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Stability to Signorini Problem with Pointwise Damping

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Abstract

We consider the Timoshenko beam model which can be in contact to two rigid obstacles and subject to a pointwise damping. We analyze the existence in time and the asymptotic behavior of solutions by using the hybrid method.

Keywords Timoshenko beams · Contact problem · Semilinear problem · Asymptotic behaviour

1 Introduction

In this paper we investigate the mechanical evolution of a Timoshenko homogeneous beam, of natural length ℓ, which may come in one-sided contact with two rigid obstacles. Let $0 < T \leq \infty$. We denote by $\varphi = \varphi(x,t) : (0,\ell) \times (0,T) \to \mathbb{R}$ the transverse displacement (vertical deflection) of the cross section at $x \in (0,\ell)$ and at time $t \in (0,T)$. Supposing that plane cross sections remain plane, the angle of rotation of a cross section is defined by $\psi = \psi(x,t) : (0,\ell) \times (0,T) \to \mathbb{R}$.

Assuming that a pointwise damping acts on the beam at $\xi \in (0,\ell)$, we describe the evolution of the system by the following equations (for details, see e.g. [12, 17, 20]), where the physical setting is represented by Fig. 1,

$$\begin{aligned} &\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \gamma_1\delta(x-\xi)\varphi_t = 0, \\ &\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma_2\delta(x-\xi)\psi_t = 0, \end{aligned} \tag{1.1}$$

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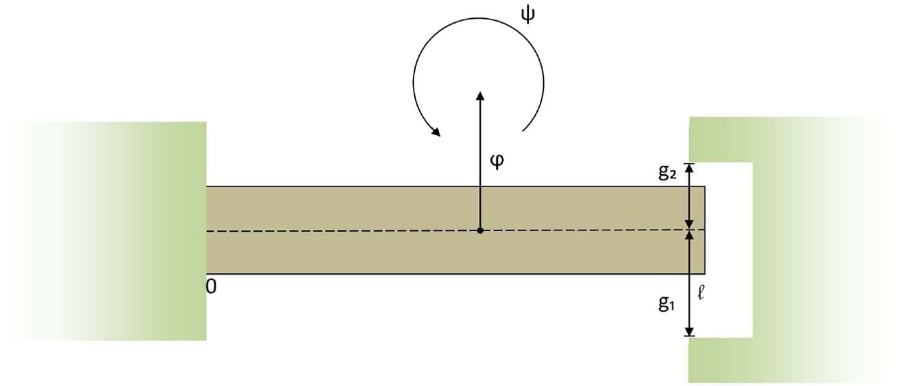

Fig. 1 Beam subject to a constraint at the free end $x = \ell$

in $(0, \ell) \times (0, T)$. Here the coefficients represent: $\rho_1 = \rho A$ the mass density, $\rho_2 = \rho I$ the moment of mass inertia, $k = \kappa GA$ the shear modulus of elasticity, $b = EI$ the rigidity coefficient of cross-section, where E is the Young's modulus, G is the modulus of rigidity and κ is the transversal shear factor, and I is the moment of inertia. Functions $S := k(\varphi_x + \psi)$ and $M := b\psi_x$ stand for the shear force and the bending moment, respectively. Moreover, $\delta(x - \xi)$ is the Dirac mass $+1$ at the point $x = \xi$. Lastly, γ_1 and γ_2 denote positive damping coefficients. Subscripts x and t represent partial derivatives with respect to x and t. The initial conditions are given by

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \forall x \in (0,\ell), \\ &\psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ \forall x \in (0,\ell), \end{aligned} \tag{1.2}$$

for some given functions $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0 : (0,\ell) \to \mathbb{R}$. In addition, we suppose that, at $x = 0$ and $x = \ell$,

$$\varphi(0,t) = 0, \quad \psi_x(0,t) = 0, \quad \psi(\ell,t) = 0, \quad \text{in } (0,T). \tag{1.3}$$

The joint at $x = \ell$ is modeled with the Signorini non penetration condition (see, e.g., [13]). In particular, the joint with gap g is asymmetrical so that $g = g_1 + g_2$, where $g_1 > 0$ and $g_2 > 0$ are, respectively, the upper and lower clearance, when the system is at rest. Then, the right end of the left beam is assumed to move vertically only between two stops, namely

$$-g_1 \le \varphi(\ell,t) \le g_2, \quad 0 \le t \le T. \tag{1.4}$$

This condition assures that the displacement at $x = \ell$ is constrained between the stops g_1 and g_2. The mathematical boundary conditions for this physical setting are as follows

$$
\begin{aligned}
&S(\ell,t) \geq 0 \text{ if } \varphi(\ell,t) = -g_1,\\
&S(\ell,t) = 0 \text{ if } -g_1 < \varphi(\ell,t) < g_2,\\
&S(\ell,t) \leq 0 \text{ if } \varphi(\ell,t) = g_2.
\end{aligned}
\tag{1.5}
$$

Before proceeding, let us recall some related results in the literature. This list is not intended to be exhaustive but only to contain a hint of the path so far made in this field.

Concerning the one dimensional quasistatic problem of thermoelastic contact, we recall the paper [4]. Furthermore, in [13] the authors consider the material constitutive law to be either elastic or viscoelastic of the Kelvin-Voigt type. Numerical aspects of the problem have been analyzed in [8, 9]. In particular, Copeti and Elliot show existence, uniqueness and regularity of solution, and they obtain error estimates using the finite element method.

The exponential energy decay rate for weak solutions of a contact problem of locally viscoelastic materials, contacting a rigid obstacle, is analyzed in [16]. In [5] a transversal contact problem has been considered and the exponential decay of the energy has been proved. Finally, in [7] the existence and exponential decay for contact problem to a thermoelastic Timoshenko beam model under a dissipative frictional type mechanism.

In all the above articles, the Signorini contact problem has been analyzed in a weak sense, and, to prove existence, the Div-Rot Lemma has been used.

In this paper we follow a new and different approach. We consider the linear Timoshenko model coupled to a dynamic boundary condition defined by an ordinary differential equation (hybrid system), the coupling is defined through a parameter ϵ, see system (2.1) below. We use semigroup theory to show the well-posedness of the problem, as well as the exponential stability of the corresponding model. We arrive at the problem of contact with normal compliance condition through a Lipschitzian perturbation. Finally, setting $\epsilon \to 0$ we get the Signorini problem. This procedure is possible thanks to the observability inequalities that the Timoshenko model possesses. We believe that this method is more effective than the usual penalty method (see [5, 13, 16] and the references therein) because we obtain more precise information about the asymptotic behavior of the solution. In particular we show that the boundary conditions of the model do not play any role in the asymptotic behavior. This means that the decay result can be proved for any boundary condition, different from the results obtained in [5, 7, 16] where particular boundary conditions were used to show the exponential decay.

The rest of this paper is organized as follows: Sect. 2 presents the existence of solution of the linear hybrid model, using the semigroup techniques. In other words, instead of directly analyzing the Signorini problem with pointwise dissipation (1.1), we will study a transmission problem, related to the associated penalized system, by formulating a model in which the singularity at $x = \xi$ is removed and replaced with two transmission conditions, see system (2.1)–(2.5) below. Section 3 is devoted to the asymptotic behaviour of the linear hybrid model, the main tool we apply is

Theorem 3.1, Theorem 3.2 and the Riemann invariants. Finally, in Sect. 4, using Lipschtizian pertubations method, we show the existence and the exponential decay of the Signorini problem (1.1)–(1.5).

2 The Hybrid Linear Model

In order to apply the semigroup theory to study the Signorini problem, we consider the linear hybrid model, approaching the penalized problem, associated to (1.1)–(1.5). For details to pass from the Signorini problem to the penalized one, see, e.g., [7]. To use the hybrid approach, let us denote by I the open set

$$I = (0, \xi) \cup (\xi, \ell).$$

Therefore, in this case it is easy to see that system (1.1)–(1.5) is equivalent to

$$\begin{aligned} &\rho_1\, \varphi_{tt} - \kappa\, (\varphi_x + \psi)_x = 0 && \text{in } I \times (0, +\infty),\\ &\rho_2\, \psi_{tt} - b\, \psi_{xx} + \kappa\, (\varphi_x + \psi) = 0 && \text{in } I \times (0, +\infty),\\ &\epsilon\varphi_{tt}(\ell, t) + \epsilon\varphi_t(\ell, t) + \epsilon\varphi(\ell, t) + S(\ell, t) = 0 \text{ in } (0, +\infty), \end{aligned} \tag{2.1}$$

satisfying the boundary conditions

$$\varphi(0, t) = 0,\ \psi_x(0, t) = 0,\ \psi(\ell, t) = 0 \text{ in } (0, +\infty). \tag{2.2}$$

Note that $\varphi(\ell, t) := v(t)$ is determined by equation $(2.1)_3$. This dynamic boundary condition can be interpreted as a beam rigidly attached at the end $x = \ell$ to a tip body of mass ϵ that models a sealed container with a granular material, for example sand. This granular material dampens the movement of the system by internal friction (for details see, e.g., [2, 3, 15]).

Additionally we consider the transmission conditions on ξ, given by

$$\varphi(\xi^-, t) = \varphi(\xi^+, t), \quad \psi(\xi^-, t) = \psi(\xi^+, t), \tag{2.3}$$

$$k\varphi_x(\xi^-, t) - k\varphi_x(\xi^+, t) = -\gamma_1\varphi_t(\xi, t), \quad b\psi_x(\xi^-, t) - b\psi_x(\xi^+, t) = -\gamma_2\psi_t(\xi, t), \tag{2.4}$$

and the initial conditions

$$\begin{gathered} \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x)\\ v(0) = v_0, \quad v_t(0) = v_1, \end{gathered} \tag{2.5}$$

This physically admissible coupling (2.3)–(2.4) represents the continuity of displacement and the discontinuity of force at $x = \xi$. We can observe that if $\gamma_i = 0, i = 1, 2$, then there is not energy dissipation at $x = \xi$ and the linkage at $x = \xi$ is conservative. Instead, if $\gamma_i > 0$, $i = 1, 2$, then the linkage is dissipative, as the case under consideration.

Putting $\Phi = \varphi_t$, $\Psi = \psi_t$ and $V = v_t$, the phase space of our problem is

$$\mathcal{H} = V_0 \times L^2(0,\ell) \times V_\ell \times L^2(0,\ell) \times \mathbb{C}^2,$$

where

$$V_0 = \Big\{ w \in H^1(0,\ell) : \quad w(0) = 0 \Big\} \quad \text{and} \quad V_\ell = \Big\{ w \in H^1(0,\ell) : \quad w(\ell) = 0 \Big\},$$

with the norm

$$\|(\varphi, \Phi, \psi, \Psi, v, V)\|^2_{\mathcal{H}} = \int_0^\ell \Big(\kappa|\varphi_x + \psi|^2 + \rho_1|\Phi|^2 + b|\psi_x|^2 + \rho_2|\Psi|^2 \Big) \\ dx + \epsilon|v|^2 + \epsilon|V|^2.$$

2.1 The C_0 Semigroup of Contractions

Denoted by $B^\top$ the transpose of a matrix B and introducing the state vector

$$U(t) = (\varphi(t), \Phi(t), \psi(t), \Psi(t), v(t), V(t))^\top := (\mathcal{U}, \mathcal{V})^\top,$$

where $\mathcal{U} = (\varphi(t), \Phi(t), \psi(t), \Psi(t))^\top$, $\mathcal{V} = (v(t), V(t))^\top$ the transmission conditions are given by

$$[k\varphi_x] = -\gamma_1 \Phi(\xi) \qquad \text{and} \qquad [b\psi_x] = -\gamma_2 \Psi(\xi), \tag{2.6}$$

where brackets mean jump, namely

$$[f] := f(\xi^-) - f(\xi^+).$$

Hence, system (2.1)–(2.5) can be written as a linear ODE in $\mathcal{H}$ of the form

$$\frac{d}{dt} U(t) = \mathcal{A} U(t), \tag{2.7}$$

where the domain $\mathcal{D}(\mathcal{A})$ of the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{D}(\mathcal{A}) = \Big\{ U \in \mathcal{H} : \ \varphi, \psi \in H^2(I), \ (\Phi, \Psi) \in V_0 \times V_\ell, \\ \text{verifying } (2.2) - (2.3) \text{ and } (2.6) \Big\},$$

and

$$\mathcal{A}U=\begin{bmatrix}\Phi\\ \dfrac{\kappa}{\rho_1}(\varphi_x+\psi)_x\\ \Psi\\ \dfrac{b}{\rho_2}\psi_{xx}-\dfrac{\kappa}{\rho_2}(\varphi_x+\psi)\\ V\\ -V-v-\frac{1}{\epsilon}S(\ell,t)\end{bmatrix}. \tag{2.8}$$

Straightforward calculations shows that the operator $\mathcal{A}$ is dissipative. Indeed, for every $U\in\mathcal{D}(\mathcal{A})$,

$$\mathrm{Re}\langle\mathcal{A}U(t),U(t)\rangle_{\mathcal{H}}=-\gamma_1|\Phi(\xi,t)|^2-\gamma_2|\Psi(\xi,t)|^2-\epsilon|V(t)|^2\leq 0. \tag{2.9}$$

Considering the resolvent equation

$$i\lambda U-\mathcal{A}U=F, \tag{2.10}$$

and taking inner product with U over the phase space $\mathcal{H}$, we get

$$\gamma_1|\Phi(\xi,t)|^2+\gamma_2|\Psi(\xi,t)|^2+\epsilon|V(t)|^2=\mathrm{Re}\langle U(t),F(t)\rangle_{\mathcal{H}}. \tag{2.11}$$

Using standard procedures we can show that $0\in\varrho(\mathcal{A})$. According to Lumer-Phillips Theorem [14, Theorem 1.2.4] the operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $\mathcal{T}(t):=e^{t\mathcal{A}}:\mathcal{H}\to\mathcal{H}$. See also [18, Theorem 1.4.3].

So, we have

Theorem 2.1 *For any $U_0\in\mathcal{H}$ there exists a unique mild solution*

$$U(t)=(\varphi(t),\varphi_t(t),\psi(t),\psi_t(t),v(t),V(t))^\top=\mathcal{T}(t)\,U_0, \tag{2.12}$$

to problem (2.1)*. Moreover, if the initial data $U_0\in D(\mathcal{A})$ there exists a strong solution satisfying*

$$U\in C^1(0,T;\mathcal{H})\cap C(0,T;D(\mathcal{A})).$$

□

3 Exponential Stability

In this section we assume that the wave speed of the model are different, that is

$$\rho_1 b-\rho_2 k\neq 0 \tag{3.1}$$

The above condition is quite natural to Timoshenko model. We would like to pointed out that condition (3.1) never happens in applications (see [11] and the references therein). The exponential decay obtained under that hypothesis is interesting only from mathematical point of view.

Here we show the exponential stability of transmission problem (2.1)–(2.5). Let us denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X a complex Banach space with norm $\|\cdot\|$.

For an operator $\mathbb{B} : D(\mathbb{B}) \subset X \to X$, we denote by $\sigma(\mathbb{B})$ its spectrum, while $\varrho(\mathbb{B}) := \mathbb{C} \setminus \sigma(\mathbb{B})$ is the resolvent set of $\mathbb{B}$.

The main tool we will apply in this paper is the following result.

Theorem 3.1 *Let $S(t) = e^{\mathbb{A}t}$ be a C_0-semigroup of contractions on Banach space. Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathbb{A}) \quad and \quad \omega_{ess}(S(t)) < 0, \tag{3.2}$$

where $\omega_{ess}(S(t))$ is the essential growth bound of the semigroup $S(t)$.

Proof Here we use [10, Corollary 2.11, p. 258] establishing that the type ω of the semigroup $e^{\mathbb{A}t}$ verifies

$$\omega = \max\{\omega_{ess}, \omega_\sigma(\mathbb{A})\}, \tag{3.3}$$

where $\omega_\sigma(\mathbb{A})$ is the upper bound of the spectrum of $\mathbb{A}$. Moreover, for any $c > \omega_{ess}$, the set $\mathcal{I}_c := \sigma(\mathbb{A}) \cap \{\lambda \in \mathbb{C} : \quad \text{Re}\lambda \geq c\}$ is finite.

Let us suppose that (3.2) is valid. Since the essential type of the semigroup ω_{ess} is negative, identity (3.3) states that the type of the semigroup will be negative provided $\omega_\sigma(\mathbb{A}) < 0$.

If $\omega_\sigma(\mathbb{A}) \leq \omega_{ess}$ then we have nothing to prove. Let us suppose that $\omega_\sigma(\mathbb{A}) > \omega_{ess}$. From (3.2) and Hille-Yosida Theorem we have $\overline{\mathbb{C}_+} \subset \varrho(\mathbb{A})$, hence $\omega_\sigma(\mathbb{A}) \leq 0$. On the other hand $\mathcal{I}_{\omega_{ess}+\delta}$ is finite for $\delta > 0$ verifying $\omega_{ess} + \delta < 0$ and $\omega_{ess} + \delta < \omega_\sigma(\mathbb{A})$. Therefore we have

$$\omega_\sigma(\mathbb{A}) = \sup \text{Re}\, \sigma(\mathbb{A}) = \sup \text{Re}\, \mathcal{I}_{\omega_{ess}+\delta} < 0.$$

Hence, the sufficient condition follows.

Reciprocally, let us suppose that the semigroup $S(t)$ is exponentially stable, in particular it goes to zero. Then, by [6, Theorem 1.1] we have that $i\mathbb{R} \subset \varrho(\mathbb{A})$. Moreover, since the type ω verifies (3.3), we have that

$$\omega_{ess} \leq \max\{\omega_{ess}, \omega_\sigma(\mathbb{A})\} = \omega < 0.$$

Then, our conclusion follows. □

Note that the above characterization is valid for any Banach space.

Other important tool we use here is the frequency domain approach, valid over Hilbert spaces (see, e.g., [19]):

Theorem 3.2 *Let $S(t) = e^{\mathbb{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

(i) *$i\mathbb{R} \subset \varrho(\mathbb{A})$, where $\varrho(\mathbb{A})$ denotes the resolvent set of $\mathbb{A}$, and*
(ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda\mathbb{I} - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty$.

Our starting point to show the exponential stability of the semigroup $S(t)$, associated to the model (2.1)–(2.5), is to prove the strong stability of $S(t)$.

Lemma 3.1 *The operator $\mathcal{A}$ defined in (2.8) satisfies $i\mathbb{R} \subset \varrho(\mathcal{A})$, provided $\xi \neq \dfrac{n}{2k+1}\ell$, $\forall n, k \in \mathbb{N}$ with n and $2k+1$ co-prime.*

Proof Because of the compacity of the resolvent family it is enough to show that there is no imaginary eigenvalues. In fact, let us suppose that $\mathcal{A}U = i\lambda U$. In terms of the components we get

$$i\lambda\varphi - \Phi = 0, \tag{3.4}$$

$$i\lambda\rho_1\,\Phi - \kappa\,(\varphi_x + \psi)_x = 0, \tag{3.5}$$

$$i\lambda\psi - \Psi = 0, \tag{3.6}$$

$$i\lambda\rho_2\,\Psi - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) = 0, \tag{3.7}$$

$$i\lambda v - V = 0, \tag{3.8}$$

$$i\epsilon\lambda V + \epsilon v + \epsilon V + S(\ell) = 0, \tag{3.9}$$

with the boundary conditions

$$\varphi(0) = 0, \quad \psi_x(0) = 0, \quad \varphi(\ell) = v, \quad \psi(\ell) = 0. \tag{3.10}$$

So we have

$$-\lambda^2\rho_1\,\varphi - \kappa\,(\varphi_x + \psi)_x = 0,$$

$$-\lambda^2\rho_2\,\psi - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) = 0.$$

From (2.11) and (3.8) we get $v = V = 0$ which together with (3.10) yields $\varphi_x(\ell) = S(\ell) = 0$. Using (2.11) once more we get

$$\Phi(\xi) = \Psi(\xi) = 0 \quad \text{which implies that} \quad [k\varphi_x] = [b\psi_x] = 0.$$

The eigenvectors of the above system must be of the form

$$\varphi_k(x) = A_k \sin\left(\frac{2k+1}{2\ell}\pi x\right), \qquad \psi_k(x) = B_k \cos\left(\frac{2k+1}{2\ell}\pi x\right).$$

Using (3.4) and (3.6) we obtain $\varphi_k(\xi) = \psi_k(\xi) = 0$. Then, we have

$$\sin\left(\frac{2k+1}{2\ell}\pi\xi\right) = 0 \quad \Leftrightarrow \quad \frac{2k+1}{2\ell}\pi\xi = j\pi \quad \Leftrightarrow \quad \xi = \frac{2j\ell}{2k+1},$$
$$\cos\left(\frac{2k+1}{2\ell}\pi\xi\right) = 0 \quad \Leftrightarrow \quad \frac{2k+1}{2\ell}\pi\xi = j\pi + \frac{\pi}{2} \quad \Leftrightarrow \quad \xi = \frac{(2j+1)\ell}{2k+1}.$$

Because of the hypothesis of this Lemma, we get $A_k = B_k = 0$. So our conclusion follows. □

To show the exponential stability we apply Theorem 3.1. Therefore, it remains to prove that the essential type ω_{ess} of the semigroup $\mathcal{T}(t)$ associated to system (2.1)–(2.5) is negative. First, let us introduce the semigroup $\mathcal{T}_0(t)$ defined by the system

$$\begin{aligned} &\rho_1\, \varphi_{tt} - \kappa\,(\varphi_x + \psi)_x = 0 && \text{in } I \times (0, +\infty), \\ &\rho_2\, \psi_{tt} - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) = 0 && \text{in } I \times (0, +\infty), \end{aligned} \tag{3.11}$$

satisfying the following boundary conditions

$$\varphi(0,t) = 0, \quad \varphi_x(\ell,t) = 0, \quad \psi_x(0,t) = 0, \quad \psi(\ell,t) = 0 \text{ in } (0,+\infty), \tag{3.12}$$

the transmission conditions (2.3)–(2.4) and the initial conditions

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \tag{3.13}$$

Note that the above problem is almost the same as system (2.1)–(2.5) except for the hybrid coupling. Let us introduce by

$$\mathcal{H}_0 = V_0 \times L^2(0,\ell) \times V_\ell \times L^2(0,\ell)$$

the phase space and by

$$\widetilde{\mathcal{H}}_0 = V_0 \times L^2(0,\ell) \times V_\ell \times L^2(0,\ell) \times \{0\} \times \{0\}$$

the extended phase space. Let us denote by Π the projection of $\mathcal{H}$ onto $\widetilde{\mathcal{H}}_0$:

$$\Pi(\varphi, \Phi, \psi, \Psi, v, V) = (\varphi, \Phi, \psi, \Psi, 0, 0).$$

Note that the composition of $\mathcal{T}_0(t)$ with Π we denote as $\mathcal{T}_0(t)\Pi$ verifies

$$\mathcal{T}_0(t)\Pi : \mathcal{H} \to \widetilde{\mathcal{H}}_0 \subset \mathcal{H}$$

It is easy to see that $\mathcal{T}_0(t)\Pi \in \mathcal{L}(\mathcal{H})$. Let us decompose the infinitesimal generator $\mathcal{A}$ in the following way

$$\mathcal{A} = \left(\begin{array}{cccc|cc} 0 & I & 0 & 0 & 0 & 0 \\ \frac{\kappa}{\rho_1}\partial_x^2 & 0 & \frac{\kappa}{\rho_1}\partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{\kappa}{\rho_2}I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I \\ \frac{\kappa}{\epsilon}\boldsymbol{\gamma} & 0 & -\frac{\kappa}{\epsilon}I & 0 & -I & -I \end{array}\right) =: \begin{pmatrix} \mathcal{A}_T & \mathbf{0} \\ B & K \end{pmatrix} \tag{3.14}$$

where $\boldsymbol{\gamma}\varphi := \varphi_x(\ell)$. Hence, recalling that $U := (\mathcal{U}, \mathcal{V})^\top$, where $\mathcal{U} := (\varphi, \Phi, \psi, \Psi)^\top$ and $\mathcal{V} := (v, V)^\top$, we obtain

$$\mathcal{A}U = \begin{pmatrix} \mathcal{A}_T\mathcal{U} \\ B\mathcal{U} + K\mathcal{V} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_T\mathcal{U} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ K\mathcal{V} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ B\mathcal{U} \end{pmatrix}.$$

Under the above conditions we have the following Lemma

Lemma 3.2 *The difference $\mathcal{T}(t) - \mathcal{T}_0(t)\Pi$ is a compact operator over $\mathcal{H}$. Hence the corresponding essential types are equal.*

Proof Note that the solution of $U_t - \mathcal{A}U = 0$ can be written as

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}_t = \begin{pmatrix} \mathcal{A}_T\mathcal{U} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ K\mathcal{V} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ B\mathcal{U} \end{pmatrix}$$

which implies that

$$\mathcal{U} = e^{t\mathcal{A}_T}\mathcal{U}_0, \quad \mathcal{V} = e^{tK}\mathcal{V}_0 + \int_0^t e^{(t-s)K} B\mathcal{U}(s)\, ds.$$

Therefore

$$U(t) - \begin{pmatrix} e^{t\mathcal{A}_T}\mathcal{U}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ e^{tK}\mathcal{V}_0 + \int_0^t e^{(t-s)K} B\mathcal{U}(s)\, ds \end{pmatrix}.$$

Note that the right hand side of the above equation is a compact operator. In fact, e^{tK} is a finite dimensional semigroup and

$$\mathfrak{G}(t) = \int_0^t e^{(t-s)K} B\mathcal{U}(s)\, ds = \int_0^t e^{(t-s)K} \begin{pmatrix} V \\ -V - v + \frac{1}{\epsilon}S(\ell, s) \end{pmatrix} ds,$$

verifies $\mathfrak{G} \in H^1(0, T)$. Therefore

$$[\mathcal{T}(t) - \mathcal{T}_0(t)\Pi]$$

is compact. □

Our next step is to prove that the essential type of $\mathcal{T}_0(t) = e^{\mathcal{A}_T t}$ is negative, where $\mathcal{A}_T$ is defined in (3.14). To do that let us introduce the semigroup $\mathcal{T}_1(t)$ defined by the system

$$\begin{aligned} &\rho_1\, \varphi_{tt} - \kappa\, (\varphi_x + \psi)_x = 0 \ \text{ in } I \times (0, +\infty), \\ &\rho_2\, \psi_{tt} - b\, \psi_{xx} + \kappa \varphi_x = 0 \text{ in } I \times (0, +\infty), \end{aligned} \tag{3.15}$$

with boundary conditions (3.12) and verifying the initial and transmission conditions (2.5) and (2.3)–(2.4), respectively.

Let us denote by $\mathfrak{B}$ the operator

$$\mathfrak{B} : \mathcal{H}_0 \to \mathcal{H}_0, \quad \mathfrak{B}U = \left(0, 0, 0, \frac{\kappa}{\rho_2}\psi\right)^\top.$$

It is easy to verify that $\mathfrak{B}$ is a compact operator over $\mathcal{H}_0$. Indeed, if $U_n = (\varphi_n, \Phi_n, \psi_n, \Psi_n)^\top$ is a bounded sequence in $\mathcal{H}_0$, in particular ψ_n is bounded in $H^1(0, \ell)$. Hence there exists a subsequence which converges strongly in $L^2(0, \ell)$. So, for any bounded sequence in $\mathcal{H}_0$ there exists a subsequence, we still denote in the same way such that $\mathfrak{B}U_n$, converges strongly in $\mathcal{H}_0$. Then, the operator

$$\mathcal{A}_0 = \mathcal{A}_T - \mathfrak{B}$$

is the infinitesimal generator of a C_0-semigroup denoted by $\mathcal{T}_1(t) = e^{\mathcal{A}_0 t}$.

Under the above conditions we have the following Lemma.

Lemma 3.3 *The difference $\mathcal{T}_0(t) - \mathcal{T}_1(t)$ is a compact operator. Hence the corresponding essential types are equal.*

Proof The equation $\mathcal{U}_t - \mathcal{A}_T \mathcal{U} = 0$ can be written as

$$\mathcal{U}_t - \mathcal{A}_0 \mathcal{U} = \mathfrak{B}\mathcal{U}.$$

Then the solution can be written as

$$\mathcal{U}(t) = e^{\mathcal{A}_0 t}\mathcal{U}_0 + \int_0^t e^{\mathcal{A}_0 (t-s)} \mathfrak{B}\mathcal{U}(s)\, ds. \tag{3.16}$$

Recalling the definition of $\mathcal{U}(t)$ and $\mathcal{T}_1(t)$, equation (3.16) implies

$$\mathcal{T}(t)\mathcal{U}_0 = \mathcal{T}_1(t)\mathcal{U}_0 + \int_0^t e^{\mathcal{A}_0 (t-s)} \mathfrak{B} e^{\mathcal{A}s}\mathcal{U}_0\, ds.$$

Since $\mathfrak{B}$ is a compact operator then the composition $e^{\mathcal{A}_0 (t-s)} \mathfrak{B} e^{\mathcal{A}s}$ is also a compact operator. Therefore, $\mathcal{T}_0(t) - \mathcal{T}_1(t)$ is a compact operator over $\mathcal{H}_0$. □

Hence, to prove the exponential decay of $\mathcal{T}(t)$ we only need to prove that the essential type of $\mathcal{T}_1$ is negative.

The One-Dimensional System Associated to (3.15)

Using the Riemann invariants

$$\begin{aligned} p_1 &= \sqrt{\rho_1}\varphi_t - \sqrt{k}\varphi_x, \quad q_1 = \sqrt{\rho_1}\varphi_t + \sqrt{k}\varphi_x, \\ p_2 &= \sqrt{\rho_2}\psi_t - \sqrt{b}\psi_x, \quad q_2 = \sqrt{\rho_2}\psi_t + \sqrt{b}\psi_x, \end{aligned}$$

we have that

$$\varphi_t = \frac{q_1 + p_1}{2\sqrt{\rho_1}}, \quad \varphi_x = \frac{q_1 - p_1}{2\sqrt{k}}, \quad \psi_t = \frac{q_2 + p_2}{2\sqrt{\rho_2}}, \quad \psi_x = \frac{q_2 - p_2}{2\sqrt{b}}.$$

Therefore, the evolution problem can be written as

$$p_{1,t} + k_1 p_{1,x} - c_1(q_2 - p_2) = 0, \tag{3.17}$$

$$q_{1,t} - k_1 q_{1,x} - c_1(q_2 - p_2) = 0, \tag{3.18}$$

$$p_{2,t} + k_2 p_{2,x} - c_2(q_1 - p_1) = 0, \tag{3.19}$$

$$q_{2,t} - k_2 q_{2,x} - c_2(q_1 - p_1) = 0, \tag{3.20}$$

where

$$k_1 = \sqrt{\frac{\kappa}{\rho_1}}, \quad k_2 = \sqrt{\frac{b}{\rho_2}}, \quad c_1 = \frac{\kappa}{2\sqrt{b\rho_1}}, \quad c_2 = -\frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}},$$

verifying the following boundary conditions

$$q_1(0) + p_1(0) = 0, \quad q_1(\ell) - p_1(\ell) = 0, \quad q_2(0) - p_2(0) = 0, \quad q_2(\ell) + p_2(\ell) = 0, \tag{3.21}$$

and transmission conditions

$$q_i(\xi^-, t) + p_i(\xi^-, t) = q_i(\xi^+, t) + p_i(\xi^+, t), \tag{3.22}$$

$$q_i(\xi^-, t) - q_i(\xi^+, t) + p_i(\xi^+, t) - p_i(\xi^-, t) = -\gamma_i k_i \left[p_i(\xi^+, t) + q_i(\xi^+, t)\right], \tag{3.23}$$

for $i = 1, 2$.

Denoting

$$\mathbf{K} = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & -k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & -k_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & c_1 & -c_1 \\ 0 & 0 & c_1 & -c_1 \\ c_2 & -c_2 & 0 & 0 \\ c_2 & -c_2 & 0 & 0 \end{pmatrix}, \quad \mathfrak{U} = \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix},$$

the system (3.17)–(3.20) can be written as

$$\mathfrak{U}_t + \mathbf{K}\mathfrak{U}_x + C\mathfrak{U} = 0, \quad \mathfrak{U}(0) = \mathfrak{U}_0\,. \tag{3.24}$$

It is not difficult to see that system (3.15) is equivalent to (3.24).

Let us denote by $\mathcal{T}_2(t)$ the semigroup defined by (3.24) over $\mathbf{H}_4 = [L^2(0,\ell)]^4$. Note that $C_0 := \operatorname{diag}(C) = \mathbf{0}$.

Let us denote by $\mathcal{T}_3(t) : \mathbf{H}_4 \to \mathbf{H}_4$ the semigroup defined by the diagonal system

$$\widetilde{\mathfrak{U}}_t + \mathbf{K}\widetilde{\mathfrak{U}}_x = 0, \quad \widetilde{\mathfrak{U}}(0) = \mathfrak{U}_0, \tag{3.25}$$

verifying the same boundary conditions (3.21) and the same transmission conditions (3.22)–(3.23).

At this point, we use the result due to Neves et al [1] that in our case implies the following result

Theorem 3.3 *Under the above notations the difference $\mathcal{T}_3(t) - \mathcal{T}_2(t)$ is a compact operator over $\mathbf{H}_4$, provided condition (3.1) holds.*

Proof Note that condition (3.1) implies that $k_i \neq k_j$ for $i \neq j$. Using [1, Theorem A] for $p = 2$, our conclusion follows. □

System (3.25) is completely decoupled and can be written as

$$p_{i,t} + k_i p_{i,x} = 0,$$

$$q_{i,t} - k_i q_{i,x} = 0,$$

$$q_1(0) + p_1(0) = 0, \quad q_1(\ell) - p_1(\ell) = 0, \tag{3.26}$$

$$q_2(0) - p_2(0) = 0, \quad q_2(\ell) + p_2(\ell) = 0, \tag{3.27}$$

together with

$$q_i(\xi^-) + p_i(\xi^-) = q_i(\xi^+) + p_i(\xi^+), \tag{3.28}$$

$$q_i(\xi^-) - q_i(\xi^+) + p_i(\xi^+) - p_i(\xi^-) = -\gamma_i k_i (p_i(\xi^+) + q_i(\xi^+)), \tag{3.29}$$

for $i = 1, 2$. The semigroup $\mathcal{T}_3(t) : \mathbf{H}_4 \to \mathbf{H}_4$ is generated by the operator

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix}, \tag{3.30}$$

where $\mathbf{0}$ is the 2×2 null matrix and $\mathbf{A}_i$ is given by

$$\mathbf{A}_i U_2 = k_i K U_{2,x}, \quad U_2 = \begin{pmatrix} p \\ q \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$D(\mathbf{A}_1) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{H}_2 : \quad p, q \in H^1((0,\xi) \cup (\xi,\ell)), \quad p(0) + q(0) = 0, \;\; q(\ell) - p(\ell) = 0 \right\},$$

and

$$\begin{aligned} D(\mathbf{A}_2) = \Bigg\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{H}_2 : \quad & p, q \in H^1((0,\xi) \cup (\xi,\ell)), \quad q(0) - p(0) \\ & = 0, \;\; p(\ell) + q(\ell) = 0 \Bigg\}. \end{aligned}$$

The resolvent system $\lambda U_2 + \mathbf{A}_i U_2 = F$ is given by

$$\lambda U_2 + k_i K U_{2,x} = F \tag{3.31}$$

where

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} p \\ q \end{pmatrix}.$$

Hence the above system can be rewritten as

$$U_{2,x} + \frac{i\lambda}{k_i} K U_2 = \frac{1}{k_i} K F, \quad i = 1, 2, \tag{3.32}$$

and, in terms of the components, it becomes

$$p_{i,x} + \frac{i\lambda}{k_i} p_i = \frac{1}{k_i} f_1^i, \quad i = 1, 2. \tag{3.33}$$

$$q_{i,x} - \frac{i\lambda}{k_i} q_{i,t} = -\frac{1}{k_i} f_2^i, \quad i = 1, 2. \tag{3.34}$$

verifying the boundary conditions (3.26)–(3.27) and the transmission conditions (3.28)–(3.29).

Lemma 3.4 *The operator* $\mathbf{A}$ *infinitesimal generator of* $\mathcal{T}_3$ *given in* (3.30) *is dissipative over the phase space* $\mathbf{H}_4$.

Proof Because of (3.30) it is enough to show that $\mathbf{A}_i$ is a dissipative operator over $\mathbf{H}_2$ for $i = 1, 2$. Here we prove only for $i = 1$, the proof to $i = 2$ is similar. For sake of simplicity the index 1 is not written in p and q. Note that

$$\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix} \right)_{\mathbf{H}_2} = \left(\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} -k_1 p_x \\ k_1 q_x \end{pmatrix} \right)_{\mathbf{H}_2}$$

$$
\begin{aligned}
&= \int_0^{\xi} (-k_1 p_x p + k_1 q_x q)\ dx + \int_{\xi}^{\ell} (-k_1 p_x p + k_1 q_x q)\ dx \\
&= \frac{k_1}{2}\int_0^{\xi} \left(-\frac{d}{dx}|p|^2 + \frac{d}{dx}|q|^2\right)\ dx + \frac{k_1}{2}\int_{\xi}^{\ell} \left(-\frac{d}{dx}|p|^2 + \frac{d}{dx}|q|^2\right)\ dx \\
&= \frac{k_1}{2}\left(-|p(\xi^-)|^2 + |p(0)|^2 + |q(\xi^-)|^2 - |q(0)|^2\right) \\
&\quad + \frac{k_1}{2}\left(-|p(\ell)|^2 + |p(\xi^+)|^2 + |q(\ell)|^2 - |q(\xi^+)|^2\right).
\end{aligned}
$$

Using the boundary conditions we get

$$
\begin{aligned}
\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix}\right)_{\mathbf{H}_2} &= \frac{k_1}{2}\left(-|p(\xi^-)|^2 + |q(\xi^-)|^2\right) + \frac{k_1}{2}\left(|p(\xi^+)|^2 - |q(\xi^+)|^2\right) \\
&= \frac{k_1}{2}\left(-p(\xi^-) + q(\xi^-)\right)\left(p(\xi^-) + q(\xi^-)\right) \\
&\quad + \frac{k_1}{2}\left(p(\xi^+) - q(\xi^+)\right)\left(p(\xi^+) + q(\xi^+)\right).
\end{aligned}
$$

Applying the continuity of the sum $p + q$ we get

$$
\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix}\right)_{\mathbf{H}_2} = \frac{k_1}{2}\left[\left(-p(\xi^-) + q(\xi^-)\right) + \left(p(\xi^+) - q(\xi^+)\right)\right]\left(p(\xi) + q(\xi)\right).
$$

Using (3.29) we obtain that

$$
\left(\begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A}_1 \begin{pmatrix} p \\ q \end{pmatrix}\right)_{\mathbf{H}_2} = -\frac{1}{2}\gamma_1 k_1^2 \left|p(\xi) + q(\xi)\right|^2, \tag{3.35}
$$

and our conclusion follows. □

Lemma 3.5 *The infinitesimal generator* $\mathbf{A}$ *of* $\mathcal{T}_3$ *given in* (3.30) *verifies*

$$
i\mathbb{R} \subset \varrho(\mathbf{A})
$$

provided $\xi \neq \dfrac{n}{2m+1}\ell$, $\forall n, m \in \mathbb{N}$ *with* n *and* $2m + 1$ *co-prime.*

Proof Since system (3.25) is fully decoupled it is enough to show that $i\mathbb{R} \subset \varrho(\mathbf{A}_i)$ for $i = 1, 2$. Because of the compacity of the resolvent family associated to $\mathbf{A}_i$ we prove that there are no imaginary eigenvalues. First, we consider the case $i = 1$, and subsequently, the case $i = 2$. For sake of simplicity the index 1 is not written in p and q. Let us suppose that for $\lambda \in \mathbb{R}$ there exists $0 \neq W \in D(\mathbf{A}_1)$ such that $\mathbf{A}_1 W = i\lambda W$. Since $\mathbf{A}_1$ is dissipative we get

$$
|p(\xi) + q(\xi)|^2 = 0,
$$

which implies that

$$p(\xi^+) - q(\xi^+) = p(\xi^-) - q(\xi^-).$$

In terms of the components of $\mathbf{A}_1 W = i\lambda W$ we find

$$p_x + i\frac{\lambda}{k_1}p = 0, \quad q_x - i\frac{\lambda}{k_1}q = 0.$$

Since $p(0) + q(0) = 0$, we obtain

$$p(x) = p(0)e^{-i\frac{\lambda}{k_1}x}, \qquad q(x) = -p(0)e^{i\frac{\lambda}{k_1}x},$$

Since $p(\ell) - q(\ell) = 0$, we have

$$p(\ell) - q(\ell) = p(0)\left(e^{-\frac{i\lambda}{k_1}\ell} + e^{\frac{i\lambda}{k_1}\ell}\right) = 0, \quad \text{and this implies} \quad \lambda = \left(m + \frac{1}{2}\right)k_1\frac{\pi}{\ell}.$$

At the point $x = \xi$ we get

$$p(\xi) = p(0)e^{-i\frac{\lambda}{k_1}\xi}, \quad q(\xi) = -p(0)e^{i\frac{\lambda}{k_1}\xi},$$

and consequently

$$p(\xi) + q(\xi) = p(0)\left(e^{-i\frac{\lambda}{k_1}\xi} - e^{i\frac{\lambda}{k_1}\xi}\right) = 0.$$

This implies that $e^{i\frac{2\lambda}{k_1}\xi} = 1$ and then we find that $\frac{2\lambda}{k_1}\xi = 2n\pi$. Substitution of λ yields

$$\frac{(2m+1)\pi}{2\ell}\xi = n\pi \quad \Rightarrow \quad \xi = \frac{2n}{2m+1}\ell,$$

but this is not possible for hypothesis, so $p(0) = 0$. Therefore $W = 0$, which is a contradiction.
Now, we prove $i\mathbb{R} \subset \varrho(\mathbf{A}_2)$. If there exists $0 \neq (p_2, q_2) = W \in D(\mathbf{A}_2)$ such that $\mathbf{A}_2 W = i\lambda W$, making a reasoning similar to case $i = 1$, we conclude, because of the boundary conditions, that

$$p_2(x) = p_2(0)e^{-i\frac{\lambda}{k_1}x}, \qquad q_2(x) = p_2(0)e^{i\frac{\lambda}{k_1}x},$$

hence we have

$$p_2(\ell) + q_2(\ell) = p_2(0)\left(e^{-\frac{i\lambda}{k_1}\ell} + e^{\frac{i\lambda}{k_1}\ell}\right) = 0, \quad \text{and this implies} \quad \lambda = \left(m + \frac{1}{2}\right)k_1\frac{\pi}{\ell},$$

also we get

$$p_2(\xi)+q_2(\xi)=p_2(0)\left(e^{-i\frac{\lambda}{k_1}\xi}+e^{i\frac{\lambda}{k_1}\xi}\right)=0.$$

Substitution of λ yields

$$\frac{(2m+1)\pi}{2\ell}\xi=\frac{(2n+1)\pi}{2\ell}\pi \quad\Rightarrow\quad \xi=\frac{2n+1}{2m+1}\ell,$$

but this is not possible for hypothesis, so $p_2(0)=0$. Therefore $W=0$, which is a contradiction. □

Let us introduce the function $\mathfrak{F}^i_\xi(\lambda)$:

$$\mathfrak{F}^1_\xi(\lambda)=\cos^2\left(\frac{\lambda}{k_1}\ell\right)+\frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda}{k_1}\xi\right)\cos^2\left(\frac{\lambda}{k_1}(\ell-\xi)\right),$$
$$\mathfrak{F}^2_\xi(\lambda)=\cos^2\left(\frac{\lambda}{k_1}\ell\right)+\frac{\gamma_1^2}{k_1^2}\cos^2\left(\frac{\lambda}{k_1}\xi\right)\sin^2\left(\frac{\lambda}{k_1}(\ell-\xi)\right),$$

and let us denote by

$$A^i=\left\{\xi\in(0,\ell):\quad \inf\mathfrak{F}^i_\xi(\mathbb{R})>0\right\},\quad i=1,2.$$

Lemma 3.6 *Let us suppose that* $\xi\in\mathbb{Q}\ell$ *such that* $\xi\neq\frac{n}{2m+1}\ell$, $\forall n,m\in\mathbb{N}$, *with* n, $2m+1$ *co-prime numbers then we have that*

$$I^i:=\inf\mathfrak{F}^i_\xi(\mathbb{R})>0,\quad i=1,2.$$

Proof We show for $i=1$, the other is similar. Note that $\frac{\ell}{2}\in A^1\neq\emptyset$. In fact, for $\xi=\frac{\ell}{2}$ we get

$$\begin{aligned}&\left[\cos^2\left(\frac{\lambda}{k_1}\ell\right)+\frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda}{k_1}\xi\right)\cos^2\left(\frac{\lambda}{k_1}(\ell-\xi)\right)\right]_{\xi=\frac{\ell}{2}}\\
&\quad=\cos^2\left(\frac{\lambda}{k_1}\ell\right)+\frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda}{k_1}\frac{\ell}{2}\right)\cos^2\left(\frac{\lambda}{k_1}\frac{\ell}{2}\right)\\
&\quad=\cos^2\left(\frac{\lambda}{k_1}\ell\right)+\frac{\gamma_1^2}{4k_1^2}\sin^2\left(\frac{\lambda}{k_1}\ell\right)\\
&\quad\geq\min\left\{1,\ \frac{\gamma_1^2}{4k_1^2}\right\}\left([\cos^2\left(\frac{\lambda}{k_1}\ell\right)+\sin^2\left(\frac{\lambda}{k_1}\ell\right)\right]\end{aligned}$$

$$\geq \min\left\{1,\ \frac{\gamma_1^2}{4k_1^2}\right\}.$$

By contradiction, suppose that $I = 0$. Since $\xi \in \mathbb{Q}\ell$ we can suppose that $\xi = \frac{m}{n}\ell$ with m and n co-prime. Therefore the function $\mathfrak{F}_\xi^1$ is periodic with period equal to

$$T = 2\pi \frac{k_1}{\ell} n.$$

Hence

$$\inf \mathfrak{F}_\xi^1(\mathbb{R}) = \inf \mathfrak{F}_\xi^1([0, T]).$$

So, there exists a sequence of elements $\lambda_n \in [0, T]$ such that

$$\cos^2\left(\frac{\lambda_n}{k_1}\ell\right) + \frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda_n}{k_1}\xi\right)\cos^2\left(\frac{\lambda_n}{k_1}(\ell-\xi)\right) \to I = 0.$$

Since λ_n is bounded, there exists a convergent subsequence (we still denote in the same way) such that $\lambda_n \to \lambda^*$ and that

$$\cos^2\left(\frac{\lambda^*}{k_1}\ell\right) + \frac{\gamma_1^2}{k_1^2}\sin^2\left(\frac{\lambda^*}{k_1}\xi\right)\cos^2\left(\frac{\lambda^*}{k_1}(\ell-\xi)\right) = 0.$$

Then we have that

$$\frac{\lambda^*}{k_1}\ell = \frac{2j+1}{2}\pi, \quad \frac{\lambda^*}{k_1}\xi = \nu\pi, \quad j, \nu \in \mathbb{N}, \tag{3.36}$$

or

$$\frac{\lambda^*}{k_1}\ell = \frac{2j+1}{2}\pi, \quad \frac{\lambda^*}{k_1}(\ell-\xi) = \frac{2\mu+1}{2}\pi, \quad j, \mu \in \mathbb{N}. \tag{3.37}$$

Let us suppose that (3.36) holds, the other is similar, taking $\xi = r\ell$ with $r \in \mathbb{Q}$, we get

$$\frac{\lambda^*}{k_1}\xi = \frac{\lambda^*}{k_1}r\ell = \nu\pi \quad \Rightarrow \quad \frac{2j+1}{2}\pi r = \nu\pi \quad \Rightarrow \quad r = \frac{2\nu}{2j+1}.$$

But this is contradictory to $\xi \neq \frac{n}{2m+1}\ell$ with $n,\ m \in \mathbb{N}$. □

Theorem 3.4 *The semigroup $\mathcal{T}_3$ is exponentially stable, provided that ξ verifies hypothesis of Lemma 3.6.*

Proof Because of (3.30) it is enough to show that $e^{\mathbf{A}_i t}$ is exponentially stable over $\mathbf{H}_2$ for $i = 1, 2$. First we prove only for $i = 1$. For convenience we denote $p_1 = p$ and $q_1 = q$. We use Theorem 3.2 to show the exponential stability. Because of Lemma 3.1 it is enough to show that the resolvent operator is uniformly bounded over the imaginary axes. So the solution of (3.32) is given by

$$p(x) = p(0)e^{-i\frac{\lambda}{k_1}x} + \frac{1}{k_1}\int_0^x e^{-i\frac{\lambda}{k_1}(x-s)} f_1(s)\, ds, \quad x \in [0, \xi], \tag{3.38}$$

$$q(x) = -p(0)e^{i\frac{\lambda}{k_1}x} - \frac{1}{k_1}\int_0^x e^{i\frac{\lambda}{k_1}(x-s)} f_2(s)\, ds, \quad x \in [0, \xi]. \tag{3.39}$$

Similarly over $[\xi, \ell]$ we have that

$$p(x) = p(\xi^+)e^{-i\frac{\lambda}{k_1}(x-\xi)} + \frac{1}{k_1}\int_\xi^x e^{-i\frac{\lambda}{k_1}(x-s)} f_1(s)\, ds, \quad x \in [\xi, \ell], \tag{3.40}$$

$$q(x) = q(\xi^+)e^{i\frac{\lambda}{k_1}(x-\xi)} - \frac{1}{k_1}\int_\xi^x e^{i\frac{\lambda}{k_1}(x-s)} f_2(s)\, ds, \quad x \in [\xi, \ell]. \tag{3.41}$$

The above solution verify equation (3.31) and also the boundary condition at $x = 0$. Using (3.38) and (3.39) we get

$$p(\xi^-) = p(0)e^{-\frac{i\lambda}{k_1}\xi} + J_1, \qquad q(\xi^-) = -p(0)e^{i\frac{\lambda}{k_1}\xi} + J_2, \quad x \in [0, \xi], \tag{3.42}$$

$$J_1 = \frac{1}{k_1}\int_0^\xi e^{-\frac{i\lambda}{k_1}(x-s)} f_1(s)\, ds, \qquad J_2 = -\frac{1}{k_1}\int_0^\xi e^{\frac{i\lambda}{k_1}(x-s)} f_2(s)\, ds. \tag{3.43}$$

Now we adjust $q(\xi^+)$ and $p(\xi^+)$ such that the transmission conditions (3.29) hold for $i = 1$.

$$\begin{aligned} p(\xi^+) + q(\xi^+) &= p(\xi^-) + q(\xi^-), \\ p(\xi^+) - q(\xi^+) &= p(\xi^-) - q(\xi^-) + \frac{\gamma_1}{k_1}(p(\xi^+) + q(\xi^+)). \end{aligned}$$

Solving the above system we get

$$p(\xi^+) = p(\xi^-) + \frac{\gamma_1}{2k_1}(p(\xi^-) + q(\xi^-)), \quad q(\xi^+) = q(\xi^-) - \frac{\gamma_1}{2k_1}(p(\xi^-) + q(\xi^-)).$$

Applying (3.42) we get

$$\begin{aligned} p(\xi^+) &= p(0)e^{-i\frac{\lambda}{k_1}\xi} - \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}) + J_1 + \frac{\gamma_1}{2k_1}(J_1 + J_2), \\ q(\xi^+) &= -p(0)e^{i\frac{\lambda}{k_1}\xi} + \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}) + J_2 - \frac{\gamma_1}{2k_1}(J_1 + J_2). \end{aligned}$$

Hence, with this choice the transmission conditions (3.28)–(3.29) hold. Finally we adjust $p(0)$ such that the boundary condition at $x=\ell$ holds.

$$\begin{aligned} p(\xi^+)e^{-i\frac{\lambda}{k_1}(x-\xi)} &= p(0)e^{-i\frac{\lambda}{k_1}x} - \frac{\gamma_1}{2k_1}p(0)\left(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}\right)e^{-\frac{i\lambda}{k_1}(x-\xi)} \\ &\quad + \left(J_1 + \frac{\gamma_1}{2k_1}(J_1+J_2)\right)e^{-\frac{i\lambda}{k_1}(x-\xi)}, \\ q(\xi^+)e^{i\frac{\lambda}{k_1}(x-\xi)} &= -p(0)e^{\frac{i\lambda}{k_1}x} + \frac{\gamma_1}{2k_1}p(0)\left(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi}\right)e^{\frac{i\lambda}{k_1}(x-\xi)} \\ &\quad + \left(J_2 - \frac{\gamma_1}{2k_1}(J_1+J_2)\right)e^{\frac{i\lambda}{k_1}(x-\xi)}. \end{aligned}$$

Using (3.40)–(3.41) we get that $q(\ell)-p(\ell)=0$ implies

$$0 = -p(0)\left(e^{i\frac{\lambda}{k_1}\ell} + e^{-i\frac{\lambda}{k_1}\ell}\right) + \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} - e^{-i\frac{\lambda}{k_1}\xi})\left(e^{\frac{i\lambda}{k_1}(\ell-\xi)} + e^{\frac{-i\lambda}{k_1}(\ell-\xi)}\right) + G.$$

So $p(0)$ has to be choosen such that

$$\begin{aligned} 0 &= -2p(0)\cos\left(\frac{\lambda}{k_1}\ell\right) + 2i\frac{\gamma_1}{k_1}p(0)\sin\left(\frac{\lambda}{k_1}\xi\right)\cos\left(\frac{\lambda}{k_1}(\ell-\xi)\right) + G \\ &= -2p(0)\left[\cos\left(\frac{\lambda}{k_1}\ell\right) + i\frac{\gamma_1}{k_1}\sin\left(\frac{\lambda}{k_1}\xi\right)\cos\left(\frac{\lambda}{k_1}(\ell-\xi)\right)\right] + G. \end{aligned}$$

The existence of solution will depend on

$$\cos\left(\frac{\lambda}{k_1}\ell\right) + i\frac{\gamma_1}{k_1}\sin\left(\frac{\lambda}{k_1}\xi\right)\cos\left(\frac{\lambda}{k_1}(\ell-\xi)\right) \neq 0.$$

The above expresion identically vanishes if and only if

$$\cos\left(\frac{\lambda}{k_1}\ell\right) = 0, \quad \cos\left(\frac{\lambda}{k_1}(\ell-\xi)\right) = 0,$$

or

$$\cos\left(\frac{\lambda}{k_1}\ell\right) = 0, \quad \sin\left(\frac{\lambda}{k_1}\xi\right) = 0,$$

simultaneously. But the above identity implies

$$\frac{\lambda}{k_1}\ell = \frac{\pi}{2} + j\pi, \quad \frac{\lambda}{k_1}(\ell-\xi) = \frac{\pi}{2} + m\pi, \quad j,m \in \mathbb{Z},$$

and consequently

$$\lambda = \frac{2j+1}{2\ell}k_1\pi, \quad \xi = \frac{2(j-m)}{2j+1}\ell, \quad j, m \in \mathbb{Z}.$$

But this is not possible because our hypothesis. Therefore we have

$$2p(0) = \frac{G}{\cos(\frac{\lambda}{k_1}\ell) + i\frac{\gamma_1}{k_1}\sin(\frac{\lambda}{k_1}\xi)\cos(\frac{\lambda}{k_1}(\ell-\xi))},$$

and we find that

$$|p(0)| \le c\frac{\|F\|_{\mathbf{H}_2}}{\sqrt{\cos^2(\frac{\lambda}{k_1}\ell) + \frac{\gamma_1^2}{k_1^2}\sin^2(\frac{\lambda}{k_1}\xi)\cos^2(\frac{\lambda}{k_1}(\ell-\xi))}} = c\frac{\|F\|_{\mathbf{H}_2}}{\sqrt{\mathfrak{F}_\xi^1(\lambda)}}.$$

Hence using Lemma 3.6 we get

$$|p(0)| \le c\|F\|_{\mathbf{H}_2},$$

from where it follows

$$\|U_2\|_{\mathbf{H}_2} = \left\|\begin{pmatrix} p \\ q \end{pmatrix}\right\|_{\mathbf{H}_2} = \|(i\lambda I - \mathbf{A}_1)^{-1}F\|_{\mathbf{H}_2} \le c\|F\|_{\mathbf{H}_2}$$

Using Theorem 3.2 we get the exponential stability.

Finally, we show that $e^{\mathbf{A}_2 t}$ is exponentially stable. The only difference from the proof of $e^{\mathbf{A}_1 t}$ is the boundary condition. This means that the solution of the corresponding resolvent system is written as

$$p(x) = p(0)e^{-i\frac{\lambda}{k_1}x} + \frac{1}{k_1}\int_0^x e^{-i\frac{\lambda}{k_1}(x-s)}f_1(s)\,ds, \quad x \in [0,\xi], \tag{3.44}$$

$$q(x) = p(0)e^{i\frac{\lambda}{k_1}x} - \frac{1}{k_1}\int_0^x e^{i\frac{\lambda}{k_1}(x-s)}f_2(s)\,ds, \quad x \in [0,\xi]. \tag{3.45}$$

Since the pointwise dissipation is the same as in the case $\mathbf{A}_1$ we conclude that the value of $p(0)$ that verifies the boundary condition at $x = \ell$ is given by

$$0 = -p(0)\left(e^{i\frac{\lambda}{k_1}\ell} + e^{-i\frac{\lambda}{k_1}\ell}\right) - \frac{\gamma_1}{2k_1}p(0)(e^{i\frac{\lambda}{k_1}\xi} + e^{-i\frac{\lambda}{k_1}\xi})\left(e^{\frac{i\lambda}{k_1}(\ell-\xi)} - e^{\frac{-i\lambda}{k_1}(\ell-\xi)}\right) + G.$$

Therefore we have

$$2p(0) = \frac{G}{\cos(\frac{\lambda}{k_1}\ell) + i\frac{\gamma_1}{k_1}\cos(\frac{\lambda}{k_1}\xi)\sin(\frac{\lambda}{k_1}(\ell-\xi))},$$

Following the same arguments as in the case of $\mathbf{A}_1$ thanks to Lemma 3.6 we conclude that $e^{\mathbf{A}_2 t}$ is exponentially stable. □

Now we are in conditions to show the main result of this paper.

Theorem 3.5 *The semigroup* $\mathcal{T}(t) = e^{\mathcal{A}t}$ *associated to system* (2.1)–(2.4) *is exponentially stable provided* ξ *verifies hypothesis of Lemma* 3.6

Proof From Lemma 3.2 the difference $\mathcal{T}(t)-\mathcal{T}_0(t)\Pi$ is a compact operator over $\mathcal{H}$. By Lemma 3.3 we get that $\mathcal{T}_0 - \mathcal{T}_1$ is a compact operator over $\mathbf{H}_4$, hence $\omega_{ess}(\mathcal{T}_0(t)) = \omega_{ess}(\mathcal{T}_1(t))$. Note that $\mathcal{T}_1$ and $\mathcal{T}_2$ are different representation of the same system, so we have $\omega_{ess}(\mathcal{T}_1(t)) = \omega_{ess}(\mathcal{T}_2(t))$. Moreover from Theorem 3.3 the operator $\mathcal{T}_2(t)-\mathcal{T}_3(t)$ is a compact operator over $\mathbf{H}_4$ hence $\omega_{ess}(\mathcal{T}_2(t)) = \omega_{ess}(\mathcal{T}_3(t))$. Finally, from Theorem 3.4 we get

$$\omega_{ess}(\mathcal{T}(t)) = \omega_{ess}(\mathcal{T}_3(t)) \leq \omega(\mathcal{T}_3(t)) = \max\{\omega_{ess}, \omega_\sigma(\mathbf{A}_i)\} < 0.$$

From Lemma 3.1 $i\mathbb{R} \subset \varrho(\mathcal{A})$ provided $\xi \neq \dfrac{n}{2k+1}\ell$, with n and $2m+1$ co-prime. Applying Theorem 3.1 our conclusion follows. □

Remark 3.1 From the above Theorem we conclude that there exists a positive constant C independent of ϵ such that

$$\|(i\lambda I - \mathcal{A})^{-1}\| \leq C$$

that implies that there exists a positive constant $\gamma > 0$ such that

$$\|T(t)U_0\|_{\mathcal{H}} \leq e^{-\gamma t}\|U_0\|_{\mathcal{H}} \tag{3.46}$$

4 The Signorini Problem

Here we prove the well possedness of an abstract semilinear problem and we show, under suitable conditions that the solution also decays polynomially to zero. So we introduce a local Lipschitz $\mathcal{F}$ function defined over a Hilbert space $\mathcal{H}$. We suppose that for any ball $B_R = \{W \in \mathcal{H} : \quad \|W\|_{\mathcal{H}} \leq R\}$, there exists a function globally of Lipschitz $\widetilde{\mathcal{F}}_R$ such that

$$\mathcal{F}(0) = 0, \quad \mathcal{F}(U) = \widetilde{\mathcal{F}}_R(U), \quad \forall U \in B_R \tag{4.1}$$

and additionally, that there exists a positive constant κ_0 such that

$$\int_0^t \big(\widetilde{\mathcal{F}}_R(U(s)), U(s)\big)_{\mathcal{H}}\, ds \leq \kappa_0 \|U(0)\|^2_{\mathcal{H}}, \qquad \forall U \in C([0,T];\mathcal{H}) \tag{4.2}$$

Under these conditions we present

Theorem 4.1 *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup of contraction, exponentially stable semigroup with infinitesimal generator $\mathbb{A}$ over the phase space $\mathcal{H}$. Let $\mathcal{F}$ locally Lipschitz on $\mathcal{H}$ satisfying conditions (4.1) and (4.2). Then there exists a global solution to*

$$U_t - \mathbb{A}U = \mathcal{F}(U), \quad U(0) = U_0 \in \mathcal{H}, \tag{4.3}$$

that decays exponentially to zero.

Proof By hypotheses, there exist positive constants c_0 and γ such that $\|T(t)\| \leq c_0 e^{-\gamma t}$, and $\widetilde{\mathcal{F}}_R$ globally Lipschitz with Lipschitz constant K_0 verifying conditions (4.1) and (4.2). Let us consider the following space.

$$E_\mu = \left\{ V \in L^\infty(0,\infty;\mathcal{H});\ \ t \mapsto e^{-\mu t}\|V(s)\| \in L^\infty(\mathbb{R}) \right\}.$$

Using standard fixed point arguments we can show that there exists only one global solution to

$$U_t^R - \mathbb{A}U^R = \widetilde{\mathcal{F}}_R(U^R), \quad U^R(0) = U_0 \in \mathcal{H}. \tag{4.4}$$

Multiplying the above equation by U^R we get that

$$\frac{1}{2}\frac{d}{dt}\|U^R(t)\|_{\mathcal{H}}^2 - (\mathbb{A}U^R, U^R)_{\mathcal{H}} = (\widetilde{\mathcal{F}}_R(U^R), U^R)_{\mathcal{H}}.$$

Since the semigroup is contractive, its infinitesimal generator is dissipative, therefore

$$\|U^R(t)\|_{\mathcal{H}}^2 \leq \|U_0\|_{\mathcal{H}}^2 + 2\int_0^t (\widetilde{\mathcal{F}}_R(U^R), U^R)_{\mathcal{H}}\, dt.$$

Using (4.2) we get

$$\|U^R(t)\|_{\mathcal{H}}^2 \leq (1+k_0)\|U_0\|_{\mathcal{H}}^2.$$

Note that for $R > (1+k_0)\|U_0\|_{\mathcal{H}}^2$, we have that

$$\widetilde{\mathcal{F}}_R(V) = \mathcal{F}(V), \quad \forall\ \ \|V\|_{\mathcal{H}} \leq R.$$

In particular we find

$$\widetilde{\mathcal{F}}_R(U^R(t)) = \mathcal{F}(U^R(t)).$$

This means that U^R is also solution of system (4.3) and because of the uniqueness we conclude that $U^R = U$. To show the exponential stability to system (4.3), it is enough to show the exponential decay to system (4.4). To do that, we use fixed points arguments. Let us consider

$$\mathcal{T}(V) = T(t)U_0 + \int_0^t T(t-s)\widetilde{\mathcal{F}}_R(V(s))\, ds.$$

Note that $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$ for δ small, $(\gamma - \delta > 0)$. In fact, for any $V \in E_{\gamma-\delta}$ we have

$$\begin{aligned}
\|\mathcal{T}(V)\|_{\mathcal{H}} &\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \int_0^t \|\widetilde{\mathcal{F}}_R(V(s))\|_{\mathcal{H}} e^{-\gamma(t-s)}\, ds, \\
&\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 \int_0^t \|V(s)\|_{\mathcal{H}} e^{-\gamma(t-s)}\, ds, \\
&\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 e^{-\gamma t} \int_0^t e^{\delta s}\, ds \sup_{s\in[0,t]} \left\{ e^{(\gamma-\delta)s} \|V(s)\|_{\mathcal{H}} \right\}, \\
&\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \frac{K_0 C}{\delta} e^{-(\gamma-\delta)t}.
\end{aligned}$$

Hence $\mathcal{T}(V) \in E_{\gamma-\delta}$. Using standard arguments we show that $\mathcal{T}^n$ satisfies

$$\|\mathcal{T}^n(W_1) - \mathcal{T}^n(W_2)\| \le \frac{(k_1 t)^n}{n!} \|W_1 - W_2\|_{\mathcal{H}}.$$

Therefore we have a unique fixed point satisfying

$$\mathcal{T}^n(U) = U = T(t)U_0 + \int_0^t T(t-s)\widetilde{\mathcal{F}}_R(U(s))\, ds,$$

that is U is a solution of (4.4), and since $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$, then the solution decays exponentially. □

Let us consider the semilinear system

$$\begin{aligned}
&\rho_1 \varphi^{\epsilon}_{tt} - k(\varphi^{\epsilon}_x + \psi^{\epsilon})_x = 0, && \text{in } I \times (0,\infty),\\
&\rho_2 \psi^{\epsilon}_{tt} - b\psi^{\epsilon}_{xx} + k(\varphi^{\epsilon}_x + \psi^{\epsilon}) = 0, && \text{in } I \times (0,\infty), \qquad (4.5)\\
&\epsilon v^{\epsilon}_{tt} + \epsilon v^{\epsilon}_t + \epsilon v^{\epsilon} + S^{\epsilon}(L,t) = -\tfrac{1}{\epsilon}\left[(v^{\epsilon} - g_2)^+ - (g_1 - v^{\epsilon})^+\right] && \text{in } (0,\infty),
\end{aligned}$$

verifying the transmission conditions (2.3)–(2.4). The above system can be written as

$$U_t - \mathcal{A}U = \mathcal{F}(U), \qquad U(0) = U_0,$$

where $\mathcal{A}$ is given by (2.8) and $\mathcal{F}$ is given by

$$\mathcal{F}(U) = (0,0,0,0,0,f(v))^{\top}, \quad f(v) = -\frac{1}{\epsilon^2}\Big[(v-g_2)^+ - (g_1 - v)^+\Big] \qquad (4.6)$$

Note that $\mathcal{F}$ is a Lipschitz function verifing hypothesis (4.1)–(4.2). In fact, $\mathcal{F}(0) = 0$. Moreover

$$\begin{aligned}\int_0^t \big(\mathcal{F}(U(s)), U(s)\big)_{\mathcal{H}}\, ds &= -\int_0^t \frac{1}{\epsilon^2}\Big[(v-g_2)^+ - (g_1 - v)^+\Big] v_t\, ds\\ &= -\frac{1}{2\epsilon^2}\int_0^t \frac{d}{dt}\Big[|(v-g_2)^+|^2 + |(g_1-v)^+|^2\Big]\, ds\\ &\le \frac{1}{2\epsilon^2}\Big[|(v_0-g_2)^+|^2 + |(g_1-v_0)^+|^2\Big].\end{aligned}$$

Theorem 4.2 *The nonlinear semigroup defined by system* (4.5) *is exponentially stable.*

Proof It is a direct consequence of Theorem 4.1. □

Next we show the energy inequality

Lemma 4.1 *The solution of system* (4.5) *satisfies*

$$E(t,\varphi^\epsilon,\psi^\epsilon) + \int_0^t \left[\gamma_1|\varphi_t^\epsilon(\xi,t)|^2 + \gamma_2|\psi_t^\epsilon(\xi,t)|^2\right] dt \le E(0,\varphi^\epsilon,\psi^\epsilon), \tag{4.7}$$

where

$$2E(t) = \int_0^\ell \left[\rho_1|\varphi_t^\epsilon|^2 + \rho_2|\psi_t^\epsilon|^2 + k|\varphi_x^\epsilon + \psi^\epsilon|^2 + b|\psi_x^\epsilon|^2\right]dx + \frac{1}{\epsilon}\mathcal{N}(t) + \epsilon|v_t^\epsilon|^2 + \epsilon|v^\epsilon|^2,$$

and

$$\mathcal{N}(t) := |(\varphi^\epsilon(\ell,t) - g_2)^+|^2 + |(g_1 - \varphi^\epsilon(\ell,t))^+|^2$$

Proof Multiplying Eq. $(4.5)_1$ by φ_t, Eq. $(4.5)_2$ by ψ_t, and Eq. $(4.5)_3$ by v_t, summing up the product result our conclusion follows. □

Let us introduce the functionals

$$\begin{aligned}\mathcal{I}(x,t) &= \rho_2 b|\psi_t(x,t)|^2 + |M(x,t)|^2 + \rho_1\kappa|\varphi_t(x,t)|^2 + |S(x,t)|^2,\\ \mathcal{L}(t) &= \int_0^\ell \left(\rho_2 b q_x|\psi_t|^2 + q_x|M|^2 + \rho_1\kappa q_x|\varphi_t|^2 + q_x|S|^2\right) dx\\ &\quad - \int_0^L \left(q\rho_1\kappa\Phi\overline{\Psi} - qS\overline{M}\right) dx,\end{aligned}$$

where q is as in (4.11) hence there exist positive constants C_0 and C_1 such that

$$C_0\int_0^\ell \mathcal{I}(x,t)\, dx \le \mathcal{L}(t) \le C_1\int_0^\ell \mathcal{I}(x,t)\, dx. \tag{4.8}$$

Under the above conditions we have

Lemma 4.2 *The solution of system* (4.5) *satisfies*

$$\left|\int_0^t q(\ell)\mathcal{I}(\ell,t)\,dx-\int_0^t \mathcal{L}(s)\,ds\right|\le cE(0),$$
$$\left|\int_0^t q(0)\mathcal{I}(0,t)\,dx-\int_0^t \mathcal{L}(s)\,ds\right|\le cE(0).$$

Proof Let us multiply equation $(4.5)_2$ by $q\overline{M}$ we get

$$\frac{d}{dt}\int_0^\ell \rho_2 q\psi_t M\,dx-\frac{1}{2}\int_0^\ell q\frac{d}{dx}\left[\rho_2 b|\psi_t|^2+|M|^2\right]dx=-\int_0^\ell q\overline{M}S\,dx. \tag{4.9}$$

Similarly, multiplying equation $(4.5)_1$ by $q\overline{S}$, we get

$$\frac{d}{dt}\int_0^\ell \rho_1 q\varphi_t S\,dx-\frac{1}{2}\int_0^\ell q\frac{d}{dx}\left[\rho_2\kappa|\varphi_t|^2+|S|^2\right]dx=\rho_1\kappa\int_0^L q\varphi_t\psi_t\,dx. \tag{4.10}$$

Therefore summing identities (4.9) and (4.10) and integrating over $[0,t]$ we get

$$\frac{1}{2}\int_0^t\int_0^\ell q\frac{d}{dx}\mathcal{I}(x,t)\,dxdt=\int_0^\ell\left(\rho_1 q\varphi_t S+\rho_2 q\psi_t M\right)\,dx\Big|_0^t$$
$$-\int_0^t\int_0^L \rho_1\kappa q\varphi_t\psi_t-q\overline{M}S\,dx,$$

performing integrations by parts and recalling the definition of $\mathcal{L}$, we get

$$\int_0^t\left[q(\ell)\mathcal{I}(\ell,s)-q(0)\mathcal{I}(0,s)\right]\,ds-\int_0^t\mathcal{L}(s)\,ds=\int_0^\ell\left(\rho_1 q\varphi_t S+\rho_2 q\psi_t M\right)dx\Big|_0^t.$$

Since

$$\left|\int_0^\ell \rho_2 q\psi_t M\,dx\right|\le cE(0),\quad \left|\int_0^\ell \rho_1 q\varphi_t S\,dx\right|\le cE(0),$$

we conclude that

$$\left|\int_0^t\left[q(\ell)\mathcal{I}(\ell,s)-q(0)\mathcal{I}(0,s)\right]ds-\int_0^t\mathcal{L}(s)\,ds\right|\le cE(0).$$

Taking

$$q(x)=\frac{e^{nx}-1}{n},\qquad q_0(x)=\frac{e^{-nx}-e^{-n\ell}}{n}. \tag{4.11}$$

Note that $q'(x)$ is large in comparison to q for n large, therefore there exist positive constants c_0 and c_1 such that

$$c_0 \int_a^b \mathcal{I}(x)\, dx \le \mathcal{L} \le c_1 \int_0^L \mathcal{I}(x)\, dx.$$

So our result follows. □

Theorem 4.3 *For any initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H}$ there exists a weak solution to Signorini problem (1.1)–(1.4) which decays as established in Theorem 4.2.*

Proof From Theorem 4.1 we have that there exists only one solution to system (4.5). Using Lemma 4.1 and Lemma 4.2 we get

$$\mathcal{I}_\epsilon(\ell, t) \quad \text{uniformly bounded in} \quad L^2(0, T), \tag{4.12}$$

which means that the first order energy is uniformly bounded for any $\epsilon > 0$. Standard procedures implies that the solution of system (4.5) converges in the distributional sense to system (1.1). It remains only to show that conditions (1.4) holds. To do that we use the observability inequality in Theorem 4.2, and we get that $\varphi_t^\epsilon(\ell, t)$ and $S^\epsilon(\ell, t)$ are bounded in $L^2(0, T)$, so is v_{tt}. Using $(4.5)_4$ we obtain

$$\int_0^T \left[\epsilon v_{tt} + \epsilon v_t + \epsilon v + S^\epsilon(\ell, t)\right](u - v)\, dt = -\frac{1}{\epsilon} \int_0^T \left[(v - g_2)^+ - (g_1 - v)^+\right](u - v)\, dt.$$

For any $u \in L^2(0, T; \mathcal{K}) \cap H^1(0, T; L^2(0, \ell))$, where $\mathcal{K} = \{w \in H^1(0, \ell), \quad g_1 \le u(x) \le g_2\}$. It is no difficult to see that

$$\lim_{\epsilon \to 0} \int_0^T \left(\epsilon v_{tt}^\epsilon + \epsilon v_t^\epsilon + \epsilon v^\epsilon\right)(u - v^\epsilon)\, dt = 0.$$

In fact, from $(4.5)_4$ ϵv_{tt}^ϵ is bounded for any $\epsilon > 0$ (by a constant depending on ϵ) in $L^2(0, T)$, from (4.12) v_t^ϵ is also uniformly bounded in $L^2(0, T)$. Therefore v_t^ϵ is a continuous function, uniformly bounded in $L^\infty(0, T)$. Making an integration by parts we find

$$\int_0^T \epsilon v_{tt}^\epsilon[u - v^\epsilon]\, dt = \epsilon\ v_t^\epsilon[u - v^\epsilon]\Big|_0^T - \int_0^T \epsilon v_t^\epsilon[u_t - v_t^\epsilon]\, dt \quad \to \quad 0.$$

Hence,

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_0^T S^\epsilon(\ell, t)[u(t) - v(t)]\, dt \\ &\quad = \lim_{\epsilon \to 0} \int_0^T -\frac{1}{\epsilon}\left[(v - g_2)^+ - (g_1 - v)^+\right][u(t) - v(t)]\, dt. \end{aligned}$$

Since

$$\begin{aligned}
&\int_0^T (v-g_2)^+[u(t)-v(t)]\,dt\\
&\quad=\int_0^T (v-g_2)^+[u(t)-g_2]\,dt-\int_0^T (v-g_2)^+(v-g_2)\,dt\\
&\quad=\int_0^T (v-g_2)^+[u(t)-g_2]\,dt-\int_0^T (v-g_2)^+(v-g_2)^+\,dt\le 0,
\end{aligned}$$

for all $g_1 \le u \le g_2$. Similarly we get

$$-\int_0^T [(g_1-v)^+[u(t)-v(t)]\,dt \le 0.$$

Therefore, from the last two inequalities we get

$$\int_0^T \frac{1}{\epsilon}\Big[(v-g_2)^+-(g_1-v)^+\Big][u(t)-v(t)]\,dt\le 0,\quad \forall\epsilon>0.$$

For any $u \in H^1(0,T;L^2(0,\ell))$ such that $g_1 \le u \le g_2$. Taking the limit $\epsilon \to 0$ we get

$$\int_0^T S(\ell,t)[u(\ell,t)-\varphi(\ell,t)]\,dt \ge 0,\quad \forall u\in L^2(0,T;\mathcal{K}).$$

From this relation we obtain (1.5). The proof of the existence is now complete. To show the asymptotic behaviour, recalling Remark 3.46, we get

$$E(t,\varphi^\epsilon,\psi^\epsilon)\le E(0,\varphi^\epsilon,\psi^\epsilon)e^{-\gamma t}.$$

Integrating over $[t_1,t_2]$ and applying the semicontinuity of the norm, we conclude the exponential stability of a solution of the Signorini problem. □

Remark 4.1 The uniqueness of the solution to Signorini problem (1.1)–(1.4) remains an open question.

The same approach can be used to show existence of the semilinear problem

$$\begin{aligned}
&\rho_1\varphi_{tt}-k(\varphi_x+\psi)_x+\mu_1\varphi|\varphi|^\alpha+\gamma_1\delta(x-\xi)\varphi_t=0,\\
&\rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi)+\mu_2\psi|\psi|^\beta+\gamma_2\delta(x-\xi)\psi_t=0.
\end{aligned}\tag{4.13}$$

Theorem 4.4 *Under the same hypothesis from Theorem 4.3, there is at least one solution to Signorini problem* (4.13) *satifying* (1.2)–(1.5).

Proof As in Theorem 4.3, we consider the function

$$\mathcal{F}(U) = (0, -\mu_1\varphi^\epsilon|\varphi^\epsilon|^\alpha, 0, -\mu_2\psi^\epsilon|\psi^\epsilon|^\beta, 0, f(v))^\top.$$

where f is the same as in (4.6). Note that $\mathcal{F}(0) = 0$. Using the mean value theorem to $g(s) = |s|^\alpha s$ we obtain the inequality

$$\Big|s|s|^\alpha - r|r|^\alpha\Big| \leq (|s|^\alpha + |r|^\alpha)|s - r|.$$

Taking the norm in $\mathcal{H}$ and since φ_i^ϵ and ψ_i^ϵ belong to $H^1(0,\ell) \subset L^\infty(0,\ell)$, then we get

$$\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\|_\mathcal{H} \leq C\|U_1 - U_2\|_\mathcal{H},$$

Therefore, $\mathcal{F}$ is locally Lipschtiz. Since

$$(\mathcal{F}U, U)_\mathcal{H} = -\frac{d}{dt}\int_0^\ell \left(\frac{\mu_1}{1+\alpha}|\varphi^\epsilon|^{\alpha+2} + \frac{\mu_2}{1+\beta}|\psi^\epsilon|^{\beta+2}\right) dx,$$

then

$$\int_0^t (\mathcal{F}U, U)_\mathcal{H} \leq \int_0^\ell \left(\frac{\mu_1}{1+\alpha}|\varphi^\epsilon(0)|^{\alpha+2} + \frac{\mu_2}{1+\beta}|\psi^\epsilon(0)|^{\beta+2}\right) dx.$$

Thus, there exists a positive constant c_0 such that

$$\int_0^t (\mathcal{F}U, U)_\mathcal{H} \leq c_0\|U\|^2_\mathcal{H}.$$

Note that for this function, there exists the cut-off function

$$f_{1,R_2} = \begin{cases} \mu_1 x|x|^\alpha & \text{if } x \leq R_2, \\ \mu_1 x|R_2|^\alpha & \text{if } x \geq R_2, \end{cases} \qquad f_{2,R_2} = \begin{cases} \mu_2 x|x|^\beta & \text{if } x \leq R_2, \\ \mu_2 x|R_2|^\beta & \text{if } x \geq R_2, \end{cases}$$

It is not difficult to check that

$$\widetilde{\mathcal{F}}_{R_2} = (0, -f_{1,R_2}, 0, -f_{2,R_2}, 0, f(v))^\top$$

is globally Lipschtiz. Using Theorem 4.1 our conclusion follows. □

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Declarations

Conflict of interest This work does not have any conflict of interest.

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