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Pointwise stabilization of Bresse systems

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Abstract. Bresse system over the interval $(0, L)$ with pointwise dissipation at $\xi \in (0, L)$ is analyzed. The exponential stability of the related semigroup is shown provided the dissipative points are of the form $\xi \in \mathbb{Q}L$ and $\xi \neq \frac{n}{2m+1}L$, where $n, m \in \mathbb{N}$ and n, and $2m + 1$ are co-prime.

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1. Introduction

In this paper we study the mechanical behaviour of a Bresse system, also known as the circular arch problem, of natural length L, with a pointwise dissipative mechanism at $\xi \in (0, L)$.

Let $0 < T \leq \infty$. We denote by $w = w(x, t) : (0, L) \times (0, T) \to \mathbb{R}$, $\varphi = \varphi(x, t) : (0, L) \times (0, T) \to \mathbb{R}$, and $\psi = \psi(x, t) : (0, L) \times (0, T) \to \mathbb{R}$, the longitudinal, vertical and shear angle displacements of the cross section at $x \in (0, L)$ and at time $t \in (0, T)$, respectively. The Bresse system (see Fig. 1) we consider here is the following (see, e.g., [5,11])

$$\begin{aligned} &\rho_1\, \varphi_{tt} - \kappa\, (\varphi_x + \psi + \ell\, w)_x - \ell\, \kappa_0\, (w_x - \ell\, \varphi) + \gamma_1 \delta(x - \xi)\varphi_t = 0 \;\text{ in }\; (0, L) \times (0, T),\\ &\rho_2\, \psi_{tt} - b\psi_{xx} + \kappa\, (\varphi_x + \psi + \ell\, w) + \gamma_2 \delta(x - \xi)\psi_t = 0 \;\text{ in }\; (0, L) \times (0, T),\\ &\rho_1\, w_{tt} - \kappa_0\, (w_x - \ell\, \varphi)_x + \ell\, \kappa\, (\varphi_x + \psi + \ell\, w) + \gamma_3 \delta(x - \xi) w_t = 0 \;\text{ in }\; (0, L) \times (0, T). \end{aligned} \tag{1.1}$$

Let us denote by $N = \kappa_0\, (w_x - \ell\, \varphi)$ the axial force, by $S = \kappa\, (\varphi_x + \psi + \ell\, w)$ the shear force and by $M = b\psi_x$, is the bending moment, where $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa_0 = EA$, $\kappa = k'GA$, $b = EI$ and $\ell = R^{-1}$. Coefficients aforementioned, all assumed positive, represent: Moreover, $\delta(x - \xi)$ is the Dirac mass $+1$ at

- ρ the density,	- E the modulus of elasticity,
- G the shear modulus,	- k' the shear factor,
- A the cross-sectional area,	- I the second moment of area of the cross-section,
- R the radius of curvature,	- ℓ the curvature.

the point $x = \xi$. Lastly, γ_i, $i = 1, 2, 3$, denote positive damping coefficients. We supplement (1.1) with the initial conditions

$$\varphi(x, 0) = \varphi_0, \;\; \varphi_t(x, 0) = \varphi_1, \;\; \psi(x, 0) = \psi_0, \;\; \psi_t(x, 0) = \psi_1, \;\; w(x, 0) = w_0, \;\; w_t(x, 0) = w_1. \tag{1.2}$$

In addition, we consider the following boundary conditions

$$\varphi(0, t) = 0, \quad S(L, t) = 0, \quad \psi_x(0, t) = \psi(L, t) = 0, \quad N(0, t) = 0, \quad w(L, t) = 0, \quad \forall t \geq 0.$$

In particular, recalling the definition of S and N we have

$$\varphi(0, t) = \varphi_x(L, t) = 0, \quad \psi_x(0, t) = \psi(L, t) = 0, \quad w_x(0, t) = w(L, t) = 0, \quad \forall t \geq 0. \tag{1.3}$$

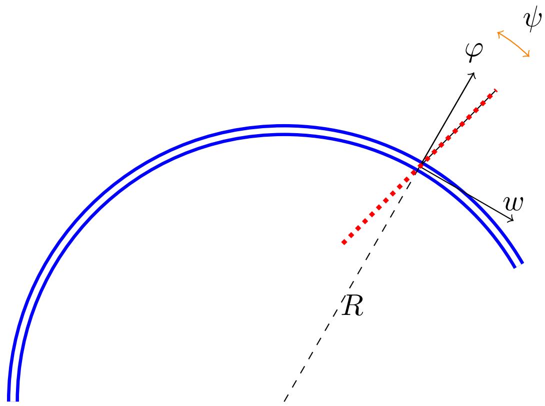

FIG. 1. The circular arch

The asymptotic behavior of Bresse systems, with different dissipative mechanism, has been studied in recent years for several authors, see e.g. [1,7,9,10,12–14,18–23] and the references therein. Alabau Boussouira et al. [1] investigated the Bresse system with frictional dissipation present only in the equation of angular displacement. The equalities

$$\frac{\rho_1}{\rho_2} = \frac{\kappa}{b} \qquad \text{and} \qquad \kappa = \kappa_0. \tag{1.4}$$

have been observed as necessary and sufficient conditions for exponential decay of the system and, in the general case, the system is polynomially stable. Liu and Rao [12] analyzed the decay of thermoelastic Bresse system, and they showed that the system is exponentially stable if and only if the propagation speeds (1.4) holds. Otherwise, the authors proved the lack of exponential stability and that the system decays polynomially to zero. The above result has been improved by Fatori and Muñoz Rivera [10] considering the thermoelastic Bresse problem with the temperature affective only over the bending moment. They found the exponential stability of the system if and only if relation (1.4) holds. If not, then the system decays polynomially. Condition (1.4) is only mathematically sound. Physically it is not acceptable.

Ammari et al. [2] studied point stabilization for the Rayleigh and Euler Bernoulli beam equation using two dissipative mechanisms at the point ξ of the interval $]0,\pi[$, one acting at the shear force and another applied at the bending moment. There, the authors proved that the Rayleigh beam equation is exponentially stable when the point, where these dissipative mechanisms are applied, is strategically chosen so that $\frac{\xi}{\pi} = \frac{n}{2m+1} \in \mathbb{Q}$ and that, in general, the system decays polynomially when these mechanisms are applied at an arbitrary point of the interval $]0,\pi[$. Instead for the Euler Bernoulli beam the system is always exponentially stable for any point ξ of the interval $]0,\pi[$.

On the other hand, Ammari and Tucsnak [3] considered a pointwise stabilization for the Euler Bernoulli beam equation using only one dissipative mechanisms at the point ξ. For this case, the stabilization depend on the choose of the point $\xi \in]0,\pi[$, that is if ξ verifies $\frac{\xi}{\pi} = \frac{n}{2m+1} \in \mathbb{Q}$ then the system decays exponentially, but unlike the previous result in [2], in general the solution does not decay to zero for an infinity of points in the interval.

In this paper we show the exponential decay of system (1.1)–(1.3) when $\xi \in \mathbb{Q}L$ satisfies the following property

$$\xi \neq \frac{n}{2k+1}L, \quad \forall n,k \in \mathbb{N}, \qquad \text{with} \quad n, \quad \text{and} \quad 2k+1 \quad \text{are co-prime.} \tag{1.5}$$

To our knowledge, this result is new in the context of Bresse systems.

We recently showed the exponential stability for the Timoshenko model [14]. However, we would like to emphasize that the Bresse curved beam model presents difficulties that go beyond being a system of partial differential equations 3 by 3. For example, the energy associated with the Bresse system does not always define a norm for the corresponding phase space see Lemma 2.1 below. Remember that the norm chooses in the phase space is important to verify if the model is dissipative or not, so it is an important piece in the study of stabilization. Other point is the complexity of the axial force and the shear force which involve the sum of two and three functions.

Plan of the paper

Instead of directly analyzing the Bresse problem with point dissipation at $x = \xi$, in Sect. 2 we study the corresponding transmission problem by eliminating the singularity at $x = \xi$ and in its instead we consider the transmission conditions at $x = \xi$ given by (2.2)-(2.4) below. In Sect. 3 we demonstrate a characterization of exponential stability in Banach spaces. Finally, in Sect. 4 we demonstrate the main result of this article about the exponential stability of system (1.1) by using the Riemann invariants and the result due to Neves et al. [15, Theorem A].

2. Preliminaries and semigroup setting

To apply the semigroup approach, we rewrite the system (1.1)–(1.3) as a transmission problem. Let us denote by $\mathbf{I}$ the open set

$$\mathbf{I} := (0, \xi) \cup (\xi, \ell).$$

It is easy to show that system (1.1)–(1.3) is equivalent to

$$\begin{array}{ll} \rho_1 \varphi_{tt} - \kappa (\varphi_x + \psi + \ell w)_x - \ell \kappa_0 (w_x - \ell \varphi) = 0 \text{ in} & \mathbf{I} \times (0, T), \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa (\varphi_x + \psi + \ell w) = 0 \quad \text{in} & \mathbf{I} \times (0, T), \\ \rho_1 w_{tt} - \kappa_0 (w_x - \ell \varphi)_x + \ell \kappa (\varphi_x + \psi + \ell w) = 0 \text{ in} & \mathbf{I} \times (0, T), \end{array} \tag{2.1}$$

verifying initial conditions (1.2), boundary conditions (1.3), and supplemented with the transmission conditions on $\xi \in (0, L)$, given by

$$\varphi(\xi^-, t) = \varphi(\xi^+, t), \quad w(\xi^-, t) = w(\xi^+, t), \quad \psi(\xi^-, t) = \psi(\xi^+, t) \tag{2.2}$$

$$k\varphi_x(\xi^-, t) - k\varphi_x(\xi^+, t) = -\gamma_1 \varphi_t(\xi, t), \quad \kappa_0 w_x(\xi^-, t) - \kappa_0 w_x(\xi^+, t) = -\gamma_3 w_t(\xi, t), \tag{2.3}$$

$$b\psi_x(\xi^-, t) - b\psi_x(\xi^+, t) = -\gamma_2 \psi_t(\xi, t). \tag{2.4}$$

Relation (2.2) represents the continuity of the vertical, longitudinal and shear angle displacements and relations (2.3)-(2.4) the discontinuity of the shear and axial force and the bending moment at $x = \xi$. Note that when $\gamma_i = 0$, $i = 1, 2, 3$, the model is conservative. Instead, if $\gamma_i > 0$, $i = 1, 2, 3$, then system (2.1) is dissipative.

Let us fix now some general notation that will be used throughout the paper. Let X be a complex Banach space with norm $\|\cdot\|_X$. In particular, we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm defined on $L^2(0, L)$, respectively. Let us denote by $\Phi = \varphi_t$, $\Psi = \psi_t$ and $W = w_t$ and let us put

$$U = (\varphi, \Phi, \psi, \Psi, w, W)^\top,$$

where $(\cdot)^\top$ denotes transpose. The phase space we consider here is given by

$$\mathcal{H} := V_0 \times L^2(0, L) \times V_L \times L^2(0, L) \times V_L \times L^2(0, L).$$

where

$$V_0 = \left\{ f \in H^1(0, L) : \quad f(0) = 0 \right\}, \quad V_L = \left\{ f \in H^1(0, L) : \quad f(L) = 0 \right\}.$$

The phase space $\mathcal{H}$ together with the "norm"

$$\|U\|_{\mathcal{H}}^2 = \rho_1 \|\Phi\|^2 + \rho_2 \|\Psi\|^2 + \rho_1 \|W\|^2 + b\|\psi_x\|^2 + \kappa \|\varphi_x + \psi + \ell w\|^2 + \kappa_0 \|w_x - \ell \varphi\|^2. \tag{2.5}$$

is a Hilbert space. Let us denote the jump of f in ξ as

$$[\![f]\!]_\xi := f(\xi^-) - f(\xi^+).$$

Under this notations we rewrite the transmission conditions (2.2)–(2.4) as

$$[\![\varphi]\!]_\xi = [\![\psi]\!]_\xi = [\![w]\!]_\xi = 0 \tag{2.6}$$

$$[\![k\varphi_x]\!]_\xi = -\gamma_1\Phi(\xi), \quad [\![b\psi_x]\!]_\xi = -\gamma_2\Psi(\xi), \quad \text{and} \quad [\![\kappa_0 w_x]\!]_\xi = -\gamma_3 W(\xi). \tag{2.7}$$

Let us denote by $\mathbb{A}$ the linear operator given by

$$\mathbb{A} = \begin{bmatrix} 0 & I_d & 0 & 0 & 0 & 0 \\ \frac{\kappa}{\rho_1}\partial_x^2 - \frac{\kappa_0\ell^2}{\rho_1}I_d & 0 & \frac{\kappa}{\rho_1}\partial_x & 0 & \frac{(\kappa+\kappa_0)\ell}{\rho_1}\partial_x & 0 \\ 0 & 0 & 0 & I_d & 0 & 0 \\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{\kappa}{\rho_2}I_d & 0 & -\frac{\kappa\ell}{\rho_2}I_d & 0 \\ 0 & 0 & 0 & 0 & 0 & I_d \\ -\frac{(\kappa_0+\kappa)\ell}{\rho_1}\partial_x & 0 & -\frac{\kappa\ell}{\rho_1}I_d & 0 & \frac{\kappa_0}{\rho_1}\partial_x^2 - \frac{\kappa\ell^2}{\rho_1}I_d & 0 \end{bmatrix}, \tag{2.8}$$

where I_d is the identity operator, $\partial_x := \frac{\partial}{\partial x}$ and $\partial_x^2 := \frac{\partial^2}{\partial x^2}$, with domain $\mathcal{D}(\mathbb{A})$ given by

$$\mathcal{D}(\mathbb{A}) = \left\{U \in \mathcal{H}: \quad \Phi \in V_0, \quad \Psi, W \in V_L, \; \varphi, \psi, w \in H^2(\mathbf{I}), \text{ verifying (1.3), (2.6), (2.7)}\right\}.$$

Then system (2.1)-(2.4) can be written as

$$\frac{d}{dt}U = \mathbb{A}\,U, \quad U(0) = U_0 \tag{2.9}$$

where $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^\top$. Indeed, the operator $\mathbb{A}$ is dissipative in $\mathcal{H}$ and verifies

$$\mathrm{Re}\langle \mathbb{A}U, \overline{U}\rangle_{\mathcal{H}} = -\gamma_1|\Phi(\xi)|^2 - \gamma_2|\Psi(\xi)|^2 - \gamma_3|W(\xi)|^2 \le 0, \quad \forall U \in \mathcal{D}(\mathbb{A}). \tag{2.10}$$

Next, we will show that the functional given in (2.5) defines a norm in the phase space only if we correctly choose the coefficients of the system. The condition we use here is

$$\left(\frac{2j+1}{2L}\right)^2 \pi^2 \ne \ell^2, \quad \forall j \in \mathbb{N}. \tag{2.11}$$

Lemma 2.1. *The functional defined in* (2.5) *is a norm provided condition* (2.11) *holds.*

Proof. We prove the positivity of the functional given in (2.5). Suppose that $\|\mathbf{U}\|_{\mathcal{H}} = 0$. So we have that $\Phi = \Psi = W = 0$ and

$$\psi_x = 0, \quad \varphi_x + \psi + \ell w = 0, \quad w_x - \ell\varphi = 0.$$

Because of the boundary condition, we have $\psi = 0$. So, we get

$$\varphi_x + \ell w = 0, \quad w_x - \ell\varphi = 0, \quad \Rightarrow \quad \varphi_{xx} + \ell^2\varphi = 0.$$

This means that ℓ^2 is an eigenvalue. Since $\mathcal{D}(\mathbb{A})$ must be dense in $\mathcal{H}$, we can assume that φ verifies $\varphi_x(L) = 0$. Therefore, we find that φ is an eigenvector of the form $\varphi(x) = A_k \sin\left(\frac{2k+1}{2L}\pi x\right)$ with the corresponding eigenvalue $(\frac{2k+1}{2L})^2\pi^2$. But this is not possible, because our assumption (2.11). Hence $\varphi = 0$. By similar procedure, we obtain $w = 0$. So, our conclusion follows.

Lemma 2.2. *Let us suppose that $\varphi(0) = 0$ with $\varphi_x + \ell w,\ w_x - \ell\varphi \in L^2(0, L)$. Then there exists $c > 0$ such that*

$$\int_0^L \left(|\varphi|^2 + |w|^2\right)\, dx \le c\int_0^x \left(|\varphi_x + \ell w|^2 + |w_x - \ell\varphi|^2\right)\, dx.$$

Proof. Let us denote by $\varphi_x + \ell w = F, \quad w_x - \ell\varphi = G$, so we get

$$(\varphi + w)_x + \ell(w - \varphi) = F + G, \quad (\varphi - w)_x + \ell(\varphi + w) = F - G.$$

multiplying the first equation by $\varphi + w$ and the second by $\varphi - w$ we arrive to

$$\frac{1}{2}\frac{d}{dx}\left(|\varphi + w|^2 + |\varphi - w|^2\right) = (F+G)(\varphi + w) + (F-G)(\varphi - w) := R$$

Integrating over $]0, x[$ and using that $\varphi(0) = 0$ we get

$$|\varphi + w|^2 + |\varphi - w|^2 - 2|w(0)|^2 = \int_0^x R\, dx$$

$$\leq c\int_0^x \left(|\varphi + w|^2 + |\varphi - w|^2\right) dx + c\int_0^x \left(|F|^2 + |G|^2\right) dx. \tag{2.12}$$

Applying the Gagliardo-Nirenberg's inequality over the interval $[0, x]$ we get

$$|w(0)| \leq c\|\varphi + w\|_{L^2(0,x)}^{1/2}\|\varphi + w\|_{H^1(0,x)}^{1/2} \leq c\|\varphi + w\|_{L^2(0,x)} + \|\varphi + w\|_{L^2(0,x)}^{1/2}\|\varphi_x + w_x\|_{L^2(0,x)}^{1/2}. \tag{2.13}$$

Using $\varphi_x + w_x = \ell(\varphi - w) + F + G$ into (2.13) and inserting into inequality (2.12) we get

$$|\varphi + w|^2 + |\varphi - w|^2 \leq c\int_0^x \left(|\varphi + w|^2 + |\varphi - w|^2\right) dx + c\int_0^x \left(|F|^2 + |G|^2\right) dx.$$

Using Gronwall's inequality we get that

$$|\varphi + w|^2 + |\varphi - w|^2 \leq c\int_0^x \left(|F|^2 + |G|^2\right) dx,$$

from where our conclusion follows.

Lemma 2.3. *The functional defined in* (2.5) *is a norm equivalent to the usual Sobolev's norm of* $\mathcal{H}$ *provided condition* (2.11) *holds.*

Proof. In fact, let U_ν be a Cauchy sequence with the norm $\|\cdot\|_{\mathcal{H}}$. Hence Φ_ν, Ψ_ν, W_ν are Cauchy sequences in $L^2(0, L)$ and ψ_ν is a Cauchy sequence in $H^1(0, L)$. This implies that $\varphi_{\nu,x} + \omega_\nu$ and $\omega_{\nu,x} - l\varphi_\nu$ are also Cauchy sequences in $L^2(0, L)$. Using Lemma 2.2 we get that φ_ν and ω_ν are Cauchy sequences in $L^2(0, L)$ and therefore also in $H^1(0, L)$. Hence the norm $\|\cdot\|_{\mathcal{H}}$ generate the space $\mathcal{H}$ that is $\overline{\mathcal{H}}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H}$ and since

$$\|U\|_{\mathcal{H}}^2 \leq c\int_0^L \left(|\Phi|^2 + |\psi|^2 + |W|^2 + |\varphi_x|^2 + |\psi|^2 + |\psi_x|^2 + |w|^2 + |w_x|^2\right) dx$$

the open mapping Theorem guarantees the equivalence of the above norms, see [6, Corollary 2.8].

Let us take $F := (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, the resolvent system is given by

$$i\lambda U - \mathbb{A}U = F, \tag{2.14}$$

namely

$$i\lambda\varphi - \Phi = f_1, \tag{2.15}$$

$$i\lambda\rho_1\Phi - S_x - \ell N = \rho_1 f_2, \tag{2.16}$$

$$i\lambda\psi - \Psi = f_3, \tag{2.17}$$

$$i\lambda\rho_2\Psi - M_x + S = \rho_2 f_4, \tag{2.18}$$

$$i\lambda w - W = f_5, \tag{2.19}$$
$$i\lambda\rho_1 W - N_x + \ell S = \rho_1 f_6, \tag{2.20}$$

satisfying boundary conditions

$$\varphi(0) = \varphi_x(L) = \psi_x(0) = \psi(L) = w_x(0) = w(L) = 0,$$

together with the transmission conditions (2.6) and (2.7).

To show that $\mathbb{A}$ is the infinitesimal generator we use the following result

Theorem 2.4. *Let $\mathcal{A}$ be dissipative with $0 \in \varrho(\mathcal{A})$. If $\mathcal{H}$ is reflexive then $\mathcal{A}$ is the infinitesimal generator of a semigroup of contractions.*

Proof. Since $\varrho(\mathcal{A})$ is an open set we have that there exists $\epsilon > 0$ such that $\epsilon \in \varrho(\mathcal{A})$. This implies that any $\lambda > 0$ belongs to $\varrho(\mathcal{A})$, in particular we have that $R(I - \mathcal{A}) = \mathcal{H}$. Using Theorem 4.6 of Pazy [16], we conclude that $\overline{D(\mathcal{A})} = \mathcal{H}$. Using Lummer-Phillips Theorem [16, Theorem 1.4.3], our conclusion follows.

Theorem 2.5. *Let us suppose that ℓ verifies condition (2.11).Then the operator $\mathbb{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\mathcal{T}(t)$ over $\mathcal{H}$.*

Proof. Since $\mathbb{A}$ is dissipative and because of Theorem 2.4 it is enough to show that $0 \in \varrho(\mathbb{A})$. In fact, we will show that for any $F \in \mathcal{H}$, there exist only one $\mathcal{U} \in D(\mathbb{A})$ such that $-\mathbb{A}\mathcal{U} = F$. Let us denote by

$$F = (f_1, f_2, f_3, f_4, f_5, f_6)^\top, \quad \mathcal{U} = (\varphi, \Phi, \psi, \Psi, \omega, W) \in D(\mathbb{A})$$

For $\lambda = 0$ the resolvent system (2.15)–(2.20) can be written as

$$\Phi = -f_1 \quad , \Psi = -f_3 \quad , W = -f_5. \tag{2.21}$$

$$\begin{aligned} -\kappa(\varphi_x + \psi + l\omega)_x - l\kappa_0(\omega_x - l\varphi) &= \rho_1 f_2, \\ -b\psi_{xx} + \kappa(\varphi_x + \psi + l\omega) &= \rho_2 f_4, \\ -\kappa_0(\omega_x - l\varphi)_x + \kappa l(\varphi_x + \psi + l\omega) &= \rho_1 f_6. \end{aligned} \tag{2.22}$$

Let us introduce the space $\mathcal{V} = V_0 \times V_L \times V_0$ denoting $\mathbf{U}^i = (\varphi^i, \psi^i, \omega^i) \in \mathcal{V}$ we conclude that the bilinear form $\mathbf{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$

$$\mathbf{a}(\mathbf{U}^1, \mathbf{U}^2) = \int_0^\ell \kappa(\varphi_x^1 + \psi^1 + l\omega^1)(\varphi_x^2 + \psi^2 + l\omega^2) + b\psi_x^1\psi_x^2 + \kappa_0(\omega_x^1 - l\varphi^1)(\omega_x^2 - l\varphi^2)\, dx$$

is continuous over $\mathcal{V}$. The coercivity is guaranteed by Lemma 2.3. The function given by

$$\mathbf{F}(\mathbf{U}) = \int_0^\ell \left(\rho_1 f_2\overline{\varphi} + \rho_2 f_4\overline{\psi} + \rho_1 f_6\overline{\omega}\right)\, dx + \gamma_1 f_1(\xi)\varphi(\xi) + \gamma_2 f_3(\xi)(\psi(\xi) + \gamma_3 f_5(\xi)\omega(\xi), \tag{2.23}$$

belongs to $\mathcal{V}^*$. So, Lax-Milgran's Lemma guarantees the existence of only one solution $U \in \mathcal{V}$ to problem

$$\mathbf{a}(\mathbf{U}, \tilde{\mathbf{U}}) = \mathbf{F}(\tilde{\mathbf{U}}), \quad \forall\ \tilde{\mathbf{U}} \in \mathcal{V}, \tag{2.24}$$

Taking $\tilde{\mathbf{U}} \in [C_0^\infty(\mathbf{I})]^3$ in (2.24) we conclude that $\mathbf{U} = (\varphi, \psi, \omega)$ verifies (2.22) in the distribucional sense in $[\mathcal{D}'(\mathbf{I})]^3$, hence $(\varphi, \psi, \omega) \in H^2(\mathbf{I})$. Finally, taking $\tilde{\mathbf{U}} \in [H_0^1(0, L)]^3$ in (2.24) using system (2.22) we conclude the solution verifies

$$[\![\varphi_x]\!]_\xi = \gamma_1 f_1(\xi) = -\gamma_1\Phi(\xi), \quad [\![\psi_x]\!]_\xi = \gamma_2 f_3(\xi) = -\gamma_2\Psi(\xi), \quad [\![\omega_x]\!]_\xi = \gamma_3 f_5(\xi) = -\gamma_3 W(\xi).$$

Then we have $\mathcal{U} \in D(\mathbb{A})$, hence $0 \in \varrho(\mathbb{A})$.

Remark 2.6. If condition (2.11) fails then $\|\cdot\|_\mathcal{H}$ is a seminorm. So, we have to consider other norm to use for the phase space, but with such a norm the system will not necessarily be dissipative.

3. A useful tool

We now denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on a complex Banach space X endowed with the operator norm, which again is denoted by $\|\cdot\|$. For an operator $\mathbb{T} \in \mathcal{L}(X)$, $\sigma(\mathbb{T})$ stands for its spectrum, while $\varrho(\mathbb{T}) := \mathbb{C} \setminus \sigma(\mathbb{T})$ is the resolvent set of $\mathbb{T}$.

The main tool we use in this paper is the following result.

Theorem 3.1. *Let $S(t) = e^{\mathbb{A}t}$ be a C_0-semigroup of contractions over a Banach space and let $\omega_{ess}(S(t))$ be the essential growth bound of $S(t)$. Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathbb{A}) \quad \textit{and} \quad \omega_{ess}(S(t)) < 0, \tag{3.1}$$

Proof. Here we use [8, Corollary 2.11, p. 258] establishing that the type ω of the semigroup $e^{\mathbb{A}t}$ verifies

$$\omega = \max\{\omega_{ess}, \omega_\sigma(\mathbb{A})\}, \tag{3.2}$$

where $\omega_\sigma(\mathbb{A})$ is the upper bound of the spectrum of $\mathbb{A}$. Moreover, for any $c > \omega_{ess}$, the set $\mathcal{I}_c := \sigma(\mathbb{A}) \cap \{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda \geq c\}$ is finite.

From (3.1) and (3.2), to show exponential stability, it is enough to prove that $\omega_\sigma(\mathbb{A}) < 0$. If $\omega_\sigma(\mathbb{A}) \leq \omega_{ess}$ then we have nothing to prove. Let us suppose that $\omega_\sigma(\mathbb{A}) > \omega_{ess}$, so the set $\mathcal{I}_{\omega_{ess}+\delta}$ is finite for $\delta > 0$ small such that $\omega_{ess} + \delta < 0$ and $\omega_{ess} + \delta < \omega_\sigma(\mathbb{A})$, moreover $\omega_\sigma(\mathbb{A}) \in \mathcal{I}_{\omega_{ess}+\delta}$. From (3.1) and Hille-Yosida's Theorem we have $\overline{\mathbb{C}_+} \subset \varrho(\mathbb{A})$ hence $\omega_\sigma(\mathbb{A}) < 0$, therefore the sufficient condition follows.

Reciprocally, let us suppose that the semigroup $S(t)$ is exponentially stable, in particular it goes to zero. Then, by [4, Theorem 1.1] we have that $i\mathbb{R} \subset \varrho(\mathbb{A})$. Moreover, since the type ω verifies (3.2), we have that

$$\omega_{ess} \leq \max\{\omega_{ess}, \omega_\sigma(\mathbb{A})\} = \omega < 0.$$

Then, our conclusion follows.

4. Decay of the energy

In what follows, we will apply the characterization due to Prüss [17], reported here below.

Theorem 4.1. *Let $S(t) = e^{\mathbb{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

(i) $i\mathbb{R} \subset \varrho(\mathbb{A})$,
(ii) $\limsup_{|\lambda|\to\infty} \|(i\lambda\mathbb{I} - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty$.

Taking inner product over the resolvent Eq. (2.14) with U, using (2.10) and then taking the real part we get

$$\gamma_1|\Phi(\xi)|^2 + \gamma_2|\Psi(\xi)|^2 + \gamma_3|W(\xi)|^2 = \mathrm{Re}\langle U, F\rangle_{\mathcal{H}}. \tag{4.1}$$

Our starting point is to prove the strong stability of $\mathcal{T}(t)$.

Lemma 4.2. *Let $\mathbb{A}$ be the infinitesimal generator given by (2.8), then $i\mathbb{R} \subset \varrho(\mathbb{A})$, provided condition (1.5) holds.*

Proof. Since $\mathcal{D}(\mathbb{A})$ has compact embedding over the phase space $\mathcal{H}$ and since $0 \in \varrho(\mathbb{A})$, then $\mathbb{A}^{-1}$ is a compact operator. Therefore the spectrum of $\mathbb{A}$ is given only for eigenvalues. Then, we will show that there are not imaginary eigenvalues. By contradiction, let us suppose that there exists an eigenvector $U \neq 0$ verifying

$$\mathbb{A}U = i\lambda U \quad \text{for some} \ \ \lambda \in \mathbb{R}.$$

From (4.1) we get

$$\gamma_1|\Phi(\xi,t)|^2+\gamma_2|\Psi(\xi,t)|^2+\gamma_3|W(\xi,t)|^2=0.$$

This implies that $\varphi(\xi)=\psi(\xi)=w(\xi)=0$. Hence the eigenvectors must be of the form

$$\varphi_k(x)=A_k\sin\left(\frac{2k+1}{2L}\pi x\right),\qquad \psi_k(x)=B_k\cos\left(\frac{2k+1}{2L}\pi x\right),\qquad w_k(x)=C_k\cos\left(\frac{2k+1}{2L}\pi x\right).$$

Using (2.15), (2.17) and (2.19), we get $\varphi_k(\xi)=\psi_k(\xi)=w_k(\xi)=0$. Hence, we have

$$\sin\left(\frac{2k+1}{2L}\pi\xi\right)=0\Leftrightarrow\frac{2k+1}{2L}\pi\xi=j\pi\qquad\Leftrightarrow\xi=\frac{2jL}{2k+1},$$
$$\cos\left(\frac{2k+1}{2L}\pi\xi\right)=0\Leftrightarrow\frac{2k+1}{2L}\pi\xi=j\pi+\frac{\pi}{2}\Leftrightarrow\xi=\frac{(2j+1)L}{2k+1}.$$

Analogously for w_k. But condition (1.5) implies that $A_k=B_k=C_k=0$. So our conclusion follows.

For the sake of simplicity, here and in what follows we shall employ the same symbol C for different constants, even in the same formula.

To show the exponential stability of $\mathcal{T}(t)$ associated to system (2.1)–(2.4), we apply Theorem 3.1. Indeed, we will prove that the essential type $\omega_{ess}(\mathcal{T})$ of $\mathcal{T}(t)$ is negative

To show that, we introduce the semigroup $\mathcal{T}_1(t)$ defined by the system

$$\begin{array}{ll}\rho_1\,\varphi_{tt}-\kappa\,(\varphi_x+\psi+\ell\,w)_x-\ell\,\kappa_0 w_x=0 & \text{in}\quad \mathbf{I}\times(0,T),\\ \rho_2\,\psi_{tt}-b\psi_{xx}+\kappa\varphi_x=0 & \text{in}\quad \mathbf{I}\times(0,T),\\ \rho_1\,w_{tt}-\kappa_0\,(w_x-\ell\,\varphi)_x+\ell\,\kappa\varphi_x=0 & \text{in}\quad \mathbf{I}\times(0,T),\end{array}\tag{4.2}$$

with boundary conditions (1.3) and verifying the initial and transmission conditions (1.2) and (2.2)–(2.4), respectively.

Let us denote by B the operator

$$B:\mathcal{H}\to\mathcal{H},\quad BU=\left(0,\frac{\kappa_0\ell^2}{\rho_1}\varphi,0,\frac{\kappa}{\rho_2}\psi+\frac{\kappa\ell}{\rho_2}w,0,\frac{\kappa\ell}{\rho_1}\psi+\frac{\kappa\ell^2}{\rho_1}w\right).$$

It is easy to verify that B is a compact operator over $\mathcal{H}$. Hence the operator

$$\mathcal{A}_0=\mathbb{A}-B$$

is the infinitesimal generator of a C_0-semigroup denoted by $\mathcal{T}_1(t)=e^{\mathcal{A}_0t}$.

Under the above conditions we have the following Lemma.

Lemma 4.3. *The difference $\mathcal{T}(t)-\mathcal{T}_1(t)$ is a compact operator. Hence the corresponding essential types are equal.*

Proof. Equation $U_t-\mathbb{A}U=0$ can be written as $U_t-\mathcal{A}_0U=BU$. Then the solution can be written as

$$U(t)=e^{\mathcal{A}_0t}U_0+\int_0^t e^{\mathcal{A}_0(t-s)}BU(s)\,ds.\tag{4.3}$$

Recalling the definition of $U(t)$ and $\mathcal{T}_1(t)$, equation (4.3) implies

$$\mathcal{T}(t)U_0=\mathcal{T}_1(t)U_0+\int_0^t e^{\mathcal{A}_0(t-s)}Be^{\mathbb{A}s}U_0\,ds.$$

Since B is a compact operator then the composition $e^{\mathcal{A}_0(t-s)}Be^{\mathbb{A}s}$ is also a compact operator. Therefore, $\mathcal{T}(t)-\mathcal{T}_1(t)$ is a compact operator over $\mathcal{H}$.

Hence, to prove the exponential decay of $\mathcal{T}(t)$ we only need to prove that the essential type of $\mathcal{T}_1$ is negative.

The one-dimensional system associated to (4.2)

In this section we assume that

$$\rho_1 b - \rho_2\kappa \neq 0 \quad \text{and} \quad \kappa_0 \neq \kappa \tag{4.4}$$

The above condition is quite natural to Bresse model. Using the Riemann invariants associated to system (4.2) we get

$$\begin{aligned} p_1 &= \sqrt{\rho_1}\varphi_t - \sqrt{\kappa}\varphi_x, \quad q_1 = \sqrt{\rho_1}\varphi_t + \sqrt{\kappa}\varphi_x,\\ p_2 &= \sqrt{\rho_2}\psi_t - \sqrt{b}\psi_x, \quad q_2 = \sqrt{\rho_2}\psi_t + \sqrt{b}\psi_x,\\ p_3 &= \sqrt{\rho_1}w_t - \sqrt{\kappa_0}w_x, \quad q_3 = \sqrt{\rho_1}w_t + \sqrt{\kappa_0}w_x, \end{aligned}$$

we have that

$$\varphi_t = \frac{q_1+p_1}{2\sqrt{\rho_1}}, \quad \varphi_x = \frac{q_1-p_1}{2\sqrt{k}}, \quad \psi_t = \frac{q_2+p_2}{2\sqrt{\rho_2}}, \quad \psi_x = \frac{q_2-p_2}{2\sqrt{b}}, \quad w_t = \frac{q_3+p_3}{2\sqrt{\rho_1}}, \quad w_x = \frac{q_3-p_3}{2\sqrt{\kappa_0}}.$$

Therefore, the evolution problem can be written as

$$p_{1,t} + k_1 p_{1,x} - c_1(q_2 - p_2) - c_3(q_3 - p_3) = 0, \tag{4.5}$$
$$q_{1,t} - k_1 q_{1,x} - c_1(q_2 - p_2) - c_3(q_3 - p_3) = 0, \tag{4.6}$$
$$p_{2,t} + k_2 p_{2,x} - c_2(q_1 - p_1) = 0, \tag{4.7}$$
$$q_{2,t} - k_2 q_{2,x} - c_2(q_1 - p_1) = 0, \tag{4.8}$$
$$p_{3,t} + k_3 p_{3,x} - c_4(q_1 - p_1) = 0, \tag{4.9}$$
$$q_{3,t} - k_3 q_{3,x} - c_4(q_1 - p_1) = 0, \tag{4.10}$$

where

$$\begin{aligned} &k_1 = \sqrt{\frac{\kappa}{\rho_1}}, \quad k_2 = \sqrt{\frac{b}{\rho_2}}, \quad k_3 = \sqrt{\frac{\kappa_0}{\rho_1}},\\ &c_1 = \frac{\kappa}{2\sqrt{b\rho_1}}, \, c_2 = -\frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}}, \, c_3 = \frac{\ell(\kappa_0+\kappa)}{2\sqrt{\rho_1\kappa_0}}, \, c_4 = -\frac{\ell(\kappa_0+\kappa)}{2\sqrt{\rho_1\kappa_0}}, \end{aligned}$$

verifying the following boundary conditions

$$q_1(0) + p_1(0) = 0, \quad q_1(L) - p_1(L) = 0, \quad q_j(0) - p_j(0) = 0, \quad q_j(L) + p_j(L) = 0, \quad j = 2, 3, \tag{4.11}$$

and the following transmission conditions

$$[\![q_i(\cdot,t) + p_i(\cdot,t)]\!]_\xi = 0, \quad i = 1, 2, 3, \tag{4.12}$$
$$[\![q_i(\cdot,t) - p_i(\cdot,t)]\!]_\xi = -\gamma_i k_i \left(p_i(\xi,t) + q_i(\xi,t)\right), \quad i = 1, 2, 3. \tag{4.13}$$

Let us denote by

$$\mathbf{K} = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 & 0\\ 0 & -k_1 & 0 & 0 & 0 & 0\\ 0 & 0 & k_2 & 0 & 0 & 0\\ 0 & 0 & 0 & -k_2 & 0 & 0\\ 0 & 0 & 0 & 0 & k_3 & 0\\ 0 & 0 & 0 & 0 & 0 & -k_3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & c_1 & -c_1 & c_3 & -c_3\\ 0 & 0 & c_1 & -c_1 & c_3 & -c_3\\ c_2 & -c_2 & 0 & 0 & 0 & 0\\ c_2 & -c_2 & 0 & 0 & 0 & 0\\ c_4 & -c_4 & 0 & 0 & 0 & 0\\ c_4 & -c_4 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{U} = \begin{pmatrix} p_1\\ q_1\\ p_2\\ q_2\\ p_3\\ q_3 \end{pmatrix},$$

hence system (4.5)–(4.8) can be written as

$$\mathfrak{U}_t + \mathbf{K}\mathfrak{U}_x + C\mathfrak{U} = 0, \quad \mathfrak{U}(0) = \mathfrak{U}_0\,. \tag{4.14}$$

Note that system (4.2) is equivalent to (4.14). Let us denote by $\mathcal{T}_2(t)$ the semigroup defined by (4.14) over $\mathbf{H}_6 := [L^2(0,L)]^6$. By definition we have $C_0 := \operatorname{diag}(C) = \mathbf{0}$.

Let us denote by $\mathcal{T}_3(t) : \mathbf{H}_6 \to \mathbf{H}_6$ the semigroup defined by the diagonal system

$$\widetilde{\mathfrak{U}}_t + \mathbf{K}\widetilde{\mathfrak{U}}_x = 0, \quad \widetilde{\mathfrak{U}}(0) = \mathfrak{U}_0, \tag{4.15}$$

verifying the boundary conditions (4.11) and the transmission conditions (4.12)–(4.13). At this point, we use the result due to Neves et al. [15] that in our case implies the following result

Theorem 4.4. *Under the above notations the difference $\mathcal{T}_3(t) - \mathcal{T}_2(t)$ is a compact operator over $\mathbf{H}_6$, provided condition (4.4) holds.*

Proof. Note that condition (4.4) implies that $k_i \neq k_j$ for $i \neq j$. Using [15, Theorem A] for $p = 2$, our conclusion follows.

System (4.15) is completely decoupled and can be written as

$$p_{i,t} + k_i p_{i,x} = 0, \quad q_{i,t} - k_i q_{i,x} = 0,$$

$$q_1(0) + p_1(0) = 0, \quad q_1(L) - p_1(L) = 0, \tag{4.16}$$

$$q_i(0) - p_i(0) = 0, \quad q_i(L) + p_i(L) = 0, \quad i = 2, 3. \tag{4.17}$$

together with

$$q_i(\xi^-) + p_i(\xi^-) = q_i(\xi^+) + p_i(\xi^+), \tag{4.18}$$

$$q_i(\xi^-) - q_i(\xi^+) + p_i(\xi^+) - p_i(\xi^-) = -\gamma_i k_i (p_i(\xi^+) + q_i(\xi^+)), \tag{4.19}$$

for $i = 1, 2, 3$. The semigroup $\mathcal{T}_3(t) : \mathbf{H}_6 \to \mathbf{H}_6$ is generated by the operator

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & 0 & 0 \\ 0 & \mathbf{A}_2 & 0 \\ 0 & 0 & \mathbf{A}_3 \end{pmatrix}, \tag{4.20}$$

where $\mathbf{A}_i$ is given by

$$\mathbf{A}_i U_2 = k_i K U_{2,x}, \quad U_2 = \begin{pmatrix} p \\ q \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Denoting by $\mathbf{I} = (0, \xi) \cup (\xi, L)$ the corresponding domains are given by

$$D(\mathbf{A}_1) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{H}_2 : \ p, q \in H^1(\mathbf{I}), \quad \text{verifying} \quad (4.16), (4.18), (4.19), \right\},$$

and

$$D(\mathbf{A}_i) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{H}_2 : \ p, q \in H^1(\mathbf{I}), \quad \text{verifying} \quad (4.17), (4.18), (4.19), \right\},$$

for $i = 2, 3$ and with $\mathbf{H}_2 := [L^2(0, L)]^2$. The resolvent system $\lambda U_2 + \mathbf{A}_i U_2 = F$ is given by

$$\lambda U_2 + k_i K U_{2,x} = F, \tag{4.21}$$

where

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} p \\ q \end{pmatrix}.$$

Hence the above system can be rewritten as

$$U_{2,x} + \frac{i\lambda}{k_i} K U_2 = \frac{1}{k_i} K F, \quad i = 1, 2, 3. \tag{4.22}$$

In terms of the components the above system can be written as

$$p_x + \frac{i\lambda}{k_i} p = \frac{1}{k_i} f_1, \tag{4.23}$$

$$q_x - \frac{i\lambda}{k_i} q = -\frac{1}{k_i} f_2, \tag{4.24}$$

verifying the boundary conditions (4.16)–(4.17) and the transmission conditions (4.18)–(4.19).

Lemma 4.5. *The operator* $\mathbf{A}$*, infinitesimal generator of* $\mathcal{T}_3$ *given in* (4.20)*, is dissipative over the phase space* $\mathbf{H}_6$*.*

Proof. Because of (4.20) it is enough to show that $\mathbf{A}_i$ is a dissipative operator over $\mathbf{H}_2$ for $i = 1, 2, 3$. Here we prove only for $i = 1$, the proof to $i = 2, 3$ is similar. For sake of simplicity, the index 1 is not written in p and q. Note that

$$\begin{aligned}\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix}\right)_{\mathbf{H}_2} &= \left(\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} -k_1 p_x \\ k_1 q_x \end{pmatrix}\right)_{\mathbf{H}_2} \\ &= \int_0^\xi (-k_1 p_x p + k_1 q_x q)\ dx + \int_\xi^L (-k_1 p_x p + k_1 q_x q)\ dx \\ &= \frac{k_1}{2}\left(-|p(\xi^-)|^2 + |p(0)|^2 + |q(\xi^-)|^2 - |q(0)|^2\right) \\ &\quad + \frac{k_1}{2}\left(-|p(L)|^2 + |p(\xi^+)|^2 + |q(L)|^2 - |q(\xi^+)|^2\right).\end{aligned}$$

Using the boundary conditions (4.16)–(4.17) and the transmission conditions (4.18)–(4.19), we get

$$\begin{aligned}\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix}\right)_{\mathbf{H}_2} &= \frac{k_1}{2}\left(-|p(\xi^-)|^2 + |q(\xi^-)|^2\right) + \frac{k_1}{2}\left(|p(\xi^+)|^2 - |q(\xi^+)|^2\right) \\ &= \frac{k_1}{2}\left[\left(-p(\xi^-) + q(\xi^-)\right) + \left(p(\xi^+) - q(\xi^+)\right)\right]\left(p(\xi) + q(\xi)\right).\end{aligned}$$

Where we used the continuity of the sum $p + q$. Finally from (4.19) we get

$$\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix}\right)_{\mathbf{H}_2} = -\frac{1}{2}\gamma_1 k_1^2 \left|p(\xi) + q(\xi)\right|^2, \tag{4.25}$$

and our conclusion follows.

Lemma 4.6. *The infinitesimal generator* $\mathbf{A}$ *of* $\mathcal{T}_3$ *given in* (4.20)*, verifies*

$$i\mathbb{R} \subset \varrho(\mathbf{A})$$

provided $\xi \neq \dfrac{n}{2m+1}L$*,* $\forall n, m \in \mathbb{N}$ *and* n *and* $2m + 1$ *co-prime.*

Proof. Since system (4.15) is fully decoupled it is enough to show that $i\mathbb{R} \subset \varrho(\mathbf{A}_i)$ for $i = 1, 2, 3$. Because of the compacity of the resolvent family associated to $\mathbf{A}_i$ it is enough to show that there is no imaginary eigenvalues. On the contrary suppose that there exists $0 \neq W \in D(\mathbf{A}_1)$ such that $\mathbf{A}_1 W = i\lambda W$, $\lambda \in \mathbb{R}$. Since $\mathbf{A}_1$ (the cases $i = 2, 3$ are similar) is dissipative we get

$$\left|p(\xi) + q(\xi)\right|^2 = 0 \tag{4.26}$$

which implies that

$$p(\xi^+) - q(\xi^+) = p(\xi^-) - q(\xi^-).$$

For $W = (p, q)$ we have that $\mathbf{A}_1 W = i\lambda W$ implies

$$p_x + i\frac{\lambda}{k_1}p = 0, \quad q_x - i\frac{\lambda}{k_1}q = 0.$$

Using the boundary condition at $x = 0$, $p(0) + q(0) = 0$ we solve the above system as

$$p(x) = p(0)e^{-i\frac{\lambda}{k_1}x}, \qquad q(x) = -p(0)e^{i\frac{\lambda}{k_1}x},$$

From the boundary condition at $x = L$, $p(L) - q(L) = 0$ we get

$$p(L) - q(L) = p(0)\left(e^{-\frac{i\lambda}{k_1}L} + e^{\frac{i\lambda}{k_1}L}\right) = 0, \quad \text{and this implies} \quad \lambda = \left(m + \frac{1}{2}\right) k_1 \frac{\pi}{L}. \tag{4.27}$$

At the transmission point $x = \xi$ we get

$$p(\xi) = p(0) e^{-i\frac{\lambda}{k_1}\xi}, \quad q(\xi) = -p(0) e^{i\frac{\lambda}{k_1}\xi},$$

and consequently using (4.26) we arrive to

$$p(\xi) + q(\xi) = p(0)\left(e^{-i\frac{\lambda}{k_1}\xi} - e^{i\frac{\lambda}{k_1}\xi}\right) = 0.$$

This implies that $e^{i\frac{2\lambda}{k_1}\xi} = 1$ hence $\frac{2\lambda}{k_1}\xi = 2n\pi$. Substitution of λ given in (4.27) yields

$$\frac{(2m+1)\pi}{2L}\xi = n\pi \quad \text{implying} \quad \xi = \frac{2n}{2m+1}L,$$

but this is not possible for hypothesis, so $p(0) = 0$, hence $W = 0$, but this is a contradiction.

Let us introduce the function $\mathfrak{F}^i_\xi(\lambda)$:

$$\mathfrak{F}^1_\xi(\lambda) = \cos^2\left(\frac{\lambda}{k_1}L\right) + \frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda}{k_1}\xi\right)\cos^2\left(\frac{\lambda}{k_1}(L-\xi)\right),$$
$$\mathfrak{F}^2_\xi(\lambda) = \cos^2\left(\frac{\lambda}{k_1}L\right) + \frac{\gamma_1^2}{k_1^2}\cos^2\left(\frac{\lambda}{k_1}\xi\right)\sin^2\left(\frac{\lambda}{k_1}(L-\xi)\right),$$

and let us denote by

$$A^i = \left\{\xi \in (0, L) : \quad \inf \mathfrak{F}^i_\xi(\mathbb{R}) > 0\right\}, \quad i = 1, 2.$$

Lemma 4.7. *For $\xi \in \mathbb{Q}L$ such that $\xi \neq \frac{n}{2m+1}L$, $\forall n, m \in \mathbb{N}$, with n and $2m+1$ co-prime we have*

$$I^i := \inf \mathfrak{F}^i_\xi(\mathbb{R}) > 0, \quad i = 1, 2.$$

Proof. Note that $\frac{L}{2} \in A^i \neq \emptyset$. In fact

$$\mathfrak{F}^i_{\frac{L}{2}}(\lambda) = \cos^2\left(\frac{\lambda}{k_1}L\right) + \frac{\gamma_1^2}{4k_1^2}\sin^2\left(\frac{\lambda}{k_1}L\right)$$
$$\geq \min\{1, \frac{\gamma_1^2}{4k_1^2}\}\left(\cos^2\left(\frac{\lambda}{k_1}L\right) + \sin^2\left(\frac{\lambda}{k_1}L\right)\right) \geq \min\{1, \frac{\gamma_1^2}{4k_1^2}\}.$$

By contradiction, let us suppose that $I = 0$. Since $\xi \in \mathbb{Q}L$ we can assume that $\xi = \frac{m}{n}L$ with m and n co-prime. Therefore the function $\mathfrak{F}^1_\xi$ is periodic with period equal to $T = 2\pi\frac{k_1}{L}n$. Hence

$$\inf \mathfrak{F}^1_\xi(\mathbb{R}) = \inf \mathfrak{F}^1_\xi([0, T]).$$

So, there exists a sequence of elements $\lambda_n \in [0, T]$ such that

$$\cos^2\left(\frac{\lambda_n}{k_1}L\right) + \frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda_n}{k_1}\xi\right)\cos^2\left(\frac{\lambda_n}{k_1}(L-\xi)\right) \to I = 0.$$

Since λ_n is bounded, there exists a convergent subsequence (we still denote in the same way) such that $\lambda_n \to \lambda^*$ and that

$$\cos^2\left(\frac{\lambda^*}{k_1}L\right) + \frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda^*}{k_1}\xi\right)\cos^2\left(\frac{\lambda^*}{k_1}(L-\xi)\right) = 0.$$

Hence we have that

$$\frac{\lambda^*}{k_1}L = \frac{2j+1}{2}\pi, \quad \frac{\lambda^*}{k_1}\xi = \nu\pi, \quad j, \nu \in \mathbb{N}, \tag{4.28}$$

or

$$\frac{\lambda^*}{k_1}L = \frac{2j+1}{2}\pi, \quad \frac{\lambda^*}{k_1}(L-\xi) = \frac{2\mu+1}{2}\pi, \quad j,\mu \in \mathbb{N}. \tag{4.29}$$

Let us suppose that (4.28) holds (the other is similar), taking $\xi = rL$ with $r \in \mathbb{Q}$, we get

$$\frac{\lambda^*}{k_1}\xi = \frac{\lambda^*}{k_1}rL = \nu\pi \quad \Rightarrow \quad \frac{2j+1}{2}\pi r = \nu\pi \quad \Rightarrow \quad r = \frac{2\nu}{2j+1}.$$

But this is contradictory to $\xi \neq \frac{n}{2m+1}L$ with $n,\ m \in \mathbb{N}$.

Theorem 4.8. *The semigroup $\mathcal{T}_3$ is exponentially stable, provided that ξ verifies hypotheses of Lemma 4.7.*

Proof. Because of (4.20) it is enough to show that $e^{\mathbf{A}_i t}$ is exponentially stable over $\mathbf{H}_2$ for $i = 1, 2, 3$. First we prove only for $i = 1$. For convenience we denote $p_1 = p$ and $q_1 = q$. We use Theorem 4.1 to show the exponential stability. Because of Lemma 4.6 it is enough to show that the resolvent operator is uniformly bounded over the imaginary axes. So the solution of (4.22) is given by

$$p(x) = p(0)e^{-i\frac{\lambda}{k_1}x} + \frac{1}{k_1}\int_0^x e^{-i\frac{\lambda}{k_1}(x-s)}f_1(s)\,ds, \quad x \in [0,\xi], \tag{4.30}$$

$$q(x) = -p(0)e^{i\frac{\lambda}{k_1}x} - \frac{1}{k_1}\int_0^x e^{i\frac{\lambda}{k_1}(x-s)}f_2(s)\,ds, \quad x \in [0,\xi]. \tag{4.31}$$

Similarly over $[\xi, L]$ we have that

$$p(x) = p(\xi^+)e^{-i\frac{\lambda}{k_1}(x-\xi)} + \frac{1}{k_1}\int_\xi^x e^{-i\frac{\lambda}{k_1}(x-s)}f_1(s)\,ds, \quad x \in [\xi, L], \tag{4.32}$$

$$q(x) = q(\xi^+)e^{i\frac{\lambda}{k_1}(x-\xi)} - \frac{1}{k_1}\int_\xi^x e^{i\frac{\lambda}{k_1}(x-s)}f_2(s)\,ds, \quad x \in [\xi, L]. \tag{4.33}$$

The above solution verify Eq. (4.21) and also the boundary condition at $x = 0$. Using (4.30) and (4.31) we get

$$p(\xi^-) = p(0)e^{-\frac{i\lambda}{k_1}\xi} + J_1, \qquad q(\xi^-) = -p(0)e^{i\frac{\lambda}{k_1}\xi} + J_2, \quad x \in [0,\xi] \tag{4.34}$$

$$J_1 = \frac{1}{k_1}\int_0^\xi e^{-\frac{i\lambda}{k_1}(x-s)}f_1(s)\,ds, \qquad J_2 = -\frac{1}{k_1}\int_0^\xi e^{\frac{i\lambda}{k_1}(x-s)}f_2(s)\,ds. \tag{4.35}$$

Now we adjust $q(\xi^+)$ and $p(\xi^+)$ such that the transmission conditions (4.19) holds for $i = 1$,

$$p(\xi^+) + q(\xi^+) = p(\xi^-) + q(\xi^-),$$
$$p(\xi^+) - q(\xi^+) = p(\xi^-) - q(\xi^-) + \frac{\gamma_1}{k_1}(p(\xi^+) + q(\xi^+)).$$

Solving the above system, we get

$$p(\xi^+) = p(\xi^-) + \frac{\gamma_1}{2k_1}(p(\xi^-) + q(\xi^-)), \quad q(\xi^+) = q(\xi^-) - \frac{\gamma_1}{2k_1}(p(\xi^-) + q(\xi^-)).$$

Applying (4.34) we obtain

$$p(\xi^+) = p(0)e^{-i\frac{\lambda}{k_1}\xi} - \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}) + J_1 + \frac{\gamma_1}{2k_1}(J_1 + J_2),$$
$$q(\xi^+) = -p(0)e^{i\frac{\lambda}{k_1}\xi} + \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}) + J_2 - \frac{\gamma_1}{2k_1}(J_1 + J_2).$$

Hence, with this choice the transmission conditions (4.18)-(4.19) holds. Finally, we take $p(0)$ such that the boundary conditions at $x = L$ hold.

$$p(\xi^+)e^{-i\frac{\lambda}{k_1}(x-\xi)} = p(0)e^{-i\frac{\lambda}{k_1}x} - \frac{\gamma_1}{2k_1}p(0)\left(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}\right)e^{-\frac{i\lambda}{k_1}(x-\xi)} + \left(J_1 + \frac{\gamma_1}{2k_1}(J_1+J_2)\right)e^{-\frac{i\lambda}{k_1}(x-\xi)},$$

$$q(\xi^+)e^{i\frac{\lambda}{k_1}(x-\xi)} = -p(0)e^{\frac{i\lambda}{k_1}x} + \frac{\gamma_1}{2k_1}p(0)\left(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}\right)e^{\frac{i\lambda}{k_1}(x-\xi)} + \left(J_2 - \frac{\gamma_1}{2k_1}(J_1+J_2)\right)e^{\frac{i\lambda}{k_1}(x-\xi)}.$$

Using (4.32)–(4.33) we get that $q(L) - p(L) = 0$ implies

$$0 = -p(0)\left(e^{i\frac{\lambda}{k_1}L} + e^{-i\frac{\lambda}{k_1}L}\right) + \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi})\left(e^{\frac{i\lambda}{k_1}(L-\xi)} + e^{\frac{-i\lambda}{k_1}(L-\xi)}\right) + G.$$

Where

$$G = \left(J_1 + \frac{\gamma_1}{2k_1}(J_1+J_2)\right)e^{-\frac{i\lambda}{k_1}(x-\xi)} + \left(J_2 - \frac{\gamma_1}{2k_1}(J_1+J_2)\right)e^{\frac{i\lambda}{k_1}(x-\xi)}$$
$$+\frac{1}{k_1}\int_\xi^L e^{-i\frac{\lambda}{k_1}(x-s)}f_1(s)\,ds - \frac{1}{k_1}\int_\xi^L e^{i\frac{\lambda}{k_1}(x-s)}f_2(s)\,ds.$$

So $p(0)$ has to be choosen such that

$$0 = -2p(0)\cos\left(\frac{\lambda}{k_1}L\right) + 2i\frac{\gamma_1}{k_1}p(0)\sin\left(\frac{\lambda}{k_1}\xi\right)\cos\left(\frac{\lambda}{k_1}(L-\xi)\right) + G$$
$$= -2p(0)\left(\cos\left(\frac{\lambda}{k_1}L\right) + i\frac{\gamma_1}{k_1}\sin\left(\frac{\lambda}{k_1}\xi\right)\cos\left(\frac{\lambda}{k_1}(L-\xi)\right)\right) + G.$$

The existence of solution will depend on

$$\cos\left(\frac{\lambda}{k_1}L\right) + i\frac{\gamma_1}{k_1}\sin\left(\frac{\lambda}{k_1}\xi\right)\cos\left(\frac{\lambda}{k_1}(L-\xi)\right) \neq 0.$$

The above expresion identically vanishes if and only if

$$\cos\left(\frac{\lambda}{k_1}L\right) = 0, \quad \cos\left(\frac{\lambda}{k_1}(L-\xi)\right) = 0, \quad \text{or} \quad \cos\left(\frac{\lambda}{k_1}L\right) = 0, \quad \sin\left(\frac{\lambda}{k_1}\xi\right) = 0,$$

simultaneously. But the above identity implies

$$\frac{\lambda}{k_1}L = \frac{\pi}{2} + j\pi, \quad \frac{\lambda}{k_1}(L-\xi) = \frac{\pi}{2} + m\pi, \quad j, m \in \mathbb{Z},$$

and consequently

$$\lambda = \frac{2j+1}{2L}k_1\pi, \quad \xi = \frac{2(j-m)}{2j+1}L, \quad j, m \in \mathbb{Z}.$$

But this is not possible because our hypothesis. Therefore we have

$$2p(0) = \frac{G}{\cos(\frac{\lambda}{k_1}L) + i\frac{\gamma_1}{k_1}\sin(\frac{\lambda}{k_1}\xi)\cos(\frac{\lambda}{k_1}(L-\xi))},$$

and we find that

$$|p(0)| \leq c\frac{\|F\|_{\mathbf{H}_2}}{\sqrt{\cos^2(\frac{\lambda}{k_1}L) + \frac{\gamma_1^2}{k_1^2}\sin^2(\frac{\lambda}{k_1}\xi)\cos^2(\frac{\lambda}{k_1}(L-\xi))}} = c\frac{\|F\|_{\mathbf{H}_2}}{\sqrt{\mathfrak{F}_\xi^1(\lambda)}}.$$

Hence using Lemma 4.7 we get $|p(0)| \leq c\|F\|_{\mathbf{H}_2}$, from where it follows

$$\|U_2\|_{\mathbf{H}_2} = \left\|\begin{pmatrix} p \\ q \end{pmatrix}\right\|_{\mathbf{H}_2} = \|(i\lambda I - \mathbf{A}_1)^{-1}F\|_{\mathbf{H}_2} \leq c\|F\|_{\mathbf{H}_2}$$

Applying Theorem 4.1 we get the exponential stability. To show that $e^{\mathbf{A}_2 t}$ we follows the same above procedure to get

$$2p(0) = \frac{\tilde{G}}{\cos(\frac{\lambda}{k_1}L) + i\frac{\gamma_1}{k_1}\cos(\frac{\lambda}{k_1}\xi)\sin(\frac{\lambda}{k_1}(L-\xi))} \quad \Rightarrow \quad |p(0)| \leq c\frac{\|F\|_{\mathbf{H}_2}}{\sqrt{\mathfrak{F}_\xi^2(\lambda)}}$$

and similar to $\mathbf{A}_3$.

We are now in a position to state the main result of the paper.

Theorem 4.9. *Under the hypothesis of Lemma 4.7, system (1.1)–(1.3) is exponentially stable, that is to say the semigroup $\mathcal{T}(t) = e^{\mathcal{A}t}$ associated to system (2.1)–(2.4) is exponentially stable.*

Proof. Theorem 4.3 implies that $\mathcal{T} - \mathcal{T}_1$ is a compact operator over $\mathbf{H}_6$, hence $\omega_{ess}(\mathcal{T}(t)) = \omega_{ess}(\mathcal{T}_1(t))$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are different representation of the same system we get $\omega_{ess}(\mathcal{T}_1(t)) = \omega_{ess}(\mathcal{T}_2(t))$. Moreover from Theorem 4.4 the operator $\mathcal{T}_2(t) - \mathcal{T}_3(t)$ is a compact operator over $\mathbf{H}_6$ hence $\omega_{ess}(\mathcal{T}_2(t)) = \omega_{ess}(\mathcal{T}_3(t))$. Finally, from Theorem 4.8 we get

$$\omega_{ess}(\mathcal{T}(t)) = \omega_{ess}(\mathcal{T}_3(t)) \leq \omega(\mathcal{T}_3(t)) = \max\{\omega_{ess}, \omega_\sigma(\mathbf{A})\} < 0.$$

The above inequality together with Lemma 4.2 and Theorem 3.1 implies the exponential stability.

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Declarations

Conflict of interest This work does not have any conflict of interest.

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