to 0, we have that $u(t,x_n) \neq 0$ and has a fixed signed for all n. Hence, $\Delta(t,x_n) = (0,p_n)$ with $p_n \neq 0$. Lemmas 5.11 and 5.12 then ensure that $p_n \in \left]0, \sqrt{2}\right[$. Up to a subsequence, $\lim_{n\to+\infty} p_n = p_*$ for a suitable $p_* > 0$ (for, otherwise, u would vanish).

$$\begin{aligned} u(t,0+) &= \lim_{n\to+\infty} u(t,x_n) \\ &= \lim_{n\to+\infty} \mathcal{F}_p\left(t, \Delta(t,x_n)\right) && \text{[By (5.45)]} \\ &= \lim_{n\to+\infty} \mathcal{F}_p(t,0,p_n) && \text{[By (5.41)–(5.42)]} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_q(t,0,p_n) && \text{[By (5.29)]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_q(t,0,p_*)\,. && \text{[By Lemma 5.1]}
\end{aligned}
$$

Note that $\mathcal{F}_q(t,0,p_*) = 0$, so that by (5.42) the map $t \mapsto \mathcal{F}_q(t,0,p_*)$ passes from positive to negative at t, showing that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_q(t,0,p_*) \leq 0$, so that $u(t,0+) \leq u(t,0-)$. By (**CVX**), we obtain the Lax Entropy Inequality [10, Section 11.9] at $x = 0$.

Consider now the initial datum. By (3) in Lemma 5.15, for $x \in \,]0,+\infty[$, $\lim_{t\to 0+} u(t,x) = \lim_{t\to 0+} \mathcal{F}_p\left(t,\Delta(t,x)\right) = \mathcal{F}_p\left(0,x,2\right) = 2 = u_o(x)$. The case $x < 0$ is entirely analogous.

Along Γ, u is continuous so that Rankine-Hugoniot and Lax entropy conditions are met. Hence, u is an entropy solution to (**CL**)–(4.1)–(5.28) both where $x > 0$ and, by symmetry, also where $x < 0$. Along $x = 0$, Rankine-Hugoniot conditions and the usual Lax entropy inequalities are met, both if u is continuous or not. As $t \to 0+$, $u(t)$ pointwise converges to the initial datum (5.28). A standard argument then ensures that u solves (**CL**)–(4.1)–(5.28) in the sense of Definition 2.1, see [5, Conclusion in the proof of Lemma 3.3] for the details. The proof is completed. □

Proof Proof of Theorem 4.1 Let $T > 0$. Set $w = u(T,\cdot)$ with u as in (5.45). By Proposition 5.16, $I_T^{CL}(w) \neq \emptyset$ and $u_o \in I_T^{CL}(w)$, where u_o is as in (5.28). Moreover, thanks to the regularity of u outside $x = 0$ proved in Proposition 2.4 and to [11, Theorem 4.1], since $\pi_w(\mathbb{R}) = \mathbb{R}$, by Theorem 3.3 we deduce that $I_T^{CL}(w) = \{u_o\}$.

To complete the proof, use $\mathcal{T}$ as defined in (5.31), note that a shock in u first arises at time $\inf_{p_o\in[0,\sqrt{2}]} \mathcal{T}(p_o)/2 = \pi/(2\sqrt{2})$ by (iii) in Lemma 5.13. The growth of the shock size follows from the fact that $\mathcal{T}$ is strictly increasing. □

A. Appendix to Section 5

Lemma A.1. *The polynomial* $P(x) \coloneqq x^8 - \frac{32}{7}x^6 + \frac{59}{7}x^4 - 8x^2 - \frac{6}{7}$ *is negative for all* $x \in [-1,1]$.

Proof of Lemma A.1 The Sturm sequence, see [36], of P is:

	sign at − 1	sign at 1
$x^8 - \frac{32}{7}x^6 + \frac{59}{7}x^4 - 8x^2 - \frac{6}{7}$	−	−
$8x^7 - \frac{192}{7}x^5 + \frac{236}{7}x^3 - 16x$	+	−
$\frac{8}{7}x^6 - \frac{59}{14}x^4 + 6x^2 + \frac{6}{7}$	+	+
$-\frac{29}{14}x^5 + \frac{58}{7}x^3 + 22x$	−	+
$-\frac{5}{14}x^4 - \frac{526}{29}x^2 - \frac{6}{7}$	−	−
$-\frac{3972}{35}x^3 - \frac{944}{35}x$	+	−

$\frac{3639116}{201579}x^2+\frac{6}{7}$	$+$	$+$
$\frac{137451770}{6368453}x$	$-$	$+$
$-\frac{6}{7}$	$-$	$-$

Sturm theorem, see [36], ensures that P has $4-4=0$ roots in $[-1,1]$. Therefore, for all $x \in [-1,1]$, $P(x)$ has the sign of $P(0)=-6/7<0$. □

Lemma A.2. *Let $\mathcal{A}$ be as in (5.36) with g as in (4.1). Then, $\mathcal{A}$ is continuous on $]0,1[$.*

Proof of Lemma A.2 For $(r,\theta) \in \,]0,1[\, \times [0,1[$, define $\mathcal{B}(r,\theta) := r\big/\sqrt{g(r)-g(\theta\, r)}$. $\mathcal{B}$ is positive. For any $\varepsilon \in \,]0,1/2[$, fix $r \in [\varepsilon, 1-\varepsilon]$. Then, for $\theta \in [0,1/2]$, $\mathcal{B}(r,\theta) \leq \max_{[\varepsilon,1-\varepsilon]\times[0,1/2]} \mathcal{B}$, while for $\theta \in [1/2,1[$

$$\mathcal{B}(r,\theta) \leq \frac{1}{\sqrt{1-\theta}\sqrt{\min_{\rho\in[\varepsilon/2,1-\varepsilon]} g'(\rho)}}.$$

Hence, $\mathcal{B}$ is continuous and dominated, therefore $\mathcal{A}$ is continuous, too. □

B. Appendix: technical results from the literature

We collect below definitions and technical statements, adapted to the present notations, that were used in the preceding proofs and can be found in the literature. Detailed references are provided.

Definition B.1. [20, Definition 1.1.1] *Let $A \subset \mathbb{R}^n$ be open. A function $u\colon A \to \mathbb{R}$ is* semiconcave *if $u \in \mathbf{C}^0(A;\mathbb{R})$ and there exists $C \geq 0$ such that $u(x+h)+u(x-h)-2u(x) \leq C\,\|h\|^2$ for all $x,h \in \mathbb{R}^n$ such that $\{\theta\, x+(1-\theta)h\colon \theta \in [0,1]\} \subset A$.*

Lemma B.2. [20, Theorem 5.3.8] *Let $U \in \mathbf{Lip}_{\mathbf{loc}}(]0,T[\, \times \mathbb{R};\mathbb{R})$ be a viscosity solution of $\partial_t U + H(x,\partial_x U) = 0$ in $]0,T[\, \times \mathbb{R}$, where $H \in \mathbf{Lip}_{\mathbf{loc}}(\mathbb{R}^2;\mathbb{R})$ is strictly convex in the last variable. Then U is locally semiconcave in $]0,T[\, \times \mathbb{R}$.*

Lemma B.3. [26, Lemma 8.1.3] *Let H satisfy (C3), (CNH), and (CVX). Then L, as defined in (2.5), also satisfies (C3), (CNH), and (CVX). Moreover,*

$$\begin{aligned}
&\forall \lambda > 0,\ \forall x \in \mathbb{R},\ \forall v \in \mathbb{R}^*, && \frac{L(x,v)}{|v|} \geq \lambda - \frac{1}{|v|}\left(\sup_{\substack{y\in\mathbb{R}\\ |u|\leq\lambda}} H(y,u)\right);\\
&\forall \lambda > 0,\ \forall x \in \mathbb{R},\ \forall u \in \mathbb{R}^*, && \frac{H(x,u)}{|u|} \geq \lambda - \frac{1}{|u|}\left(\sup_{\substack{y\in\mathbb{R}\\ |v|\leq\lambda}} L(y,v)\right).
\end{aligned} \tag{B.1}$$

Lemma B.4. [26, Corollary 8.1.4] *Let H satisfy (C3), (CNH), and (CVX). Then, H and L, as defined in (2.5), have Nagumo growth, i.e., there exists a $\phi \in \mathbf{C}^0(\mathbb{R}^+;\mathbb{R})$ such that $\frac{1}{r}\phi(r) \xrightarrow[r\to+\infty]{} +\infty$ and $H(x,v) \geq \phi(|v|)$ for all $x,v \in \mathbb{R}$.*

Lemma B.5. [26, Corollary 8.3.7] *Let H satisfy* (**C3**), (**CNH**), *and* (**CVX**). *Fix $t > 0$, $x \in \mathbb{R}$ and $U_0 \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$. With the notation* (2.4), *call y a Lipschitz minimizer of $\mathcal{J}_t$ in $\left\{y \in \mathbf{W}^{1,1}(]0,t[;\mathbb{R}) \colon y(t)=x\right\}$. Then, y is in $\mathbf{W}^{2,\infty}(]0,t[;\mathbb{R})$ and satisfies the weak form of the Euler-Lagrange equations, i.e,*

$$\forall \phi \in \mathbf{C}_{\mathbf{c}}^{\infty}(]0,t[;\mathbb{R}), \qquad \int_0^t \dot\phi(s)\, \partial_v L\left(y(s), \dot y(s)\right) \mathrm{d}s = -\int_0^t \phi(s)\, \partial_x L\left(y(s), \dot y(s)\right) \mathrm{d}s\ .$$

Lemma B.6. [26, Theorem 8.3.9] *Let H satisfy* (**C3**) *and* (**CVX**). *There exists a constant $C_H > 0$ such that for all $t > 0$, for all $x \in \mathbb{R}$ and for all Lipschitz minimizers y of $\mathcal{J}_t$ in $\left\{y \in \mathbf{W}^{1,1}(]0,t[;\mathbb{R}) \colon y(t)=x\right\}$, we have $\left\|\dot y\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \leq C_H$.*

Lemma B.7. [26, Theorem 8.3.12] *Let H satisfy* (**C3**) *and* (**CVX**). *With the notation* (2.4), *define $U(t,x) := \min\left\{\mathcal{J}_t(y) \colon y \in \mathbf{W}^{1,1}(]0,t[;\mathbb{R}) \text{ and } y(t)=x\right\}$, for $(t,x) \in \,]0,T[\times \mathbb{R}$. Then, $U \in \mathbf{Lip}(]0,T[\times \mathbb{R};\mathbb{R})$, U is a viscosity solution to* (**HJ**) *with initial data U_0 and for all $(t,x),(\tau,\xi) \in \,]0,T[\times \mathbb{R}$, using the constant C_H from* Lemma B.6,

$$\begin{aligned} \left|U(x,t) - U(\xi,\tau)\right| \leq{} & T\left(\sup_{\substack{y\in\mathbb{R}\\ |v|\leq C_H}} \left|\partial_x L(y,v)\right| + \left\|U_0'\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\right)|x-\xi| \\ & + \left(\sup_{\substack{y\in\mathbb{R}\\ |v|\leq C_H}} \left|L(y,v)\right| + \left\|U_0'\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} C_H\right)|t-\tau|\,. \end{aligned}$$

Lemma B.8. [26, Lemma 8.3.13] *Let $U \in \mathbf{Lip}(]0,T[\times \mathbb{R};\mathbb{R})$ be the viscosity solution to* (**HJ**) *with initial data $U_0 \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$. Fix $\xi, \zeta \in \mathbf{Lip}([0,T];\mathbb{R})$, $\xi \leq \zeta$. Then, for all $s, \tau \in [0,T]$ with $s < \tau$, we have:*

$$\begin{aligned} & \int_{\xi(\tau)}^{\zeta(\tau)} U(x,\tau)\,\mathrm{d}x - \int_{\xi(s)}^{\zeta(s)} U(x,s)\,\mathrm{d}x + \int_s^\tau \int_{\xi(t)}^{\zeta(t)} H\left(x, \partial_x U(x,t)\right) \mathrm{d}x\,\mathrm{d}t \\ = {} & \int_s^\tau \left(\dot\zeta(t)\; U\left(\zeta(t),t\right) - \dot\xi(t)\; U\left(\xi(t),t\right)\right) \mathrm{d}t\ . \end{aligned}$$

The previous Lemma is analogous to [11, Lemma 3.2].

Lemma B.9. [26, Corollary 8.3.15] *Let H satisfy* (**C3**) *and* (**CVX**). *Fix $t > 0$, $x \in \mathbb{R}$ and $U_0 \in \mathbf{Lip}(\mathbb{R};\mathbb{R})$. Then, with the notations* (2.4) *and* (2.2),

$$\begin{aligned} \min\left\{\mathcal{J}_t(y) \colon y \in \mathbf{W}^{1,1}(]0,t[;\mathbb{R}) \text{ and } y(t)=x\right\} & = \min\left\{\mathcal{J}_t(y) \colon y \in \mathbf{Lip}(]0,t[;\mathbb{R}) \text{ and } y(t)=x\right\} \\ & = \min\left\{\mathcal{J}_t(y) \colon y \in \mathcal{R}_t \text{ and } y(t)=x\right\}\ . \end{aligned}$$

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