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The inventory routing problem with split deliveries

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Abstract

We study the benefit of introducing split deliveries in the inventory routing problem (IRP), both when the order-up-to level (OU) and the maximum level replenishment policies are applied. We first propose a mathematical formulation and solve it by implementing a branch-and-cut algorithm. Then, we carry out a worst-case analysis to show the cost increase we have in the worst case by using unsplit deliveries instead of split deliveries, both for the OU and the maximum-level replenishment policies. Extensive computational results on benchmark instances allow us to evaluate the benefit of introducing split deliveries. Finally, a sensitivity analysis on customer demands, initial inventory levels, maximum inventory levels and distance to the depot allows us to understand the instance features that make split deliveries effective in IRPs.

KEYWORDS

branch-and-cut algorithm, inventory routing problems, maximum-level, order-up-to-level, split deliveries, worst-case analysis

1 | INTRODUCTION

The inventory routing problem (IRP) is a combinatorial optimization problem in which the demands of a set of customers are served by one supplier over a time horizon. The goal is to determine the distribution plan that minimizes the sum of inventory and routing cost. Its notability came from the rising popularity of the vendor managed inventory (VMI) practice, in which the supplier takes over inventory management of customers. In a VMI system, contrary to the classical retailer managed inventory (RMI) systems, the logistics manager monitors the inventory level of the customers. Hence, she can arrange a delivery policy so that the total cost of inventory holding and vehicle routing is minimized, without the need to wait for the orders placed by the customers. Archetti and Integration [5] argued that the VMI systems can provide significant total cost savings of up to 33% over the RMI systems.

From a mathematical perspective, the IRP is a generalization of the well-known vehicle routing problems (VRP): while the VRP concerns only about the delivery of products to the customers so as to satisfy demands in a given time period, the IRPs also consider the cost of keeping such products in inventory over a time horizon. Bell et al. [13] were the first to conceptualize what would become the IRPs, initially as a variant of the VRP. Over the years, they received a lot of attention and had become a major area of interest with many studies conducted on its variants. Coelho et al. [22] defined a basic version of the IRP on the variant introduced by Archetti et al. [7] and then categorized the different problems studied starting from this variant according to seven criteria, which include: time horizon, structure, routing, inventory policy, inventory decisions, fleet composition, and fleet size. More specifically, the literature on these problems is prolific of both exact and heuristic approaches. Exact approaches are mainly based on branch-and-cut schemes. We refer to [1, 8, 18-20, 26], for recent branch-and-cut algorithms for the multi-vehicle case, with Manousakis et al. [26] being the current best-in-class exact approach for the IRP. To the best of our knowledge, only two contributions proposed branch-and-price algorithms for the IRP, namely Desaulniers et al. [23] and Bard and Nananukul [12]. In addition, [3] provide an interesting study on the equivalence of different formulations for the IRP in terms of value of the linear relaxation. Recent heuristic approaches are based on matheuristic and metaheuristic schemes, see [2, 9, 11, 17, 21, 31]. For recent surveys and tutorials on IRPs we refer to [14-16, 22, 30].

According to the classification proposed in [22], in this article we study a problem with finite time horizon, one-supplier-to-many-customers structure with multiple customers per route. The fleet is composed of multiple homogenous vehicles. Inventory level must be nonnegative with both maximum level (ML) and order-up-to level (OU) inventory policies. Specifically, we are interested in studying the impact of allowing split deliveries to customers, that is, allowing each customer to be visiting multiple times in each time period. This problem is referred to as the IRP with split deliveries. Note that all contributions mentioned above study the case in which split deliveries are not allowed, that is, the entire quantity received by each customer in each time period has to be delivered by a single vehicle.

Split delivery is a well studied area in the VRP literature. Specifically, the split delivery vehicle routing problem (SDVRP) is a generalization of the capacitated vehicle routing problem (CVRP) in which multiple visits to customers are allowed. The problem was introduced in [24, 25]. Archetti et al. [6] has proved that split delivery can save up to at most half the cost, that is, that the cost increase of unsplit deliveries with respect to split deliveries is 100% in the worst case. We refer to [4] for a survey on the SDVRP.

On the contrary, very little has been done on the IRPs with split deliveries. The literature so far has been mainly focused on the different algorithms to solve them, as done by Yu et al. [34] and Wong and HasnahMoin [33]. Yu et al. [34] developed a stochastic IRP with split deliveries, and proposed an approximate method to solve this problem. Meanwhile, Wong and HasnahMoin [33] proposed a three-steps ant colony optimization algorithm for a split IRP. Both of these approaches provide good results in a reasonable amount of solving time. Concerning algorithms to solve these problems, [32] developed a matheuristic for the IRP with ML replenishment, using a path-flow formulation with a route set updated after every iteration, which has overall better results than exact and heuristics methods. Moin et al. [28] utilized a hybrid genetic algorithm to generate solutions for a many-to-one IRP. A two-phase variable neighborhood search algorithm was studied by Mjirda et al. [27], with the first phase solving only the transportation portion, and the second phase improving the received solution to achieve overall cost minimization.

Despite its cost improvement, split deliveries are often considered an inconvenience for the customers, as it is easier to arrange for one shipment per time period than two or more. The question then becomes whether or not the overall cost reduction is worth the effort, which is based subjectively on the perception of the operation manager. Therefore, the aim of this study is to understand the improvement in term of cost saving by using split delivery over the unsplit delivery policy in the IRP, and provide a quantifiable basis for operation managers.

More specifically, the contributions of this article can be summarized as follows:

- We are the first to approach the split IRP with an exact method.
- We propose mathematical formulations for the IRP with split deliveries, both under the ML and the OU replenishment policies.
- We solve the formulations, enriched with valid inequalities, through a branch-and-cut algorithm.
- We carry out a worst-case analysis to show the cost increase in worst case by using unsplit deliveries instead of split deliveries, both for the ML and OU replenishment policies.
- We perform extensive computational tests on benchmark IRP instances with the aim of evaluating the benefits of split deliveries.
- We perform a sensitivity analysis, by varying input parameters of the benchmark instances, to identify the instance features that make split deliveries effective.

The article is organized as follows. In Section 2, we present the problem description. In Section 3, we provide the formulations for all the policies we analyze, together with the valid inequalities we use, and briefly describe the branch-and-cut algorithm we implement to solve all formulations. A worst-case analysis is carried out in Section 4 to show the maximum cost increase of unsplit deliveries with respect to split deliveries, both for the ML and OU replenishment policies. Computational results are presented in Section 6, while conclusions are drawn in Section 7.

2 | PROBLEM DESCRIPTION

With a time horizon of H time periods, the problem we study is defined over a complete undirected graph $G = (N, E)$, where node 0 is the supplier and nodes in $N' = \{1, \dots, n\}$ are the customers. Each customer $i \in N'$ has a maximum inventory level U_i. Each node $i \in N$ also has a unit inventory holding cost per time period h_i and starts the first period with an initial inventory level I_{i0}. Let T denote the set of time periods $\{1, \dots, H\}$. In each time period $t \in T$, an amount of r_{0t} units of the product are made available at the supplier, and r_{it} units are consumed at customer $i \in N'$. No stockout is allowed at customers and supplier at any time period. A non-negative cost c_{ij} is associated to each edge $\{i,j\}$ denoting the distance between locations i and j. We assume that costs c_{ij} satisfy the triangle inequality. The product is delivered by a fleet K of identical vehicles with capacity Q. Each route starts and ends at the supplier (which corresponds to the vehicles' depot). At each time period, inventory holding costs are to be paid at both the supplier and the customers, and routing costs are charged if a delivery to any customer occurs. The decisions to be made are the quantity of product to deliver to each customer and the routes the vehicles travel at each time period $t \in T$, so as to minimize the overall total cost.

We consider different replenishment and delivery policies. The replenishment policy can be either the OU or the ML policy. In the OU policy, the inventory level of each customer i must reach its ML U_i at each time period t in which customer i is visited. In the ML policy, any quantity can be delivered to a customer i, provided that the level U_i is not exceeded. The delivery policy can be either the unsplit (US) or the split (SP) policy. The US policy is when no customer is visited by more than one vehicle at each time period t. The SP policy is when more than one vehicle can serve any customer at each time period t.

Given these policies, we provide four different formulations: maximum level-split (ML-SP), maximum level-unsplit (ML-US), order-up-to level-split (OU-SP), and order-up-to level-unsplit (OU-US). Since the least constrained formulation is ML-SP, we first introduce it and then we provide the additional constraints to obtain the other formulations.

3 | MATHEMATICAL FORMULATIONS

We now present the formulation for ML-SP. To this end, the following variables are used: z_i^{tk} is a binary variable equal to 1 if node $i \in N$ is visited by vehicle $k \in K$ at time period $t \in T$, q_i^{tk} is the quantity of product delivered to $i \in N'$ by vehicle $k \in K$ at time period $t \in T$, y_{ij}^{tk} is the number of times edge $\{i,j\} \in E$ is travelled in time period $t \in T$ by vehicle $k \in K$, and variable I_i^t represents the inventory level at node $i \in N$ at the end of time period $t \in T$.

The ML-SP formulation is as follows:

$$\min \quad \sum_{t\in T} h_0 I_0^t + \sum_{i\in N'}\sum_{t\in T} h_i I_i^t + \sum_{k\in K}\sum_{\{i,j\}\in E}\sum_{t\in T} c_{ij} y_{ij}^{kt}, \tag{1}$$

$$\text{subject to} \quad I_0^t = I_0^{t-1} + r_0 - \sum_{k\in K}\sum_{i\in N'} q_i^{tk}, \qquad t \in T, \tag{2}$$

$$I_i^t = I_i^{t-1} + \sum_{k\in K} q_i^{tk} - r_{it}, \qquad i \in N',\ t \in T, \tag{3}$$

$$I_i^t \geq 0, \qquad i \in N,\ t \in T, \tag{4}$$

$$\sum_{k\in K} q_i^{tk} \leq U_i - I_i^{t-1}, \qquad i \in N',\ t \in T, \tag{5}$$

$$q_i^{tk} \leq U_i z_i^{tk}, \qquad i \in N',\ t \in T,\ k \in K, \tag{6}$$

$$\sum_{i\in N'} q_i^{tk} \leq Q z_0^{tk}, \qquad t \in T,\ k \in K, \tag{7}$$

$$\sum_{j:\{i,j\}\in E} y_{ij}^{tk} = 2 z_i^{tk}, \qquad i \in N, k \in K, t \in T, \tag{8}$$

$$\sum_{\{i,j\}\in E(S)} y_{ij}^{tk} \leq \sum_{i\in S} z_i^{tk} - z_s^{tk}, \qquad S \subseteq N', s \in S, k \in K, t \in T, \tag{9}$$

$$z_i^{tk} \in \{0,1\}, \qquad i \in N, k \in K, t \in T, \tag{10}$$

$$q_i^{tk} \geq 0, \qquad i \in N', k \in K, t \in T, \tag{11}$$

$$y_{ij}^{tk} \in \{0,1\}, \qquad \{i,j\} \in E, k \in K, t \in T, \tag{12}$$

$$y_{0j}^{tk} \in \{0,1,2\}, \qquad j \in N', k \in K, t \in T. \tag{13}$$

The objective function (1) aims at minimizing the total cost, which includes the inventory cost at the supplier, inventory cost at the customers and the routing cost. Constraints (2) and (3) define the inventory level over time at both the supplier and the customers, respectively, while constraints (4) prevent stockout in all nodes at all times. Constraints (5) enforce the ML policy requirements imposing that, if a customer is visited, the quantity delivered is such that the inventory level does not exceed the maximum level. Constraints (6) link quantity and visiting variables. Constraints (7) are vehicle capacity constraints. Constraints (8) are degree constraints, while (9) are generalized subtour elimination constraints. Finally, constraints (10)–(13) define variables domain.

3.1 | Maximum level-unsplit

By adding the following constraints to the ML-SP formulation, we get the formulation for the ML-US:

$$\sum_{k\in K} z_i^{tk} \le 1, \qquad i \in N', t \in T. \tag{14}$$

These constraints ensure that there is at most one vehicle visiting each customer at each time period.

3.2 | Order-up-to level-split

For the OU-SP, we need to introduce a new binary variable v_i^t denoting whether customer i is visited at time t. Then, the following constraints are added to ML-SP:

$$\sum_{k\in K} q_i^{tk} \ge U_i v_i^t - I_i^{t-1}, \qquad i \in N', t \in T, \tag{15}$$

$$\sum_{k\in K} z_i^{tk} \le |K| v_i^t, \qquad i \in N', t \in T, \tag{16}$$

$$v_i^t \le \sum_{k\in K} z_i^{tk}, \qquad i \in N', t \in T. \tag{17}$$

Constraints (15) force delivery to reach the maximum inventory level at the customer. Constraints (16)–(17) are logical constraints linking variables v and z.

3.3 | Order-up-to level-unsplit

For the OU-US, in addition to constraints (14), the following constraints are added to the ML-SP formulation

$$\sum_{k\in K} q_i^{tk} \ge U_i \sum_{k\in K} z_i^{tk} - I_i^{t-1}, \qquad i \in N', t \in T. \tag{18}$$

Constraint (18), together with (14), ensures that each time a delivery is made, the inventory level at the customer reaches the maximum level.

4 | WORST-CASE ANALYSIS

In this section, our aim is to compare the US and the SP delivery policies in the worst case, both when the ML and the OU replenishment policies are applied. For the scope of the analysis in this section, the number of vehicles in the fleet is assumed to be sufficient to guarantee the existence of feasible solutions for all policies, i.e. $K = n$. Doing this, we can either have a guarantee on the maximum cost increase of the US policy with respect to the SP policy or we can show that the cost increase of the US policy with respect to the SP policy is unbounded in the worst case.

Let us first consider the ML replenishment policy. We define with $z^{\text{ML-US}}$ and $z^{\text{ML-SP}}$ the optimal cost of the US and SP delivery policies with ML replenishment policy, respectively. The following theorem shows that, if the maximum inventory level U_i of each customer i is not greater than the transportation capacity Q, then the US delivery policy has a maximum cost increase of 100% with respect to the SP delivery policy; otherwise, the cost increase is unbounded in the worst case.

Theorem 1. *If $U_i \le Q$ for each customer $i \in N'$, then $\frac{z^{ML\text{-}US}}{z^{ML\text{-}SP}} \le 2$ and the bound is tight. Otherwise, there exists an instance such that $\frac{z^{ML\text{-}US}}{z^{ML\text{-}SP}} \to \infty$.*

Proof. Let us first focus on the case in which $U_i \le Q$ for each customer $i \in N'$. Consider an optimal solution of the ML-SP. Let $\overline{q}_{it}^k$ be the optimal quantity delivered to each customer i at each time period t by using each vehicle k and $\overline{z}_t^{\text{ML-SP-I}}$ and $\overline{z}_t^{\text{ML-SP-R}}$ be the corresponding inventory and routing cost at time period t, respectively. Note that $\sum_k \overline{q}_{it}^k \le U_i$ for each customer i. Since $U_i \le Q$, then $\sum_k \overline{q}_{it}^k \le Q$.

We now build a feasible solution for the ML-US as follows: (1) we set the delivery quantities equal to $\sum_k \overline{q}_{it}^k$ for each customer i and time period t and compute the corresponding inventory levels; (2) for each time period t, we optimally solve the corresponding VRP. Let $\overline{z}_t^{\text{ML-US-R}}$ be the corresponding routing cost.

Since the delivery quantities in the solution of the ML-US are equal to the ones of the ML-SP, the inventory cost $\overline{z}_t^{\text{ML-US-I}}$ of the ML-US is equal to $\overline{z}_t^{\text{ML-SP-I}}$ on each time period t. Moreover, since Archetti et al. [6] proved that $\overline{z}_t^{\text{ML-US-R}} \le 2\overline{z}_t^{\text{ML-SP-R}}$, then

$$\frac{z^{\text{ML-US}}}{z^{\text{ML-SP}}} \leq \frac{\sum_{t\in T} \overline{z}_t^{\text{ML-SP-I}} + \sum_{t\in T} 2\overline{z}_t^{\text{ML-SP-R}}}{\sum_{t\in T} \overline{z}_t^{\text{ML-SP-I}} + \sum_{t\in T} \overline{z}_t^{\text{ML-SP-R}}} \leq \frac{2\sum_{t\in T} \overline{z}_t^{\text{ML-SP-R}}}{\sum_{t\in T} \overline{z}_t^{\text{ML-SP-R}}} = 2.$$

We now prove that the bound is tight, that is, that it is not overestimated in the worst case, as there exists an instance in which the ratio $\frac{z^{\text{ML-US}}}{z^{\text{ML-SP}}}$ tends to 2 when $H \to \infty$. Consider the following instance: $H = 2$, n customers such that $n \geq \frac{2}{Q}$ is an even number, $h_i = 0$ for $i \in N$, $r_{it} = \frac{Q}{2}+\frac{1}{n}$ for $i \in N'$ and $t \in T$, $I_{00} = \sum_{i\in N'}(\frac{Q}{2}+\frac{1}{n})$, $I_{i0} = 0$ for $i \in N'$, $c_{ii} = 0$ for $i \in N$, $c_{0i} = 1$ for $i \in N'$, $c_{ij} = \frac{1}{n}$ otherwise, $U_i = \frac{Q}{2} + \frac{1}{n}$ for $i \in N'$, $|K| = n$. An example of such an instance with four customers is depicted in Figure 1, together with the corresponding split and unsplit solutions.

Since $I_{0i} = 0$ and $r_{it} = \frac{Q}{2} + \frac{1}{n}$ for each $i \in N'$ and time period t, and $U_i = \frac{Q}{2} + \frac{1}{n}$, the only feasible solution is to deliver the quantity $\frac{Q}{2} + \frac{1}{n}$ at each time period to each customer, both for US and SP.

In the US, since $r_{it} = \frac{Q}{2} + \frac{1}{n}$ on each time period t, only direct shipments can be used on each time period. The corresponding cost is $z^{\text{ML-US}} = 2nH$.

In the SP, a feasible solution is to use $\frac{n}{2} + 1$ vehicles on each time period t, where the first $\frac{n}{2}$ vehicles deliver $\frac{Q}{2}$ units to two customers each and the additional vehicle delivers $\frac{1}{n}$ units to each customer. The corresponding cost is $z^{\text{ML-SP}} = \left[\left(2+\frac{1}{n}\right)\frac{n}{2} + 2 + (n-1)\frac{1}{n}\right]H = \left(n+\frac{7}{2}-\frac{1}{n}\right)H$. Therefore

$$\frac{z^{\text{ML-US}}}{z^{\text{ML-SP}}} \geq \frac{2n}{n+\frac{7}{2}-\frac{1}{n}} \to 2 \qquad \text{for } n \to \infty.$$

Consider now the case in which $U_i > Q$ for at least one customer i. The following instance allows us to show that in this case $\frac{z^{\text{ML-US}}}{z^{\text{ML-SP}}} \to \infty$: time horizon H, n customers, $h_0 = 1$, $h_i = 0$ for $i \in N'$, $r_{0t} = 0$ for $t \in T$, $r_{it} = Q$ for $i \in N'$ and $t \in T$, $I_{00} = nQH$, $I_{i0} = 0$ for $i \in N'$, $c_{ii} = 0$ for $i \in N$, $c_{ij} = 1$ otherwise, $U_i = QH$ for $i \in N'$, $|K| = Hn$.

In the US, since $I_{i0} = 0$ and $r_{it} = Q$ for each customer i and time period t, the only feasible solution is to deliver Q units to each customer i at each time period t by direct shipping. The corresponding inventory levels at the supplier are: $nQ(H-1)$ at time period 1, $nQ(H-2)$ at time period 2, …, nQ at time period $H-1$ and 0 at time period H. Therefore, $h_0\sum_t I_{0t} = nQ\frac{(H-1)H}{2}$. Since the corresponding routing cost is $2nH$, then $z^{ML-US} = nQ\frac{(H-1)H}{2} + 2nH$.

In the SP, a feasible solution is to deliver QH units to each customer i at time period 1, using H vehicles per customer. The corresponding inventory cost at the supplier is 0, while the routing cost is $2nH$. A visualization of both US and SP solutions can be seen in Figure 2. Therefore

$$\frac{z^{ML-US}}{z^{ML-SP}} \geq \frac{nQ\frac{(H-1)H}{2} + 2nH}{2nH} \to \infty \qquad \text{for } H \to \infty.$$

■

Let us now consider the OU replenishment policy. We define with $z^{\text{OU-US}}$ and $z^{\text{OU-SP}}$ the optimal cost of the US and the SP delivery policies, when the OU replenishment policy is applied, respectively. The following theorem shows that the US delivery policy has a maximum cost increase of 100% with respect of the SP delivery policy.

Theorem 2. $\frac{z^{OU\text{-}US}}{z^{OU\text{-}SP}} \leq 2$ *and the bound is tight.*

Proof. First note that when the OU policy is applied, $U_i \leq Q$ for each customer i as otherwise no feasible solution exists for the unsplit case. The proof of the worst-case performance bound follows the same lines as the ones of Theorem 1. Moreover, note that the first instance used in the proof of Theorem 1 can be used to prove tightness when the OU policy is applied.

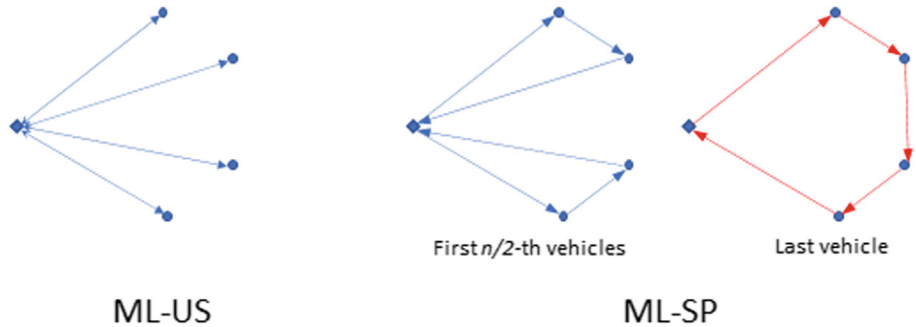

FIGURE 1 Graphical example of an instance related to the tight bound of Theorem 1: Case $U_i \leq Q$.

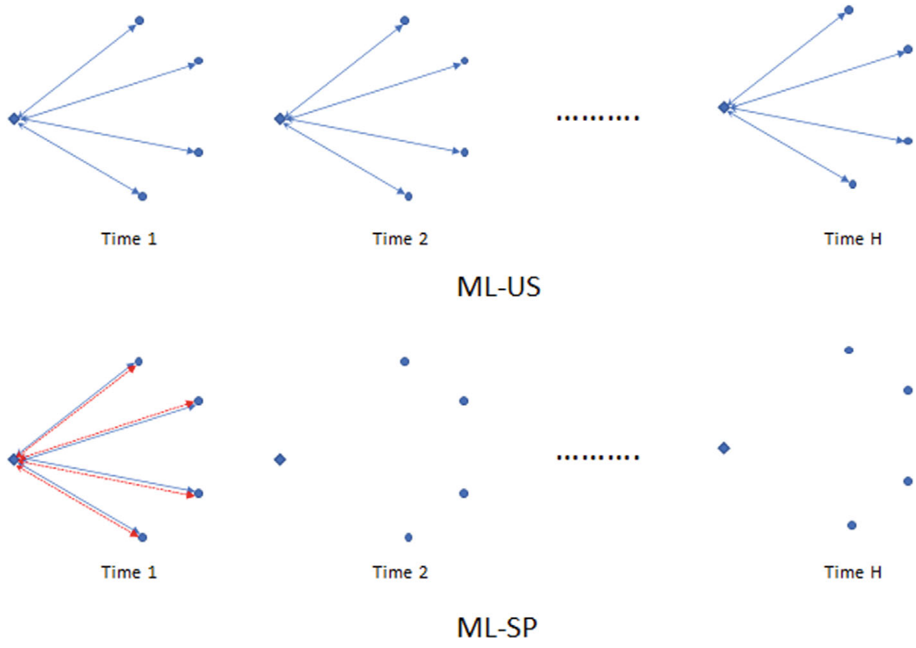

FIGURE 2 Graphical example of an instance related to the tight bound of Theorem 1: Case $U_i > Q$. ■

To complete our worst-case analysis, we also consider the comparison of the OU-US delivery policy with respect to the ML-SP. The following theorem shows that the cost increase of the former with respect to the cost of the latter is unbounded in the worst case.

Theorem 3. *There exists an instance such that* $\frac{z^{OU\text{-}US}}{z^{ML\text{-}SP}} \to \infty$.

Proof. Consider the following instance: $H = \frac{1}{\epsilon}$, where $0 < \epsilon < 1$ is such that $\frac{1}{\epsilon}$ is an integer number, $h_0 = 0$, $h_i = 1$ for $i \in N'$, $r_{it} = 1$ for $i \in N'$, $r_{0t} = n$, $U_i = \frac{1}{\epsilon}$, $I_{00} = \frac{n}{\epsilon}$, $I_{i0} = 0$ for $i \in N'$, $c_{ii} = 0$ for $i \in N$, $c_{ij} = \epsilon$ for $i \neq j$ and $Q = \frac{1}{\epsilon}$.

Since $I_{i0} = 0$ for $i \in N'$, all customers must be served at time 1. Therefore, in any optimal solution of the OU-US, given that $I_{i0} = 0$ and $U_i = \frac{1}{\epsilon}$, $\frac{1}{\epsilon}$ units are delivered to each customer at time period 1. Thus, the inventory levels at all customers i are $\frac{1}{\epsilon} - 1$ at time period 1, $\frac{1}{\epsilon} - 2$ at time period 2, …, 1 at time period $\frac{1}{\epsilon} - 1$ and 0 at time period $\frac{1}{\epsilon}$. The total inventory cost is $\frac{(\frac{1}{\epsilon}-1)\frac{1}{\epsilon}}{2}n$. Each customer is served by a full load direct shipment at time period 1. Therefore, the routing cost is $2\epsilon n$. Hence, $z^{OU-US} \geq \frac{(\frac{1}{\epsilon}-1)\frac{1}{\epsilon}}{2}n + 2\epsilon n$.

A feasible solution of the ML-SP is to deliver 1 unit to each customer at each time period by direct shipment. The corresponding inventory cost is 0, while the corresponding routing cost is $2\epsilon n\frac{1}{\epsilon} = 2n$. Figure 3 shows both the US and SP solutions mentioned above, for an instance with four customers. Therefore

$$\frac{z^{\text{OU-US}}}{z^{\text{ML-SP}}} \geq \frac{\frac{(\frac{1}{\epsilon}-1)\frac{1}{\epsilon}}{2}n + 2\epsilon n}{2n} = \frac{(H-1)H}{4} + \frac{1}{H} \to \infty \qquad \text{for } H \to \infty.$$ ■

Finally, we consider the comparison of the OU-SP with respect to the ML-US. The following theorem shows that the cost increase of the former with respect to the cost of the latter is unbounded in the worst case.

Theorem 4. *There exists an instance such that* $\frac{z^{OU\text{-}SP}}{z^{ML\text{-}US}} \to \infty$.

Proof. Let us consider the same instance used in Theorem 3. Following the same reasoning, the optimal solution in Theorem 3 for the OU-US is also the optimal solution for the OU-SP. Hence $z^{\text{OU-SP}} \geq \frac{(\frac{1}{\epsilon}-1)\frac{1}{\epsilon}}{2}n + 2\epsilon n$.

The feasible ML-US solution in Theorem 3 is a feasible ML-SP solution by definition. Thus $z^{\text{ML-SP}} \leq 2n$. Therefore

$$\frac{z^{\text{OU-SP}}}{z^{\text{ML-US}}} \geq \frac{\frac{(\frac{1}{\epsilon}-1)\frac{1}{\epsilon}}{2}n + 2\epsilon n}{2n} = \frac{(H-1)H}{4} + \frac{1}{H} \to \infty \qquad \text{for } H \to \infty.$$ ■

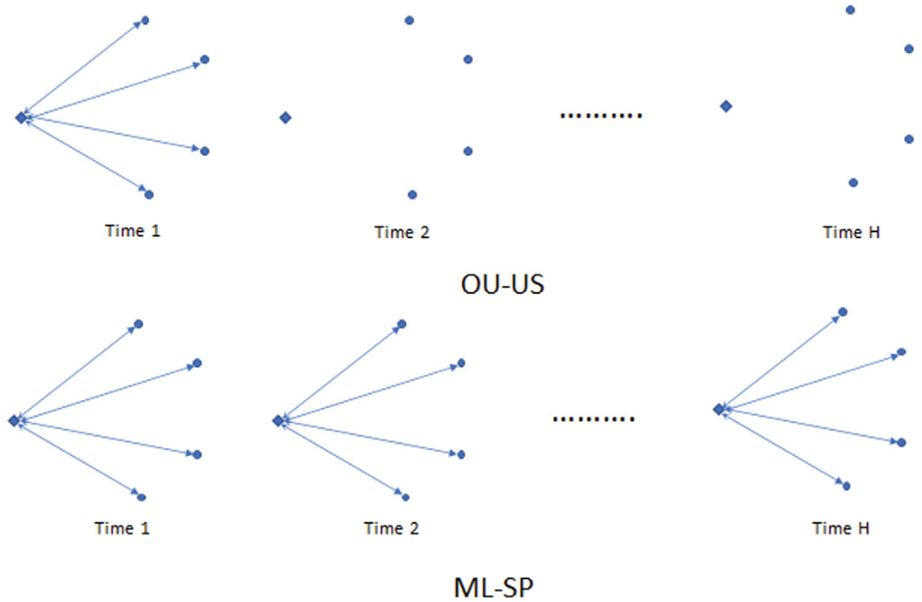

FIGURE 3 Graphical example of an instance related to the tight bound of Theorem 3.

In the VRP, Archetti et al. [6] has proved that $\frac{z^{\text{US}}}{z^{\text{SP}}} \leq 2$ in all cases, where z^{US} is the optimal cost of the unsplit VRP and z^{SP} is the optimal cost of the split VRP, and this bound is tight; when vehicle capacity is sufficiently small $\frac{z^{\text{US}}}{z^{\text{SP}}} \leq \frac{3}{2}$. Meanwhile, in the IRP, only under the OU policy we can prove that $\frac{z^{\text{OU-US}}}{z^{\text{OU-SP}}} \leq 2$ in all cases. For the ML policy, if the vehicle capacity is smaller than the maximum inventory level at every customer, $\frac{z^{\text{ML-US}}}{z^{\text{ML-SP}}} \leq 2$. In all other cases, it is unbounded.

5 | VALID INEQUALITIES AND BRANCH AND CUT ALGORITHM

In this section, we first present valid inequalities that have been added to the formulations presented above to strengthen the corresponding relaxation. Then, the overall solution approach, namely the branch-and-cut algorithm, is presented.

5.1 | Valid inequalities

We now present different classes of valid inequalities that are added to strengthen the previous formulations. They are all inherited from previous works on IRPs. Thus, for each class of inequalities, we provide the reference to the contribution (or one of the contributions) introducing it and we refer the reader to the corresponding paper for the proof of validity and more details about inequalities meaning.

The following are classes of inequalities proposed in [8].

- Logical constraints:

$$z_i^{tk} \leq z_0^{tk}, \qquad i \in N', t \in T, k \in K, \tag{19}$$

$$y_{ij}^{tk} \leq z_i^{tk}, y_{ij}^{tk} \leq z_j^{tk}, \qquad i \in N', j \in N', i < j, t \in T, k \in K, \tag{20}$$

$$y_{i0}^{tk} \leq 2z_0^{tk}, y_{i0}^{tk} \leq 2z_i^{tk}, \qquad i \in N', t \in T, k \in K. \tag{21}$$

  This class of valid inequalities aims to improve consistency between the z and y variables, strengthening the corresponding link.
- Replenishment policy constraints:

$$I_{it-\tau} \geq \left(1 - \sum_{k \in K} \sum_{t'=t-\tau+1}^{t} z_i^{t'k}\right)\left(\sum_{t'=t-\tau+1}^{t} r_i^t\right), \quad i \in N', t \in T, \tau = 0, \ldots, t-1. \tag{22}$$

  Constraint (22) states that, if a customer is not visited between time periods $t-\tau$ and t, the inventory level at $t-\tau$ must be enough to satisfy the demands between $t-\tau$ and t.

Adulyasak et al. [1] proposed the following symmetry breaking constraints:

$$\sum_{i=1}^{j} 2^{(j-i)} z_i^{tk} \geq \sum_{i=1}^{j} 2^{(j-i)} z_i^{t,k+1}, \qquad j \in N', 1 \leq k \leq m-1, t \in T. \tag{23}$$

Constraints (23) assign a coefficient to each customer and the vehicles are dispatched according to the lexicographic ordering of the sum of the coefficients of the customers served.

Coelho and Laporte [20] proposed the following additional class of inequalities on the minimum number of routes over a time interval:

$$\sum_{t'=1}^{t} \sum_{j \in N} \sum_{k \in K} y_{0j}^{t'k} \geq \left\lceil \frac{\sum_{i \in N} \max\left\{0, \sum_{t'=1}^{t} r_i^{t'} - I_i^0\right\}}{Q} \right\rceil, \qquad t \in T. \tag{24}$$

Finally, the following class of valid inequalities were proposed in [10] for the case in which demands are constant over time, that is, $r_i^t = r_i$, and maximum inventory levels are multiple of demands. The logic behind these constraints is similar to the one of inequalities (22), that is, they set a lower bound on inventory level for the case in which customers are not visited for a given number of consecutive periods. In particular, let $U_i = V_i r_i$. The following lot size inequalities are proposed for the cases $V_i = 2$ and $V_i = 3$:

- Case $V_i = 3$:

$$I_i^t \leq r_i \left(1 + \sum_{k \in K} z_i^{tk}\right), \qquad i \in N', t \in T, \tag{25}$$

$$I_i^{t-1} + r_i \sum_{k \in K} z_i^{tk} \geq r_i, \qquad i \in N', t \in T, \tag{26}$$

$$I_i^{t-1} + 3r_i \sum_{k \in K} z_i^{tk} \geq r_i + I_i^t, \qquad i \in N', t \in T, \tag{27}$$

$$r_i \sum_{k \in K} (z_i^{t-1k} + 2z_i^{tk}) \geq I_i^t, \qquad i \in N', t \in T, t \geq 2, \tag{28}$$

$$\sum_{k \in K} (z_i^{t-2k} + z_i^{t-1k} + z_i^{tk}) \geq 1, \qquad i \in N', t \in T, t \geq 3, \tag{29}$$

$$I_i^{t-2} + r_i \sum_{k \in K} (2z_i^{t-1k} + z_i^{tk}) \geq 2r_i, \qquad i \in N', t \in T, t \geq 2, \tag{30}$$

$$I_i^{t+1} + r_i \sum_{k \in K} (z_i^{t-1k} + z_i^{tk}) \geq I_i^t, \qquad i \in N', t \in T, 2 \leq t \leq T-1, \tag{31}$$

$$I_i^{t-1} + I_i^{t+1} + 2r_i \sum_{k \in K} z_i^{tk} \geq r_i + I_i^t, \qquad i \in N', t \in T, t \leq T-1, \tag{32}$$

$$r_i + I_i^t + 2r_i \sum_{k \in K} z_i^{t+1k} \geq I_i^{t-1} + I_i^{t+1}, \qquad i \in N', t \in T, t \leq T-1. \tag{33}$$

- Case $V_i = 2$:

$$r_i \sum_{k \in K} z_i^{tk} + I_i^{t-1} \geq r_i, \qquad i \in N', t \in T, \tag{34}$$

$$r_i \sum_{k \in K} z_i^{tk} \geq I_i^t, \qquad i \in N', t \in T. \tag{35}$$

5.2 | A branch-and-cut algorithm

All formulations, enriched with the valid inequalities introduced in this section, were solved through a branch-and-cut algorithm. Specifically, constraints (9) are initially removed and dynamically inserted when violated through the classical min-cut separation algorithm proposed in [29]. Since the valid inequalities proposed in Section 5.1 are all in polynomial number, they are added to the initial formulation.

6 | COMPUTATIONAL EXPERIMENTS

In this section, we present the experiments we made to assess the benefits of split deliveries in the IRP, both for OU and ML replenishment policies. As the difference of OU and ML policies in the unsplit setting is already well researched in the literature, we will only compare the OU and ML replenishment policies in the split setting.

The formulations were implemented in C++ in Microsoft Visual Studio 2019 using CPLEX 12.10.0 with Concert Technology. The machine used for the experiments is a 64-bit Windows 10 workstation with the following specifications: Intel Xeon ES-1650 processor at 3.50 GHz with 64 GB of RAM. We set a maximum time limit of 2 h for the solution of each instance.

The section is organized as follows. We first describe the benchmark IRP instances used in our tests in Section 6.1. Section 6.2 is focused on evaluating the performance of the branch-and-cut algorithm benefits of split deliveries on the benchmark instances. We then perform additional tests by modifying some instance parameters, to perform a sensitivity analysis of the benefits of split deliveries with respect to problem features. Results are presented in Section 6.3.

6.1 | Data set

The data set used in the experiments is composed of the benchmark instances proposed by Archetti et al. [7] for the single vehicle case and adapted in following studies to the multi-vehicle case (see, e.g., [8, 18]).

For the experiment, we used the instances with the following data:

1. Time horizon $H = 3$ and $H = 6$.
2. Number of customers $n = 5l$, with $l = 1, \dots, 6$ for $H = 3$ and $l = 1, \dots, 4$ for $H = 6$.
3. Number of vehicles: from 2 to 5.
4. High and low inventory cost.

More details about instance features can be found in [7]. With these specifications, for each policy combination, there are a total of 400 instances. The data set and detailed results can be found and downloaded at https://or-brescia.unibs.it/instances.

6.2 | Performance on benchmark instances

In this section, we present the benefits of split deliveries in both ML and OU replenishment policies. We also analyse the difficulty in solving the problems, measured in terms of computing time and number of instances solved to optimality. Tests are made on the benchmark instances described above.

Table 1 presents the average percent cost increase of the solution value of the ML and OU replenishment policies when split deliveries are not allowed with respect to the case when they are allowed. Moreover, it shows the percent cost increase of OU with respect to ML, when split deliveries are allowed in both problems. In particular, in the second column we show the results obtained for the ML policy, while in the third column the ones obtained for the OU policy. Finally, in the last column, we compare the solution of OU policy versus the ML policy, when split deliveries are allowed. In all cases, the comparison is made on the subset of instances solved to optimality in both problems. Results are aggregated over number of vehicles first, then number of customers, class of inventory cost and time horizon. The results show that there is little improvement when applying split deliveries in the ML replenishment policy, while the benefits increase, even if just slightly, in the OU policy, especially for instances with few customers and many vehicles. When the number of vehicles increases, the gaps between US and SP also increase, while the trend is not clear in case of more customers. This might be due to the fact that instances with a higher number of customers are harder to solve to optimality and therefore statistics are less representative of the average behavior. The last column shows that the main advantage comes from the more flexible replenishment policy, that is, ML versus OU. This is in line with the analysis made in [7] for the unsplit case.

Table 2 reports the average computing time in seconds, restricted to instances solved to optimality, and the number of instances solved to optimality out of 400 for each replenishment and delivery policy. Results are aggregated as in Table 1. As expected, we note that the split delivery version of the problem is more difficult to solve than the unsplit version, for both replenishment policies. This is due to having a larger solution space. Moreover, OU policies are more difficult to solve than ML policies, both in the unsplit and the split deliveries cases. It is also noted that instances with larger number of customers, vehicles and time horizon are more difficult to solve, as seen in the larger computational time required and the smaller number of optimal solutions found. Changes in unit inventory cost do not provide much difference in this regard.

Finally, Table 3 reports the average optimality gap at termination for all policies. Again, results are aggregated as in Table 1 and they confirmed what examined from Table 2. In fact, optimality gaps are on average higher for split problems with respect to the unsplit counterpart. Moreover, these results confirm that OU policies are more difficult to solve than ML policies, both in the unsplit and the split deliveries cases. Table 3 also reinforces the difficulty of solving instances with larger number of customers, vehicles and time horizon, similar to Table 1. In addition, we also observe that low inventory cost instances have larger gaps than those with high inventory cost. This might be due to the higher impact of routing cost on total cost in this case, which amplifies the effect of relaxation on binary x variables in the calculation of the lower bound.

TABLE 1 Comparison of the optimal cost of the different delivery and replenishment policies.

No. of vehicles	Gap ML-US/ML-SP	Gap OU-US/OU-SP	Gap OU-SP/ML-SP
2	0.42%	0.24%	8.03%
3	0.67%	3.20%	6.63%
4	0.94%	1.99%	5.90%
5	1.14%	4.78%	6.01%
No. of customers	**Gap ML-US/ML-SP**	**Gap OU-US/OU-SP**	**Gap OU-SP/ML-SP**
5	0.18%	4.16%	5.67%
10	1.30%	1.06%	6.98%
15	1.01%	0.37%	7.49%
20	0.42%	0.06%	11.08%
25	0.41%	0.15%	8.08%
30	0.00%	0.08%	6.97%
Inventory cost	**Gap ML-US/ML-SP**	**Gap OU-US/OU-SP**	**Gap OU-SP/ML-SP**
High	0.49%	1.65%	5.33%
Low	0.83%	2.00%	8.67%
Time horizon	**Gap ML-US/ML-SP**	**Gap OU-US/OU-SP**	**Gap OU-SP/ML-SP**
3	0.82%	0.97%	7.18%
6	0.07%	5.00%	6.38%
Average	0.66%	1.83%	7.01%
Max	6.48%	19.63%	25.30%

TABLE 2 Running times and number of instances solved to optimality.

Number of vehicles	ML-SP	ML-US	OU-SP	OU-US
2	336 (86)	351 (86)	727 (69)	628 (76)
3	545 (57)	590 (60)	365 (48)	414 (51)
4	1006 (43)	403 (48)	1120 (39)	681 (49)
5	984 (23)	1271 (39)	1106 (28)	990 (42)
Number of customers	**ML-SP**	**ML-US**	**OU-SP**	**OU-US**
5	312 (69)	34 (78)	171 (75)	0 (80)
10	738 (51)	458 (54)	801 (45)	550 (54)
15	531 (39)	854 (43)	1156 (32)	1362 (44)
20	554 (18)	1252 (22)	1525 (12)	1471 (17)
25	1293 (19)	1251 (22)	1681 (14)	955 (16)
30	881 (13)	1097 (14)	2444 (6)	1982 (7)
Inventory cost	**ML-SP**	**ML-US**	**OU-SP**	**OU-US**
High	654 (106)	564 (117)	710 (91)	609 (108)
Low	548 (103)	590 (116)	835 (93)	709 (110)
Time horizon	**ML-SP**	**ML-US**	**OU-SP**	**OU-US**
3	449 (159)	629 (175)	800 (142)	599 (160)
6	1089 (50)	422 (58)	683 (42)	826 (58)
Average time (total number)	602 (209)	577 (233)	774 (184)	660 (218)

6.3 | Sensitivity analysis

In this section, we provide the results related to the sensitivity analysis we performed in order to analyse the impact of instance features on the benefits of split deliveries. In the following, we focus on the ML policy only.

The study is performed on instances which are obtained by modifying the benchmark instances described in Section 6.1. Tests are made on the subset of instances with time horizon $H = 3$, high inventory cost and number of customers equal to at most 20.

We modified the following instance features: customer demands, initial inventory levels, maximum inventory levels and distance to the depot. For each feature, we considered two cases as described in the following:

TABLE 3 Optimality gaps at termination.

No. of vehicles	ML_SP	ML_US	OU_SP	OU_US
2	1.22%	1.05%	3.45%	2.88%
3	5.94%	5.59%	11.98%	9.66%
4	13.60%	11.65%	22.79%	16.33%
5	23.26%	17.53%	34.74%	24.76%
No. of customers	**ML_SP**	**ML_US**	**OU_SP**	**OU_US**
5	0.38%	0.00%	0.20%	0.00%
10	7.36%	4.55%	11.70%	4.21%
15	9.98%	8.22%	16.45%	10.68%
20	25.68%	21.09%	40.35%	30.31%
25	10.47%	9.09%	20.70%	13.86%
30	12.75%	11.84%	24.42%	21.57%
Inventory cost	**ML_SP**	**ML_US**	**OU_SP**	**OU_US**
High	7.71%	5.93%	12.29%	8.72%
Low	14.30%	11.89%	24.14%	17.35%
Time horizon	**ML_SP**	**ML_US**	**OU_SP**	**OU_US**
3	5.75%	4.79%	11.33%	8.19%
6	18.89%	15.18%	28.56%	20.19%
Average	11.00%	8.91%	18.20%	13.04%

1. Customer demands:
   - all customer demands are randomized to be between 50% and 90% of the vehicle capacity (denoted as All);
   - demands as modified as above only for half of the customers. For the remaining ones, demands correspond to the ones of the benchmark instances. The customers for which the demands are modified are chosen at random (denoted as Half).

   In both cases, the number of vehicles is increased in such a way that a feasible solution exists according to the new value of demands.
2. Maximum inventory level:
   - low maximum inventory level, that is, $U_i = r_i$, where r_i is the constant demand of customer i over time in the benchmark instances (denoted as L);
   - high inventory level, that is, U_i is randomized to be either 2 or 3 times the customer's demand (denoted as H).
3. Initial inventory level:
   - initial inventory at the customers equal to the demand (denoted as D);
   - initial inventory at the customers equal to 0 (denoted as 0).
4. Distance to the supplier:
   - equal to the one of the base instance (denoted as B).
   - equal to the one of the base instance multiplied by 100 (denoted as 100).

We experimented on the following combinations: All-L-D-100, Half-L-D-B, All-L-0-100, All-H-D-100, and All-L-D-B.

Table 4 presents the average percent cost increase of US with respect to SP. The comparison is made on the best solution value obtained at termination. Results are aggregated over the number of customers. The last three rows presents the average, maximum, and minimum gap, respectively. We first present the results for instances All-L-D-100, followed by Half-L-D-B, then All-L-0-100, All-H-D-100 and finally All-L-D-B. The results in columns 2 and 3 show that customer demands have a high impact on the benefit of split deliveries. Indeed, when the number of customers with large demand is reduced to a half, the average benefit drops from almost 37% to 18%. This is in line with what found for the SDVRP, as demonstrated in [6]. The results in columns 2 and 4 show that the value of the initial inventory at customers has a very small impact. The comparison of the results in columns 2 and 5 show the huge impact of the maximum inventory level. Indeed, we observe that the increase of the level almost nullify all benefits. This is due to the fact that consolidation gains, obtained by delivering a larger quantity at each visit, overcome split delivery savings. Finally, the comparison of the results in columns 2 and 6 show that the distance to

TABLE 4 Benefit of split deliveries.

n	All-L-D-100	Half-L-D-B	All-L-0-100	All-H-D-100	All-L-D-B
5	24.29%	2.30%	24.46%	0.19%	9.47%
10	42.96%	19.19%	43.60%	1.10%	27.23%
15	36.97%	21.90%	37.79%	1.53%	30.35%
20	42.89%	28.96%	44.34%	2.05%	40.16%
Average	36.78%	18.09%	37.55%	1.22%	26.80%
Max	45.14%	57.18%	45.78%	2.92%	67.30%
Min	22.36%	0.00%	22.87%	0.01%	3.66%

TABLE 5 Gap between upper and lower bounds at termination.

	All-L-D-100		Half-L-D-B		All-L-0-100	
No. customers	Gap US	Gap SP	Gap US	Gap SP	Gap US	Gap SP
5	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%
10	0.00%	0.55%	0.00%	0.99%	0.00%	0.74%
15	0.00%	0.81%	0.00%	7.04%	0.00%	0.89%
20	0.00%	1.17%	0.06%	9.68%	0.00%	1.37%
Average	0.00%	0.63%	0.02%	4.43%	0.00%	0.75%
Max	0.00%	1.69%	0.59%	17.24%	0.00%	1.91%
Min	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

	All-H-D-100		All-L-D-B	
No. customers	Gap US	Gap SP	Gap US	Gap SP
5	0.01%	0.01%	0.00%	0.00%
10	0.07%	0.25%	0.00%	8.29%
15	0.43%	0.83%	0.00%	12.23%
20	0.98%	1.20%	0.00%	11.71%
Average	0.37%	0.57%	0.00%	8.06%
Max	1.67%	2.87%	0.00%	18.36%
Min	0.00%	0.01%	0.00%	0.00%

the depot has a relative impact on savings: when the distance is decreased by 100, the average benefits drops from almost 37% to almost 27%, but remains remarkable.

Table 5 shows the average gaps between upper and lower bounds at termination. It is observed that US solutions are associated with very small gaps, while SP solutions tend to have much larger gaps, especially on instances with the original distances to supplier (B) and with initial inventory level at customers equal to the demand (D). Therefore, the gaps reported in Table 4 may actually be smaller than the true cost improvement.

Thus, we can conclude that the two main features that determine the advantage of split deliveries versus unsplit deliveries are:

- customer demands: larger benefits are observed when demands are large, specifically, larger than half of vehicle capacity;
- maximum inventory level: smaller values generate larger savings.

7 CONCLUSIONS

In this article, we presented mathematical formulations for solving the IRPs with split deliveries, when the ML and OU replenishment policies are applied. The worst-case analysis we carried out showed that the cost increase of the unsplit case with respect to the split case can be very large in the worst case. In fact, for the OU policy the maximum cost increase is 100%, while for the ML policy it is 100% when the maximum inventory level of each customer is not greater than the transportation capacity, while it is unbounded in the opposite case. This is quite different than what is known for the CVRP, where the maximum cost increase is 100% over all instances. Extensive computational results on benchmark instances showed that, on average, the cost increase of the unsplit case with respect to the split case is much lower than what we can have in the worst case: 0.66% for the

ML policy and 1.83% for the OU policy. However, since the maximum cost increase is 6.48% for ML and 19.63% for OU, split deliveries can provide significant savings, mainly when then OU is applied. Moreover, the computational results showed that solution times and optimality gaps are much higher in the split case with respect to the unsplit case. Therefore, the split case is more difficult to solve than the unsplit case. As expected, there is a trade-off between obtained savings and the computational time required to find an optimal solution. Finally, a sensitivity analysis on customer demands, initial inventory levels, maximum inventory levels and distance to the depot allowed us to understand that high customer demands and low maximum inventory levels are the two main instance features that make split deliveries effective in IRPs. Future research could be devoted to design effective heuristic algorithms for the IRPs with split deliveries.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are openly available in OR-Brescia at https://or-brescia.unibs.it/.

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