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Unfitted mixed finite element methods for elliptic interface problems

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Abstract

In this article, new unfitted mixed finite elements are presented for elliptic interface problems with jump coefficients. Our model is based on a fictitious domain formulation with distributed Lagrange multiplier. The relevance of our investigations is better seen when applied to the framework of fluid-structure interaction problems. Two finite element schemes with piecewise constant Lagrange multiplier are proposed and their stability is proved theoretically. Numerical results compare the performance of those elements, confirming the theoretical proofs and verifying that the schemes converge with optimal rates.

KEYWORDS

discontinuous Lagrange multiplier, fictitious domains problems, finite elements, fluid-structure interactions, immersed domain, interface problems, stability and inf-sup condition

1 | INTRODUCTION

Elliptic interface problems with jump coefficients are important and widely used in applications including bio-science and fluid-dynamics applications.

We consider a problem where the coefficients in the governing partial differential equation may jump across the interface that separates two or more sub-domains. There are various possibilities for the decomposition of the domain. In this article, we consider the case where one subdomain is immersed into another one. Other cases might be considered as well [19] and limiting to two subdomains does not affect the generality of our discussion.

One way to address interface problems is to use fitted meshes. This approach involves generating a mesh that conforms to the interface between different materials or regions within the domain,

as illustrated in Figure 1a. Through the years, several methods have been developed within this approach, including the extended finite element method (FEM) using fixed-grids which is based on an extended FEM strategy [15] and multiscale FEM [21], as well as others [4,11,23,31]. One common technique used to deal with interface problems within this category involves utilizing an arbitrary Lagrangian–Eulerian (ALE) coordinate system [13,20,22]. While this method can provide accurate solutions to such problems, it can be challenging, particularly for time-dependent problems like fluid-structure interaction (FSI) problems. In these cases, the mesh must track the system evolution and interface explicitly, which can result in ill-shaped meshes around the interface due to large displacements and deformations. One way to address this issue is to update the mesh at each time step, although this approach can be computationally expensive and difficult to use.

An alternative approach is to use unfitted meshes. In this approach, the meshes are independent of the interface, which is allowed to cut through the interior of elements, as shown in Figure 1b. A popular way to enforce the jump conditions in the unfitted mesh approach is to modify the finite element basis near the interface. Many numerical methods have been proposed in this direction, such as the immersed boundary method (IBM), immersed interface method, immersed FEM, multiscale FEM, extended FEM, and many others, see, for example, [12,24–27,30]. The jump condition can also be imposed by introducing penalty terms across interfaces [2] or applying the Nitsche’s method [33] as in the cut finite element methods (CUT-FEM), see, for example, [17,18,29]. This article introduces the fictitious domain with distributed Lagrange multiplier method (FD-DLM), which adapts the unfitted mesh approach. This method is based on the IBM, which was introduced by Peskin in [34] for the simulation of blood flow in the heart in the early seventies using finite differences. Later, in 2003, IBM was established in the framework of finite elements in [7]. This research led to the FD-DLM method, which avoids mesh regeneration by using fixed meshes. It makes use of the fictitious domain approach introduced in [16,35] to fictitiously extend one mesh into the other. Then, the two meshes are considered independent of each other and constructed only once. To enforce that the solution in the extended domain coincides with the solution in the immersed domain, we add a coupling term, and in our model, a Lagrange multiplier term is responsible for that. The advantage of the FD-DLM method is that it avoids the need for re-meshing and simplifies the mesh generation process, especially when the interface is moving in time.

Our focus on studying the elliptic interface problem in this formulation goes beyond the development of an advanced method for approximating interface problems and finding a novel approach with optimal solutions for them. Rather, our research represents a theoretical exploration of this

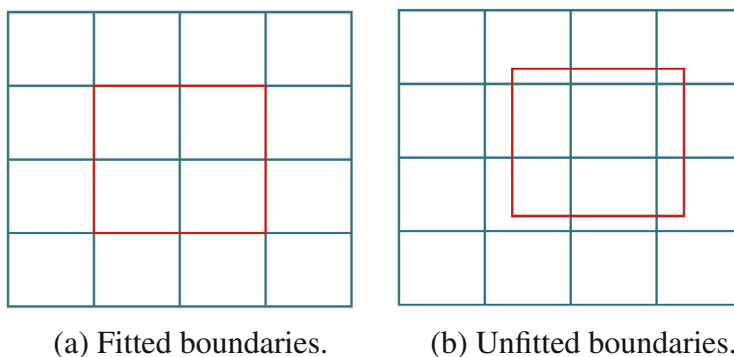


FIGURE 1 Fitted versus unfitted boundaries.

issue, which serves as a foundation for modeling FSI problems where the interface is dynamic and possibly deforming over time. The FD-DLM formulation is advantageous in these cases. As a result, finding alternative combinations of finite element spaces that can provide a stable discretization of our problem is valuable in the context of using these same spaces for more intricate scenarios.

A crucial aspect of this method is how to deal with the coupling term. Such a term is represented by a bilinear form defined on suitable Hilbert spaces. For instance, in [1], the discretization of this term was represented as the scalar product of L^2 . In [9] the coupling term was represented as the scalar product either in L^2 or H^1 . Furthermore, the computation of this term involves the evaluation of the integral on the immersed domain of shape functions that are supported on two different meshes. In [6] it is discussed how to implement this term in practice. It is shown that, in order to achieve the optimal convergence rate of the method, one has to perform the integral exactly by examining the intersection of the two meshes. Moreover, it is observed that finding the geometric intersection cannot be avoided even if the precision of the used quadrature rule is increased. Hence, in our numerical tests we will follow the intersection approach. More details on the coupling terms are given in Section 4.

In [1] continuous piecewise linear finite element spaces are considered for the discretization of the problem on triangular meshes. In [9] continuous piecewise bilinear finite element spaces are considered on quadrilateral meshes. Recently, [8] showed, in the framework of FSI, the stability of a linearization of the continuous problem and introduced a unified setting for the choice of the finite element spaces. This setting allows for more general choices of spaces. However, so far only continuous finite element spaces are considered for the multiplier responsible for the coupling term.

Our work is an extension to those papers where we are interested in finding stable elements with more flexibility in the choice of the multiplier. In particular, we addressed for the first time the question of whether piecewise discontinuous elements can be used for the approximation of the Lagrange multiplier. The motivation for our choice originates from FSI problems, where having a discontinuous Lagrange multiplier could improve the local mass conservation properties.

In Section 2.1, we introduce in detail the problem in the continuous setting. This problem is well-posed; we then consider its discretization and propose possible choices of elements that are the main object of our study in Section 2.2. Next, we prove the well-posedness of our discrete schemes in Section 3. The stability of our schemes is based on the presence of interior degrees of freedom (so-called bubble functions) in the space approximating the solution where the Lagrange multiplier is distributed. We are showing numerically in Section 4.1 that the presence of the bubble functions is necessary for the discrete inf-sup condition. Section 4.2 reports a series of numerical tests which confirm the theoretical results.

2 | FORMULATION OF THE METHOD

2.1 | Model problem

Let Ω be a domain in $\mathbb{R}^d$, $d = 1, 2, 3$ with a bounded Lipschitz boundary $\partial\Omega$. We assume that Ω is subdivided into two subdomains Ω_i , $i = 1, 2$ so that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. The subdomains are separated by a Lipschitz continuous interface $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$. In order to simplify the presentation, we assume that Ω_2 is immersed in Ω so that $\bar{\Gamma} \cap \partial\Omega = \emptyset$. Figure 2 reports a sketch of the situation in 2D. Then, we consider the following problem:

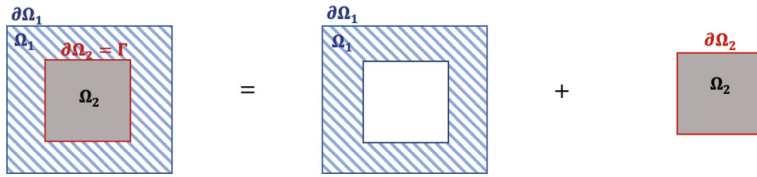
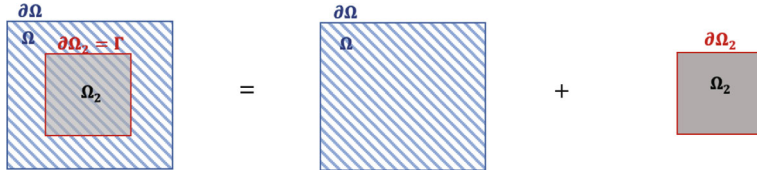


FIGURE 2 Domain decomposition in 2D.

FIGURE 3 Ω_1 fictitiously extended in Ω_2 in 2D.

Problem 2.1. Given $f_1 : \Omega_1 \rightarrow \mathbb{R}, f_2 : \Omega_2 \rightarrow \mathbb{R}$, find $u_1 : \Omega_1 \rightarrow \mathbb{R}$ and $u_2 : \Omega_2 \rightarrow \mathbb{R}$ such that:

$$-\nabla \cdot (\beta_i \nabla u_i) = f_i \quad \text{in } \Omega_i, \quad i = 1, 2, \quad (1a)$$

$$u_1 = u_2 \quad \text{on } \Gamma, \quad (1b)$$

$$\beta_1 \nabla(u_1) \cdot \mathbf{n}_1 = -\beta_2 \nabla(u_2) \cdot \mathbf{n}_2 \quad \text{on } \Gamma, \quad (1c)$$

$$u_1 = 0 \quad \text{on } \partial\Omega_1, \quad (1d)$$

where $\mathbf{n}_i, i = 1, 2$, is the unit vector pointing out of Ω_i and normal to Γ .

This is an elliptic interface problem with a jump in the coefficients and homogeneous Dirichlet boundary condition. We assume that the coefficients β_i belong to $L^\infty(\Omega_i)$ ($i = 1, 2$) and that they are bounded by below as follows:

$$\beta_1 > \underline{\beta}_1 > 0, \quad (2)$$

$$\beta_2 > \underline{\beta}_2 > 0. \quad (3)$$

In 2D, this model typically describes the displacement of a membrane made of two materials. The coefficients β_i stand for the stiffness of the materials, f_i $i = 1, 2$ for the loads applied to the membrane, and u_i for the vertical displacement in Ω_i ($i = 1, 2$), respectively. Equation (1b) guarantees the continuity of the solutions u_1 and u_2 on the interface Γ . This means that we are considering connected materials that do not break. Moreover, Equation (1c) prescribes a jump of the normal derivatives of u_1 and u_2 at the interface that is inversely proportional to the ratio of the coefficients.

In this article, we consider a fictitious domain approach, therefore we reformulate Problem 2.1 following [1,9]. More precisely, we extend u_1 , β_1 , and f_1 to Ω and denote such extensions by u , β , and f , respectively, so that $u|_{\Omega_1} = u_1, f|_{\Omega_1} = f_1$, and $\beta|_{\Omega_1} = \beta_1$. Moreover, we enforce the extended solution u to coincide with u_2 in Ω_2 , that is, $u|_{\Omega_2} = u_2$ by introducing a Lagrange multiplier. The resulting formulation will be called FD-DLM (Figure 3).

In view of the introduction of the variational formulation of the problem, we recall some notations. For any open connected domain, or part of a domain, $\omega \subset \mathbb{R}^d$ for $d = 2, 3$, we denote the standard Lebesgue and Sobolev spaces by $L^2(\omega)$ and $H^1(\omega)$, respectively. Those spaces are endowed with their norms; $\|\cdot\|_{L^2(\omega)} = \|\cdot\|_{0,\omega}$ and $\|\cdot\|_{H^1(\omega)} = \|\cdot\|_{1,\omega}$. Moreover, $(\cdot, \cdot)_\omega$ stands for the scalar product in $L^2(\omega)$.

Let us consider the spaces:

$$\begin{aligned} V &= H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\} \\ V_2 &= H^1(\Omega_2), \end{aligned}$$

endowed with their natural norms $\|v\|_V = |v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}$ and $\|v_2\|_{V_2} = \|v_2\|_{1,\Omega_2}$, respectively.

We denote by Λ the dual space of V_2 , that is, $\Lambda = [H^1(\Omega_2)]^*$, endowed with the following dual norm:

$$\|\mu\|_\Lambda = \sup_{v_2 \in V_2} \frac{\langle \mu, v_2 \rangle}{\|v_2\|_{V_2}},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between V_2 and its dual space Λ .

In [1] it has been proved that Problem 2.1 is equivalent to the following fictitious domain formulation with distributed Lagrange multiplier.

Problem 2.2. Given $f \in L^2(\Omega)$, $f_2 \in L^2(\Omega_2)$, $\beta \in L^\infty(\Omega)$ and $\beta_2 \in L^\infty(\Omega_2)$ with $f|_{\Omega_1} = f_1$ and $\beta|_{\Omega_1} = \beta_1$, find $(u, u_2, \lambda) \in V \times V_2 \times \Lambda$ such that

$$\begin{aligned} (\beta \nabla u, \nabla v)_\Omega + \langle \lambda, v|_{\Omega_2} \rangle &= (f, v)_\Omega & \forall v \in V \\ ((\beta_2 - \beta) \nabla u_2, \nabla v_2)_{\Omega_2} - \langle \lambda, v_2 \rangle &= (f_2 - f, v_2)_{\Omega_2} & \forall v_2 \in V_2 \\ \langle \mu, u|_{\Omega_2} - u_2 \rangle &= 0 & \forall \mu \in \Lambda. \end{aligned}$$

After some standard calculations, one can obtain the following characterization of λ , that will be useful to estimate the approximation error, see [1]:

$$\langle \lambda, v_2 \rangle = - \int_{\Omega_2} \left(\frac{\beta}{\beta_2} f_2 - f \right) v_2 \, dx + \int_\Gamma (\beta_2 - \beta) \nabla u_2 \cdot \mathbf{n}_2 \, v_2 \, d\gamma. \quad (4)$$

Clearly, Problem 2.2 is a saddle point problem that can be written in operator form as follow:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \\ C_1 & -C_2 \end{pmatrix} \begin{pmatrix} u \\ u_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}, \quad (5)$$

where A_1 and A_2 are the operators associated with the bilinear forms $(\beta \nabla u, \nabla v)_\Omega$ and $((\beta_2 - \beta) \nabla u_2, \nabla v_2)_{\Omega_2}$, respectively. Moreover, $(C_1, -C_2)$ is the operator pair that is associated with the bilinear form $\langle \mu, u|_{\Omega_2} - u_2 \rangle$ with kernel:

$$\mathbb{K} = \{(u, u_2) \in V \times V_2 : \langle \mu, u|_{\Omega_2} - u_2 \rangle = 0, \forall \mu \in \Lambda\}.$$

Notice that due to the definition of Λ we have that $u|_{\Omega_2} = u_2$ for all $(u, u_2) \in \mathbb{K}$.

Typically, in order to prove the well-posedness of a continuous saddle point problem like Problem 2.2, one needs to verify the following two sufficient conditions (see [3,5,10]).

- **Continuous elker condition:** There exists a constant $\overline{\gamma}_1 > 0$ such that for all (v, v_2) in $\mathbb{K}$ the following inequality holds true.

$$(\beta \nabla v, \nabla v)_\Omega + ((\beta_2 - \beta) \nabla v_2, \nabla v_2)_{\Omega_2} \geq \overline{\gamma}_1 (\|v\|_V^2 + \|v_2\|_{V_2}^2).$$

- **Continuous inf-sup condition:** There exists a constant $\overline{\gamma}_2 > 0$, such that for all $\mu \in \Lambda$ the following bound holds true.

$$\sup_{(v,v_2) \in V \times V_2} \frac{\langle \mu, v|_{\Omega_2} - v_2 \rangle}{(\|v\|_V^2 + \|v_2\|_{V_2}^2)^{\frac{1}{2}}} \geq \overline{\gamma}_2 \|\mu\|_{\Lambda}.$$

Both conditions were shown to hold in [1]. Therefore, a unique stable solution exists for Problem 2.2 and we can state the following proposition (see [3,5,10]).

Proposition 2.1 (Stability). *Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, there exists a unique solution (u, u_2, λ) in $V \times V_2 \times \Lambda$ for Problem 2.2 which satisfies the following a priori estimate:*

$$\|u\|_V + \|u_2\|_{V_2} + \|\lambda\|_{\Lambda} \leq C(\|f\|_{0,\Omega} + \|f_2\|_{0,\Omega_2}).$$

Remark 1. It is known that the regularity of the solution (u_1, u_2) of an elliptic interface problem with discontinuous coefficients and a Lipschitz interface Γ , such as Problem 2.1, is limited by the presence of re-entrant corners of the interface and of the external boundary. Hence, we have that there exists s with $\frac{3}{2} < s \leq 2$ such that $u_i \in H^s(\Omega_i)$ for $i = 1, 2$, (see [32]). Moreover, since the solution $u \in H_0^1(\Omega)$ of Problem 2.2 exhibits jump in the derivative normal to the interface, it belongs to $H^r(\Omega)$, with $1 < r < \frac{3}{2}$.

2.2 | Finite element discretization

Let $\mathcal{T}$ and $\mathcal{T}_2$ be two shape-regular meshes of the fictitiously extended domain Ω and the immersed domain Ω_2 , respectively. Here, we are considering quadrilateral meshes in 2D and hexahedral in 3D, with h and h_2 denoting the maximum mesh size of $\mathcal{T}$ and $\mathcal{T}_2$, respectively. We introduce the finite element spaces $V_h \subset V$, $V_{2h} \subset V_2$, and $\Lambda_h \subset \Lambda$. V_h and V_{2h} contain piecewise polynomials continuous across the interelement boundaries, while for Λ_h we choose discontinuous finite elements. However, since $\Lambda_h \subset L^2(\Omega_2)$, the duality pairing in Problem 2.2 can be evaluated using the L^2 scalar product in Ω_2 . Then, the discrete version of Problem 2.2 reads:

Problem 2.3. Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, find $(u_h, u_{2h}, \lambda_h) \in V_h \times V_{2h} \times \Lambda_h$ such that

$$\begin{aligned} (\beta \nabla u_h, \nabla v_h)_{\Omega} + (\lambda_h, v_h|_{\Omega_2})_{\Omega_2} &= (f, v_h)_{\Omega} & \forall v_h \in V_h \\ ((\beta_2 - \beta) \nabla u_{2h}, \nabla v_{2h})_{\Omega_2} - (\lambda_h, v_{2h})_{\Omega_2} &= (f_2 - f, v_{2h})_{\Omega_2} & \forall v_{2h} \in V_{2h} \\ (\mu_h, u_h|_{\Omega_2} - u_{2h})_{\Omega_2} &= 0 & \forall \mu_h \in \Lambda_h. \end{aligned}$$

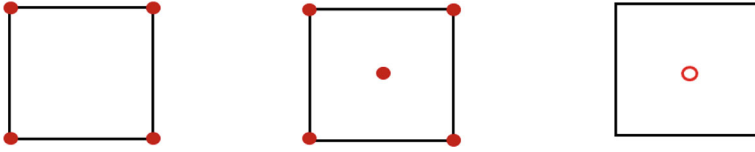
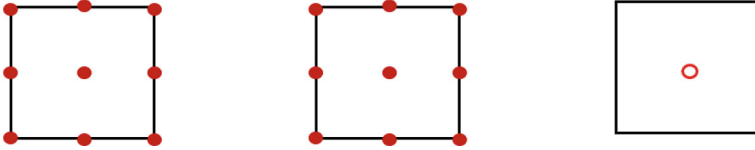
For K an element of $\mathcal{T}$ or $\mathcal{T}_2$, we define $Q_k(K)$, $k \geq 1$, to be the space of finite elements that are polynomials of degree at most k , separately in each variable on K . If $k = 0$ then $Q_0(K)$ is the space of constant polynomials and will be denoted by $P_0(K)$. Moreover, let $B(K) \in Q_2(K)$ be a bubble function defined on K and vanishing at the boundary of the element K . This function will be used in this work to enrich the space $Q_1(K)$.

In the following, we are going to discretize the Lagrange multiplier by piecewise constants, hence we introduce the following natural choices for $V_h \times V_{2h} \times \Lambda_h$.

- **Element 1:** $Q_1 - (Q_1 + B) - P_0$

In this element we define the discrete subspaces V_h, V_{2h}, Λ_h to be

$$V_h = \{v_h \in V : v_h|_K \in Q_1(K), \forall K \in \mathcal{T}\}$$


 FIGURE 4 Element 1 in 2D, $Q_1 - (Q_1 + B) - P_0$.

 FIGURE 5 Element 2 in 2D, $Q_2 - Q_2 - P_0$.

$$V_{2h} = \{v_{2h} \in V_2 : v_{2h}|_K \in Q_1(K) + B(K), \forall K \in \mathcal{T}_2\} \quad (6)$$

$$\Lambda_h = \{\mu_h \in \Lambda : \mu_h|_K \in P_0(K), \forall K \in \mathcal{T}_2\}.$$

Hence, the solution u_h is approximated by piecewise bilinear polynomials and u_{2h} by piecewise bilinear polynomials enriched by bubble functions, so that, five degrees of freedom are used per element in the space V_{2h} (see Figure 4).

• **Element 2:** $Q_2 - Q_2 - P_0$

Here, we define the discrete subspaces V_h, V_{2h}, Λ_h to be

$$\begin{aligned} V_h &= \{v_h \in V : v_h|_K \in Q_2(K), \forall K \in \mathcal{T}\} \\ V_{2h} &= \{v_{2h} \in V_2 : v_{2h}|_K \in Q_2(K), \forall K \in \mathcal{T}_2\} \\ \Lambda_h &= \{\mu_h \in \Lambda : \mu_h|_K \in P_0(K), \forall K \in \mathcal{T}_2\}, \end{aligned} \quad (7)$$

where we approximate both u_h, u_{2h} by piecewise biquadratic polynomials. Hence, we use nine nodes for each element in the spaces V_h and V_{2h} (see Figure 5). It is clear that compared to Element 1, V_{2h} uses four extra nodes at the middle of each edge, while V_h requires five extra degrees of freedom, so this element is computationally more expensive than Element 1.

3 | ERROR ESTIMATE

In this section, we study the existence of a unique stable solution of Problem 2.3. Since it is a discrete saddle point problem, sufficient conditions for its well-posedness are the discrete elker and inf-sup conditions (see [3,5,10]).

Let the discrete kernel $\mathbb{K}_h$ associated with the bilinear form $(\mu_h, u_h|_{\Omega_2} - u_{2h})_{\Omega_2}$ be defined as follow:

$$\mathbb{K}_h = \{(u_h, u_{2h}) \in V_h \times V_{2h} : (\mu_h, u_h|_{\Omega_2} - u_{2h})_{\Omega_2} = 0, \forall \mu_h \in \Lambda_h\}.$$

Then, the following proposition states the discrete elker condition.

Proposition 3.1 (Discrete elker condition). *Let us consider V_h, V_{2h}, Λ_h defined in (6) and (7) and assume that the coefficients satisfy $\beta_2 > \beta > \underline{\beta} > 0$ in Ω_2 , then there exists a constant $\gamma_1 > 0$ independent of the discretization parameters h, h_2 , such that for all $(v_h, v_{2h}) \in \mathbb{K}_h$ the following inequality holds true.*

$$(\beta \nabla v_h, \nabla v_h)_\Omega + ((\beta_2 - \beta) \nabla v_{2h}, \nabla v_{2h})_{\Omega_2} \geq \gamma_1 [\|v_h\|_V^2 + \|v_{2h}\|_{V_2}^2].$$

Proof. For all $(v_h, v_{2h}) \in \mathbb{K}_h$ we have:

$$\begin{aligned} (\beta \nabla v_h, \nabla v_h)_\Omega + ((\beta_2 - \beta) \nabla v_{2h}, \nabla v_{2h})_{\Omega_2} &= \beta \|\nabla v_h\|_{0,\Omega}^2 + (\beta_2 - \beta) \|\nabla v_{2h}\|_{0,\Omega_2}^2 \\ &\geq \underline{\beta} \|\nabla v_h\|_{0,\Omega}^2 + \eta_0 \|\nabla v_{2h}\|_{0,\Omega_2}^2, \end{aligned}$$

where η_0 is such that $\beta_2 - \beta \geq \eta_0 > 0$.

Since $V = H_0^1(\Omega)$, we can apply the Poincaré inequality $\|v_h\|_{0,\Omega} \leq C_\Omega \|\nabla v_h\|_{0,\Omega}$, and we have that:

$$(\beta \nabla v_h, \nabla v_h)_\Omega \geq \frac{\underline{\beta}}{2} \left(1 + \frac{1}{C_\Omega^2}\right) \|v_h\|_{1,\Omega}^2.$$

It remains to bound by below $\|\nabla v_{2h}\|_{0,\Omega_2}$ by means of $\|v_{2h}\|_{1,\Omega_2}$. In order to use the Poincaré–Wirtinger inequality, we split v_{2h} as follows

$$v_{2h} = E(v_{2h}) + [v_{2h} - E(v_{2h})],$$

where $E(v_{2h}) \in \mathbb{R}$ is the mean value of v_{2h} . Then,

$$\begin{aligned} \|v_{2h}\|_{0,\Omega_2} &\leq \|E(v_{2h})\|_{0,\Omega_2} + \|v_{2h} - E(v_{2h})\|_{0,\Omega_2} \\ &\leq \|E(v_{2h})\|_{0,\Omega_2} + C \|\nabla v_{2h}\|_{0,\Omega_2}. \end{aligned}$$

Now, since $(v_h, v_{2h}) \in \mathbb{K}_h$, and Λ_h contains constant functions on Ω_2 , we have

$$\begin{aligned} (E(v_{2h}), E(v_{2h}))_{\Omega_2} &= (E(v_{2h}), v_{2h})_{\Omega_2} - (E(v_{2h}), v_{2h} - E(v_{2h}))_{\Omega_2} \\ &= (E(v_{2h}), v_h|_{\Omega_2})_{\Omega_2} - (E(v_{2h}), v_{2h} - E(v_{2h}))_{\Omega_2} \\ &= (E(v_{2h}), v_h|_{\Omega_2})_{\Omega_2}. \end{aligned} \quad (8)$$

Then, (8) gives the bound

$$\|E(v_{2h})\|_{0,\Omega_2}^2 \leq \|E(v_{2h})\|_{0,\Omega_2} \|v_h\|_{0,\Omega_2}.$$

Hence,

$$\|E(v_{2h})\|_{0,\Omega_2} \leq \|v_h\|_{0,\Omega_2},$$

which concludes the proof. $\blacksquare$

Remark 2. In a particular case and under the additional condition that h_2/h^d is sufficiently small, in [9] it has been shown that the discrete elker condition holds true even if the constraint $\beta_2 > \beta$ is not verified. In that case triangular linear elements were used for the approximation of both V_2 and Λ .

In the next proposition, we prove the discrete inf-sup condition.

Proposition 3.2 (Discrete inf-sup condition). *Let V_h, V_{2h}, Λ_h be the spaces defined in (6) and (7) for Elements 1 and 2, respectively. Then there exists a positive constant $\gamma_2 > 0$ independent of the discretization parameters h, h_2 , such that the following discrete inf-sup condition holds.*

$$\sup_{(v_h, v_{2h}) \in V_h \times V_{2h}} \frac{(\mu_h, v_h|_{\Omega_2} - v_{2h})_{\Omega_2}}{(\|v_h\|_V^2 + \|v_{2h}\|_{V_2}^2)^{\frac{1}{2}}} \geq \gamma_2 \|\mu_h\|_\Lambda. \quad (9)$$

Proof. We prove this proposition for Element 1. The same proof carries on for Element 2.

Since $v_h = 0$ is a possible choice in V_h , we have that

$$\sup_{(v_h, v_{2h}) \in V_h \times V_{2h}} \frac{(\mu_h, v_h|_{\Omega_2} - v_{2h})_{\Omega_2}}{(\|v_h\|_V^2 + \|v_{2h}\|_{V_2}^2)^{\frac{1}{2}}} \geq \sup_{v_{2h} \in V_{2h}} \frac{(\mu_h, v_{2h})_{\Omega_2}}{\|v_{2h}\|_{V_2}}.$$

Hence, it is enough to prove the following bound:

$$\sup_{v_{2h} \in V_{2h}} \frac{(\mu_h, v_{2h})_{\Omega_2}}{\|v_{2h}\|_{V_2}} \geq \gamma_2 \|\mu_h\|_{\Lambda}. \quad (10)$$

In order to show that (10) is satisfied, we use a Fortin trick (see [5, Prop. 5.4.2]). Due to the fact that the continuous inf-sup condition is satisfied, the aim of this proof is to find a linear Fortin operator $\Pi_h : V_2 \rightarrow V_{2h}$ that satisfies the following relations for all $v_2 \in V_2$:

$$\begin{aligned} \text{i) } & \|\Pi_h v_2\|_{V_2} \leq C \|v_2\|_{V_2}, \\ \text{ii) } & (\mu_h, v_2)_{\Omega_2} = (\mu_h, \Pi_h v_2)_{\Omega_2} \quad \forall \mu_h \in \Lambda_h. \end{aligned} \quad (11)$$

We introduce the following subspaces of V_{2h} :

$$\begin{aligned} \overline{V_{2h}} &= \{v_{2h} \in V_2 : v_{2h}|_K \in Q_1(K), \forall K \in \mathcal{T}_2\} \subset V_{2h} \\ \overline{V_{2h}} &= \{v_{2h} \in V_2 : v_{2h}|_K \in B(K), \forall K \in \mathcal{T}_2\} \subset V_{2h}. \end{aligned}$$

Let $\Pi_h = \Pi_1 + \Pi_2(I - \Pi_1)$ where $\Pi_1 : V_2 \rightarrow \overline{V_{2h}} \subset V_{2h}$ is the Cl  ment's operator, such that for all $v_2 \in V_2$

$$\sum_{K \in \mathcal{T}_2} h_K^{-2} \|v_2 - \Pi_1 v_2\|_{0,K}^2 \leq C \|v_2\|_{V_2}^2, \quad (12a)$$

$$\sum_{K \in \mathcal{T}_2} \|v_2 - \Pi_1 v_2\|_{1,K}^2 \leq C \|v_2\|_{V_2}^2, \quad (12b)$$

and $\Pi_2 : V_2 \rightarrow \overline{V_{2h}} \subset V_{2h}$ such that $\Pi_2(p_i) = 0$ for all nodes p_i at vertices of the element $K \in \mathcal{T}_2$ and

$$\int_K \Pi_2 v_2 = \int_K v_2 \quad \forall v_2 \in V_2 \text{ and } K \in \mathcal{T}_2. \quad (13)$$

Let $\mathcal{F}_K^{-1}$ be the affine mapping that maps objects from the current element, $K \in \mathcal{T}_2$, to the reference element denoted by $\hat{K}$, that is the unit square in 2D and the unit cube in 3D. Then, $\hat{v}_2 = v_2 \circ \mathcal{F}_K$ where symbols with hat refer to quantities evaluated in the reference domain $\hat{K}$.

Let $|K|$ be the measure of the element K . Then,

$$\int_{\hat{K}} \widehat{\Pi_2 v_2} = |K|^{-1} \int_K \Pi_2 v_2 = |K|^{-1} \int_K v_2 = |K|^{-1} |K| \int_{\hat{K}} \hat{v}_2 = \int_{\hat{K}} \hat{v}_2,$$

which means that $\widehat{\Pi_2}$ also satisfies Equation (13). Moreover, let us define the following:

$$\|\widehat{\Pi_2 v_2}\| = \left| \int_{\hat{K}} \widehat{\Pi_2 v_2} \right|.$$

It is clear that this is a norm in this specific case, since the function $\widehat{\Pi_2 v_2}$ by definition is the bubble in the reference element. Hence, $\left| \int_{\hat{K}} \widehat{\Pi_2 v_2} \right| = 0$ if and only if $\widehat{\Pi_2 v_2} = 0$.

Using the definition of Π_h and the equality in (13), we get

$$(\mu_h, v_2 - \Pi_h v_2)_{\Omega_2} = (\mu_h, (v_2 - \Pi_1 v_2) - \Pi_2(v_2 - \Pi_1 v_2))_{\Omega_2} = 0 \quad \forall \mu_h \in \Lambda_h,$$

which proves that the proposed operator Π_h satisfies condition ii) in (11).

Now, we need to show condition i) for all $v_2 \in V_2$, that is:

$$\|\Pi_h v_2\|_{V_2} = \|\Pi_1 v_2 + \Pi_2(v_2 - \Pi_1 v_2)\|_{V_2} \leq C\|v_2\|_{V_2}. \quad (14)$$

It is clear that (12b) implies

$$\|\Pi_1 v_2\|_{V_2} \leq C\|v_2\|_{V_2}. \quad (15)$$

For the sake of simplicity, let $w = v_2 - \Pi_1 v_2$ and write the H^1 -norm as follows:

$$\|\Pi_2 w\|_{1,\Omega_2}^2 = \|\Pi_2 w\|_{0,\Omega_2}^2 + |\Pi_2 w|_{1,\Omega_2}^2. \quad (16)$$

For all $K \in \mathcal{T}_2$, we estimate the first term in (16) as follows:

$$\begin{aligned} \|\Pi_2 w\|_{0,K} &\leq C |K|^{\frac{1}{2}} \left\| \widehat{\Pi_2 w} \right\|_{0,\hat{K}} && \text{mapping to the reference element } \hat{K} \\ &\leq C |K|^{\frac{1}{2}} \left| \int_{\hat{K}} \widehat{\Pi_2 w} \right| && \text{by equivalence of norms in finite dimensions} \\ &= C |K|^{\frac{1}{2}} \left| \int_{\hat{K}} \hat{w} \right| && \text{since } \widehat{\Pi_2} \text{ satisfies (13)} \\ &\leq C |K|^{\frac{1}{2}} \|\hat{w}\|_{0,\hat{K}} && \text{by Cauchy-Schwarz inequality} \\ &\leq C |K|^{\frac{1}{2}} |K|^{-\frac{1}{2}} \|w\|_{0,K} && \text{mapping back to the physical element } K \\ &= C \|w\|_{0,K}. \end{aligned}$$

Similarly, using the same argument as before we have

$$\begin{aligned} \|\nabla \Pi_2 w\|_{0,K} &\leq C h_K^{-1} |K|^{\frac{1}{2}} \left| \widehat{\Pi_2 w} \right|_{1,\hat{K}} \leq C h_K^{-1} |K|^{\frac{1}{2}} \left| \int_{\hat{K}} \widehat{\Pi_2 w} \right| \\ &\leq C h_K^{-1} |K|^{\frac{1}{2}} \left| \int_{\hat{K}} \hat{w} \right| \leq C h_K^{-1} |K|^{\frac{1}{2}} \|\hat{w}\|_{0,\hat{K}}. \end{aligned}$$

Mapping back to the current element K , we obtain:

$$\|\nabla \Pi_2 w\|_{0,K} \leq C h_K^{-1} |K|^{\frac{1}{2}} |K|^{-\frac{1}{2}} \|w\|_{0,K} = C h_K^{-1} \|w\|_{0,K}. \quad (17)$$

By substituting (17) in (16) and using the bounds of Π_1 in (12a), we get

$$\|\Pi_2(v_2 - \Pi_1 v_2)\|_{V_2}^2 \leq C \sum_{K \in \mathcal{T}_2} h_K^{-2} \|v_2 - \Pi_1 v_2\|_{0,K}^2 \leq C \|v_2\|_{V_2}^2. \quad (18)$$

Using the triangle inequality, and applying the results of (15) and (18), yield that condition i) holds, in fact

$$\begin{aligned} \|\Pi_h v_2\|_{V_2} &= \|\Pi_1 v_2 + \Pi_2(v_2 - \Pi_1 v_2)\|_{V_2} \\ &\leq \|\Pi_1 v_2\|_{V_2} + \|\Pi_2(v_2 - \Pi_1 v_2)\|_{V_2} \\ &\leq C\|v_2\|_{V_2}. \end{aligned}$$

This concludes the proof and shows that the H^1 -stability of the constructed Fortin operator Π_h is satisfied. Therefore, the discrete inf-sup condition (9) holds. ■

Remark 3. Proposition 3.2 shows rigorously that the inf-sup condition for Elements 1 and 2 is satisfied uniformly. The bubble function in the space V_{2h} is necessary for the stability

of the element. In fact, in Section 4.1 we give numerical evidence that we do not have a uniform inf-sup bound if we modify Element 1 and remove the bubble to become $Q_1 - Q_1 - P_0$.

Thanks to Propositions 3.1 and 3.2 there exists a unique stable solution (u_h, u_{2h}, λ_h) for Problem 2.3 in the spaces $V_h \times V_{2h} \times \Lambda_h$ that are defined in (6) and (7). Moreover, by taking into consideration the regularity results of the solutions u and u_2 recalled in Remark 1, we can state the following a priori error estimate.

Proposition 3.3 (Error estimate). *Given $(f, f_2) \in L^2(\Omega) \times L^2(\Omega_2)$. Let $(u, u_2, \lambda) \in V \times V_2 \times \Lambda$ and $(u_h, u_{2h}, \lambda_h) \in V_h \times V_{2h} \times \Lambda_h$ be the solutions of Problems 2.2 and 2.3, respectively. We consider Elements 1 and 2 defined in (6) and (7). Then, under the assumption that the mesh $\mathcal{T}_2$ is quasi-uniform, the following error estimate holds true:*

$$\begin{aligned} & \|u - u_h\|_V + \|u_2 - u_{2h}\|_{V_2} + \|\lambda - \lambda_h\|_\Lambda \\ & \leq C \left(h^{r-1} \|u\|_{r,\Omega} + \max(h_2^{s-1}, h_2^{1-t}) \|u_2\|_{s,\Omega_2} + h_2 \left\| \frac{\beta}{\beta_2} f_2 - f \right\|_{0,\Omega_2} \right) \end{aligned}$$

with $1 < r < 3/2$ and $3/2 < s \leq 2$, defined in Remark 1, and for $1/2 < t < 1$.

Proof. Thanks to the continuous elker and inf-sup conditions and their discrete version in Propositions 3.1 and 3.2, the theory of the saddle point problem yields the usual quasi-optimal error estimate (see [5, Th. 5.2.2]), that in this case, reads as follows

$$\begin{aligned} & \|u - u_h\|_V + \|u_2 - u_{2h}\|_{V_2} + \|\lambda - \lambda_h\|_\Lambda \\ & \leq C \left(\inf_{v \in V_h} \|u - v\|_V + \inf_{v_2 \in V_{2h}} \|u_2 - v_2\|_{V_2} + \inf_{\mu \in \Lambda_h} \|\lambda - \mu\|_\Lambda \right). \end{aligned}$$

As a consequence of the regularity of u and u_2 reported in Remark 1, standard arguments on the approximation error for the spaces V_h and V_{2h} imply that

$$\begin{aligned} \inf_{v \in V_h} \|u - v\|_V & \leq Ch^{r-1} |u|_{r,\Omega} \\ \inf_{v_2 \in V_{2h}} \|u_2 - v_2\|_{V_2} & \leq Ch_2^{s-1} |u_2|_{s,\Omega_2}. \end{aligned}$$

The estimate of the best approximation of λ requires a more careful analysis. A similar consideration has been performed in [1, Prop. 6] in a different setting. Actually, taking into account (4), the Lagrange multiplier can be represented as the sum of two pieces $\lambda = \lambda_1 + \lambda_2$ with

$$\begin{aligned} \langle \lambda_1, v_2 \rangle & = \int_{\Omega_2} ((\beta/\beta_2)f_2 - f) v_2 \, dx \\ \langle \lambda_2, v_2 \rangle & = \int_{\Gamma} (\beta_2 - \beta) \nabla u_2 \cdot \mathbf{n}_2 v_2 \, d\gamma. \end{aligned} \quad (19)$$

We treat separately the two pieces and look for two elements in Λ_h which approximate λ_1 and λ_2 .

Since $f \in L^2(\Omega)$, $f_2 \in L^2(\Omega_2)$ and since we assumed the bound (2), the first equality in (19) implies that $\lambda_1 \in L^2(\Omega_2)$ with $\|\lambda_1\|_{0,\Omega_2} \leq \|(\beta/\beta_2)f_2 - f\|_{0,\Omega_2}$. Let $P_0 : L^2(\Omega_2) \rightarrow \Lambda_h$ be the L^2 -projection onto Λ_h , we set $\lambda_{1h} = P_0 \lambda_1 \in \Lambda_h$ with

$$(\lambda_1 - \lambda_{1h}, \mu_h) = 0 \quad \forall \mu_h \in \Lambda_h.$$

Then we observe that

$$\|\lambda_1 - \lambda_{1h}\|_\Lambda = \sup_{v_2 \in V_2} \frac{\langle \lambda_1 - \lambda_{1h}, v_2 \rangle}{\|v_2\|_{V_2}} = \sup_{v_2 \in V_2} \frac{(\lambda_1 - \lambda_{1h}, v_2)}{\|v_2\|_{V_2}},$$

but

$$\begin{aligned} (\lambda_1 - \lambda_{1h}, v_2) &= (\lambda_1 - \lambda_{1h}, v_2 - P_0 v_2) = (\lambda_1, v_2 - P_0 v_2) \\ &\leq \|\lambda_1\|_{0,\Omega_2} \|v_2 - P_0 v_2\|_{0,\Omega_2} \leq C \|(\beta/\beta_2)f_2 - f\|_{0,\Omega_2} h_2 \|v_2\|_{V_2}. \end{aligned}$$

Therefore we end up with

$$\|\lambda_1 - \lambda_{1h}\|_\Lambda \leq Ch_2 \|(\beta/\beta_2)f_2 - f\|_{0,\Omega_2}. \quad (20)$$

Let us now construct an approximation of λ_2 and bound the approximation error. Remark 1 states that $u_2 \in H^s(\Omega_2)$ for $3/2 < s \leq 2$. Therefore the trace of the normal derivative of u_2 belongs to $H^{s-3/2}(\Gamma)$ and we can infer from the second equality in (19) that $\lambda_2 \in H^{-t}(\Omega_2)$ with $1/2 < t < 1$, namely by trace inequality we have

$$\begin{aligned} \|\lambda_2\|_{H^{-t}(\Omega_2)} &= \sup_{v_2 \in H^t(\Omega_2)} \frac{\langle \lambda_2, v_2 \rangle}{\|v_2\|_{H^t(\Omega_2)}} = \sup_{v_2 \in H^t(\Omega_2)} \frac{\int_\Gamma (\beta_2 - \beta) \nabla u_2 \cdot \mathbf{n}_2 v_2 \, d\gamma}{\|v_2\|_{H^t(\Omega_2)}} \\ &\leq |\beta_2 - \beta| \sup_{v_2 \in H^t(\Omega_2)} \frac{\|\nabla u_2 \cdot \mathbf{n}_2\|_{H^{s-3/2}(\Gamma)} \|v_2\|_{H^{3/2-s}(\Gamma)}}{\|v_2\|_{H^t(\Omega_2)}} \\ &\leq C \sup_{v_2 \in H^t(\Omega_2)} \frac{\|u_2\|_{H^s(\Omega_2)} \|v_2\|_{H^{-1/2}(\Gamma)}}{\|v_2\|_{H^t(\Omega_2)}} \\ &\leq C \sup_{v_2 \in H^t(\Omega_2)} \frac{\|u_2\|_{H^s(\Omega_2)} \|v_2\|_{H^t(\Omega_2)}}{\|v_2\|_{H^t(\Omega_2)}} = C \|u_2\|_{H^s(\Omega_2)}. \end{aligned}$$

Let $\lambda_{2h} \in \Lambda_h$ be such that

$$\langle \lambda_{2h}, v_{2h} \rangle = \int_\Gamma (\beta_2 - \beta) \nabla u_2 \cdot \mathbf{n}_2 v_{2h} \, d\gamma \quad \forall v_{2h} \in V_{2h}.$$

Then, it is easily seen that $\langle \lambda_2 - \lambda_{2h}, v_{2h} \rangle = 0$ for all $v_{2h} \in V_{2h}$ so that we have for $v_{2h} \in V_{2h}$

$$\langle \lambda_2 - \lambda_{2h}, v_2 \rangle = \langle \lambda_2 - \lambda_{2h}, v_2 - v_{2h} \rangle = \langle \lambda_2, v_2 - v_{2h} \rangle - \langle \lambda_{2h}, v_2 - v_{2h} \rangle. \quad (21)$$

We now choose v_{2h} so that $v_{2h}|_\Gamma$ is the $L^2(\Gamma)$ projection of $v_2|_\Gamma$ and at the interior nodes of Ω_2 coincides with the Clément interpolant of v_2 . In particular, we have the estimates

$$\begin{aligned} \|v_2 - v_{2h}\|_{0,\Gamma} &\leq Ch_2^{1/2} \|v_2\|_{H^{1/2}(\Gamma)} \\ \|v_2 - v_{2h}\|_{0,\Omega_2} &\leq Ch_2 \|v_2\|_{1,\Omega_2}. \end{aligned}$$

Hence, using (19), we get

$$\begin{aligned} \langle \lambda_2, v_2 - v_{2h} \rangle &= \int_\Gamma (\beta_2 - \beta) \nabla u_2 \cdot \mathbf{n}_2 (v_2 - v_{2h}) \, d\gamma \\ &\leq |\beta_2 - \beta| \|\nabla u_2 \cdot \mathbf{n}_2\|_{H^{s-3/2}(\Gamma)} \|v_2 - v_{2h}\|_{H^{3/2-s}(\Gamma)}. \end{aligned}$$

The last norm can be estimated as follows

$$\begin{aligned} \|v_2 - v_{2h}\|_{H^{3/2-s}(\Gamma)} &= \sup_{w \in H^{s-3/2}(\Gamma)} \frac{(v_2 - v_{2h}, w)_\Gamma}{\|w\|_{H^{s-3/2}(\Gamma)}} \\ &= \sup_{w \in H^{s-3/2}(\Gamma)} \frac{(v_2 - v_{2h}, w - w^I)_\Gamma}{\|w\|_{H^{s-3/2}(\Gamma)}} \end{aligned}$$

$$\begin{aligned} &\leq Ch_2^{s-3/2} \|v_2 - v_{2h}\|_{0,\Gamma} \\ &\leq Ch_2^{s-1} \|v_2\|_{H^{1/2}(\Gamma)} \leq Ch_2^{s-1} \|v_2\|_{1,\Omega_2}, \end{aligned}$$

where w^I is the Clément interpolant of w satisfying

$$\|w - w^I\|_{0,\Gamma} \leq Ch_2^{s-3/2} \|w\|_{H^{s-3/2}(\Gamma)}.$$

It remains to bound the second term in (21): we have

$$\langle \lambda_{2h}, v_2 - v_{2h} \rangle \leq \|\lambda_{2h}\|_{0,\Omega_2} \|v_2 - v_{2h}\|_{0,\Omega_2} \leq Ch_2 \|\lambda_{2h}\|_{0,\Omega_2} \|v_2\|_{1,\Omega_2}.$$

In order to conclude the proof, we now estimate $\|\lambda_{2h}\|_{0,\Omega_2}$. Recalling the Fortin operator Π_h satisfying (11), we have

$$\begin{aligned} \|\lambda_{2h}\|_{0,\Omega_2} &= \sup_{v_2 \in V_2} \frac{\langle \lambda_{2h}, v_2 \rangle}{\|v_2\|_{0,\Omega_2}} = \sup_{v_2 \in V_2} \frac{\langle \lambda_{2h}, \Pi_h v_2 \rangle}{\|v_2\|_{0,\Omega_2}} = \sup_{v_2 \in V_2} \frac{\langle \lambda_2, \Pi_h v_2 \rangle}{\|v_2\|_{0,\Omega_2}} \\ &\leq \sup_{v_2 \in V_2} \frac{\|\lambda_2\|_{H^{-t}(\Omega_2)} \|\Pi_h v_2\|_{H^t(\Omega_2)}}{\|v_2\|_{0,\Omega_2}} \\ &\leq Ch_2^{-t} \|u_2\|_{H^s(\Omega_2)} \sup_{v_2 \in V_2} \frac{\|\Pi_h v_2\|_{0,\Omega_2}}{\|v_2\|_{0,\Omega_2}}. \end{aligned}$$

The last supremum can be bounded if we provide a uniform $L^2(\Omega_2)$ estimate of the Fortin operator. Looking at the construction of the Fortin operator as $\Pi_h v_2 = \Pi_1 v_2 + \Pi_2(v_2 - \Pi_1 v_2)$, and inspecting the proof of its stability, it turns out that this can be done if we replace the Clément interpolation Π_1 with an interpolation that is bounded in $L^2(\Omega_2)$. This can be done by adopting the quasi-interpolation operator introduced in [14], denoted again Π_1 for simplicity, and which has been proved to have the required stability

$$\|\Pi_1 v_2\|_{0,\Omega_2} \leq C \|v_2\|_{0,\Omega_2}.$$

We finally get

$$\langle \lambda_{2h}, v_2 - v_{2h} \rangle \leq Ch_2 \|\lambda_{2h}\|_{0,\Omega_2} \|v_2\|_{1,\Omega_2} \leq Ch_2^{1-t} \|u_2\|_{H^s(\Omega_2)} \|v_2\|_{H^1(\Omega_2)}.$$

Putting together all the pieces we see that the best approximations of u , u_2 , and λ converge with order h^{r-1} , h_2^{s-1} , and h_2^{1-t} , respectively. In the case when $h_2 < 1$, since $s-1 > 1-t$, we get that h_2^{s-1} is dominated by h_2^{1-t} . Therefore, the final rate of convergence is given by the maximum between h^{r-1} and h_2^{1-t} . ■

Remark 4. This article only considers quadrilateral and hexahedra mesh. However, all results and proofs might be easily extended to triangular and tetrahedra mesh with the corresponding element choices $P_1 - (P_1 + \bar{B}) - P_0$ and $P_2 - (P_2 + \bar{B}) - P_0$ where $\bar{B}$ is a proper bubble function associated with an internal node.

4 | NUMERICAL RESULTS

In this section, we delve into a comprehensive numerical study of the elliptic interface problem in the FDDLM formulation. Our primary objective is to investigate the accuracy, stability, and convergence rate of the two proposed elements.

We consider a two-dimensional setting, and we fix the right-hand sides of the problem as $f = 1$ and $f_2 = 1$.

Our simulations involve various cases, including different ratios between the mesh sizes of the two domains, as well as different combinations of coefficients between the two domains, including cases where the sufficient condition of the elker, $\beta_2 - \beta > 0$, is violated. In particular, we investigate the three cases:

Case 1: $\beta = 1, \beta_2 = 10$.

Case 2: $\beta = 1, \beta_2 = 10,000$.

Case 3: $\beta = 10, \beta_2 = 1$.

To further assess the performance of the proposed elements, we consider four examples involving various immersed shapes. Specifically, we consider a square, an L-shape, a circle, and a flower, each equipped with suitable Dirichlet boundary conditions. These examples were chosen to demonstrate the robustness of our proposed method in dealing with different geometries. In each example, we choose the domains Ω and Ω_2 as follows:

Example 1. We consider a rectangular mesh with a square immersed shape mesh, where $\Omega = [0, 6]^2$ and $\Omega_2 = [e, 1 + \pi]^2$.

Example 2. We investigate a rectangular mesh with an immersed L-shape mesh, where $\Omega = [0, 6]^2$ and $\Omega_2 = [1, 3]^2 \setminus [2, 3]^2$.

Example 3. We study a rectangular mesh with a circular immersed shape mesh, where $\Omega = [-1.4, 1.4]^2$ and $\Omega_2 = B((0, 0), 1)$.

Example 4. We examine a rectangular mesh with an immersed flower shape mesh, where $\Omega = [-2, 3]^2$ and Ω_2 is a flower with radius $1 + 0.1 \cos(5\theta)$ and center $(0, 0)$.

For the first, second, and fourth examples, we obtain reference solutions by solving Laplace problems with jumping coefficients on a very fine mesh (up to 16 million degrees of freedom) using a high-order mixed FEM. In the case of the circle, we have an exact solution in the different three cases given as follows:

$$\begin{aligned} \text{Case 1: } & \begin{cases} U_1 = \frac{4 - x^2 - y^2}{4} & \text{in } \Omega_1 \\ U_2 = \frac{31 - x^2 - y^2}{40} & \text{in } \Omega_2, \end{cases} \\ \text{Case 2: } & \begin{cases} U_1 = \frac{4 - x^2 - y^2}{4} & \text{in } \Omega_1 \\ U_2 = \frac{30001 - x^2 - y^2}{40000} & \text{in } \Omega_2, \end{cases} \\ \text{Case 3: } & \begin{cases} U_1 = \frac{4 - x^2 - y^2}{40} & \text{in } \Omega_1 \\ U_2 = \frac{13 - 10x^2 - 10y^2}{40} & \text{in } \Omega_2. \end{cases} \end{aligned}$$

We plot in Figures 6–9 the exact\reference solutions for each of the examples in the first case to provide a visual representation of the problems.

In any case, dealing with the coupling term arising from non-matching meshes between the two domains is one of the main challenges in interface problem simulations. To tackle this issue, we describe the numerical methods and techniques utilized in our study. Additionally, we analyze the need for the bubble function to ensure the stability of the proposed elements, and further investigate the inf-sup condition numerically. Lastly, we present a detailed convergence rate analysis and discuss the

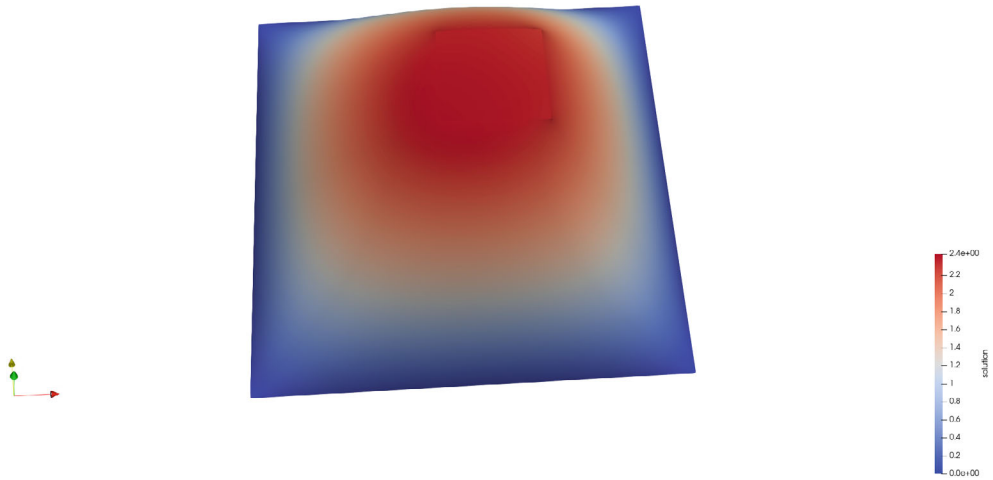


FIGURE 6 Exact solution of Example 1.

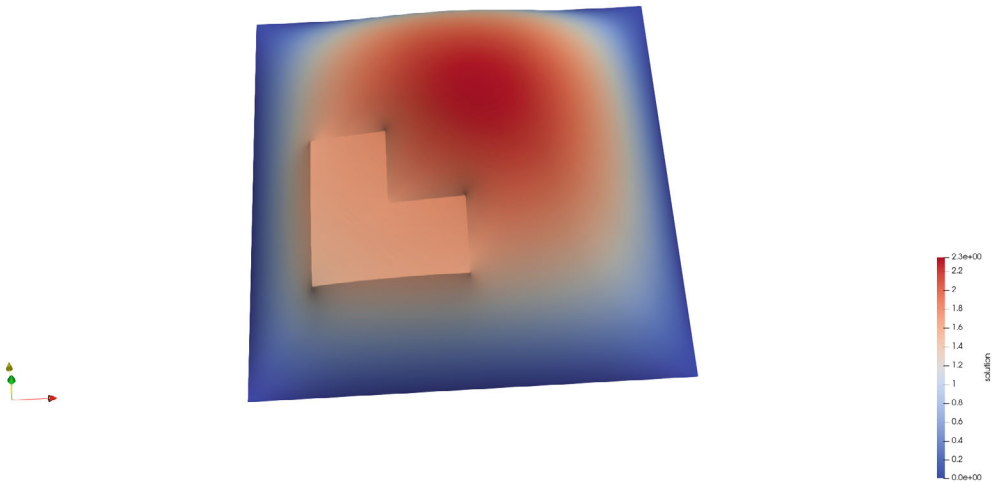


FIGURE 7 Exact solution of Example 2.

performance of the two proposed elements in all cases. We present some numerical tests that confirm the expected convergence rate of our schemes.

In the following, we will describe how we handle the numerical treatment of the coupling term $(\lambda_h, v_h)_{\Omega_2}$ in Problem 2.3 for Example 2 (without loss of generality), where $\lambda_h \in \Lambda_h$ and $v_h \in V_h$. This term is represented as an L^2 scalar product in Ω_2 . Evaluating this term requires the evaluation of the following integral:

$$\int_{\Omega_2} \phi_i \psi_j |_{\Omega_2} dx = \sum_{K \in \mathcal{T}_2} \int_K \phi_i \psi_j |_{\Omega_2} dx, \quad (22)$$

where ϕ_i $i = 1, \dots, \dim(\Lambda_h)$ and ψ_j $j = 1, \dots, \dim(V_h)$ are the basis functions that span Λ_h and V_h , respectively. Thanks to the choice of Λ_h , $\phi_i = 1$ on the element $K_i \in \mathcal{T}_2$ and vanishes elsewhere.

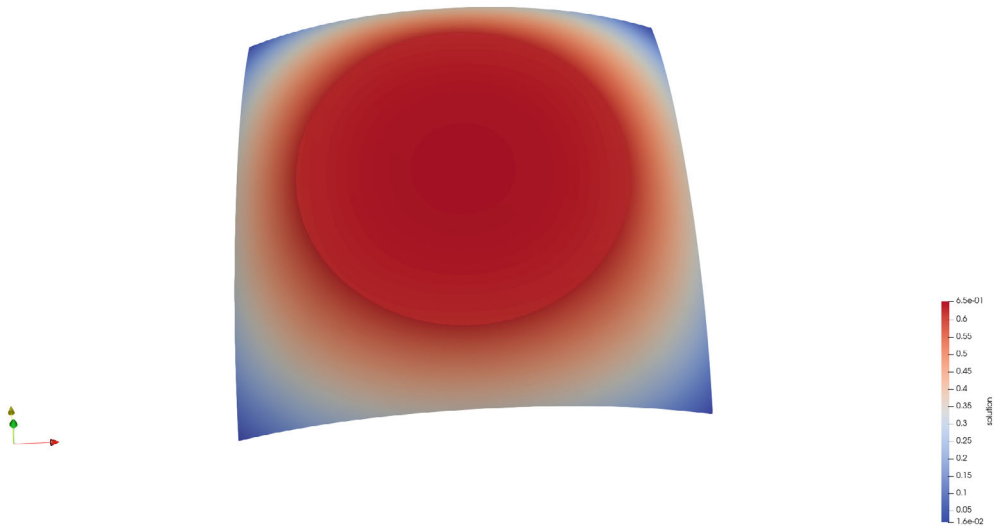


FIGURE 8 Exact solution of Example 3.

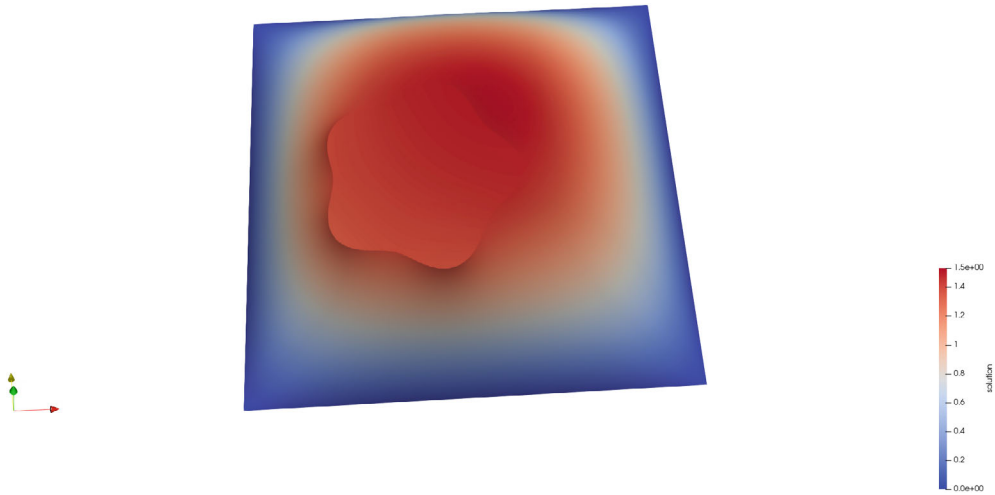


FIGURE 9 Exact solution of Example 4.

Hence the integral in (22) reduces to the integral on the intersection between K_i with the support of ψ_j .

To simplify the idea, consider the 2D case and assume that the support of ϕ_i is the element $K_i \in \mathcal{T}_2$ colored with brown in Figure 10a. We start by finding the geometric intersections of elements in $\mathcal{T}$ with K_i . This leads us to the introduction of a new mesh in Ω_2 , that we denote by $\bar{\mathcal{T}}_2$ with the property that each element $\bar{K} \in \bar{\mathcal{T}}_2$ is contained in one element in $\mathcal{T}$ as in Figure 10b. This new mesh is also a quadrilateral mesh but finer than $\mathcal{T}_2$. Therefore we have

$$\int_{\Omega_2} \phi_i \psi_j |_{\Omega_2} dx = \int_{K_i \cap \text{supp}(\psi_j)} \psi_j dx,$$

where $K_i \cap \text{supp}(\psi_j)$ is the rectangle colored in red in Figure 10c.

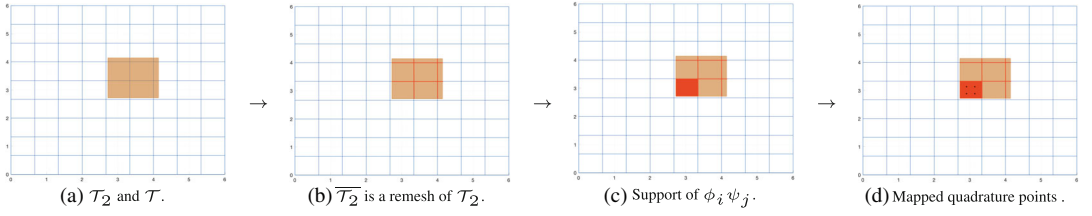


FIGURE 10 Intersection approach.

This approach will give us an exact evaluation of the scalar product since it takes into the account the exact region for which the two shape functions interact [6]. The procedure for adapting this approach is briefly summarized below:

- Choose the order of the quadrature rule depending on the degree of ψ_j .
- Find the corresponding quadrature points and weights in the reference element $\hat{K}$. Denote by $\hat{q}_k$ the k th quadrature point and by $\hat{w}_k$ the associated quadrature weight.
- Map the quadrature points and weights to $\bar{K}$ where $\bar{K} = K_i \cap \text{supp}(\psi_j)$. So, we have $w_k = |\bar{K}| \hat{w}_k$, and $\psi_j(q_k) = \hat{\psi}_j(\hat{q}_k)$, see, for example, Figure 10d.
- Evaluate the integral as follows:

$$\int_{\Omega_2} \phi_i \psi_j = \int_{\bar{K}} \psi_j = |\bar{K}| \int_{\hat{K}} \hat{\psi}_j = |\bar{K}| \sum_k \hat{w}_k \hat{\psi}_j(\hat{q}_k) = \sum_k w_k \psi_j(q_k).$$

4.1 | Numerical discrete inf-sup test

In this section, numerical tests on the discrete inf-sup bound of the proposed elements is presented to validate the theoretical proofs given in Section 3.

For the same reason explained in Section 3, it is enough to show that there exists a constant $\gamma_2 > 0$ such that the following inf-sup bound is satisfied:

$$\sup_{v_{2h} \in V_{2h}} \frac{(\mu_h, v_{2h})_{\Omega_2}}{\|v_{2h}\|_{V_2}} \geq \gamma_2 \|\mu_h\|_{\Lambda}. \quad (23)$$

In order to estimate numerically the constant γ_2 , we use the following standard procedure.

Let N_1 and N_2 be the two matrices associated with the following norms:

$$\begin{aligned} \|v_2\|_{V_2}^2 &= |v_2|_{V_2}^2 + \|v_2\|_{0,\Omega_2}^2 = v_2^T N_2 v_2 & v_2 &\in V_{2h} \\ \|\mu_h\|_{\Lambda}^2 &\approx h_2^2 \|\mu_h\|_{0,\Omega_2}^2 = \mu_h^T (h_2^2 N_1) \mu_h & \mu_h &\in \Lambda_h. \end{aligned}$$

Arguing as in [28], the eigenvalue equation associated with this inf-sup condition is

$$(C_2)(h_2^2 N_1)^{-1} (C_2^T) v_2 = \sigma N_2 v_2, \quad (24)$$

where C_2 is the operator associated with the bilinear form $(\mu_h, v_{2h})_{\Omega_2}$. If the considered finite element satisfies the inf-sup condition, then, with increase refinements, the square root of the smallest eigenvalue σ_1 is bounded from below away from zero independently from the mesh sizes. This bound is the desired inf-sup bound.

The problem is solved in a sequence of five refinements. The results are plotted using logarithmic scaling. In Figure 11, we report the results of our test for the two elements presented in Section 3 together with the test for the element $Q_1 - Q_1 - P_0$, where we did not add the bubble to the space V_{2h} .

It is clear that Elements 1 and 2 are stable as the mesh is refined, that is the inf-sup constant does not degenerate, while the element without the bubble is not, that is, the inf-sup constant tends to zero as h goes to zero.

4.2 | Rate of convergence

The rate of convergence is a critical aspect in numerical methods for solving partial differential equations as it measures how quickly the numerical solution approaches the exact solution with mesh refinement which depends on the smoothness of the solution and the efficiency of the method. In this section, we analyze the rate of convergence of our numerical methods for Examples 1–4, comparing the convergence rates with the theoretical rates predicted by the FEM to assess the effectiveness of our proposed method in dealing with different immersed shapes.

To gain a better understanding of the performance of our numerical method, we conduct extensive testing for each example, examining three different choices of the ratio between mesh sizes. Furthermore, we explore the performance of our method by allowing the diffusion coefficients to vary considering Cases 1–3 introduced at the beginning of this section. We analyze the behavior of the solution using the two proposed elements, namely $(Q_1, Q_1 + B, P_0)$ and (Q_2, Q_2, P_0) . In addition, we investigate the numerical behavior of the (Q_1, Q_1, P_0) element, which we previously demonstrated that it fails to pass the numerical inf-sup test.

Given the multitude of possible combinations to test, we have obtained a substantial number of results for each example. While the outcomes are undoubtedly valuable, they tend to be very similar to each other. To ensure a comprehensive coverage of our findings, we will begin by discussing all the different possibilities we have attempted for Example 3 presented in Figures 12–17. To avoid overwhelming the reader with a deluge of redundant information, we will focus on presenting a representative subset of the most relevant results for each of the other examples. In particular, we present the results using one different ratio between mesh sizes for each as seen in Figures 18–23. Lastly, we present three cases using the unstable (Q_1, Q_1, P_0) , to show that the rate of convergence does not converge in some situations as illustrated by Figure 24.

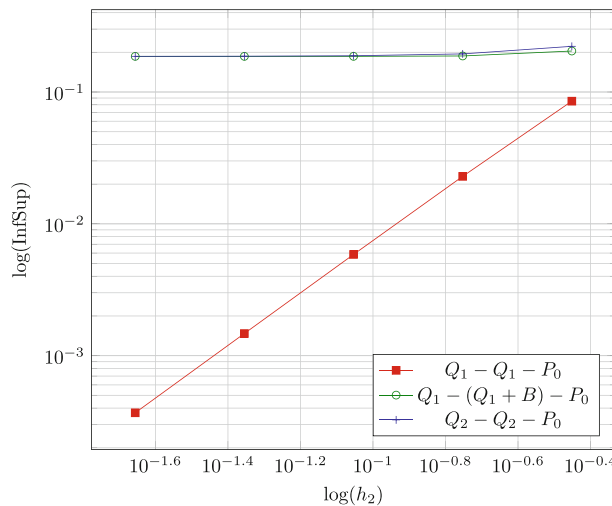


FIGURE 11 Numerical discrete inf-sup test.

The order of convergence of the L^2 and H^1 norms of the error provide information about how quickly the error decreases as the mesh size is refined. In general, a higher order of convergence indicates that the error decreases more rapidly, which implies a smoother solution. All examples considered encounter jumps in the coefficients that can be quite large. These jumps can pose a challenge for numerical methods, as they can lead to numerical instabilities and affect the accuracy of the solution. Moreover, Examples 1 and 2 exhibit singularities in the geometry due to the re-entrant corners unlike Examples 3 and 4, which can further complicate the problem and possibly lead to numerical instabilities.

Despite these challenges in the problem of our interest, our proposed numerical scheme is able to achieve a reasonable rate of convergence, demonstrating its stability, robustness, and effectiveness. In general, the results from Cases 1 and 2 for all examples are consistent with our theoretical expectations. Furthermore, the findings from Case 3 indicate that the elker condition constraint may be relaxed to some extent.

Specifically, we know that the solution belongs to $H^r(\Omega)$ with $1 < r < \frac{3}{2}$. Consequently, we would expect the order of convergence to represent a solution that at least lives in H^1 . The observed orders of convergence of almost $O(h^1)$ for the L^2 norm and almost $O(h^{\frac{1}{2}})$ for the H^1 norm of the error are consistent with this expectation.

The convergence rate plots depicted in Figures 12–21 reveal interesting patterns. Specifically, we observed that in Examples 3 and 4, where the interface is smoother, the convergence plots exhibit more linear behavior. In contrast, Examples 1 and 2, which feature re-entrant corners, exhibit more oscillations in their convergence plots. Interestingly, these oscillations only appear when the ratio between mesh sizes is greater than or equal to 1 in Cases 1 and 2, and the opposite situation in Case 3. Additionally, we found that the (Q_2, Q_2, P_0) element demonstrated linear convergence, whereas the $(Q_1, Q_1 + B, P_0)$ element exhibited these oscillations, which were more pronounced in Example 2 than in Example 1. On the other hand, in Examples 3 and 4, we observed that both elements produced similar results. It is important to note that the rate of convergence is not significantly affected by these oscillations and our scheme is achieving an optimal rate of convergence in all cases.

In the first example, where we have an immersed square shape, we benchmark our results against those obtained in [9], when an L^2 scalar product is used to evaluate the duality term numerically and found that our results are similar to those reported there.

We conclude this section by going back to the element $Q_1 - Q_1 - P_0$ due to its appeal coming from its easier implementation. Recall that, in Section 4.1, we showed numerically that this element does not pass the numerical inf-sup test. Unexpectedly, we have an optimal rate of convergence when the

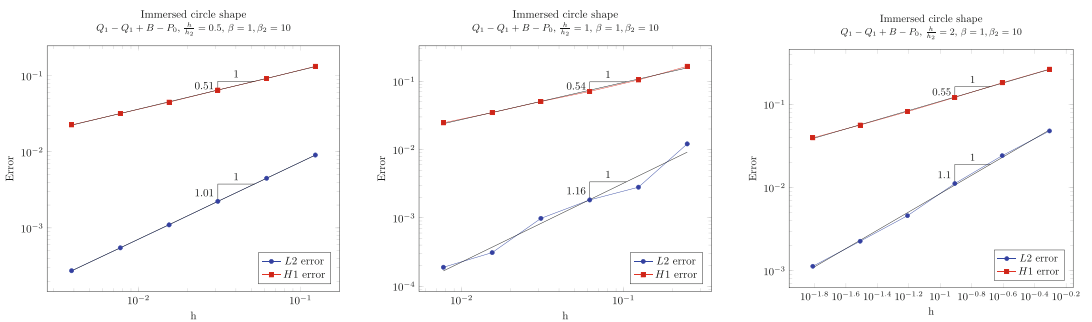


FIGURE 12 Convergence of $Q_1 - (Q_1 + B) - P_0$ FDDLM: $\beta = 1$ $\beta_2 = 10$.

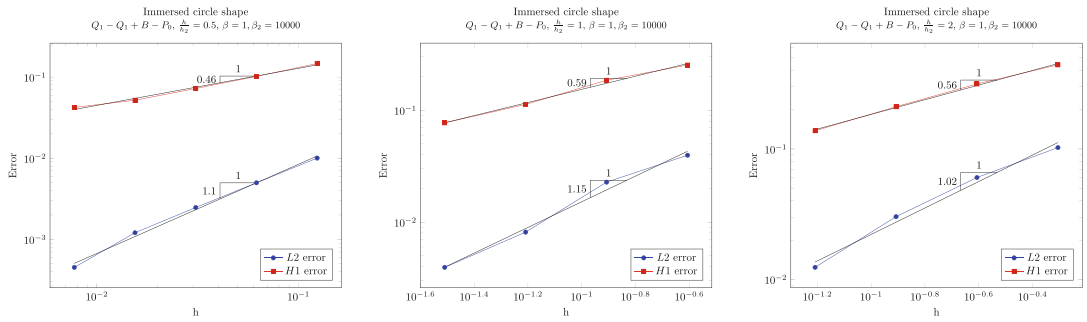


FIGURE 13 Convergence of $Q_1 - (Q_1 + B) - P_0$ FDDLM: $\beta = 1$ $\beta_2 = 10,000$.

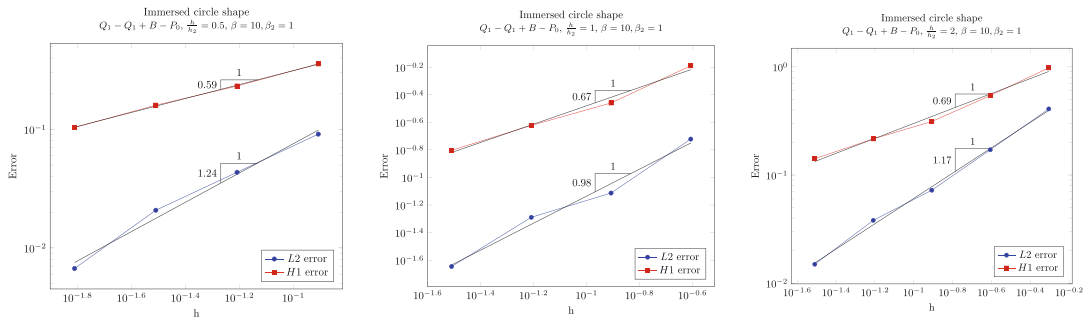


FIGURE 14 Convergence of $Q_1 - (Q_1 + B) - P_0$ FDDLM: $\beta = 10$ $\beta_2 = 1$.

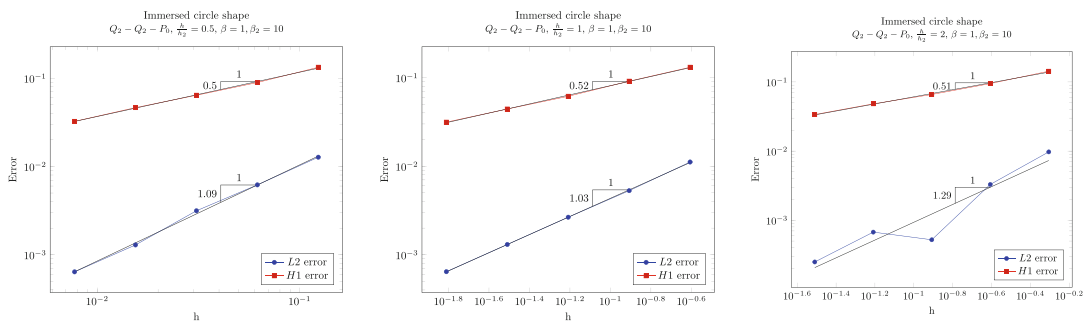


FIGURE 15 Convergence of $Q_2 - Q_2 - P_0$ FDDLM: $\beta = 1$ $\beta_2 = 10$.

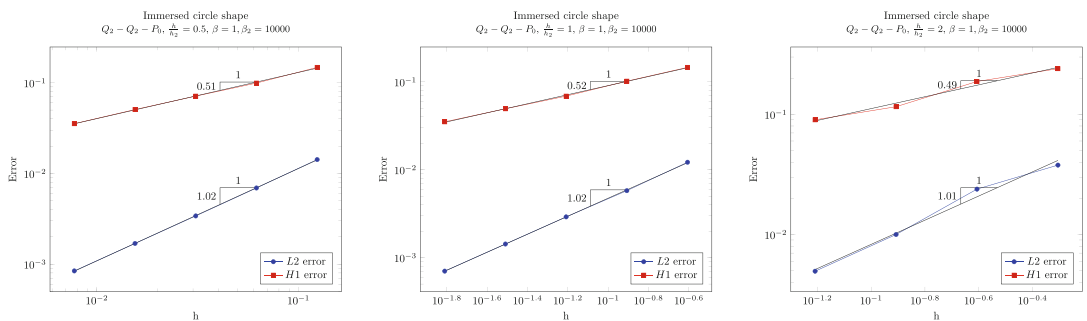


FIGURE 16 Convergence of $Q_2 - Q_2 - P_0$ FDDLM: $\beta = 1$ $\beta_2 = 10,000$.

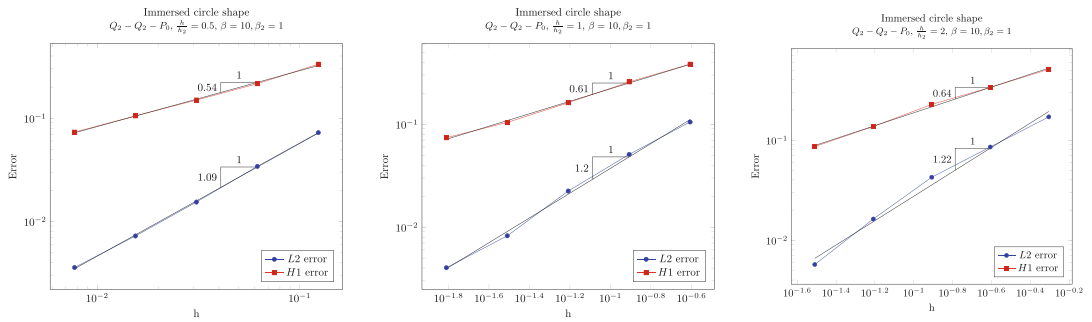


FIGURE 17 Convergence of $Q_2 - Q_2 - P_0$ FDDL M: $\beta = 10$ $\beta_2 = 1$.

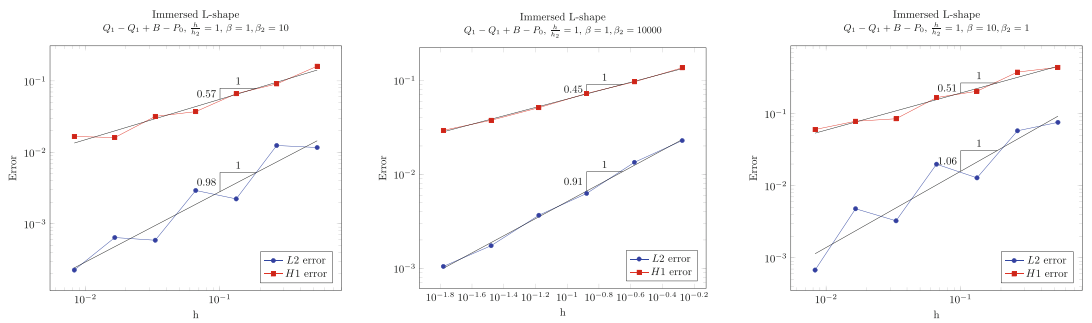


FIGURE 18 Convergence of $Q_1 - (Q_1 + B) - P_0$ FDDL M: Immersed L-shape.

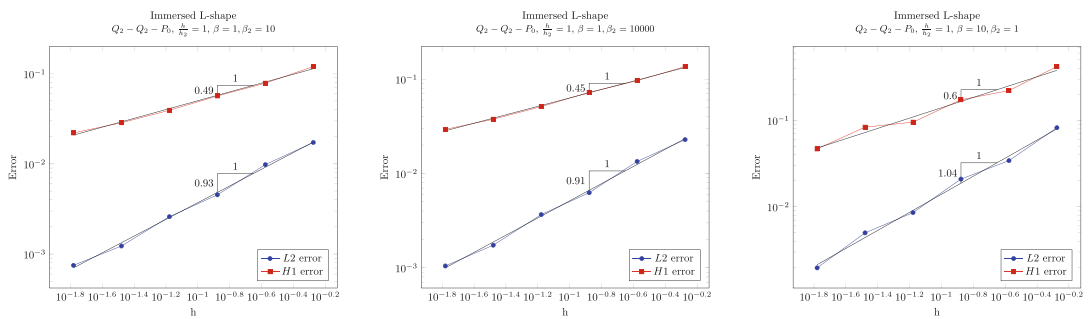


FIGURE 19 Convergence of $Q_2 - Q_2 - P_0$ FDDL M: Immersed L-shape.

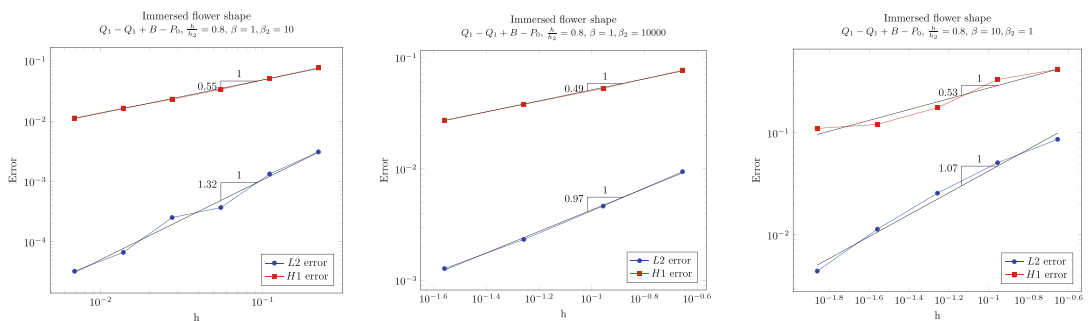
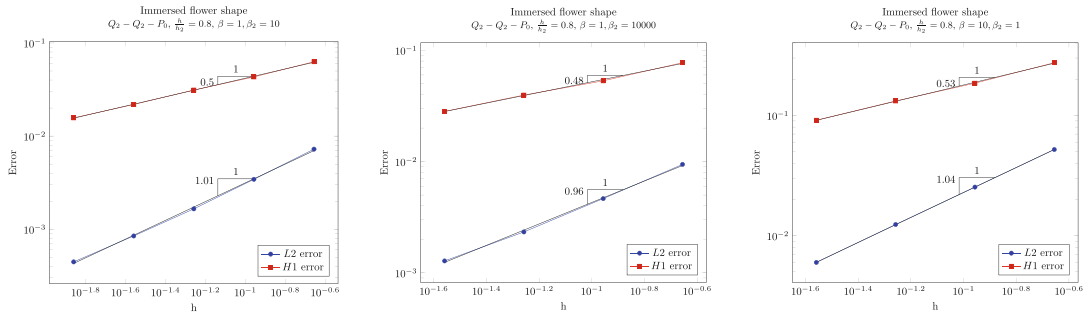
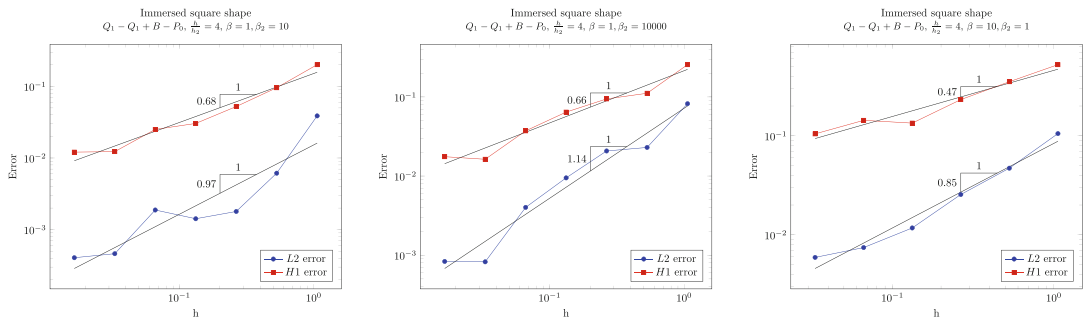
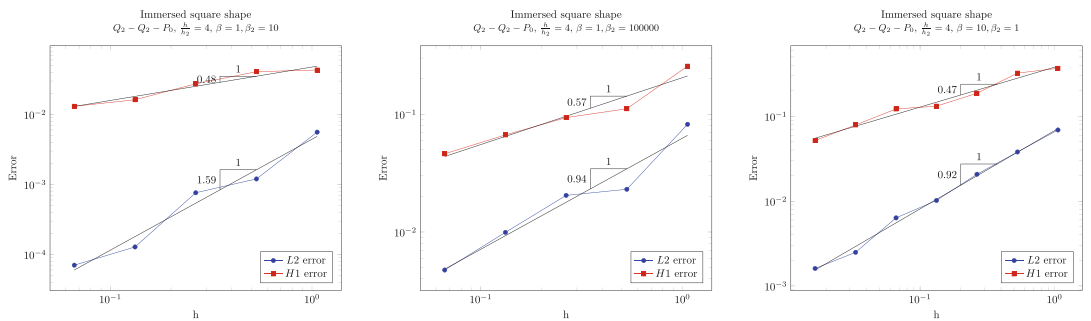
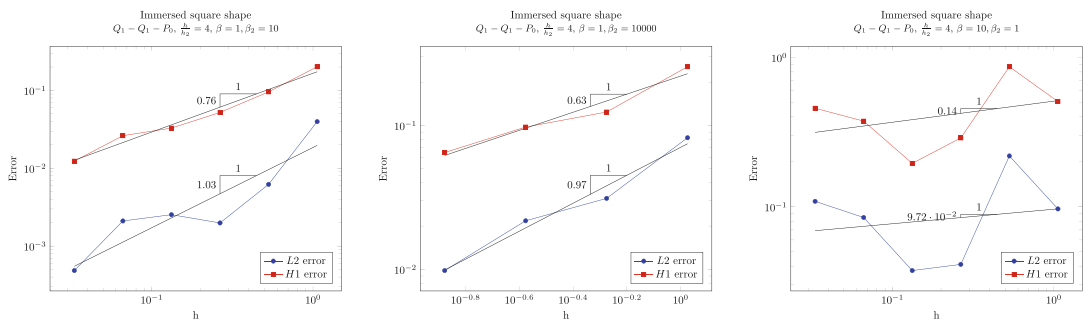


FIGURE 20 Convergence of $Q_1 - (Q_1 + B) - P_0$ FDDL M: Immersed flower shape.

FIGURE 21 Convergence of $Q_2 - Q_2 - P_0$ FDDL: Immersed flower shape.FIGURE 22 Convergence of $Q_1 - (Q_1 + B) - P_0$ FDDL: Immersed square shape.FIGURE 23 Convergence of $Q_2 - Q_2 - P_0$ FDDL: Immersed square shape.FIGURE 24 Non-convergence of $Q_1 - Q_1 - P_0$ FDDL: Immersed square shape.

difference $\beta_2 - \beta > 0$. Lastly, when the difference $\beta_2 - \beta < 0$, this element fails to converge as shown in Figure 24.

5 | CONCLUSION

This study proposes two stable elements for solving elliptic interface problems with jump coefficients. Our numerical results confirm that the discrete inf-sup is bounded away from zero for these proposed elements, and the convergence rate is consistent with our theoretical expectations. Specifically, we observe that all cases converge with the optimal convergence rate known in the literature of FEM, but less oscillation appears in the particular ratios between the mesh sizes. Furthermore, we demonstrate the robustness of our scheme when dealing with different immersed shapes. We observe that Element 2 consistently outperforms Element 1 in all cases. These results extend the findings of [9] to the case where a discontinuous Lagrange multiplier is used. In future work, it would be interesting to study the a posteriori error estimate for these schemes and use the results to apply local refinements to control the overall error and reduce computational costs by refining only where necessary. Additionally, this work could be extended to a FSI problem, which was the main motivation for our study.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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