

# Schwarzian derivative, Painlevé XXV–Ermakov equation, and Bäcklund transformations

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## Abstract

The role of Schwarzian derivative in the study of nonlinear ordinary differential equations is revisited. Solutions and invariances admitted by Painlevé XXV–Ermakov equation, Ermakov equation, and third-order linear equation in a normal form are shown to be based on solutions of the Schwarzian equation. Starting from the Riccati equation and the second-order element of the Riccati chain as the simplest examples of linearizable equations, by introducing a suitable change of variables, it is shown how the Schwarzian derivative represents a key tool in the construction of solutions. Two families of Bäcklund transformations, which link the linear and nonlinear equations under investigation, are obtained. Some analytical examples are given and discussed.

## KEYWORDS

Bäcklund transformations, Ermakov equation, Painlevé XXV–Ermakov equation, Schwarzian derivative

## 1 | INTRODUCTION

The importance of nonlinear Ermakov-type equations as well as of Painlevé equations is well known and motivates the present investigation. The hybrid Ermakov–Painlevé systems were introduced in [17]. Later, Ermakov–Painlevé II–IV were extensively investigated since, in particular, their physical applications in cold plasma physics, nonlinear Korteweg capillarity theory, and multi-ion electro-diffusion. Notably, such systems inherit the characteristic-type invariants of Ermakov systems as well as properties of the connected Painlevé equation. Key notions in the present investigation are, on one

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side, nonlinear ODEs of Ermakov type and, on the other one, Bäcklund transformations. Indeed, Bäcklund transformations represent a powerful tool to connect different equations as well as to reveal invariances they enjoy. The interested reader is referred to the books [7, 9, 19, 20] where Bäcklund transformation are studied, while in [18] the focus is on their applications.

The Riccati equation

$$\frac{dv}{dz} = a_2(z)v(z)^2 + a_1(z)v(z) + a_0(z) \quad (1.1)$$

can be considered as the simplest nonlinear ordinary differential equation (ODE), since among ODEs of the form  $\frac{dv}{dz} = \frac{P(v,z)}{Q(v,z)}$ , with  $P(v,z)$  and  $Q(v,z)$  polynomial in  $v$  and analytic in  $z$ , it is the only first-order nonlinear ODE that possesses the Painlevé property [12, 13]. It has many applications in different areas: from control theory to the theory of random processes, diffusion problems, orbiting satellites, and seasonal phenomena [14, 16]. Also, it plays a very important role in the solution of integrable nonlinear partial differential equations. As an example, the simplest Bäcklund transformation of the Korteweg-de Vries equation is represented by a Riccati equation [8]. The Riccati equation (1.1) can be linearized: Indeed by setting

$$v = -\frac{\gamma'}{a_2\gamma}, \quad (1.2)$$

it follows that  $\gamma = \gamma(z)$  solves a linear second-order differential equation. Linear second-order differential equations are strictly related to the Schwarzian derivative. This differential operator is invariant under linear fractional transformations and plays a fundamental role in a different area of mathematics: besides the theory of linear second-order differential equations, there are numerous applications in the theory of modular forms, hypergeometric functions, univalent functions, and conformal mappings. Given any smooth enough function  $f(z)$ , its Schwarzian derivative  $\{f, z\}$  is defined via

$$\{f, z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \quad (1.3)$$

The link between the Schwarzian derivative and the theory of linear differential equations is given by the following result.

**Theorem 1.1.** (see, e.g., [12], Theorem 10.1.1): *If  $B(z)$  is analytic in a simply connected domain  $\Omega \subset \mathbb{C}$ , then for any two linearly independent solutions  $\eta_1$  and  $\eta_2$  of*

$$\eta''(z) = B(z)\eta(z), \quad (1.4)$$

*their quotient  $\Omega = \eta_1/\eta_2$  is locally injective and satisfies the differential equation*

$$\{\Omega, z\} = -2B(z). \quad (1.5)$$

The converse of this statement is also true.

Higher order linear differential equations can be associated to generalized Riccati equations: all together these equations constitute the *Riccati chain* [8]. In this paper, we discuss the links between the second member of the Riccati chain (corresponding to a third-order linear equation) and the Schwarzian derivative: in particular, the connections with the Ermakov equation and with the Painlevé XXV-Ermakov equation studied in [4] are further investigated.

Unlike the hybrid Ermakov systems, inheriting both the characteristic-type invariants of Ermakov systems together with the properties of the underlying base Painlevé equation, the Painlevé XXV-Ermakov equation, studied in this paper, contains Painlevé XXV and the Ermakov equation as reductions. The Painlevé equations I–VI are not C-integrable (i.e., solvable with a change of variables) like the Ermakov systems [1]. In this work we show how the Painlevé XXV-Ermakov equation can be linearized to a third-order equation written in a normal form and hence is C-integrable. Thanks to the properties of this linear equation, in Section 3, we find a class of auto-Bäcklund transformations for the Ermakov equation and for a particular case of the Painlevé XXV-Ermakov equation. We would like to stress that, by Bäcklund transformations, we mean any implicit or explicit map, which links one or more solutions of a differential equation to a solution of a second differential equation. The map can be algebraic or differential or also, as in the present case, can

involve some function satisfying a suitable functional equation. Accordingly, we term auto-Bäcklund transformation any transformation linking different solutions of the same differential equation. In Section 3, polynomial relations among different solutions of the third-order equation are also introduced. In Section 4, a second class of auto-Bäcklund transformations for the Ermakov equation and the Painlevé XXV–Ermakov equation is given: as we show this new class of transformations is strictly related to the properties of the Schwarzian derivative. In Section 5, algebraic relations among the solutions of the Schwarzian equation

$$\{\Omega, z\} = -2B(z), \quad (1.6)$$

and its derivative (i.e.,  $\Omega$  and  $\Omega'$ ) and the solutions of the Ermakov equation and of the Painlevé XXV–Ermakov equation are derived. These relations and the use of the second class of auto-Bäcklund transformations give the possibility to write the general solution of the Ermakov equation and a particular case of the Painlevé XXV–Ermakov equation in terms of  $\Omega$  and  $\Omega'$ . Finally, in Section 6, two examples are discussed: the first one involving the Weierstrass elliptic function and the second one involving a rational function with an arbitrary number of double poles.

## 2 | THE SECOND MEMBER OF THE RICCATI CHAIN AND THE PAINLEVÉ XXV–ERMAKOV EQUATION

The properties of the solutions of the Riccati equation can be generalized to higher order equations: in this case, the corresponding equations constitute the so-called Riccati-chain [8]. By defining the differential operator

$$L = \frac{d}{dz} + v(z), \quad (2.1)$$

the  $n$ -th order equation of the chain is represented by

$$L^n v(z) + \sum_{k=1}^n a_k(z)(L^{k-1}v(z)) + a_0(z) = 0, \quad (2.2)$$

which can be linearized to an  $(n + 1)$ -th-order differential equation via the change of variables  $v = \gamma'/\gamma$ , giving

$$\frac{d^{n+1}\gamma}{dz^{n+1}} + \sum_{k=0}^n a_k(z)\frac{d^k\gamma}{dz^k} = 0. \quad (2.3)$$

The second-order element of the Riccati chain (2.2) is considered in [4]. When  $n = 2$ , Equation (2.2) reads:

$$\frac{d^2v}{dz^2} + 3v\frac{dv}{dz} + v^3 + p(z)(v' + v^2) + q(z)v + r(z) = 0, \quad (2.4)$$

where, with the notations of [4], we set  $a_2(z) = p(z)$ ,  $a_1(z) = q(z)$ , and  $a_0(z) = r(z)$ . The corresponding linear equation is then given by

$$\gamma''' + p(z)\gamma'' + q(z)\gamma' + r(z)\gamma = 0. \quad (2.5)$$

Like for the Riccati equation (i.e., the first member of the chain), it is possible to consider the Schwarzian derivative of the ratio of two independent solutions of the corresponding linear equation. In this case, by setting

$$\gamma_1(z) = \phi(z)\gamma_2(z), \quad (2.6)$$

where both  $\gamma_1$  and  $\gamma_2$  satisfy (2.5), we define the function  $\xi(z)$  to be the Schwarzian derivative of  $\phi(z)$ , that is,

$$\xi(z) \doteq \{\phi(z), z\}. \quad (2.7)$$

In [4], it is shown that  $\xi(z)$  satisfies the following nonlinear second-order differential equation

$$(12\xi(z) + b(z))\xi'' = 15\xi'^2 - h_0\xi' - 8\xi^3 - h_1\xi^2 - h_2\xi - h_3, \quad (2.8)$$

where the functions  $h_i$ ,  $i = 0 \dots 3$ , are determined in terms of the functions  $p$ ,  $q$ ,  $r$ , and their derivatives as

$$\begin{aligned} b &= 2(p^2 - 3q + 3p'), \quad h_1 = 4b, \\ h_0 &= 2(4pp' - 3p'' - 9pq + 2p^3 - 6q' + 27r), \\ h_2 &= 2(-4pp'' - 12qp' + 2p^2p' + 5p'^2 + 6qp' - 6p^2q + p^4 + 6q'' + 9q^2 - 18r'), \\ h_3 &= -6p''q' - 2pqp'' + 18rp'' + 6p'q'' + 2pp'q' - 2p^2qp' - 2qp'^2 + 6q^2p' - 18p'r' + \\ &\quad + 2p^2q'' + 2p^3q' - 6pqq' + 18pqr + p^2q^2 - 6p^2r' - 4p^3r - 6qq'' + 3q'^2 + \\ &\quad + 18qr' - 4q^3 - 27r^2. \end{aligned} \quad (2.9)$$

The following change of dependent variable

$$12\xi(z) + b(z) = 12y(z), \quad (2.10)$$

together with the definition of the functions  $A(z)$  and  $B(z)$

$$\begin{aligned} A(z) &= \frac{1}{4}p'' + \frac{1}{2}pp' - \frac{3}{4}q' + \frac{1}{9}p^3 - \frac{1}{2}pq + \frac{3}{2}r, \\ 4B(z) &= p' + \frac{1}{3}p^2 - q, \end{aligned} \quad (2.11)$$

allow to recast Equation (2.8) for  $\xi(z)$  in a more concise form:

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 3Ay' + 4By^2 - 2A'y - A^2 = 0. \quad (2.12)$$

The previous equation is the Painlevé XXV–Ermakov equation [4]: The name is due to the fact that, as shown here, by a suitable change of variables, (2.12) gives both the Painlevé XXV equation (according to Ince's numbering [13]) and the Ermakov equation. Indeed the following proposition holds:

**Proposition 2.1.** *In Equation (2.12), by setting*

$$y(z) = \frac{g(z)}{u(z)^4}, \quad \text{with } g' = 2A(z)u(z)^4, \quad (2.13)$$

$u(z)$  satisfies the following generalized Ermakov equation:

$$u'' = B(z)u(z) + \frac{g(z)}{6u(z)^3}. \quad (2.14)$$

In this proposition, the adjective *generalized* referred to the Ermakov equation means that the coefficient of  $u^{-3}$  depends on  $z$ , in contrast to the Ermakov equation where there is a constant term instead. Also, as it can be seen from (2.13), the function  $g(z)$  is related to  $u(z)$  and hence (2.14) is not strictly an Ermakov equation. Clearly, in the case  $A = 0$ , (2.13) implies that  $g(z)$  is constant, and in this case, (2.14) is a proper Ermakov equation.

A particular case of the previous transformation, leading to the Painlevé XXV equation, is given in the following proposition.

**Proposition 2.2.** In Proposition 2.1, by setting  $g(z) = 2A(z)u(z)^3$  and not considering the further constraint  $g' = 2Au^4$ , that is, by setting

$$y(z) = \frac{2A(z)}{u(z)} \quad (2.15)$$

in Equation (2.12), the Painlevé XXV equation is obtained for  $u(z)$ :

$$u'' = \frac{3u'^2}{4u} + \left( \frac{A'}{2A} - \frac{3u}{2} \right) u' - \frac{1}{4}u^3 + \frac{A'}{2A}u^2 + \left( 4B - \frac{5A'^2}{4A^2} + \frac{A''}{A} \right) u + \frac{4}{3}A. \quad (2.16)$$

From another point of view, we notice that in the Proposition 2.1, it is possible to look at Equation (2.14) as an equation defining the function  $g(z)$ . The constraint (2.13), that is,  $g' = 2A(z)u(z)^4$ , then gives the following quadratic equation for  $u(z)$ :

$$u'''u + 3u''u' - \frac{A}{3}u^2 - 4Buu' - B'u^2 = 0, \quad (2.17)$$

which gives a linear equation in the new variable  $w = u^2$ :

$$w''' - 4Bw' - 2\left( B' + \frac{A}{3} \right) w = 0. \quad (2.18)$$

The previous observation allows to get a linearization of Equation (2.12): Indeed it holds the following:

**Proposition 2.3.** Let  $w(z)$  be a solution of the linear equation (2.18). Then, the function

$$y = \frac{3w''}{w} - \frac{3}{2} \left( \frac{w'}{w} \right)^2 - 6B \quad (2.19)$$

is a solution of the Painlevé XXV–Ermakov equation (2.12).

**Proposition 2.4.** If  $A = 0$ , the linear equation (2.18) reduces to an equation considered also by Gambier in relation to the Ermakov equation. Indeed, in this case, Equation (2.18) possesses a first integral given by

$$w''w - \frac{1}{2}(w')^2 - 2Bw^2 = 2I, \quad (2.20)$$

where  $I$  is a constant. From Proposition 2.1 and Equation (2.19), it follows that the function  $u(z)$  defined by  $w = u^2$  solves the Ermakov equation:

$$u'' = B(z)u + \frac{I}{u^3}. \quad (2.21)$$

In Proposition 2.2, we considered the equation for  $g(z)$ , without the further constraint  $g' = 2Au^4$ . If the derivative of  $g$  satisfies this last constraint too, then it is possible to get a family of solutions of Equation (2.12), corresponding also to a family of solutions of a linear equation and of the Painlevé XXV equation. Indeed, from the equations  $g(z) = 2A(z)u^3$  and  $g' = 2A(z)u^4$ , it follows that  $u(z)$  satisfies a Riccati equation:

$$3Au' - Au^2 + A'u = 0. \quad (2.22)$$

By setting  $u = v'$ , the function  $A$  can be expressed in terms of  $v$  and its derivative as:

$$A(z) = \frac{ce^v}{(v')^3}, \quad (2.23)$$

where  $c$  is an arbitrary constant. By Proposition 2.1, it follows that  $u = v'$  satisfies also Equation (2.14) (a linear equation in this case). This equation fixes the value of  $B(z)$  in terms of  $v$  and its derivatives. Indeed one has:

$$B(z) = \frac{v'''}{v'} - \frac{ce^v}{3(v')^4}. \quad (2.24)$$

Finally, the function  $y(z)$  satisfying the Painlevé XXV–Ermakov equation is given by:

$$y(z) = \frac{2ce^v}{(v')^4}. \quad (2.25)$$

The above equations implies that the following proposition holds.

**Proposition 2.5.** *Suppose that the functions  $A(z)$  and  $B(z)$  in (2.12) are given by the expressions (2.23) and (2.24) for some function  $v(z)$ . Then, a solution of the Painlevé XXV–Ermakov equation (2.12) is given by the expression (2.25). Further, the function  $u = v'$  satisfies the Painlevé XXV equation (2.16), Equation (2.17), and the linear equation  $u'' = Bu + A/3$ .*

### 3 | WRONSKIAN AND ALGEBRAIC RELATIONS AMONG SOLUTIONS

The third-order linear equation (2.18) is in *normal form* [6]. The function  $A(z)$  is usually called the Laguerre invariant (see, e.g., [6]). It is known (see again [6]) that, if  $w_1$  and  $w_2$  are two solutions of (2.18), that is,

$$w_i''' - 4Bw_i' - 2\left(B' + \frac{A}{3}\right)w_i = 0, \quad i = 1, 2, \quad (3.1)$$

then their Wronskian

$$w = w_1w_2' - w_1'w_2 \quad (3.2)$$

is a solution of

$$w''' - 4Bw' - 2\left(B' - \frac{A}{3}\right)w = 0. \quad (3.3)$$

In the case  $A = 0$ , the transformation (3.2) represents an auto-Bäcklund transformation admitted by Equation (2.18). This case is interesting since, as discussed in Proposition 2.4, it is related to the Ermakov equation. Indeed, it holds the following

**Proposition 3.1.** *Suppose that  $u_1$  and  $u_2$  are two solutions of the Ermakov equation*

$$u'' = B(z)u + \frac{c}{u^3}, \quad (3.4)$$

then the function  $u$  defined by

$$u^2 = 2u_1u_2(u_1u_2' - u_2u_1') \quad (3.5)$$

is a solution of the following Ermakov equation:

$$u'' = B(z)u + \frac{k}{u^3}. \quad (3.6)$$

The constants  $c$  and  $k$  are conserved quantities for Equation (2.18) in the case  $A = 0$ , that is, if  $u^2 = w$ ,  $u_1^2 = w_1$ , and  $u_2^2 = w_2$ , then one has

$$w_i''w_i - \frac{1}{2}(w_i')^2 - 2Bw_i^2 = 2c, \quad i = 1, 2,$$

$$w''w - \frac{1}{2}(w')^2 - 2Bw^2 = 2k. \tag{3.7}$$

*Proof.* The result follows as an application of the Bäcklund transformation (3.2) to Proposition 2.4: In particular, with the changes of variables  $w = u^2$ ,  $w_i = u_i^2$ ,  $i = (1, 2)$ , and from (3.2) and (3.3), we get (3.5) and (3.6). Equations (3.7) also follow from Proposition 2.4.  $\square$

**Proposition 3.2.** *From Equations (3.4) and (3.6) and relation (3.5), it follows that the functions  $u_1, u_2$ , and  $u$  of Proposition 3.1 satisfy the following polynomial equation:*

$$\frac{u^4}{4} + c(u_1^2 - u_2^2)^2 - au_1^2u_2^2 = 0, \tag{3.8}$$

where the constant  $a$  is related to  $k$  and  $c$  by  $k = -a(a + 4c)$ .

*Proof.* By differentiating Equation (3.5) and taking into account Equation (3.5) itself and Equation (3.4), gives

$$uu' = c\left(\frac{u_1^2}{u_2^2} - \frac{u_2^2}{u_1^2}\right) + \frac{u^2}{2}\left(\frac{u_1'}{u_1} + \frac{u_2'}{u_2}\right). \tag{3.9}$$

For the second-order derivative  $u''$ , by differentiating the previous equation and taking into account (3.5), (3.4), and (3.9), we get

$$u'' = B(z)u - \frac{(u^4 + 4c(u_1^2 - u_2^2)^2)(u^4 + 4c(u_1^2 + u_2^2)^2)}{16u_1^4u_2^4u^3}. \tag{3.10}$$

By comparison with Proposition 3.1, the ratio on the right-hand side of the previous equation must be equal to  $k/u^3$ , that is,

$$\frac{(u^4 + 4c(u_1^2 - u_2^2)^2)}{4u_1^2u_2^2} \frac{(u^4 + 4c(u_1^2 + u_2^2)^2)}{4u_1^2u_2^2} + k = 0. \tag{3.11}$$

By differentiating and by using Equations (3.9) and (3.5), it is possible to check that both the factors in Equation (3.11) are constants, that is, there exist two constants  $a$  and  $b$  such that:

$$\begin{aligned} \frac{u^4}{4} + c(u_1^2 - u_2^2)^2 &= au_1^2u_2^2, \\ \frac{u^4}{4} + c(u_1^2 + u_2^2)^2 &= bu_1^2u_2^2. \end{aligned} \tag{3.12}$$

By subtracting the previous equations, it follows that  $-4c = a - b$ , whereas by multiplying them one has  $ab = -k$ , giving the result (3.8).  $\square$

Propositions 3.1 and 3.2 can be generalized also to the case when  $u_1$  and  $u_2$  solve two Ermakov equations with two different constants as coefficients of the  $u^{-3}$  term. The following proposition holds.

**Proposition 3.3.** *Suppose that  $u_1$  and  $u_2$  are two solutions of the Ermakov equation*

$$u_i'' = B(z)u_i + \frac{c_i}{u_i^3}, \quad i = 1, 2. \tag{3.13}$$

Then, the function  $u$  defined by

$$u^2 = 2u_1u_2(u_1u_2' - u_2u_1') \tag{3.14}$$

is a solution of the Ermakov equation

$$u'' = B(z)u + \frac{k}{u^3}. \quad (3.15)$$

The constants  $c_i$  and  $k$  are conserved quantities for Equation (2.18) in the case  $A = 0$ , that is, if  $u^2 = w$ ,  $u_1^2 = w_1$ , and  $u_2^2 = w_2$ , then one has

$$\begin{aligned} w_i'' w_i - \frac{1}{2}(w_i')^2 - 2Bw_i^2 &= 2c_i, \quad i = 1, 2, \\ w'' w - \frac{1}{2}(w')^2 - 2Bw^2 &= 2k. \end{aligned} \quad (3.16)$$

Furthermore, the solutions  $u$ ,  $u_1$ , and  $u_2$  and the constants  $c_1$ ,  $c_2$ , and  $k$  are related by the following polynomial relation:

$$\frac{u^8}{16} + \frac{(c_1 u_2^4 + c_2 u_1^4)}{2} u^4 + (c_1 u_2^4 - c_2 u_1^4)^2 + k u_1^4 u_2^4 = 0. \quad (3.17)$$

*Proof.* Let us look again at the Bäcklund transformation (3.2) and Proposition 2.4. From the equations

$$w_i'' w_i - \frac{1}{2}(w_i')^2 - 2Bw_i^2 = 2c_i, \quad i = 1, 2, \quad (3.18)$$

and the change of variables  $w_i = u_i^2$ ,  $i = (1, 2)$ , we get that the functions  $u_i$ ,  $i = 1, 2$ , solve the Ermakov equations

$$u_i'' = B(z)u_i + \frac{c_i}{u_i^3}, \quad i = 1, 2, \quad (3.19)$$

whereas the function  $u$ , defined by

$$u^2 = 2u_1 u_2 (u_1 u_2' - u_2 u_1'), \quad (3.20)$$

solves the equation

$$u'' = B(z)u + \frac{k}{u^3} \quad (3.21)$$

for a suitable value of  $k$ . This value of  $k$  is such that, if  $w = u^2$ , then

$$w'' w - \frac{1}{2}(w')^2 - 2Bw^2 = 2k. \quad (3.22)$$

Differentiating Equation (3.20) and by taking into account Equation (3.20) itself and Equations (3.19), we get:

$$uu' = \left( c_2 \frac{u_1^2}{u_2^2} - c_1 \frac{u_2^2}{u_1^2} \right) + \frac{u^2}{2} \left( \frac{u_1'}{u_1} + \frac{u_2'}{u_2} \right). \quad (3.23)$$

For the second-order derivative  $u''$ , by differentiating the previous equation and taking into account (3.20), (3.21), and (3.23), we get

$$u'' = B(z)u - \left( \frac{u^8 + 8(c_1 u_2^4 + c_2 u_1^4)u^4 + 16(c_1 u_2^4 - c_2 u_1^4)^2}{16u_1^4 u_2^4 u^3} \right). \quad (3.24)$$

From Equation (3.21), it follows that the ratio on the right-hand side of the previous equation must be equal to  $k/u^3$ , that is,

$$\frac{u^8}{16} + \frac{(c_1 u_2^4 + c_2 u_1^4)}{2} u^4 + (c_1 u_2^4 - c_2 u_1^4)^2 + k u_1^4 u_2^4 = 0. \quad (3.25)$$

□

Propositions (3.2) and (3.3), in particular the polynomial relations among solutions, can be extended to the solutions of the third-order equation (2.18) and to the solutions of the Painlevé XXV–Ermakov equation (2.12) when the function  $A(z)$  is equal to zero. We have indeed the following:

**Proposition 3.4.** *Suppose that  $w_1$  and  $w_2$  are two solutions of the linear equation*

$$w''' - 4Bw' - 2B'w = 0. \quad (3.26)$$

*Then, the function  $w$  defined by the Wronskian*

$$w = w_1 w_2' - w_2 w_1' \quad (3.27)$$

*is a solution of the same equation (3.26). There is a polynomial relation among  $w$ ,  $w_1$ , and  $w_2$ :*

$$\frac{w^4}{16} + \frac{(c_1 w_2^2 + c_2 w_1^2)}{2} w^2 + (c_2 w_1^2 - c_1 w_2^2)^2 + k w_1^2 w_2^2 = 0. \quad (3.28)$$

*The constants  $c_1$ ,  $c_2$ , and  $k$  are conserved quantities for Equation (3.26), that is,*

$$\begin{aligned} w_i'' w_i - \frac{1}{2} (w_i')^2 - 2B w_i^2 &= 2c_i, \quad i = 1, 2, \\ w'' w - \frac{1}{2} (w')^2 - 2B w^2 &= 2k. \end{aligned} \quad (3.29)$$

*If the two constants  $c_1$  and  $c_2$  are equal, that is,  $c_1 = c_2 = c$ , then the polynomial equation (3.28) reduces to the quadratic equation*

$$\frac{w^2}{4} + c(w_1 - w_2)^2 - a w_1 w_2 = 0, \quad (3.30)$$

*wherein the constants  $a$ ,  $k$ , and  $c$  are related via  $k = -a(a + 4c)$ .*

*Proof.* The previous proposition comes directly from Equations (3.1)–(3.3) and Propositions (3.2) and (3.3) by substituting  $u_i^2 = w_i$ ,  $i = 1, 2$ , and  $u^2 = w$ . □

Now, let us consider Equation (2.12) in the case  $A = 0$ . From Proposition 2.3 and Proposition 2.4, it follows that, if the functions  $w_1$ ,  $w_2$ , and  $w$  satisfy Equations (3.7), then the functions  $y_1$ ,  $y_2$ , and  $y$  defined by the relations  $y w^2 = 6k$ ,  $y_1 w_1^2 = 6c_1$ , and  $y_2 w_2^2 = 6c_2$  satisfy Equation (2.12) in the case  $A = 0$ . Indeed, let us consider Equation (2.18) in the case  $A = 0$ , like in Proposition 2.4:

$$w''' - 4Bw' - 2B'w = 0. \quad (3.31)$$

If  $w$  is a solution of (3.31), then the function defined by the transformation (2.19), that is,

$$y = \frac{3w''}{w} - \frac{3}{2} \left( \frac{w'}{w} \right)^2 - 6B \quad (3.32)$$

is a solution of the Painlevé XXV–Ermakov equation (2.12) for  $A = 0$ . Equation (3.31) has a first integral, that is,

$$w''w - \frac{1}{2}(w')^2 - 2Bw^2 = 2c, \quad (3.33)$$

where  $c$  is a constant. By comparing Equations (3.32) and (3.33), we see that Equation (3.32) can be rewritten as

$$y = \frac{6c}{w^2}. \quad (3.34)$$

Now, by considering the maps  $yw^2 = 6k$ ,  $y_1w_1^2 = 6c_1$ , and  $y_2w_2^2 = 6c_2$  and expressing the relations (3.27) and (3.28) in terms of the new variables  $y_1$ ,  $y_2$ , and  $y$  we get the following:

**Proposition 3.5.** *Suppose that  $y_1$  and  $y_2$  are two solutions of the equation*

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 4By^2 = 0. \quad (3.35)$$

*Then, the function  $y$  defined by:*

$$y = \frac{2k(y_1y_2)^3}{3c_1c_2(y_1y_2' - y_2y_1')} \quad (3.36)$$

*is a solution of the same Equation (3.35). The constants  $c_1$ ,  $c_2$ , and  $k$  are such that the functions  $w_1$ ,  $w_2$ , and  $k$ , defined by  $yw^2 = 6k$ ,  $y_1w_1^2 = 6c_1$ , and  $y_2w_2^2 = 6c_2$ , satisfy Equations (3.29). There is a polynomial relation among  $y$ ,  $y_1$ , and  $y_2$ , given by:*

$$16c_1c_2(c_1c_2(y_1 - y_2)^2 + ky_1y_2)y^2 + 8c_1c_2ky_1y_2(y_1 + y_2)y + k^2y_1^2y_2^2 = 0. \quad (3.37)$$

## 4 | FURTHER BÄCKLUND TRANSFORMATIONS

In the previous work [2] (see also [3]), another family of Bäcklund transformations for Equation (2.21) is presented. These transformations depend on a certain function  $f(z)$  that solve a suitable functional equation that is closely related with the properties of the Schwarzian derivative. For completeness, we report the result given in [2]:

**Proposition 4.1.** *Suppose that  $u_0$  is a solution of Equation (2.21). Define the function  $\Omega(z) = \frac{\eta_0}{\eta_1}$ , where  $\eta_0$  and  $\eta_1$  are two independent solutions of the linear differential equation  $\eta'' = B\eta$ . Then, if it is possible to find a function  $f(z)$  such that the following equation,*

$$\Omega(f(z)) = \frac{a\Omega(z) + b}{c\Omega(z) + d}, \quad ad - bc \neq 0, \quad (4.1)$$

*holds for some set of constants  $a, b, c, d$ , then the map*

$$u_1(z)^2 = \frac{u_0(f(z))^2}{f'(z)} \quad (4.2)$$

*is a Bäcklund transformation for Equation (2.21).*

*Proof.* Let us assume that  $u_0(z)$  is a solution of Equation (2.21) and, hence, a solution of the following equation:

$$\begin{aligned} \frac{d^2u(\tilde{z})}{d\tilde{z}^2} &= B(\tilde{z})u(\tilde{z}) + \frac{I}{u(\tilde{z})^3}, \\ \tilde{z} &= f(z). \end{aligned} \quad (4.3)$$

Differentiating Equation (4.2) with respect to  $z$ , we find (as in Equation (4.3)  $\tilde{z} = f(z)$ )

$$u_1 \frac{du_1}{dz} = u_0(\tilde{z}) \frac{du_0}{d\tilde{z}} - \frac{1}{2} \frac{\frac{d^2 f}{dz^2}}{\left(\frac{df}{dz}\right)^2} u_0(\tilde{z})^2. \quad (4.4)$$

The second-order derivative, by using Equation (4.4) for  $u_1'$ , Equation (4.2) for  $u_0(\tilde{z})$ , and Equation (4.3) for  $\frac{d^2 u_0(\tilde{z})}{d\tilde{z}^2}$ , can be written as:

$$\frac{d^2 u_1}{dz^2} = \left(\frac{df}{dz}\right)^2 B(\tilde{z}) u_1(z) + \frac{I}{u_1(z)^3} - \frac{1}{2} \{f(z), z\} u_1(z). \quad (4.5)$$

It follows that  $u_1(z)$  is a solution of Equation (2.21) whenever

$$\left( \left(\frac{df}{dz}\right)^2 B(\tilde{z}) - B(z) - \frac{1}{2} \{f(z), z\} \right) u_1(z) = 0. \quad (4.6)$$

The previous equation is an equation in the unknown  $B(z)$  and can be rewritten as

$$(-2B(z)) = (-2B(f(z))) \left(\frac{df(z)}{dz}\right)^2 + \{f(z), z\}. \quad (4.7)$$

Remember the transformation law of the Schwarzian derivative under composition of functions. If the function  $\Omega$  depends on  $z$  through the function  $\psi(z)$ , that is,  $\hat{\Omega}(z) = \Omega(\psi(z))$ , then the Schwarzian derivative of the composition of functions behaves as (see, e.g., [5])

$$\{\hat{\Omega}, z\} = \{\Omega, \psi\} \left(\frac{d\psi}{dz}\right)^2 + \{\psi, z\}. \quad (4.8)$$

Notably, the latter reminds Equation (4.7) when the function  $f$  is identified with the function  $\psi$  and the function  $-2B$  with the Schwarzian derivative  $\{\Omega, z\}$ . So we identify  $-2B(z)$  with a suitable Schwarzian derivative by setting

$$-2B(z) := \{\Omega(z), z\}. \quad (4.9)$$

In terms of  $\Omega$ , Equation (4.7) then becomes

$$\{\Omega(z), z\} = \{\Omega(f(z)), z\}. \quad (4.10)$$

Further, the previous implies that  $\Omega(f(z))$  and  $\Omega(z)$  are related by a fractional linear transformation, that is, there are four constants  $a, b, c$ , and  $d$  such that

$$\Omega(f(z)) = \frac{a\Omega(z) + b}{c\Omega(z) + d}, \quad ad - bc \neq 0. \quad (4.11)$$

The function  $\Omega(z)$  is defined by Equation (4.9). Equations (1.4) and (1.5) however imply that it can be also expressed by the ratio between two independent solutions of the equation  $\eta'' = B\eta$ . Proposition 4.1 then follows.  $\square$

Due to the relations between the solutions of Equation (3.35) and those of Equation (2.21), that is,  $y = 6I/u^4$  (see Proposition 2.1), from Proposition 4.1, one has immediately the following:

**Proposition 4.2.** *Suppose that  $y_0$  is a solution of Equation (3.35). Define the function  $\Omega(z) = \frac{\eta_0}{\eta_1}$ , where  $\eta_0$  and  $\eta_1$  are two independent solutions of the linear differential equation  $\eta'' = B\eta$ . Then, if it is possible to find a function  $f(z)$  such that the*

following equation,

$$\Omega(f(z)) = \frac{a\Omega(z) + b}{c\Omega(z) + d}, \quad ad - bc \neq 0, \quad (4.12)$$

holds for some set of constants  $a, b, c, d$ , then the map

$$y_1(z) = y_0(f(z))f'^2 \quad (4.13)$$

is a Bäcklund transformation admitted by Equation (3.35).

Actually, this family of Bäcklund transformations can be extended to the more general Equation (2.12) as well by a further constraint on the function  $A(z)$ , again expressed in terms of a functional equation. That is, the following result holds.

**Proposition 4.3.** Suppose that  $y_0$  is a solution of Equation (2.12). Define the function  $\Omega(z) = \frac{\eta_0}{\eta_1}$ , where  $\eta_0$  and  $\eta_1$  are two independent solutions of the linear differential equation  $\eta'' = B\eta$ . Then, if it is possible to find a function  $f(z)$  such that the following equation,

$$\Omega(f(z)) = \frac{a\Omega(z) + b}{c\Omega(z) + d}, \quad ad - bc \neq 0, \quad (4.14)$$

holds for some set of constants  $a, b, c, d$ , and  $A(z)$  satisfies

$$A(f(z)) \left( \frac{df(z)}{dz} \right)^3 = A(z), \quad (4.15)$$

then the map

$$y_1(z) = y_0(f(z))f'^2 \quad (4.16)$$

is a Bäcklund transformation for Equation (2.12).

*Proof.* To prove this statement, let us assume that the map (4.16) defines a new solution of Equation (2.12). Inserting (4.16) in (2.12) and using the following equation for  $y_0(f(z))$ ,

$$\begin{aligned} y(\tilde{z}) \frac{d^2 y(\tilde{z})}{d\tilde{z}^2} - \frac{5}{4} \left( \frac{dy(\tilde{z})}{d\tilde{z}} \right)^2 + 3A(\tilde{z}) \frac{dy(\tilde{z})}{d\tilde{z}} + \frac{2}{3} y(\tilde{z})^3 + 4B(\tilde{z}) y(\tilde{z})^2 + \\ - 2 \frac{dA(\tilde{z})}{d\tilde{z}} y(\tilde{z}) - A(\tilde{z})^2 = 0, \\ \tilde{z} = f(z), \end{aligned} \quad (4.17)$$

we find that  $y_1(z)$  is again a solution of Equation (2.12) provided  $A(z)$  and  $B(z)$  satisfy the following relations:

$$\{f(z), z\} + 2B(z) - 2B(f(z)) \left( \frac{df(z)}{dz} \right)^2 = 0, \quad A(f(z)) \left( \frac{df(z)}{dz} \right)^3 - A(z) = 0. \quad (4.18)$$

The equation for  $A(z)$  is exactly (4.15). The equation for  $B(z)$  is exactly the equation given in (4.6): Again, with the same line of reasoning after Equation (4.6), we conclude that for a suitable  $\Omega(z)$

$$-2B(z) = \{\Omega(z), z\}, \quad (4.19)$$

and if

$$\Omega(f(z)) = \frac{a\Omega(z) + b}{c\Omega(z) + d}, \quad ad - bc \neq 0, \quad (4.20)$$

Proposition 4.3 follows.  $\square$

## 5 | LINEARIZATION OF THE WRONSKIAN EQUATION

The previous sections show a deep connection between the Schwarzian equation (1.5), the linear equation (3.26), and the Ermakov equation (2.21). These links are made more explicit in this section. It is known (see, e.g., [6]) that a solution of Equation (3.26)

$$w''' - 4Bw' - 2B'w = 0 \quad (5.1)$$

can be written as the product of two independent solutions of  $\eta'' = B\eta$ . If  $\eta_1$  and  $\eta_2$  are two such solutions, for any choice of the constant  $(a, b, c, d)$ , one has that the function

$$w = (a\eta_1 + b\eta_2)(c\eta_1 + d\eta_2) \quad (5.2)$$

solves Equation (5.1). If  $ad - bc \neq 0$ , then the previous is the general solution of Equation (5.1) (conversely, if  $ad - bc = 0$ , then the expression on the right-hand side of (5.2) reduces to a square of a single function). We remember that Equation (5.1) can be integrated once, giving

$$ww'' - \frac{1}{2}(w')^2 - 2Bw^2 = 2c, \quad (5.3)$$

where  $c$  is an integration constant. By inserting the general solution (5.2) into (5.1), we find that the integration constant  $c$  is explicitly given by

$$c = -\frac{1}{2}W^2(ad - bc)^2, \quad (5.4)$$

where  $W$  is the Wronskian of  $\eta_1$  and  $\eta_2$ , that is,  $W = \eta_1\eta_2' - \eta_1'\eta_2$ .

On the other hand, (1.4) and (1.5) imply that if  $\eta_1$  and  $\eta_2$  are two linearly independent solutions of the second-order equation

$$\eta''(z) = B(z)\eta(z), \quad (5.5)$$

then the quotient

$$\Omega = \frac{a\eta_1 + b\eta_2}{c\eta_1 + d\eta_2}, \quad ad - bc \neq 0, \quad (5.6)$$

satisfies the Schwarzian equation

$$\{\Omega, z\} = -2B(z). \quad (5.7)$$

We notice that the product defined by the right-hand side of Equation (5.2) is proportional to  $\Omega/\Omega'$ , where  $\Omega$  is given by (5.6). Specifically one has:

$$\frac{\Omega}{\Omega'} = -(a\eta_1 + b\eta_2)(c\eta_1 + d\eta_2)/(W(ad - bc)). \quad (5.8)$$

The previous equations, together with Proposition 2.4 and Equation (3.34), give the following:

**Proposition 5.1.** *Suppose that the function  $\Omega(z)$  solves the Schwarzian equation*

$$\{\Omega, z\} = -2B(z). \quad (5.9)$$

*Then, the following statements hold:*

1. *The function  $w = \Omega/\Omega'$  solves the linear equation*

$$w''' - 4Bw' - 2B'w = 0. \quad (5.10)$$

2. *The function defined by  $u^2 = 2r (\Omega/\Omega')$  solves the Ermakov equation*

$$u'' = Bu - \frac{r^2}{u^3}. \quad (5.11)$$

3. *The function  $y(z) = -\frac{3}{2} (\Omega'/\Omega)^2$  satisfies Equation (2.12) for  $A = 0$ , that is,*

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 4By^2 = 0. \quad (5.12)$$

In Proposition 5.1, particular solutions of Equations (5.10)–(5.12) depend on the function  $\Omega(z)$  and its first-order derivative: This function is the same appearing in the Bäcklund transformations found in Section 4. This observation helps us to find the *general* solutions of Equations (5.10)–(5.12) in terms of  $\Omega(z)$  and its first-order derivative. Indeed, let us consider, for example, the Ermakov equation (5.11). From Proposition 4.1 we know that if  $u_0(z)$  is a solution of (5.21), then

$$u_1(z)^2 = \frac{u_0(f(z))^2}{f'(z)} \quad (5.13)$$

is another solution. Here, the function  $f(z)$  is such that

$$\Omega(f(z)) = \frac{a\Omega(z) + b}{c\Omega(z) + d}, \quad ad - bc \neq 0. \quad (5.14)$$

Differentiating the previous equation, we get

$$\Omega'(f(z))f'(z) = (ad - bc) \frac{\Omega'(z)}{(c\Omega(z) + d)^2}, \quad ad - bc \neq 0. \quad (5.15)$$

Setting  $u_0^2(z) = 2r (\Omega/\Omega')$  in (5.13), it follows

$$u_1(z)^2 = 2r \frac{\Omega(f)}{\Omega'(f)f'(z)}, \quad (5.16)$$

and, with the help of Equation (5.15), we get

$$u_1(z)^2 = 2r \frac{(a\Omega(z) + b)(c\Omega(z) + d)}{(ad - bc)\Omega'(z)}. \quad (5.17)$$

Equation (5.17) represents a solution of the Ermakov equation (5.11) for any choice of the constants  $(a, b, c, d)$  s.t.  $ad - bc \neq 0$  and then it is the general solution of Equation (5.11). The same line of reasoning can be applied to Equations (5.10) and (5.12): The results are summarized in the following

**Proposition 5.2.** *Suppose that the function  $\Omega(z)$  solves the Schwarzian equation*

$$\{\Omega, z\} = -2B(z), \quad (5.18)$$

and let the arbitrary constants  $(a, b, c, d)$  be such that  $ad - bc \neq 0$ . Then,

1. *the general solution of the linear equation*

$$w''' - 4Bw' - 2B'w = 0 \quad (5.19)$$

is given by

$$w(z) = \frac{(a\Omega(z) + b)(c\Omega(z) + d)}{(ad - bc)\Omega'(z)}; \quad (5.20)$$

2. *the general solution of the Ermakov equation*

$$u'' = Bu - \frac{r^2}{u^3} \quad (5.21)$$

is given by

$$u(z)^2 = 2r \frac{(a\Omega(z) + b)(c\Omega(z) + d)}{(ad - bc)\Omega'(z)}; \quad (5.22)$$

3. *The general solution of Equation (2.12) for  $A = 0$ , that is,*

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 4By^2 = 0 \quad (5.23)$$

is given by

$$y(z) = -\frac{3}{2} \left( \frac{(ad - bc)\Omega'(z)}{(a\Omega(z) + b)(c\Omega(z) + d)} \right)^2. \quad (5.24)$$

The results given in this section are applied in the examples of the next section.

## 6 | TWO ANALYTICAL EXAMPLES

This section collects two analytical examples.

### 6.1 | An example involving the Weierstrass $\wp$ function

Let us consider the duplication formula for the Weierstrass elliptic function  $\wp(z)$  (see, e.g., [15], formula 23.10.7):

$$\wp(2z) = \frac{1}{4} \left( \frac{\wp''(z)}{\wp'(z)} \right)^2 - 2\wp(z). \quad (6.1)$$

From the differential equation satisfied by  $\wp(z)$ , that is,

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \quad (6.2)$$

where  $g_2$  and  $g_3$  are the invariants, and the differential consequences of (6.2), that is,

$$\wp'' = 6\wp^2 - \frac{1}{2}g_2, \quad \wp''' = 12\wp\wp', \quad (6.3)$$

we get

$$\frac{\wp'''}{\wp'} - \frac{3}{2} \left( \frac{\wp''}{\wp'} \right)^2 = \{\wp(z), z\} = -6\wp(2z). \quad (6.4)$$

The previous equation and Proposition 5.2 suggest to set  $B(z) = 3\wp(2z)$ . Hence, we consider the linear third-order equation:

$$w''' - 12\wp(2z)w' - 12\wp'(2z)w = 0. \quad (6.5)$$

From Equation (6.4), we see that

$$\Omega(z) = \wp(z), \quad (6.6)$$

whereas Proposition 5.2 tells us that the general solution of (6.5) is given by

$$w(z) = \frac{(a\wp(z) + b)(c\wp(z) + d)}{(ad - bc)\wp'(z)}. \quad (6.7)$$

If we consider the Ermakov equation (5.21) in the case  $B(z) = 3\wp(2z)$ , that is,

$$u'' = 3\wp(2z)u - \frac{r^2}{u^3}, \quad (6.8)$$

we get, from Proposition 5.2 the general solution

$$u^2(z) = 2r \frac{(a\wp(z) + b)(c\wp(z) + d)}{\wp'(z)(ad - bc)}. \quad (6.9)$$

In addition, the general solution of the equation

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 12\wp(2z)y^2 = 0, \quad (6.10)$$

is given by

$$y(z) = -\frac{3}{2} \left( \frac{(ad - bc)\wp'(z)}{(a\wp(z) + b)(c\wp(z) + d)} \right)^2. \quad (6.11)$$

Finally note that Equation (6.5) is related to a particular case of the Lamé equation considered by Halphen (see [10, p. 105]):

$$\eta'' - 3/4\wp(z)\eta = 0. \quad (6.12)$$

Halphen showed that this equation is such that every solution is multivalued but the ratio of any two solutions is single-valued (a *fonction uniforme* according to Halphen). Further, he gives explicitly two independent solutions of (6.12) as

$$\eta_1 = \wp'(z/2)^{-\frac{1}{2}}, \quad \eta_2 = \wp'(z/2)^{-\frac{1}{2}}\wp(z/2). \quad (6.13)$$

The link between Equations (6.5) and (6.12) is obtained via the change of variables  $2z \rightarrow z$  in (6.5), which gives

$$w''' - 3\wp(z)w' - 3/2\wp'(z)w = 0, \quad (6.14)$$

and also recalling the comment after Equation (5.1) and Equation (5.2). Indeed, from Equation (5.2), the general solution of (6.14) can be written as

$$w(z) = (a\eta_1(z) + b\eta_2)(c\eta_2(z) + d\eta_1), \quad ad - bc \neq 0, \quad (6.15)$$

where  $\eta_1$  and  $\eta_2$  are two independent solutions of the Lamé equation (6.12). Explicitly, one has

$$w(z) = \frac{1}{\wp'(z)}(a\wp(z) + b)(c\wp(z) + d), \quad ad - bc \neq 0. \quad (6.16)$$

## 6.2 | An example involving rational functions

The Ermakov equation with a rational potential  $B(z)$ , like  $z^{-2}$ , appears in the theory of scalar field cosmologies (see, e.g., [2, 11]). It is well known [12] that the Schwarzian derivative of a power, say  $(z - \alpha)^{n+1}$ , is proportional to  $(z - \alpha)^{-2}$ : more precisely,

$$\{(z - \alpha)^{n+1}, z\} = -\frac{n(n+2)}{2} \frac{1}{(z - \alpha)^2}. \quad (6.17)$$

Actually it is possible to generalize the previous equation to a product of such power functions. Let us define the function  $P(z)$  by

$$P'(z) = \prod_{k=1}^N (z - a_k)^n, \quad (6.18)$$

where all  $a_k$ s are supposed to be different from each other. It is easy to show that

$$\frac{P'''}{P'} = n(n-1) \sum_{k=1}^N \frac{1}{(z - a_k)^2} + 2n^2 \sum_{k=1}^N \sum_{j \neq k}^N \frac{1}{a_j - a_k} \frac{1}{z - a_k}, \quad (6.19)$$

and

$$\left(\frac{P''}{P'}\right)^2 = n^2 \sum_{k=1}^N \frac{1}{(z - a_k)^2} + 2n^2 \sum_{k=1}^N \sum_{j \neq k}^N \frac{1}{a_j - a_k} \frac{1}{z - a_k}. \quad (6.20)$$

Equations (6.19) and (6.20) give

$$\{P(z), z\} = -\frac{n(n+2)}{2} \sum_{k=1}^N \frac{1}{(z - a_k)^2} - n^2 \sum_{k=1}^N \sum_{j \neq k}^N \frac{1}{a_j - a_k} \frac{1}{z - a_k}. \quad (6.21)$$

From Proposition 5.2, we get the following result. Let the function  $P(z)$  be defined by (6.18) and consider the following third-order linear equation:

$$w''' - 4Bw' - 2B'w = 0, \quad (6.22)$$

where

$$B(z) = \frac{n(n+2)}{4} \sum_{k=1}^N \frac{1}{(z - a_k)^2} + \frac{n^2}{2} \sum_{k=1}^N \sum_{j \neq k}^N \frac{1}{a_j - a_k} \frac{1}{z - a_k}. \quad (6.23)$$

Then, the general solution of Equation (6.22) is given by

$$\begin{aligned} w(z) &= \frac{(aP(z) + b)(cP(z) + d)}{(ad - bc)P'(z)} \\ &= \frac{(a \int \prod_{k=1}^N (z - a_k)^n dz + b)(c \int \prod_{k=1}^N (z - a_k)^n dz + d)}{(ad - bc) \prod_{k=1}^N (z - a_k)^n}. \end{aligned} \quad (6.24)$$

By considering the Ermakov equation

$$u'' = Bu - \frac{r^2}{u^3}, \quad (6.25)$$

with  $B(z)$  defined by (6.23), its general solution is defined by the equation

$$u(z)^2 = 2r \frac{(aP(z) + b)(cP(z) + d)}{(ad - bc)P'(z)} = 2r \frac{(a \int \prod_{k=1}^N (z - a_k)^n dz + b)(c \int \prod_{k=1}^N (z - a_k)^n dz + d)}{(ad - bc) \prod_{k=1}^N (z - a_k)^n}. \quad (6.26)$$

Finally, the general solution of the following equation:

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 + 4By^2 = 0, \quad (6.27)$$

where  $B(z)$  is defined by (6.23), is given by

$$\begin{aligned} y(z) &= -\frac{3}{2} \left( \frac{(ad - bc)P'(z)}{(aP(z) + b)(cP(z) + d)} \right)^2 \\ &= -\frac{3}{2} \left( \frac{(ad - bc) \prod_{k=1}^N (z - a_k)^n}{(a \int \prod_{k=1}^N (z - a_k)^n dz + b)(c \int \prod_{k=1}^N (z - a_k)^n dz + d)} \right)^2. \end{aligned} \quad (6.28)$$

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## CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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