

# Well posedness and control in renewal equations with nonlocal boundary conditions

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A large class of biological models leads to initial boundary value problems for nonhomogeneous balance laws, possibly with nonlocal boundary conditions. Here, for these equations, a general well posedness result is proved, a full set of stability estimates is provided, and sample control problems are tackled.

## KEYWORDS

control of biological model, stability of general renewal equation, well posedness in IBVPs for balance laws

## MSC CLASSIFICATION

35L50; 35Q92; 92D30

## 1 | INTRODUCTION

Several biological models lead to systems of renewal equations that fit into the following general initial boundary value problem (IBVP)

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) + m_i(t, x) u_i = f_i(t, x) & (t, x) \in I \times \mathbb{R}_+ \\ g_i(t, 0) u_i(t, 0+) = \mathcal{B}_i(t, u_1(t), \dots, u_n(t)) & t \in I \\ u_i(0, x) = u_i^o(x) & x \in \mathbb{R}_+ \end{cases} \quad i = 1, \dots, n, \quad (1.1)$$

where  $t \in I$  is time,  $I$  being an interval with  $\min I = 0$ , and  $x \in \mathbb{R}_+$  is a structural variable, for example, age or size or both, as in Ackleh and Deng.<sup>1</sup> The IBVP (1.1) describes the evolution of biological entities quantified through  $u_1, \dots, u_n$ , with growth functions  $g_1, \dots, g_n$ , mortalities  $m_1, \dots, m_n$ , and source terms  $f_1, \dots, f_n$ .

Equations like the one in the first line of (1.1), or *variazioni* on that theme, have been studied for years from a variety of different points of view. A relatively recent reference is, for instance, by Perthame,<sup>2</sup> but we also recall the much older collection by Metz and Diekmann<sup>3</sup> or the classical paper of Gurtin and MacCamy.<sup>4</sup> Here, we mostly focus on a quite general type of boundary conditions, see also Colombo and Garavello,<sup>5</sup> since we choose as boundary terms in (1.1) general transmission coefficients of the type

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$$\begin{aligned} \mathcal{B}_i(t, u_1, \dots, u_n) = & \alpha_i(t, u_1(\bar{x}_1-), \dots, u_n(\bar{x}_n-)) \\ & + \beta_i \left( t, \int_{\mathbb{R}_+} w_1^i(t, x) u_1(x) dx, \dots, \int_{\mathbb{R}_+} w_n^i(t, x) u_n(x) dx \right) \end{aligned} \quad (1.2)$$

for  $i = 1, \dots, n$ .

Here, the second summand  $\beta_i$  may describe a typical natality term, depending on the whole distribution of  $u$  over all the admissible values of the structural variable  $x$ . In fact, in (1.2),  $w_1^i, \dots, w_n^i$  are suitable weight functions that allow to comprehend, for instance, the case where fertility age intervals are present. We recall, however, that these nonlocal terms are also used to model the evolution across different phases of cell development, see for instance Billy et al.<sup>6</sup>

The first summand  $\alpha_i$  in (1.2), on the contrary, describes if and how the various individuals evolve or get transformed into the state, or population,  $u_i$  through any sort of (time-dependent) selection or metamorphosis that for the  $j$  population takes place at the stage, or stadium,  $\bar{x}_j$ . In other instances, these terms may also model some sort of external intervention aimed at the control of the system's evolution, see Colombo and Garavello.<sup>7,8</sup> For instance,  $\alpha_i$  terms are used to describe the juvenile–adult transformation, see Ackleh and Deng.<sup>1</sup>

Thus, in general, the  $\alpha_i$  term typically describes a local, or pointwise, interaction while the  $\beta_i$  is of a nonlocal nature. When only the latter term is present, we recall that well posedness results for the IBVP (1.1)–(1.2) can also be obtained by means of the entirely different technique based on the *adjoint equation*, described in other studies,<sup>2, section 3.2</sup> see in particular Clairambault et al.<sup>9</sup> Here we provide a unified approach that yields well posedness and stability estimates amenable to be applied, for instance, to general control problems.

Models of this kind are rather classical, and we recall among the pioneering works in this connection.<sup>3,4,10–12</sup> This paper provides a unified environment for all these models, ensures the well posedness of (1.1)–(1.2), equips it with a full set of stability estimates, and introduces to optimal control problems based on it. The key technical analytic tools rely on precise results about scalar renewal equations, on careful *a-priori* estimates, and on a final extension through  $\mathbf{L}^1$  convergence, taking advantage of an  $\mathbf{L}^1$ -stable definition of solution.

In the applications, suitable control parameters enter the functions  $m_i, f_i, \alpha_i$  or  $\beta_i$  and have to be chosen to optimize *costs* or *gains* resulting from the evolution described by (1.1)–(1.2). These problems, too, have been widely considered: refer for instance to the monograph by Anița<sup>13</sup> or to the classical work of Brokate.<sup>14</sup> The general form we choose for the functional  $\mathcal{J}$  to be optimized is

$$\mathcal{J} = \mathcal{J}_1(u(T)) + \mathcal{J}_2(u) \quad \text{for suitable } \begin{aligned} \mathcal{J}_1 &: \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n) \rightarrow \mathbb{R}, \\ \mathcal{J}_2 &: \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)) \rightarrow \mathbb{R}, \end{aligned} \quad (1.3)$$

although the general stability result in Theorem 2.3 comprises other forms, too. Indeed, the full set of estimates provided allows to consider problems where the control parameters enter in any of the terms in (1.1)–(1.2). In particular, the choice (1.3) allows to comprehend the applications in Section 3, where the controls are in the mortality terms and, in Section 3.1, also in the boundary term.

Both from the modeling and from the analytical points of view, problem (1.1)–(1.2) extends the one considered in Colombo and Garavello<sup>5</sup>: the general source term  $f_i$  is now present, and all  $\beta_i$  and  $w_j^i$  now depend explicitly also on  $t$ . These extensions permit to comprehend, for instance, the specific applications considered below. However, more relevant is the present introduction of the cost  $\mathcal{J}$  in (1.3) and of its optimization. Indeed, the stability estimates below provide sufficient conditions for the existence of optimal controls in a variety of models, such as other studies,<sup>1,5–8</sup> for instance.

The overall structure of the formal proofs is somewhat traditional, with Banach contraction theorem playing a key role. Note, however, that the whole analytic framework is settled in  $\mathbf{L}^1 \cap \mathbf{BV}$ , so that punctual values are not always defined. Nevertheless, the  $\alpha_i$  terms in the boundary conditions (1.2) are imposed through punctual values of the solution. Key ingredients are a careful use of different notions of solutions, see Section 4.4, and a technical trick, see (4.36). Note that the obtained estimates also allow to relax the  $\mathbf{BV}$  requirement on the initial datum, see Section 2.

The paper is structured in the following way. The well posedness and stability results are stated in Section 2. Section 3 is devoted to applications of (1.1)–(1.2). All proofs and technical analytic details are deferred to the final Section 4.

## 2 | WELL POSEDNESS AND STABILITY

Throughout, we fix  $t_* > 0$  and denote  $I = [0, t_*]$ , comprising also the case  $t_* = +\infty$ , where we set  $I = [0, +\infty[$ . If  $J$  is a real interval and  $u : J \rightarrow \mathbb{R}^n$ , then  $\text{TV}(u)$  stands for the total variation of  $u$  on  $J$ , while  $\text{TV}(u; J')$  is the total variation of  $u$  on

a subinterval  $J' \subseteq J$ , refer to Section 4 for further information on  $\mathbf{BV}$  functions. As it is usual in the present context, we regard  $\mathbf{BV}$  as a subset of  $\mathbf{L}^\infty$ , so that  $\mathbf{BV}$  functions need not be in  $\mathbf{L}^1$ . When a  $\mathbf{BV}$  function is considered, we refer to its right continuous representative, and, where appropriate, we adopt the usual notation  $u(x_\pm) = \lim_{\xi \rightarrow x_\pm} u(\xi)$ .

We now deal with an analytic framework where the models described in Sections 3.1 and 3.2 can be settled, their well posedness proved, and the corresponding optimization problems tackled. Indeed, the models in Section 3 all fit in the general IBVP (1.1)–(1.2) while the quantities to be optimized can be written as in (1.3).

Our assumptions on (1.1)–(1.2) are the following, for fixed values of the positive (dimensional) constants  $F_1, F_\infty, G_1, G_\infty, M, W$  and  $\check{g}, \hat{g}$ , with  $\check{g} < \hat{g}$ :

- (b)  $\alpha, \beta \in \mathbf{C}^{0,1}(I \times \mathbb{R}^n; \mathbb{R}^n)$ ,  $w \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}^{n \times n}))$  with  $\|w(t, x)\| \leq W$  and there exists  $\xi > 0$  such that, for all  $t \in I$  and  $x \geq \xi$ ,  $\alpha(t, 0) = \beta(t, 0) = w^i(t, x) = 0$ .
- (f)  $f \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}))$  and, for all  $t \in I$ ,  $\begin{cases} \|f(t, \cdot)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} & \leq F_1; \\ \|f(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + TV(f(t, \cdot)) & \leq F_\infty. \end{cases}$
- (g)  $g \in \mathbf{C}^{0,1}(I \times \mathbb{R}_+; [\check{g}, \hat{g}])$ , for  $(t, x) \in I \times \mathbb{R}_+$   $\begin{cases} TV(g(t, \cdot)) + TV(g(\cdot, x); I) & \leq G_\infty; \\ \|\partial_x g(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + TV(\partial_x g(t, \cdot)) & \leq G_1. \end{cases}$
- (m)  $m \in \mathbf{C}^0(I \times \mathbb{R}_+; \mathbb{R})$  and, for all  $t \in I$ ,  $\|m(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + TV(m(t, \cdot)) \leq M$ .

We provide here the basic statements ensuring the well posedness and the stability of (1.1)–(1.2), by which we mean the existence of a solution, its uniqueness, and continuous dependence both with respect to the initial datum and to the terms appearing in the equation. Below, *solution to (1.1)* is understood in the following sense, see also Colombo and Garavello<sup>5</sup> or the more general Bardos et al<sup>15</sup> and Rossi.<sup>16</sup>

**Definition 2.1.** Assume that (f), (g), (m), and (b) hold. Choose an initial datum  $u_o \in \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ . The function  $u \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))$ , with  $u \equiv (u_1, \dots, u_n)$ , is a *solution to (1.1)–(1.2)* if,  $u(t) \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}^n)$  for all  $t \in I$  and setting

$$b_i(t) = B_i(t, u_1(t), \dots, u_n(t))$$

for  $i \in \{1, \dots, n\}$ , we have

1. for all  $i \in \{1, \dots, n\}$  and for all  $\varphi \in \mathbf{C}_c^1(\mathbb{R}^+; \mathbb{R})$ ,

$$\int_{\mathbb{R}_+} \int_I [u_i(t, x) \partial_t \varphi(t, x) + g_i(t, x) u_i(t, x) \partial_x \varphi(t, x) - m_i(t, x) u_i(t, x) \varphi(t, x) + f_i(t, x) \varphi(t, x)] dt dx = 0;$$

2.  $u(0, x) = u_o(x)$  for a.e.  $x \in \mathbb{R}_+$ ;
3. for every  $i \in \{1, \dots, n\}$  and for a.e.  $t \in I$ ,  $\lim_{x \rightarrow 0^+} g_i(t, x) u_i(t, x) = b_i(t)$ .

The following result ensures the well posedness of the IBVP for (1.1)–(1.2).

**Theorem 2.2.** Let  $n \in \mathbb{N} \setminus \{0\}$ ,  $\bar{x}_1, \dots, \bar{x}_n \in \mathbb{R}^+$ ,  $f_1, \dots, f_n$  satisfy (f),  $g_1, \dots, g_n$  satisfy (g),  $m_1, \dots, m_n$  satisfy (m), and the maps  $\alpha, \beta, w$ ; and satisfy (b). Then

1. For any  $u_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R}^n)$ , the problem (1.1)–(1.2) admits a unique solution in the sense of Definition 2.1.
2. There exists an increasing function  $\mathcal{K} \in \mathbf{C}^0(I; \mathbb{R}_+)$  dependent only on  $\mathbf{Lip}(\alpha), \mathbf{Lip}(\beta), W$ , on the norms and total variations of  $g$  and  $m$ , such that for any initial data  $u'_o, u''_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R}^n)$ , the corresponding solutions  $u'$  and  $u''$  satisfy

$$\|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \leq \mathcal{K}(t) \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}, \tag{2.1}$$

$$\|u'(t) - u''(t)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} \leq \mathcal{K}(t) \left( \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} + \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} \right). \tag{2.2}$$

3. If  $u_o = 0$  and  $f_i = 0$  for every  $i \in \{1, \dots, n\}$ , then the solution to (1.1) is  $u(t, x) = 0$  for all  $(t, x) \in I \times \mathbb{R}_+$ .

4. If the boundary functions and the initial data  $u'_0, u''_0 \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R}^n)$  are such that

$$\begin{aligned} \partial_{u_j} \alpha_i &\geq 0 && \text{for all } i, j = 1, \dots, n, \\ \partial_{\omega_j} \beta_i &\geq 0 && \text{for all } i, j = 1, \dots, n, \\ w_j^i &\geq 0 && \text{for all } i, j = 1, \dots, n, \\ (u'_0)_i &\geq (u''_0)_i && \text{for all } i = 1, \dots, n, \end{aligned}$$

then, the corresponding solutions satisfy  $u'_i(t, x) \geq u''_i(t, x)$  for all  $(t, x) \in I \times \mathbb{R}_+$  and  $i = 1, \dots, n$ . In particular, if  $(u'_0)_i \geq 0$  for  $i = 1, \dots, n$ , then  $u'_i(t, x) \geq 0$  for  $i = 1, \dots, n$ . (Here, we denote  $\partial_{\omega_j} \beta = \partial_{\omega_j} \beta(t, \omega_1, \dots, \omega_n)$ .)

The above result admits an immediate extension to the case  $u_0$  with possibly unbounded total variation. Indeed, in the case  $\alpha \equiv 0$ , a straightforward approximation argument based on (2.1) ensures the existence of a solution to (1.1)–(1.2). This argument works also in the case  $\alpha \neq 0$ ; however, a merely  $\mathbf{L}^1$  function can hardly be defined a *solution* to (1.1)–(1.2), due to the lack of meaning of punctual values such as  $u(t, \bar{x}_i)$ .

We now state the stability of solutions to (1.1) with respect to initial and boundary data, extending Theorem 2.5 by Colombo and Garavello.<sup>5</sup>

**Theorem 2.3.** *Let both systems*

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) + m'_i(t, x) u_i = f'_i(t, x) \\ g_i(t, 0) u_i(t, 0+) = B'_i(t, u_1(t), \dots, u_n(t)) \\ u_i(0, x) = u_i^0(x) \end{cases} \quad \begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) + m''_i(t, x) u_i = f''_i(t, x) \\ g_i(t, 0) u_i(t, 0+) = B''_i(t, u_1(t), \dots, u_n(t)) \\ u_i(0, x) = u_i^0(x) \end{cases} \quad (2.3)$$

with

$$\begin{aligned} B'_i(t, u_1, \dots, u_n) &= \alpha'_i(t, u_1(\bar{x}_1-), \dots, u_n(\bar{x}_n-)) \\ &\quad + \beta'_i \left( t, \int_{\mathbb{R}} w_1^i(t, x) u_1(x) dx, \dots, \int_{\mathbb{R}} w_n^i(t, x) u_n(x) dx \right), \\ B''_i(t, u_1, \dots, u_n) &= \alpha''_i(t, u_1(\bar{x}_1-), \dots, u_n(\bar{x}_n-)) \\ &\quad + \beta''_i \left( t, \int_{\mathbb{R}} w_1^i(t, x) u_1(x) dx, \dots, \int_{\mathbb{R}} w_n^i(t, x) u_n(x) dx \right) \end{aligned}$$

satisfy the assumptions of Theorem 2.2. Then, the corresponding solutions  $u'$  and  $u''$  satisfy

$$\begin{aligned} \|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} &\leq \mathcal{H}(t) \|\alpha' - \alpha''\|_{\mathbf{C}^0([0, t] \times \mathbb{R}^n; \mathbb{R}^n)} \\ &\quad + \mathcal{H}(t) \|\beta' - \beta''\|_{\mathbf{C}^0([0, t] \times \mathbb{R}^n; \mathbb{R}^n)} \\ &\quad + \mathcal{H}(t) \|w' - w''\|_{\mathbf{C}^0([0, t]; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^{n \times n}))} \\ &\quad + \mathcal{H}(t) \|m' - m''\|_{\mathbf{L}^1([0, t] \times \mathbb{R}_+; \mathbb{R}^n)} \\ &\quad + \mathcal{H}(t) \|f' - f''\|_{\mathbf{L}^1([0, t] \times \mathbb{R}_+; \mathbb{R}^n)}, \end{aligned} \quad (2.4)$$

where  $\mathcal{H} \in \mathbf{C}^0(I; \mathbb{R}_+)$  is such that  $\mathcal{H}(0) = 0$ .

The proofs, which extend those in Colombo and Garavello<sup>5</sup>, Theorem 2.4, proposition 2.6, are deferred to Section 4.

Observe that in view of the application of the above results to control problems, we need to relax the regularity assumption on  $f$  and  $m$  to

$$\mathbf{(F)} \quad f \in \mathbf{L}^1(I \times \mathbb{R}_+; \mathbb{R}) \text{ and for all } t \in I \begin{cases} \|f(t, \cdot)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} &\leq F_1; \\ \|f(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + \text{TV}(f(t, \cdot)) &\leq F_\infty. \end{cases}$$

$$\mathbf{(M)} \quad m \in \mathbf{L}^1(I \times \mathbb{R}_+; \mathbb{R}) \text{ and for all } t \in I, \|m(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + \text{TV}(m(t, \cdot)) \leq M.$$

Note also that it is possible to trivially extend Definition 2.1 with assumptions **(f)** and **(m)** replaced respectively by **(F)** and **(M)**.

**Corollary 2.4.** *Theorems 2.2 and 2.3 remain true if assumption **(f)** is replaced by **(F)** and assumption **(m)** is replaced by **(M)**.*

The proof is deferred to Section 4.

Once the Lipschitz estimate (2.4) is available, various techniques can be followed to actually search for the control parameters that optimize a Lipschitz cost of the type (1.3), see the next section for related examples. We refer, for instance, to Jones et al<sup>17</sup> and Malherbe and Vayatis<sup>18</sup> for samples of algorithms devoted to the optimization of Lipschitz functionals.

### 3 | APPLICATIONS

The present construction, based on Theorems 2.2 and 2.3, comprehends and extends previous results found in the literature. Indeed, a variety of models fit in (1.1), such as the one devoted to the management of renewable resources in.<sup>5,7,8</sup> We refer also to other studies<sup>1,6,19,20</sup> for other situations fitting into (1.1)–(1.2). Further examples also comprised in (1.1)–(1.2) are found, for instance, in Perthame.<sup>2, Chapter 3</sup>

We consider below in detail applications taken from the more applied literature that specifically take advantage of the extensions introduced in the present paper.

#### 3.1 | Cell growth and cancer control

Consider the following model for the evolution of cancer cells developed in Billy et al<sup>20</sup>:

$$\left\{ \begin{array}{l} \partial_t \rho_1^h + \partial_a \rho_1^h + d_1^h(t, a) \rho_1^h + K_{1 \rightarrow 2}^h(t, a) \rho_1^h = 0 \\ \partial_t \rho_2^h + \partial_a \rho_2^h + d_2^h(t, a) \rho_2^h + K_{2 \rightarrow 1}^h(t, a) \rho_2^h = 0 \\ \partial_t \rho_1^c + \partial_a \rho_1^c + d_1^c(t, a) \rho_1^c + K_{1 \rightarrow 2}^c(t, a) \rho_1^c = 0 \\ \partial_t \rho_2^c + \partial_a \rho_2^c + d_2^c(t, a) \rho_2^c + K_{2 \rightarrow 1}^c(t, a) \rho_2^c = 0 \\ \rho_1^h(t, 0+) = 2 \int_{\mathbb{R}_+} K_{2 \rightarrow 1}^h(t, a) \rho_2^h(t, a) da \\ \rho_2^h(t, 0+) = \int_{\mathbb{R}_+} K_{1 \rightarrow 2}^h(t, a) \rho_1^h(t, a) da \\ \rho_1^c(t, 0+) = 2 \int_{\mathbb{R}_+} K_{2 \rightarrow 1}^c(t, a) \rho_2^c(t, a) da \\ \rho_2^c(t, 0+) = \int_{\mathbb{R}_+} K_{1 \rightarrow 2}^c(t, a) \rho_1^c(t, a) da \end{array} \right. , \quad (3.1)$$

where  $\rho_i^h = \rho_i^h(t, a)$  is the density of healthy cells in stage  $i$  at time  $t$  of age  $a$ , with  $i = 1, 2$ , while  $\rho_i^c = \rho_i^c(t, a)$  is the analogous density for cancer cells. The functions  $K_{i \rightarrow j}^h, K_{i \rightarrow j}^c$  quantify how many cells pass from stage  $i$  to stage  $j$ , with  $\{i, j\} = \{1, 2\}$ . Note that in passing from Stage 2 to Stage 1, meiosis doubles the number of cells, accounting for the Factors 2 appearing in the corresponding right-hand sides. Finally,  $d_i^h = d_i^h(t, a)$  describes the mortality of the healthy cells at stage  $i$ , and similarly for  $d_i^c$ .

The use of a drug enters (3.1) modifying the transformation of cells, that is,

$$\begin{array}{ll} K_{1 \rightarrow 2}^h(t, a) = \kappa_1(a) \psi_1^h(t) & K_{1 \rightarrow 2}^c(t, a) = \kappa_1(a) \psi_1^c(t) \\ K_{2 \rightarrow 1}^h(t, a) = \kappa_2(a) \psi_2^h(t) (1 - G(t)) & K_{2 \rightarrow 1}^c(t, a) = \kappa_2(a) \psi_2^c(t) (1 - G(t)), \end{array} \quad (3.2)$$

where  $\kappa_i$  is the transition rate in the type  $i$  cells without circadian control;  $\psi_i^h, \psi_i^c$  are the effects of the natural 24-h-periodic circadian cycle and  $G$  is the control at the cell level of the drug infusion, which we also assume to be 24-h-periodic.

The transition functions for healthy and cancer cells  $\psi_i^h$  and  $\psi_i^c$  are provided by experimental measurements, see for instance the green lines in Billy et al,<sup>6, figs 9 and 10</sup> respectively.

**Proposition 3.1.** *Let  $d_1^h, d_2^h, d_1^c, d_2^c$  satisfy **(M)**. Choose  $\kappa_1$  and  $\kappa_2$  in  $\mathbf{BV}(\mathbb{R}_+; \mathbb{R})$  such that for a given  $\xi \in \mathbb{R}$ ,  $\kappa_1(a) = \kappa_2(a) = 0$  for all  $a > \xi$ . Let  $\psi_1^h, \psi_2^h, \psi_1^c, \psi_2^c \in \mathbf{C}^0(I; \mathbb{R})$  be bounded and  $G \in \mathbf{C}^0(I; [0, 1])$ . Then, with the definitions (3.2) and setting*

$$\begin{array}{l} n = 4, \quad g_i = 1, \quad f_i = 0, \quad \alpha_i = 0 \text{ for } i = 1, \dots, 4; \quad z = (z_1, z_2, z_2, z_4), \\ u_1 = \rho_1^h \quad m_1(t, a) = d_1^h(t, a) + K_{1 \rightarrow 2}^h(t, a) \quad w_1^1(t, a) = 2K_{2 \rightarrow 1}^h(t, a) \quad \beta_1(t, z) = z_2 \\ u_2 = \rho_2^h \quad m_2(t, a) = d_2^h(t, a) + K_{2 \rightarrow 1}^h(t, a) \quad w_1^2(t, a) = K_{1 \rightarrow 2}^h(t, a) \quad \beta_2(t, z) = z_1 \\ u_3 = \rho_1^c \quad m_3(t, a) = d_1^c(t, a) + K_{1 \rightarrow 2}^c(t, a) \quad w_2^3(t, a) = 2K_{2 \rightarrow 1}^c(t, a) \quad \beta_3(t, z) = z_2 \\ u_4 = \rho_2^c \quad m_4(t, a) = d_2^c(t, a) + K_{2 \rightarrow 1}^c(t, a) \quad w_1^4(t, a) = K_{1 \rightarrow 2}^c(t, a) \quad \beta_4(t, z) = z_1 \end{array} \quad (3.3)$$

with all other  $w_j^i$  being set to 0, system (3.1) fits into (1.1)–(1.2) and satisfies **(b)**, **(f)**, **(g)**, and **(m)** so that Theorems 2.2 and 2.3 can be applied.

Once (3.2) and (3.3) are given, verifying that **(b)**, **(f)**, **(g)**, and **(m)** hold is immediate and, hence, omitted. For completeness, we recall that Clairambault et al,<sup>9, Theorem 3.1</sup> at least in the autonomous case, covers the well posedness of (3.1). The proof therein is obtained by means of an entirely different technique, relying on the *adjoint equation*, see Perthame,<sup>2, section 3.2</sup>

Note that the assumption that  $\kappa_1$  and  $\kappa_2$  vanish for  $a > \xi$  is obviously satisfied in any real situation, since it is clearly consistent with the finite life span of cells.

Our task is to choose the function  $G$  so that the target functional defined by

$$\mathcal{J}(G) = \frac{\int_{\mathbb{R}_+} \rho_1^h(T, x) dx + \int_{\mathbb{R}_+} \rho_2^h(T, x) dx}{\int_{\mathbb{R}_+} \rho_1^c(T, x) dx + \int_{\mathbb{R}_+} \rho_2^c(T, x) dx} \quad (3.4)$$

is maximal for a given  $T$ . For instance, in Billy et al,<sup>6</sup> the choice  $T = 12$  days was adopted.

An application of Corollary 2.4, in particular of the estimate (2.4), ensures that the variation in the cost (3.4) due to different drug infusion schedules  $G_1$  and  $G_2$  can be estimated by

$$\begin{aligned} |\mathcal{J}(G_1) - \mathcal{J}(G_2)| &= \mathcal{O}(1) \left( \|G_1 - G_2\|_{L^1([0, T]; \mathbb{R})} + \|G_1 - G_2\|_{C^0([0, T]; \mathbb{R})} \right) \\ &= \mathcal{O}(1) \|G_1 - G_2\|_{C^0([0, T]; \mathbb{R})} \end{aligned}$$

for a suitable constant  $\mathcal{O}(1)$  depending on  $\mathcal{H}(T)$  in (2.4), on  $T$  and on norms of  $\kappa_1$ ,  $\kappa_2$ ,  $\psi_1^h$ ,  $\psi_2^h$ ,  $\psi_1^c$ ,  $\psi_2^c$ . Therefore, any strongly convergent minimizing sequence of drug infusion schedules actually yields a minimal cost.

### 3.2 | Age-structured population economics

The model introduced in Feichtinger et al<sup>21</sup> fits into the form

$$\begin{cases} \partial_t N + \partial_a N + \mu(t, a)N = v(t) v_o(a) \\ \partial_t K + \partial_a K + \delta(t, a)K = 0 \\ N(t, 0+) = \int_{\mathbb{R}_+} f(t, a)N(t, a) da \\ K(t, 0+) = \mathcal{F} \left( t, \int_{\mathbb{R}_+} p(t, a)N(t, a) da, \int_{\mathbb{R}_+} q(t, a)K(t, a) da \right) \end{cases}, \quad (3.5)$$

where  $N = N(t, a)$  is the number of individuals of age  $a$  at time  $t$ ,  $K = K(t, a)$  is the physical capital stock,  $f = f(t, a)$  is the fertility rate at time  $t$  of individuals of age  $a$ , so that the integral  $\int_{\mathbb{R}_+} f(t, a)N(t, a) da$  measures the amount of newborn individuals. The term  $\mathcal{F}$  is the investment in capital goods, the dependence of  $\mathcal{F}$  on  $K$  and  $N$  being nonlocal, with assigned weights  $p$  and  $q$ . The functions  $\mu = \mu(t, a)$  is the mortality rate and  $\delta = \delta(t, a)$  is the depreciation rate of the physical capital. The term  $v(t) v_o(a)$  is the (positive) net migration, with  $v_o(a)$  assigned while  $v(t)$  acts as a control parameter, see Feichtinger et al.<sup>21, Section 2</sup>

We now show that the model (3.5) fits in (1.1)–(1.2).

**Proposition 3.2.** Choose functions  $v \in L^1(I; \mathbb{R})$ ,  $v_o \in (L^1 \cap BV)(\mathbb{R}; \mathbb{R})$ ,  $\mu, \delta$  that satisfy **(M)**,  $p, q, f \in C^0(I; L^1(\mathbb{R}_+; \mathbb{R}))$ , with  $p(t, a) = q(t, a) = f(t, a) = 0$  for all  $t$  and all  $a > \xi$ , for a given  $\xi$ , and  $\mathcal{F} \in C^{0,1}(I \times \mathbb{R}^2; \mathbb{R})$ . Set  $n = 2$  and define

$$\begin{aligned} u_1 &= N & g_1(t, a) &= 1 & f_1(t, a) &= v(t) v_o(a) & m_1(t, a) &= \mu(t, a) \\ u_2 &= K & g_2(t, a) &= 1 & f_2(t, a) &= 0 & m_2(t, a) &= \delta(t, a) \\ \alpha_1 &= 0 & \beta_1(t, z) &= z_1 & w_1^1(t, a) &= f(t, a) & w_1^2(t, a) &= 0 \\ \alpha_2 &= 0 & \beta_2(t, z) &= \mathcal{F}(t, z) & w_2^1(t, a) &= p(t, a) & w_2^2(t, a) &= q(t, a) \end{aligned} \quad (3.6)$$

Then, Corollary 2.4 applies to (3.5).

Following Feichtinger et al,<sup>21</sup> the function  $v$  is chosen to maximize the social welfare, measured through a functional of the general type

$$\mathcal{J}(v) = \int_0^T \mathcal{W}(t, N(t, \cdot), K(t, \cdot)) dt. \quad (3.7)$$

Here,  $\mathcal{W}$  is related to the instantaneous welfare and depends on the functions  $a \rightarrow N(t, a)$  and  $a \rightarrow K(t, a)$ , defined for  $a \in \mathbb{R}_+$ . Refer to Feichtinger et al<sup>21</sup> for a more detailed expression of the cost functional. Obvious regularity assumptions on  $\mathcal{W}$  ensure the strong  $\mathbf{L}^1$  continuity of  $\mathcal{J}$ , thanks to Corollary 2.4. Once Lipschitz regularity is available, various procedures to actually find optimal controls are outlined in the literature, see for instance Jones et al<sup>17</sup> and Malherbe and Vayatis.<sup>18</sup> In specific applications, where  $v$  depends on a finite number of parameters, this procedure also ensures the existence of an optimal control.

### 3.3 | A juvenile–adult model with metamorphosis

Consider a species where juveniles ( $J$ ) develop into adults ( $A$ ) through a metamorphosis at age  $a = a_{\max}$ . Calling as usual  $g$ , the growth function and  $\mu_J$ , respectively  $\mu_A$ , the juvenile, respectively adult, mortality, both time and age dependent, we are lead to the following system, which was introduced in Ackleh and Deng<sup>1, Formula (2.1)</sup>:

$$\begin{cases} \partial_t J + \partial_a J + \mu_J(t, a)J = 0 & (t, a) \in \mathbb{R}_+ \times [0, a_{\max}] \\ \partial_t A + \partial_x (g(t, a)A) + \mu_A(t, x)A = 0 & (t, a) \in \mathbb{R}_+ \times [a_{\min}, a_{\max}] \\ J(t, 0) = \int_{a_{\min}}^{a_{\max}} \beta(t, \alpha)A(t, \alpha) d\alpha & t \in \mathbb{R}_+ \\ g(t, a_{\min})A(t, a_{\min}) = J(t, a_{\max}) & t \in \mathbb{R}_+ \\ J(0, a) = J_0(a) & a \in [0, a_{\max}] \\ A(0, a) = A_0(a) & a \in [a_{\min}, a_{\max}]. \end{cases} \quad (3.8)$$

The general form of (1.1)–(1.2) also comprises (3.8). Indeed, problem (3.8) fits into (1.1)–(1.2) setting  $n = 2$ ,  $x = a$ , and

$$\begin{aligned} u_1(t, x) &= J(t, a) & g_1(t, x) &= 1 & \alpha_1(t, u_1, u_2) &= 0 & f_1 &\equiv 0 \\ u_2(t, x) &= A(t, a + a_{\min}) & g_2(t, x) &= g(t, a + a_{\min}) & \alpha_2(t, u_1, u_2) &= u_1 & f_2 &\equiv 0 \\ w_1^2(t, x) &= 0 & m_1(t, x) &= \mu_J(t, a) & \beta_1(t, U_1, U_2) &= U_2 & \bar{x}_1 &= a_{\max} \\ w_2^2(t, x) &= \beta(t, a) \chi_I(a) & m_2(t, x) &= \mu_A(t, a + a_{\min}) & \beta_2(t, U_1, U_2) &= 0 & \bar{x}_2 &= 0 \end{aligned} \quad (3.9)$$

where  $I = [0, a_{\max} - a_{\min}]$ . Both Theorems 2.2 and 2.3 then apply and ensure the well posedness and the stability of (3.8) under assumptions slightly different from those in Ackleh and Deng,<sup>1</sup> simplifying the result in to Colombo and Garavello,<sup>5, section 3.1</sup> to which we refer for the details.

### 3.4 | Optimal control in biological resources' management

An extension of the above system (3.8) allows to model an economic/industrial exploitation of a biological resource. Assume that at age  $a = \bar{a}$ , juveniles ( $J$ ) are selected into those that are going to be used for reproduction ( $R$ ) and those that are bred to be sold ( $S$ ). Along the lines of Colombo and Garavello,<sup>5, section 3.3</sup> one is thus lead to the following model:

$$\begin{cases} \partial_t J + \partial_a (g_J(t, a)J) + \mu_J(t, a)J = 0 & (t, a) \in \mathbb{R}_+ \times [0, \bar{a}] \\ \partial_t S + \partial_a (g_S(t, a)S) + \mu_S(t, a)S = 0 & (t, a) \in \mathbb{R}_+ \times [\bar{a}, +\infty[ \\ \partial_t R + \partial_a (g_R(t, a)R) + \mu_R(t, a)R = 0 & (t, a) \in \mathbb{R}_+ \times [\bar{a}, +\infty[ \\ g_J(t, 0)J(t, 0) = \beta \left( \int_{\bar{a}}^{a_{\max}} R(t, x) dx \right) & t \in \mathbb{R}_+ \\ g_S(t, \bar{a})S(t, \bar{a}) = \eta(t)g_J(t, \bar{a})J(t, \bar{a}) & t \in \mathbb{R}_+ \\ g_R(t, \bar{a})R(t, \bar{a}) = (1 - \eta(t))g_J(t, \bar{a})J(t, \bar{a}) & t \in \mathbb{R}_+ \\ J(0, a) = J_0(a) & a \in [0, \bar{a}] \\ S(0, a) = S_0(a) & a \in [\bar{a}, +\infty[ \\ R(0, a) = R_0(a) & a \in [\bar{a}, +\infty[ \end{cases} \quad (3.10)$$

where we used essentially the same notation as in the preceding Section 3.1. The percentage of juveniles selected for the market is quantified by the, here time dependent, parameter  $\eta = \eta(t)$  selected in  $[0, 1]$ .

System (3.10) fits into (1.1)–(1.2) setting  $n = 3$ ,  $x = a$ ,  $U \equiv (U_1, U_2, U_3)$  and

$$\begin{aligned}
 u_1(t, x) &= J(t, a) & g_1(t, x) &= g_J(t, a) & \alpha_1(t, U) &= 0 \\
 u_2(t, x) &= S(t, a + \bar{a}) & g_2(t, x) &= g_S(t, a + \bar{a}) & \alpha_2(t, U) &= \eta(t) U_1 g_1(t, \bar{a}) \\
 u_3(t, x) &= R(t, a + \bar{a}) & g_3(t, x) &= g_R(t, a + \bar{a}) & \alpha_3(t, U) &= (1 - \eta(t)) U_1 g_1(t, \bar{a}) \\
 \bar{x}_1 &= \bar{a} & m_1(t, x) &= \mu_J(t, a) & \beta_1(t, U) &= \beta(U_3) \\
 w_1^1 &= w_2^1 \equiv 0 & m_2(t, x) &= \mu_S(t, a + \bar{a}) & \beta_2(t, U) &= 0 \\
 w_3^1(t, x) &= \chi_{[\bar{a}, a_{\max}]}(x) & m_3(t, x) &= \mu_R(t, x + \bar{a}) & \beta_3(t, U) &= 0
 \end{aligned} \tag{3.11}$$

The same remarks in Colombo and Garavello,<sup>5</sup> but based on the present extension provided by Theorems 2.2 and 2.3, ensure the well posedness of (3.10) and its stability with respect to boundary data and parameters. Note in particular that this opens the way to discussing optimal control problems where  $\eta$  has to be chosen to maximize a suitable functional modeling the long-term income due to selling the selected  $S$  individuals.

## 4 | TECHNICAL DETAILS

This section contains the technical details, both proofs of Theorems 2.2 and 2.3, and the proof of Corollary 2.4.

### 4.1 | Elementary estimates on BV functions

We now recall elementary estimates on **BV** functions, see also Colombo and Garavello,<sup>5</sup> section 4 22, section 4.2

$$\left. \begin{array}{l} u \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}) \\ w \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}) \end{array} \right\} \Rightarrow \text{TV}(uw) \leq \text{TV}(u) \|w\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + \|u\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} \text{TV}(w); \tag{4.1}$$

$$\left. \begin{array}{l} \varphi \in \mathbf{C}^{0,1}(\mathbb{R}^n; \mathbb{R}) \\ u \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}^n) \end{array} \right\} \Rightarrow \text{TV}(\varphi \circ u) \leq \mathbf{Lip}(\varphi) \text{TV}(u); \tag{4.2}$$

$$\left. \begin{array}{l} u \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}) \\ w \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}) \\ w(x) \geq \check{w} > 0 \end{array} \right\} \Rightarrow \text{TV}\left(\frac{u}{w}\right) \leq \frac{1}{\check{w}} \text{TV}(u) + \frac{1}{\check{w}^2} \text{TV}(w) \|u\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})}; \tag{4.3}$$

$$\left. \begin{array}{l} u \in \mathbf{L}^1(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R})) \\ u(t) \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}) \end{array} \right\} \Rightarrow \text{TV}\left(\int_0^t u(\tau, \cdot) d\tau\right) \leq \int_0^t \text{TV}(u(\tau)) d\tau; \tag{4.4}$$

$$\left. \begin{array}{l} u \in \mathbf{BV}(\mathbb{R}_+; \mathbb{R}) \\ \delta \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}_+) \end{array} \right\} \Rightarrow \int_{\mathbb{R}_+} |u(x + \delta(x)) - u(x)| dx \leq \text{TV}(u) \|\delta\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})}; \tag{4.5}$$

$$\left. \begin{array}{l} u \in \mathbf{L}^\infty(I \times I; \mathbb{R}) \\ \sup_{t \in I} \text{TV}(u(t, \cdot; I)) < \infty \end{array} \right\} \Rightarrow \text{TV}\left(\int_0^\cdot u(t, \cdot) dt\right) \leq \|u\|_{\mathbf{L}^\infty(I \times I; \mathbb{R})} l_* + \int_I \text{TV}(u(t, \cdot; I)) dt. \tag{4.6}$$

Inequality (4.1) follows from Ambrosio et al.<sup>23</sup>, Formula (3.10) The definition of total variation directly implies (4.2), (4.3), and (4.4). For a proof of (4.5) see for instance Bressan.<sup>24</sup>, Lemma 2.3 The integral estimate (4.6) is proved in Colombo and Garavello,<sup>22</sup>, Lemma 1

### 4.2 | Characteristic curves

If  $g$  satisfies **(g)**, we introduce the globally defined maps, see Bressan and Piccoli,<sup>25</sup> Chapter 3

$$\begin{aligned}
 t &\rightarrow X(t; t_0, x_0) \text{ that solves } \begin{cases} \dot{x} = g(t, x) \\ x(t_0) = x_0, \end{cases} & \text{and } \begin{cases} \gamma(t) = X(t; 0, 0), \\ \Gamma(x) = T(x; 0, 0). \end{cases} \\
 x &\rightarrow T(x; t_0, x_0) \text{ that solves } \begin{cases} t' = \frac{1}{g(t, x)} \\ t(x_0) = t_0, \end{cases}
 \end{aligned} \tag{4.7}$$

Note that  $x \geq \gamma(t)$  if and only if  $X(0; t, x) \in [0, x]$ , while  $x < \gamma(t)$  if and only if  $T(0; t, x) \in [0, t]$ .

We collect here results about the differentiability of the maps (4.7), deduced by classical results about ordinary differential equations.

**Lemma 4.1.** *Let  $g$  satisfy **(g)**. With the notation (4.7), for all  $t, t_0 \in I$  and  $x, x_0 \in \mathbb{R}_+$ , the following relations hold:*

$$\partial_t X(t; t_0, x_0) = g(t, X(t; t_0, x_0)) \quad \partial_x T(x; t_0, x_0) = \frac{1}{g(T(x; t_0, x_0), x)}, \tag{4.8}$$

$$TV(T(0; t, \cdot); [0, \gamma(t)]) = t. \tag{4.9}$$

If moreover  $g$  is of class  $\mathcal{C}^1$ ,

$$\begin{aligned} \partial_{x_0} X(t; t_0, x_0) &= \exp \int_{t_0}^t \partial_x g(\tau, X(\tau; t_0, x_0)) \, d\tau \\ \partial_{t_0} T(x; t_0, x_0) &= \frac{g(t_0, x_0)}{g(T(x; t_0, x_0), x)} \exp \left( \int_{t_0}^{T(x; t_0, x_0)} \partial_x g(\tau, X(\tau; t_0, x_0)) \, d\tau \right) \\ \partial_{t_0} X(t; t_0, x_0) &= -g(t_0, x_0) \exp \int_{t_0}^t \partial_x g(\tau, X(\tau; t_0, x_0)) \, d\tau \\ \partial_{x_0} T(x; t_0, x_0) &= -\frac{1}{g(T(x; t_0, x_0), x)} \exp \int_{t_0}^{T(x; t_0, x_0)} \partial_x g(s, X(s; t_0, x_0)) \, ds. \end{aligned} \tag{4.10}$$

*Proof.* The relations (4.8) and (4.10) are classical, see for instance Hartman.<sup>26</sup> Chapter 5, Section 3 The equality (4.9) follows from the monotonicity of  $x \rightarrow T(0; t, x)$ .  $\square$

### 4.3 | The scalar renewal equation

We consider the following IBVP for a linear nonhomogeneous scalar balance law, or *renewal equation*, see also Perthame<sup>2</sup>, Chapter 3:

$$\begin{cases} \partial_t u + \partial_x (g(t, x)u) + m(t, x)u = f(t, x) & (t, x) \in I \times \mathbb{R}_+ \\ u(0, x) = u_0(x) & x \in \mathbb{R}_+ \\ g(t, 0)u(t, 0+) = b(t) & t \in I \end{cases} \tag{4.11}$$

under the assumptions **(f)**, **(g)**, and **(m)** from Section 2, together with  $b \in \mathbf{BV}_{\text{loc}}(I; \mathbb{R})$ .

There is a wide literature on (4.11), we refer in particular to Colombo and Garavello<sup>22</sup>, Definition 2; see also other studies.<sup>2,15,16,24,27,28</sup> Recall the expression of the solution, based on the notation (4.7):

$$u(t, x) = \begin{cases} u_0(X(0; t, x)) \mathcal{E}(0, t, x) + \int_0^t f(\tau, X(\tau; t, x)) \mathcal{E}(\tau, t, x) \, d\tau & x > \gamma(t) \\ \frac{b(T(0; t, x))}{g(T(0; t, x), 0)} \mathcal{E}(T(0; t, x), t, x) + \int_{T(0; t, x)}^t f(\tau, X(\tau; t, x)) \mathcal{E}(\tau, t, x) \, d\tau & x < \gamma(t), \end{cases} \tag{4.12}$$

where

$$\mathcal{E}(\tau, t, x) = \exp \left[ - \int_{\tau}^t (m(s, X(s; t, x)) + \partial_x g(s, X(s; t, x))) \, ds \right]. \tag{4.13}$$

**Lemma 4.2** (<sup>22</sup>, Lemma 2). *Let **(g)** and **(m)** hold. Then,  $\mathcal{E}$  defined in (4.13) satisfies the following estimates, for  $x \in \mathbb{R}_+$  and  $\tau, t \in I$  with  $\tau \leq t$ :*

$$\mathcal{E}(\tau, t, x) \leq e^{(G_1+M)(t-\tau)}, \tag{4.14}$$

$$TV(\mathcal{E}(\tau, t, \cdot); \mathbb{R}_+) \leq (G_1 + M)(t - \tau) e^{(G_1+M)(t-\tau)}, \tag{4.15}$$

$$TV(\mathcal{E}(\tau, \cdot, x); [0, t]) \leq (G_1 + M)(t - \tau) e^{(G_1+M)(t-\tau)}, \tag{4.16}$$

$$TV(\mathcal{E}(\cdot, t, x); [0, t]) \leq (G_1 + M) t e^{(G_1+M)t}. \tag{4.17}$$

The following Lemma summarizes various properties of the solution to (4.11), see also Perthame.<sup>2</sup>

**Lemma 4.3** (<sup>22</sup>, Lemma 3). Let  $b \in \mathbf{BV}_{loc}(I; \mathbb{R})$ ,  $(\mathbf{f})$ ,  $(\mathbf{g})$ , and  $(\mathbf{m})$  hold and choose  $u_0 \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R})$ . Then, with reference to the scalar problem (4.11),

(SP.1) The map  $u : I \times \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by (4.12) solves (4.11) in the sense of Colombo and Garavello<sup>22</sup>, Definition 2:

- (a) for all  $\varphi \in \mathbf{C}_c^1(I \times \mathbb{R}_+; \mathbb{R})$ ,  $\int_{\mathbb{R}_+} \int_I [u \partial_t \varphi + g u \partial_x \varphi + (f - m u) \varphi] dt dx = 0$ ;
- (b)  $u(0, x) = u_0(x)$  for a.e.  $x \in \mathbb{R}_+$ ;
- (c) for a.e.  $t \in I$ ,  $\lim_{x \rightarrow 0^+} g(t, x)u(t, x) = b(t)$ .

(SP.2) For every  $t \in I$ , the following a-priori estimates hold:

$$\begin{aligned} \sup_{\tau \in [0, t]} \|u(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} &\leq \left( \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + \frac{1}{\check{g}} \|b\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} + F_\infty t \right) e^{(G_1 + M)t}, \\ \sup_{\tau \in [0, t]} \|u(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} &\leq (\|u_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + \|b\|_{\mathbf{L}^1([0, t]; \mathbb{R})} + F_1 t) e^{Mt}. \end{aligned}$$

(SP.3) For every  $t \in I$ , the following total variation estimate holds:

$$TV(u(t); \mathbb{R}_+) \leq H(t) \left( F_\infty t + \frac{\|b\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} + TV(b; [0, t])}{\check{g}} + \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + TV(u_0; \mathbb{R}_+) \right), \quad (4.18)$$

where  $H(t)$  is a nondecreasing continuous function of  $t$ , depending also on  $\check{g}$ ,  $G_1$ ,  $G_\infty$  and  $M$ , satisfying  $H(0) \leq 5 + G_\infty/\check{g}$ .

(SP.4) Fix  $t \in I$  and  $x \in \mathbb{R}_+$ . If  $x > \gamma(t)$ , then

$$\begin{aligned} TV(u(\cdot, x); [0, t]) &\leq [TV(u_0; \mathbb{R}_+) + 2(G_1 + M) \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} t] e^{(G_1 + M)t} \\ &\quad + 4 [1 + (G_1 + M)t] F_\infty t e^{(G_1 + M)t}. \end{aligned} \quad (4.19)$$

If  $x < \gamma(t)$ , then

$$\begin{aligned} TV(u(\cdot, x); [0, t]) &\leq \left[ TV(u_0; \mathbb{R}_+) + \frac{1}{\check{g}} TV(b(\cdot); [0, t]) \right] e^{(G_1 + M)t} \\ &\quad + 2 [1 + (G_1 + M)t] \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} e^{(G_1 + M)t} \\ &\quad + \frac{1}{\check{g}} \left[ 2 + 3(G_1 + M)t + \frac{G_\infty}{\check{g}} \right] \|b\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} e^{(G_1 + M)t} \\ &\quad + 2(7 + 6(G_1 + M)t) F_\infty t e^{(G_1 + M)t}. \end{aligned} \quad (4.20)$$

(SP.5) Fix a positive  $W$ . For any  $w \in (\mathbf{C}^1 \cap \mathbf{BV})(I; [-W, W])$ ,

$$\begin{aligned} TV \left( \int_{\mathbb{R}_+} w(\cdot, x) u(\cdot, x) dx; [0, t] \right) &\leq \|u\|_{\mathbf{L}^\infty([0, t] \times \mathbb{R}_+; \mathbb{R})} \int_{\mathbb{R}_+} TV(w(\cdot, x); [0, t]) dx \\ &\quad + W \int_{\mathbb{R}_+} TV(u(\cdot, x); [0, t]) dx \end{aligned}$$

(SP.6) For every  $t \in I$ , there exists a positive  $\mathcal{L}$  dependent on  $\|u_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})}$  and on the constants in  $(\mathbf{f})$ ,  $(\mathbf{g})$ ,  $(\mathbf{m})$ , and  $b \in \mathbf{BV}_{loc}(I; \mathbb{R})$ , such that, for  $t', t'' \in [0, t]$ ,

$$\|u(t') - u(t'')\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \leq \mathcal{L} |t'' - t'|. \quad (4.21)$$

(SP.7) If  $u_0 \geq 0$ ,  $f \geq 0$  and  $b \geq 0$ , then  $u(t) \geq 0$  for all  $t$ .

The next result deals with the stability properties of (4.11).

**Lemma 4.4** (22, Lemma 4). Let  $(\mathbf{g})$  hold. Fix data  $u'_0, u''_0 \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R})$ ,  $b', b'' \in \mathbf{BV}_{loc}(I; \mathbb{R})$ ,  $m', m''$  satisfying  $(\mathbf{m})$  and  $f', f''$  satisfying  $(\mathbf{f})$ . Call  $u'$  and  $u''$  the solutions to

$$\begin{cases} \partial_t u + \partial_x (g(t, x)u) + m'(t, x)u = f'(t, x) \\ u(0, x) = u'_0(x) \\ g(t, 0)u(t, 0+) = b'(t) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t u + \partial_x (g(t, x)u) + m''(t, x)u = f''(t, x) \\ u(0, x) = u''_0(x) \\ g(t, 0)u(t, 0+) = b''(t). \end{cases} \quad (4.22)$$

Then

**(SP.8)** The following stability conditions hold:

$$\begin{aligned} & \|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \\ & \leq e^{Mt} \|u'_0 - u''_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + e^{2(G_1+M)t} \left( 2\|f' - f''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + \|b' - b''\|_{\mathbf{L}^1([0,t]; \mathbb{R})} \right) \\ & \quad + e^{(2G_1+M)t} \left[ \|u''_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + 2tF_1 + \|b''\|_{\mathbf{L}^1([0,t]; \mathbb{R})} \right] t \|m' - m''\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}_+; \mathbb{R})}; \end{aligned} \quad (4.23)$$

$$\begin{aligned} & \|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \\ & \leq e^{Mt} \|u'_0 - u''_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + e^{2(G_1+M)t} \left( 2\|f' - f''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + \|b' - b''\|_{\mathbf{L}^1([0,t]; \mathbb{R})} \right) \\ & \quad + e^{(2G_1+M)t} \left[ \|u''_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + 2tF_\infty + \frac{\|b''\|_{\mathbf{L}^\infty([0,t]; \mathbb{R})}}{\check{g}} \right] \|m' - m''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})}. \end{aligned} \quad (4.24)$$

**(SP.9)** The following monotonicity property holds:

$$\left. \begin{array}{l} f'(t, x) \leq f''(t, x) \quad (t, x) \in I \times \mathbb{R}_+ \\ u'_0(x) \leq u''_0(x) \quad x \in \mathbb{R}_+ \\ b'(t) \leq b''(t) \quad t \in I \end{array} \right\} \Rightarrow u'(t, x) \leq u''(t, x) \quad \forall (t, x) \in I \times \mathbb{R}_+. \quad (4.25)$$

**(SP.10)** If  $\bar{x} > 0$  and  $\gamma(t) < \bar{x}$ , then

$$\begin{aligned} \|u'(\cdot, \bar{x}) - u''(\cdot, \bar{x})\|_{\mathbf{L}^1([0,t]; \mathbb{R})} & \leq e^{(G_1+M)t} \left[ e^{G_1 t} \|u'_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + t^2 F_\infty \right] \|m' - m''\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}_+; \mathbb{R})} \\ & \quad + e^{Mt} \|u'_0 - u''_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + e^{(2G_1+M)t} \|f' - f''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})}; \end{aligned} \quad (4.26)$$

$$\begin{aligned} \|u'(\cdot, \bar{x}) - u''(\cdot, \bar{x})\|_{\mathbf{L}^1([0,t]; \mathbb{R})} & \leq \frac{e^{(G_1+M)t}}{G_\infty} \left[ e^{G_1 t} \|u'_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + tF_\infty \right] \|m' - m''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\ & \quad + e^{Mt} \|u'_0 - u''_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + e^{(2G_1+M)t} \|f' - f''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})}. \end{aligned} \quad (4.27)$$

#### 4.4 | Different notions of solutions for (4.11)

We now deal with the different definitions of solutions available in the  $\mathbf{BV}_{loc}$  select (4.12) as solution to (4.11).

**Lemma 4.5.** Let  $b \in \mathbf{BV}_{loc}(I; \mathbb{R})$ ,  $(\mathbf{f})$ ,  $(\mathbf{g})$ , and  $(\mathbf{m})$  hold, and choose  $u_0 \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R})$ . Then, any two solutions to (4.11) in the sense of  $(\mathbf{a})$ ,  $(\mathbf{b})$ , and  $(\mathbf{c})$  in **(SP.1)** coincide.

*Proof.* Let  $u_1$  and  $u_2$  be two weak solutions to (4.11) and call  $w = u_2 - u_1$ . Then, by  $(\mathbf{a})$  in **(SP.1)**, for all  $\varphi \in \mathbf{C}_c^1(\overset{\circ}{I} \times \overset{\circ}{\mathbb{R}}^+; \mathbb{R})$ ,  $\int_{\mathbb{R}_+} \int_I [w \partial_t \varphi + g w \partial_x \varphi - m w \varphi] dt dx = 0$ .

**Step 1:** Fix  $\bar{t} \in \overset{\circ}{I}$  and  $x_1, x_2 \in \mathbb{R}_+$  with  $x_2 > x_1 \geq \gamma(\bar{t})$ . We prove that for all  $\vartheta \in \mathbf{C}_c^1(\mathbb{R}_+; \mathbb{R})$  with  $\text{spt} \vartheta \subseteq [x_1, x_2]$ , we have  $\int_{x_1}^{x_2} w(\bar{t}, x) \vartheta(x) dx = 0$ , assuming that  $m$  is of class  $\mathbf{C}^0$ .

Choose a test function  $\varphi(t, x) = \eta_\varepsilon(t)\psi(t, x)$ , where  $\psi \in \mathbf{C}_c^1(\overset{\circ}{I} \times \overset{\circ}{\mathbb{R}}^+; \mathbb{R})$  solves

$$\begin{cases} \partial_t \psi(t, x) + g(t, x) \partial_x \psi(t, x) = m(t, x) \psi(t, x) \\ \psi(\bar{t}, x) = \vartheta(x) \end{cases} \quad (t, x) \in [0, \bar{t}] \times \mathbb{R}_+,$$

while  $\eta_\varepsilon$  approximates the characteristic function of the time interval  $[0, \bar{t}]$ , that is,

$$\eta_\varepsilon(0) = 0, \quad \begin{aligned} \eta'_\varepsilon(t) &\in [0, 2/\varepsilon] & t \in [0, \varepsilon]; & \quad \eta_\varepsilon(t) = 1 & t \in [\varepsilon, \bar{t} - \varepsilon]; \\ \eta'_\varepsilon(t) &\in [-2/\varepsilon, 0] & t \in [\bar{t} - \varepsilon, \bar{t}]; & \quad \eta_\varepsilon(t) = 0 & t > \bar{t}. \end{aligned}$$

Choosing  $\varphi$  as test function, we have that for all positive and sufficiently small  $\varepsilon$ ,

$$\begin{aligned} 0 &= \int_I \eta_\varepsilon(t) \int_{\mathbb{R}_+} w(t, x) (\partial_t \psi(t, x) + g(t, x) \partial_x \psi(t, x) - m(t, x) \psi(t, x)) dx dt \\ &\quad + \int_0^\varepsilon \eta'_\varepsilon(t) \int_{\mathbb{R}_+} w(t, x) \psi(t, x) dx dt + \int_{\bar{t}-\varepsilon}^{\bar{t}} \eta'_\varepsilon(t) \int_{\mathbb{R}_+} w(t, x) \psi(t, x) dx dt \\ &= \int_0^1 \varepsilon \eta'_\varepsilon(\varepsilon s) \int_{\mathbb{R}_+} w(\varepsilon t, x) \psi(\varepsilon t, x) dx ds + \int_{\bar{t}-\varepsilon}^{\bar{t}} \eta'_\varepsilon(t) \int_{\mathbb{R}_+} w(t, x) \psi(t, x) dx dt \\ &\xrightarrow{\varepsilon \rightarrow 0} 0 + \int_{\mathbb{R}_+} w(\bar{t}, x) \vartheta(x) dx, \end{aligned}$$

where we used **(b)** in **(SP.1)**.

**Step 2:** Fix  $\bar{x} \in \overset{\circ}{\mathbb{R}}^+$  and  $t_1, t_2 \in I$  with  $t_2 > t_1 \geq \Gamma(\bar{x})$ . We now prove that for all  $\vartheta \in \mathbf{C}_c^1(\overset{\circ}{I}; \mathbb{R})$  with  $\text{spt} \vartheta \subseteq [t_1, t_2]$ , we have  $\int_{t_1}^{t_2} w(t, \bar{x}) \vartheta(t) dt = 0$ , assuming that  $m$  is of class  $\mathbf{C}^0$ .

Proceed as in the step above. Define  $\varphi(t, x) = \eta_\varepsilon(x)\psi(t, x)$ , where  $\psi \in \mathbf{C}_c^1(\overset{\circ}{I} \times \overset{\circ}{\mathbb{R}}^+; \mathbb{R})$  solves

$$\begin{cases} \partial_t \psi(t, x) + g(t, x) \partial_x \psi(t, x) = m(t, x) \psi(t, x) \\ \psi(\bar{t}, x) = \vartheta(x) \end{cases} \quad (t, x) \in [0, \bar{t}] \times \mathbb{R}_+,$$

while  $\eta_\varepsilon$  approximates the characteristic function of the space interval  $[0, \bar{x}]$ , that is,

$$\eta_\varepsilon(0) = 0, \quad \begin{aligned} \eta'_\varepsilon(x) &\in [0, 2/\varepsilon] & x \in [0, \varepsilon]; & \quad \eta_\varepsilon(x) = 1 & x \in [\varepsilon, \bar{x} - \varepsilon]; \\ \eta'_\varepsilon(x) &\in [-2/\varepsilon, 0] & x \in [\bar{x} - \varepsilon, \bar{x}]; & \quad \eta_\varepsilon(x) = 0 & x > \bar{x}. \end{aligned}$$

Choosing  $\varphi$  as a test function, we have that for all positive and sufficiently small  $\varepsilon$ ,

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+} \eta_\varepsilon(x) \int_I w(t, x) (\partial_t \psi(t, x) + g(t, x) \partial_x \psi(t, x) - m(t, x) \psi(t, x)) dt dx \\ &\quad + \int_0^\varepsilon \eta'_\varepsilon(x) \int_I g(t, x) \psi(t, x) dt dx + \int_{\bar{x}-\varepsilon}^{\bar{x}} \eta'_\varepsilon(x) \int_I g(t, x) \psi(t, x) dt dx \\ &= \int_0^1 \varepsilon \eta'_\varepsilon(\varepsilon x) \int_I g(t, \varepsilon s) w(t, \varepsilon s) dt ds + \int_{\bar{x}-\varepsilon}^{\bar{x}} \eta'_\varepsilon(x) \int_I g(t, x) \psi(t, x) dt dx \\ &\xrightarrow{\varepsilon \rightarrow 0} 0 + \int_I w(t, \bar{x}) \vartheta(t) dt, \end{aligned}$$

where we used **(c)** in **(SP.1)**.

**Step 3:** Extension to a general  $m$ . Fix a sequence  $m_k$  of continuous maps converging to  $m$  in  $\mathbf{L}^1$  and apply the above **Steps 1** and **2** to each  $m_k$ . Call  $w_k$  the corresponding difference of solutions as above, and in both cases

$\Gamma(\bar{x}) > t_2 > t_1$  and  $t_2 > t_1 > \Gamma(\bar{x})$ , one gets

$$\int_{t_1}^{t_2} w(t, \bar{x}) \vartheta(t) dt = \lim_{k \rightarrow +\infty} \int_{t_1}^{t_2} w_k(t, \bar{x}) \vartheta(t) dt = 0.$$

□

The next lemma bridges (4.11) to IBVPs set on bounded domains, so that we can then exploit results proved in this latter setting.

**Lemma 4.6.** *Let  $b \in \mathbf{BV}_{loc}(I; \mathbb{R})$ ,  $(f)$ ,  $(g)$ , and  $(m)$  hold and choose  $u_0 \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R})$ . Then, the following statements are equivalent.*

1. *The map  $u$  solves (3.3) in the sense of<sup>22, Definition 2</sup>:*

- (a) *for all  $\varphi \in \mathbf{C}_c^1(\dot{I} \times \dot{\mathbb{R}}^+; \mathbb{R})$ ,  $\int_{\mathbb{R}_+} \int_I [u \partial_t \varphi + g u \partial_x \varphi + (f - m u) \varphi] dt dx = 0$ ;*
- (b)  *$u(0, x) = u_0(x)$  for a.e.  $x \in \mathbb{R}_+$ ;*
- (c) *for a.e.  $t \in I$ ,  $\lim_{x \rightarrow 0^+} g(t, x) u(t, x) = b(t)$ .*

2. *The map  $u : I \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is such that for all  $\ell > 0$ , the restriction  $u_\ell = u|_{I \times [0, \ell]}$  solves*

$$\begin{cases} \partial_t u + \partial_x (g(t, x) u) + m(t, x) u = f(t, x) & (t, x) \in I \times [\ell, +\infty[ \\ u(0, x) = u_0(x) & x \in [\ell, +\infty[ \\ g(t, 0) u(t, 0+) = b(t) & t \in I \\ g(t, \ell) u(t, \ell-) = 0 & t \in I \end{cases} \quad (4.28)$$

*in the sense that*

- (a) *for all  $\varphi \in \mathbf{C}_c^1(\dot{I} \times ]0, \ell[; \mathbb{R})$ ,  $\int_0^\ell \int_I [u \partial_t \varphi + g u \partial_x \varphi + (f - m u) \varphi] dt dx = 0$ ;*
- (b)  *$u(0, x) = u_0(x)$  for a.e.  $x \in \mathbb{R}_+$ ;*
- (c) *for a.e.  $t \in I$ ,  $\lim_{x \rightarrow 0^+} g(t, x) u(t, x) = b(t)$ .*

3. *The map  $u : I \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is such that for all  $\ell > 0$ , the restriction  $u_\ell = u|_{I \times [0, \ell]}$  is a Kružkov solution to (4.28) in the sense that for a.e.  $t \in I$ , the traces  $u(t, 0+)$  and  $u(t, \ell-)$  are well defined, and moreover, for all  $k \in \mathbb{R}$  and for any test function  $\varphi \in \mathbf{C}_c^1(-\infty, \sup I] \times \mathbb{R}; \mathbb{R}_+)$ ,*

$$\begin{aligned} & \int_I \int_0^\ell |u(t, x) - k| \partial_t \varphi(t, x) dx dt \\ & + \int_I \int_0^\ell \operatorname{sgn}(u(t, x) - k) g(t, x) (u(t, x) - k) \partial_x \varphi(t, x) dx dt \\ & + \int_I \int_0^\ell \operatorname{sgn}(u(t, x) - k) (f(t, x) - m(t, x) u(t, x) - \partial_x g(t, x) k) \varphi(t, x) dx dt \\ & + \int_0^\ell |u_0(x) - k| \varphi(0, x) dx \\ & + \int_I \operatorname{sgn}(b(t) - k g(t, 0)) g(t, 0) (u(t, 0+) - k) \varphi(t, 0) dt \\ & + \int_I \operatorname{sgn}(k) g(t, \ell-) (u(t, \ell-) - k) \varphi(t, \ell) dt \geq 0. \end{aligned}$$

The proof is standard and hence omitted.

The above allows to exploit known results on the solutions to IBVP for balance laws on bounded domains. In particular, the total variation estimates **(SP.3)** and **(SP.4)**, together with Ambrosio et al,<sup>23, Remark 3.104, section 3.11</sup> ensure that  $u$  as defined by (4.12) is in  $\mathbf{BV}$  of both variables. Hence, for any positive  $\ell$ , it belongs to the space  $\mathcal{TR}^\infty(I \times [0, \ell]; \mathbb{R})$ , see Rossi.<sup>16, Definition 5.1</sup> An application of Rossi<sup>16, Theorem 5.8</sup> ensures that  $u$  is a *regular entropy* solution to (4.28) in the sense of Rossi.<sup>16, Definition 3.3</sup> This latter definition is stable with respect to  $\mathbf{L}^1$  convergence.

## 4.5 | A-priori estimates

**Lemma 4.7.** Assume **(b)**, **(f)**, **(g)**, and **(m)** hold. Suppose that  $u$  is a solution to (1.1)-(1.2) in the sense of Definition 2.1 on the time interval  $I$ . Let

$$T_1 = \min \left\{ \frac{\bar{x}_1}{2\bar{g}}, \dots, \frac{\bar{x}_n}{2\bar{g}} \right\}. \quad (4.29)$$

Then, the following a-priori estimates hold.

1. For every  $i \in \{1, \dots, n\}$  and  $t \in [0, T_1]$ ,

$$\|u_i(\cdot, \bar{x}_i)\|_{L^1([0,t];\mathbb{R})} \leq \frac{e^{Mt}}{\bar{g}} \|u_{o,i}\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \frac{1}{M\bar{g}} (e^{Mt} - 1) F_1. \quad (4.30)$$

2. There exist nondecreasing and continuous functions  $\mathcal{L}_1, \mathcal{L}_2 : [0, T_1] \rightarrow \mathbb{R}_+$ , depending only on the constants defined in **(b)**, **(f)**, **(g)**, and **(m)**, such that  $\mathcal{L}_1(0) = 0$ ,  $\mathcal{L}_2(0) > 1$ , and, for every  $t \in [0, T_1]$ ,

$$\|u(t)\|_{L^1(\mathbb{R}_+;\mathbb{R}^n)} \leq \mathcal{L}_1(t) + \mathcal{L}_2(t) \|u_o\|_{L^1(\mathbb{R}_+;\mathbb{R}^n)}. \quad (4.31)$$

3. There exists a positive constant  $\mathcal{L}$ , depending only on  $T_1$  and on the constants defined in **(b)**, **(f)**, **(g)**, and **(m)**, such that

$$\|u\|_{L^\infty([0,T_1]\times\mathbb{R}_+;\mathbb{R}^n)} \leq \mathcal{L} (1 + \|u_o\|_{L^\infty(\mathbb{R}_+;\mathbb{R}^n)} + \|u_o\|_{L^1(\mathbb{R}_+;\mathbb{R}^n)}). \quad (4.32)$$

*Proof.* The proof is divided into several parts.

**Proof of (4.30).** Fix  $t \in [0, T_1]$  and  $i \in \{1, \dots, n\}$ . Define  $\gamma_i(s) = X_i(s; 0, 0)$  for every  $s \in [0, t]$ . Assumption (4.29) implies that  $\gamma_i(s) \leq \bar{x}_i$  for every  $s \in [0, t]$ . Consequently,

$$\begin{aligned} & \int_0^t |u_i(\tau, \bar{x}_i)| \, d\tau && \text{[Use (4.12)]} \\ & \leq \int_0^t \mathcal{E}_i(0, \tau, \bar{x}_i) |u_{o,i}(X_i(0; \tau, \bar{x}_i))| \, d\tau && \text{[Set } X_i(0; \tau, \bar{x}_i) = \xi] \\ & \quad + \int_0^t \int_0^\tau \mathcal{E}_i(s, \tau, \bar{x}_i) |f_i(s, X_i(s; \tau, \bar{x}_i))| \, ds \, d\tau && \text{[Set } X_i(s; \tau, \bar{x}_i) = \xi] \\ & = \int_{X_i(0;t;\bar{x}_i)}^{\bar{x}_i} \mathcal{E}_i(0, T_i(\bar{x}_i; 0, \xi), \bar{x}_i) |u_{o,i}(\xi)| \frac{1}{g_i(T_i(\bar{x}_i; 0, \xi), \bar{x}_i)} && \text{[Use (4.13)]} \\ & \quad \times \exp\left(\int_0^{T_i(\bar{x}_i; 0, \xi)} \partial_x g_i(s, X_i(s; 0, \xi)) \, ds\right) \, d\xi \\ & \quad + \int_0^t \int_{X_i(s;t;\bar{x}_i)}^{\bar{x}_i} \mathcal{E}_i(s, T_i(\bar{x}_i; s, \xi), \bar{x}_i) |f_i(s, \xi)| \frac{1}{g_i(T_i(\bar{x}_i; s, \xi), \bar{x}_i)} && \text{[Use (4.13)]} \\ & \quad \times \exp\left(\int_s^{T_i(\bar{x}_i; s, \xi)} \partial_x g_i(r, X_i(r; s, \xi)) \, dr\right) \, d\xi \, ds \\ & = \int_{X_i(0;t;\bar{x}_i)}^{\bar{x}_i} |u_{o,i}(\xi)| \frac{1}{g_i(T_i(\bar{x}_i; 0, \xi), \bar{x}_i)} && \text{[Use (g)]} \\ & \quad \times \exp\left(-\int_0^{T_i(\bar{x}_i; 0, \xi)} m_i(s, X_i(s; 0, \xi)) \, ds\right) \, d\xi && \text{[Use (m)]} \\ & \quad + \int_0^t \int_{X_i(s;t;\bar{x}_i)}^{\bar{x}_i} |f_i(s, \xi)| \frac{1}{g_i(T_i(\bar{x}_i; s, \xi), \bar{x}_i)} && \text{[Use (f)]} \\ & \quad \times \exp\left(-\int_s^{T_i(\bar{x}_i; s, \xi)} m_i(r, X_i(r; s, \xi)) \, dr\right) \, d\xi \, ds && \text{[Use (m)]} \\ & \leq \frac{e^{Mt}}{\bar{g}} \|u_{o,i}\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \frac{1}{M\bar{g}} (e^{Mt} - 1) F_1. \end{aligned}$$

This completes the proof of (4.30).

**Proof of (4.31).** Fix  $t \in [0, T_1]$  and  $i \in \{1, \dots, n\}$ . Denoting  $\gamma_i(t) = X_i(t; 0, 0)$ , we have

$$\|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} = \int_0^{\gamma_i(t)} |u_i(t, x)| dx + \int_{\gamma_i(t)}^{+\infty} |u_i(t, x)| dx. \tag{4.33}$$

We estimate separately both terms in the right-hand side of (4.33), starting from the second one:

$$\begin{aligned} & \int_{\gamma_i(t)}^{+\infty} |u_i(t, x)| dx && \text{[Use (4.12)]} \\ &= \int_{\gamma_i(t)}^{+\infty} \mathcal{E}_i(0, t, x) |u_{o,i}(X_i(0; t, x))| dx && \text{[Set } X_i(0; t, x) = \xi] \\ &+ \int_{\gamma_i(t)}^{+\infty} \int_0^t \mathcal{E}_i(\tau, t, x) |f_i(\tau, X_i(\tau; t, x))| d\tau dx && \text{[Set } X_i(\tau; t, x) = \xi] \\ &= \int_0^{+\infty} \mathcal{E}_i(0, t, X_i(t; 0, \xi)) |u_{o,i}(\xi)| \exp\left(\int_0^t \partial_x g_i(s; X_i(s; 0, \xi)) ds\right) d\xi && \text{[Use (4.13)]} \\ &+ \int_0^{+\infty} \int_0^t \mathcal{E}_i(s, t, X_i(t; s, \xi)) |f_i(s, \xi)| \\ &\times \exp\left(\int_s^t \partial_x g_i(r; X_i(r; s, \xi)) dr\right) ds d\xi && \text{[Use (4.13)]} \\ &= \int_0^{+\infty} \exp\left(-\int_0^t m_i(s; X_i(s; 0, \xi)) ds\right) |u_{o,i}(\xi)| d\xi && \text{[Use (m)]} \\ &+ \int_0^{+\infty} \int_0^t \exp\left(-\int_s^t m_i(r; X_i(r; s, \xi)) dr\right) |f_i(s, \xi)| ds d\xi && \text{[Use (m) and (f)]} \\ &\leq e^{Mt} \|u_{o,i}\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + \frac{1}{M} (e^{Mt} - 1) F_1. \end{aligned}$$

Pass to the first term in the right-hand side of (4.33). By **(b)**, we can define, for all  $s \in [0, t]$ ,

$$\begin{aligned} b_i(s) &= \alpha_i(s, u_1(s, \bar{x}_1-), \dots, u_n(s, \bar{x}_n-)) \\ &+ \beta_i\left(s, \int_{\mathbb{R}_+} w_1^i(s, x) u_1(s, x) dx, \dots, \int_{\mathbb{R}_+} w_n^i(s, x) u_n(s, x) dx\right), \end{aligned} \tag{4.34}$$

we have

$$\begin{aligned} & \int_0^{\gamma_i(t)} |u_i(t, x)| dx && \text{[Use (4.12)]} \\ &\leq \int_0^{\gamma_i(t)} \mathcal{E}_i(T_i(0; t, x), t, x) \frac{|b_i(T_i(0; t, x))|}{g_i(T_i(0; t, x), 0)} dx && \text{[Set } T_i(0; t, x) = s] \\ &+ \int_0^{\gamma_i(t)} \int_{T_i(0; t, x)}^t \mathcal{E}_i(\tau, t, x) |f_i(\tau, X_i(\tau; t, x))| d\tau dx && \text{[Set } X_i(\tau; t, x) = \xi] \\ &= \int_0^t \mathcal{E}_i(s, t, X_i(t; s, 0)) |b_i(s)| \exp\left(\int_s^t \partial_x g_i(r; X_i(r; s, 0)) dr\right) ds && \text{[Use (4.13)]} \\ &+ \int_0^t \int_0^{\gamma_i(s)} \mathcal{E}_i(s, t, X_i(t; s, \xi)) |f_i(s, \xi)| && \text{[Use (4.13)]} \\ &\times \exp\left(\int_s^t \partial_x g_i(r; X_i(r; s, \xi)) dr\right) d\xi ds \end{aligned}$$

$$\begin{aligned}
&= \int_0^t |b_i(s)| \exp\left(-\int_s^t m_i(r, X_i(r; s, 0)) dr\right) ds && \text{[Use(m)]} \\
&\quad + \int_0^t \int_0^{\gamma_i(s)} |f_i(s, \xi)| \exp\left(-\int_s^t m_i(r, X_i(r; s, \xi)) dr\right) d\xi ds && \text{[Use(m)and(f)]} \\
&\leq e^{Mt} \int_0^t |b_i(s)| ds + \frac{F_1}{M} (e^{Mt} - 1).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_0^t |b_i(s)| ds && \text{[Use(4.34)]} \\
&\leq \int_0^t \alpha_i(s, u_1(s, \bar{x}_1), \dots, u_n(s, \bar{x}_n)) ds && \text{[Use(b)]} \\
&\quad + \int_0^t \beta_i\left(s, \int_{\mathbb{R}_+} w_1^i(s, x) u_1(s, x) dx, \dots, \int_{\mathbb{R}_+} w_1^i(s, x) u_1(s, x) dx\right) ds && \text{[Use(b)]} \\
&\leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t |u_j(s, \bar{x}_j)| ds && \text{[Use(4.30)]} \\
&\quad + \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_+} |w_j^i(s, x) u_j(s, x)| dx ds && \text{[Use(b)]} \\
&\leq n\mathbf{Lip}(\alpha) \left(\frac{e^{Mt}}{\check{\xi}} \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} + \frac{1}{M\check{\xi}} (e^{Mt} - 1) F_1\right) \\
&\quad + W\mathbf{Lip}(\beta) \int_0^t \sum_{j=1}^n \|u_j(s)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} ds.
\end{aligned}$$

Using (4.33), we deduce that

$$\begin{aligned}
&\sum_{i=1}^n \|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \\
&\leq e^{Mt} \sum_{i=1}^n \int_0^t |b_i(s)| ds + n\frac{F_1}{M} (e^{Mt} - 1) + e^{Mt} \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} + \frac{n}{M} (e^{Mt} - 1) F_1 \\
&\leq 2n\frac{F_1}{M} (e^{Mt} - 1) + e^{Mt} \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \\
&\quad + e^{Mt} n^2 \mathbf{Lip}(\alpha) \left(\frac{e^{Mt}}{\check{\xi}} \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} + \frac{1}{M\check{\xi}} (e^{Mt} - 1) F_1\right) \\
&\quad + e^{Mt} nW\mathbf{Lip}(\beta) \int_0^t \sum_{j=1}^n \|u_j(s)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} ds \\
&= \frac{F_1}{M} (e^{Mt} - 1) \left(2n + \frac{e^{Mt} n^2}{\check{\xi}} \mathbf{Lip}(\alpha)\right) + e^{Mt} \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \left(1 + \frac{e^{Mt} n^2}{\check{\xi}} \mathbf{Lip}(\alpha)\right) \\
&\quad + e^{Mt} nW\mathbf{Lip}(\beta) \int_0^t \sum_{i=1}^n \|u_i(s)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} ds.
\end{aligned}$$

Applying Grönwall lemma, we obtain that

$$\begin{aligned}
\sum_{i=1}^n \|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} &\leq \frac{F_1}{M} (e^{Mt} - 1) \left(2n + \frac{e^{Mt} n^2}{\check{\xi}} \mathbf{Lip}(\alpha)\right) \exp(e^{Mt} nW\mathbf{Lip}(\beta)t) \\
&\quad + e^{Mt} \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \left(1 + \frac{e^{Mt} n^2}{\check{\xi}} \mathbf{Lip}(\alpha)\right) \exp(e^{Mt} nW\mathbf{Lip}(\beta)t),
\end{aligned}$$

proving (4.31).

**Proof of (4.32).** Fix  $t \in [0, T_1]$  and  $i \in \{1, \dots, n\}$ . Denoting, for every  $j \in \{1, \dots, n\}$  and  $s \in [0, t]$ , with  $\gamma_j(s)$  the quantity  $X_j(s; 0, 0)$ , we have

$$\|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} = \max \left\{ \|u_i(t)\|_{\mathbf{L}^\infty((0, \gamma_i(t)); \mathbb{R})}, \|u_i(t)\|_{\mathbf{L}^\infty((\gamma_i(t), +\infty); \mathbb{R})} \right\}. \quad (4.35)$$

We estimate both terms in (4.35).

$$\begin{aligned} & \|u_i(t)\|_{\mathbf{L}^\infty((\gamma_i(t), +\infty); \mathbb{R})} && \text{[Use (4.12)]} \\ & \leq \text{ess sup}_{x > \gamma_i(t)} |u_{0,i}(X_i(0; t, x)) \mathcal{E}_i(0, t, x)| && \text{[Use (4.14)]} \\ & \quad + \text{ess sup}_{x > \gamma_i(t)} \left| \int_0^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau \right| && \text{[Use (4.14) and (f)]} \\ & \leq \left( \|u_{0,i}\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + F_\infty t \right) e^{(G_1+M)t}. \end{aligned}$$

Moreover, defining  $b_i$  as in (4.34), we deduce that

$$\begin{aligned} & \|u_i(t)\|_{\mathbf{L}^\infty((0, \gamma_i(t)); \mathbb{R})} && \text{[Use (4.12)]} \\ & \leq \text{ess sup}_{x < \gamma_i(t)} \left| \frac{b_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}_i(T_i(0; t, x), t, x) \right| && \text{[Use (4.14) and (g)]} \\ & \quad + \text{ess sup}_{x < \gamma_i(t)} \left| \int_{T_i(0; t, x)}^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau \right| && \text{[Use (4.14) and (f)]} \\ & \leq \left( \frac{1}{g} \text{ess sup}_{s \in (0, t)} |b_i(s)| + F_\infty t \right) e^{(G_1+M)t}. \end{aligned}$$

Note that (4.29) implies that  $\bar{x}_1 \geq \gamma_j(s)$  for every  $j \in \{1, \dots, n\}$  and  $s \in [0, t]$ . Thus,

$$\begin{aligned} & \text{ess sup}_{s \in [0, t]} |b_i(s)| && \text{[Use (4.34)]} \\ & \leq \text{ess sup}_{s \in [0, t]} \alpha_i(s, u_1(s, \bar{x}_1), \dots, u_n(s, \bar{x}_n)) && \text{[Use (b)]} \\ & \quad + \text{ess sup}_{s \in [0, t]} \beta_i \left( s, \int_{\mathbb{R}_+} w_1^i(s, x) u_1(s, x) dx, \dots, \int_{\mathbb{R}_+} w_n^i(s, x) u_n(s, x) dx \right) && \text{[Use (b)]} \\ & \leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \text{ess sup}_{s \in [0, t]} |u_j(s, \bar{x}_j)| \\ & \quad + \mathbf{Lip}(\beta) \sum_{j=1}^n \text{ess sup}_{s \in [0, t]} \int_{\mathbb{R}_+} |w_j^i(s, x) u_j(s, x)| dx && \text{[Use (b)]} \\ & \leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \sup_{s \in [0, t]} \|u_j(s)\|_{\mathbf{L}^\infty([\gamma_j(s), +\infty]; \mathbb{R})} \\ & \quad + W \mathbf{Lip}(\beta) \sum_{j=1}^n \sup_{s \in [0, t]} \|u_j(s)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} && \text{[Use (4.31)]} \\ & \leq \mathbf{Lip}(\alpha) \left( \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + n F_\infty t \right) e^{(G_1+M)t} \\ & \quad + W \mathbf{Lip}(\beta) \left( \mathcal{L}_1(t) + \mathcal{L}_2(t) \|u_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right). \end{aligned}$$

Therefore, using (4.35),

$$\begin{aligned} \|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} &\leq \left( \|u_{0,i}\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} + 2F_\infty t + \frac{1}{\check{g}} \operatorname{ess\,sup}_{s \in ]0, t[} |b_i(s)| \right) e^{(G_1+M)t} \\ &\leq \left( 1 + \frac{\mathbf{Lip}(\alpha)}{\check{g}} e^{(G_1+M)t} \right) e^{(G_1+M)t} \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} \\ &\quad + \left( 2 + n \frac{\mathbf{Lip}(\alpha)}{\check{g}} e^{(G_1+M)t} \right) F_\infty t e^{(G_1+M)t} \\ &\quad + \frac{W\mathbf{Lip}(\beta)\mathcal{L}_1(t)}{\check{g}} e^{(G_1+M)t} + \frac{W\mathbf{Lip}(\beta)\mathcal{L}_2(t)}{\check{g}} e^{(G_1+M)t} \|u_0\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}, \end{aligned}$$

proving (4.32).  $\square$

## 4.6 | Proofs of the main results

This part deals with the proof of Theorems 2.2 and 2.3.

*Proof of Theorem 2.2 and proof of Theorem 2.3.* We first prove both Theorems 2.2 and 2.3 under the assumptions **(b)**, **(f)**, and **(m)**, and moreover,

$$\text{There exists } \varepsilon > 0 \text{ such that } w_j^i(t, x) = 0 \text{ for all } (t, x) \in I \times [0, \varepsilon] \text{ and all } i, j = 1, \dots, n. \quad (4.36)$$

**A recursive construction of the solution to (1.1).** Let

$$T_1 = \min \left\{ \frac{\bar{x}_1}{2\check{g}}, \dots, \frac{\bar{x}_n}{2\check{g}}, t_* \right\} \quad \text{and} \quad T_1^\varepsilon = \min \left\{ \frac{\varepsilon}{2\check{g}}, T_1 \right\}. \quad (4.37)$$

Note that this choice implies that  $\gamma_i(T_1^\varepsilon) \leq \gamma_i(T_1) < \bar{x}_i$  for every  $i = 1, \dots, n$ , and that  $\gamma_i(T_1^\varepsilon) < \varepsilon$ , where we use the notation (4.7). Below, we inductively construct the solution  $u$  to (1.1) on the time intervals  $[(k-1)T_1^\varepsilon, kT_1^\varepsilon]$ , with  $k \in \mathbb{N} \setminus \{0\}$ .

Consider  $k = 0$ , and for every  $i \in \{1, \dots, n\}$ , define the sets

$$A_1^i = \{(t, x) : t \in [0, T_1^\varepsilon], 0 < x < \gamma_i(t)\}, \quad B_1^i = \{(t, x) : t \in [0, T_1^\varepsilon], x > \gamma_i(t)\},$$

and define, for every  $i \in \{1, \dots, n\}$  and  $(t, x) \in B_1^i$ , using the notation (4.13),

$$u_i(t, x) = u_o^i(X_i(0; t, x)) \mathcal{E}_i(0, t, x) + \int_0^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau, \quad (4.38)$$

which depends only on  $u_o^i$ ,  $g_i$ ,  $m_i$ , and  $f_i$ . For every  $i \in \{1, \dots, n\}$  and  $t \in [0, T_1^\varepsilon]$ , define

$$b_i(t) = \mathcal{B}_i(t, u_1(t), \dots, u_n(t)), \quad (4.39)$$

where  $\mathcal{B}_i$  is defined in (1.2). Note that (4.39) is well defined since **(b)** holds and since **(SP.3)**. Finally, for all  $i \in \{1, \dots, n\}$  and  $(t, x) \in A_1^i$ , let

$$u_i(t, x) = \frac{b_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}_i(T_i(0; t, x), t, x) + \int_{T_i(0; t, x)}^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau, \quad (4.40)$$

which completes the construction of  $u \equiv (u_1, \dots, u_n)$  in  $[0, T_1^\varepsilon]$ .

Assume now  $k > 0$ , with  $(k + 1)T_1^\varepsilon < t_*$ , and suppose that  $u$  is defined in the time interval  $[0, kT_1^\varepsilon]$ . Define, for every  $i \in \{1, \dots, n\}$ , the sets

$$\begin{aligned} A_{k+1}^i &= \{(t, x) : t \in [kT_1^\varepsilon, (k + 1)T_1^\varepsilon], 0 < x < X_i(t; kT_1^\varepsilon, 0)\} \\ B_{k+1}^i &= \{(t, x) : t \in [kT_1^\varepsilon, (k + 1)T_1^\varepsilon], x > X_i(t; kT_1^\varepsilon, 0)\}, \end{aligned}$$

and for every  $i \in \{1, \dots, n\}$  and  $(t, x) \in B_{k+1}^i$ , using the notation (4.13),

$$u_i(t, x) = u_i(kT_1^\varepsilon, X_i(kT_1^\varepsilon; t, x)) \mathcal{E}_i(kT_1^\varepsilon; t, x) + \int_{kT_1^\varepsilon}^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau,$$

which depends only on  $u_i(kT_1^\varepsilon, \cdot)$ ,  $g_i$ ,  $m_i$ , and  $f_i$ . For every  $i \in \{1, \dots, n\}$  and  $t \in [kT_1^\varepsilon, (k + 1)T_1^\varepsilon]$ , define the boundary datum as in (4.39), where  $\mathcal{B}_i$  is defined in (1.2). Finally, for all  $i \in \{1, \dots, n\}$  and  $(t, x) \in A_{k+1}^i$ , set

$$u_i(t, x) = \frac{b_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}_i(T_i(0; t, x), t, x) + \int_{T_i(0; t, x)}^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau,$$

which completes the construction of  $u$  in  $[0, (k + 1)T_1^\varepsilon]$ . Hence, point 1 of Theorem 2.2 follows immediately. Moreover, by the construction of  $u$ , also point 3 of Theorem 2.2 is proved. Remark that for every  $\ell > 0$ , each component  $u_i$  of  $u$  is a *regular entropy* solution to

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) + m_i(t, x) u_i = f_i(t, x) & (t, x) \in I \times [0, \ell[ \\ u_i(0, x) = u_i^0(x) & x \in [0, \ell[ \\ g_i(t, 0) u_i(t, 0+) = b_i(t) & t \in I \\ g_i(t, \ell) u_i(t, \ell-) = 0 & t \in I \end{cases} \quad (4.41)$$

with  $b_i$  as in (4.39), see Lemma 4.6.

**If a solution to (1.1) exists, then it is unique.** Recall  $T_1^\varepsilon$  as defined in (4.37). Assume that  $u'$  and  $u''$  both solve (1.1) in the sense of Definition 2.1. Then, for instance by Colombo and Garavello,<sup>22, section 4.3</sup> for all  $x \geq \gamma_i(t)$ , both  $u_i'$  and  $u_i''$  are given by (4.38). Assumption (4.36) then ensures that  $\mathcal{B}_i(t, u_1'(t), \dots, u_n'(t)) = \mathcal{B}_i(t, u_1''(t), \dots, u_n''(t))$ . Then, by (4.40) and Colombo and Garavello,<sup>22, section 4.3</sup> necessarily  $u_i'(t, x) = u_i''(t, x)$  for all  $t \in [0, T_1^\varepsilon]$  and  $x \in [0, \gamma_i(t)]$ . Iterate now on  $[T_1^\varepsilon, 2T_1^\varepsilon]$ ,  $[2T_1^\varepsilon, 3T_1^\varepsilon]$ , ... to obtain global uniqueness on  $I \times \mathbb{R}_+$ .

**L<sup>1</sup> well posedness and stability estimates w.r.t.  $\alpha, \beta, w$ .** Fix  $t \in I$ ,  $t \leq T_1$  as defined in (4.37). Consider two different initial conditions, namely,  $u'_0$  and  $u''_0$ , and two different sets of data, namely,  $\alpha', \beta', w'$  and  $\alpha'', \beta'', w''$  with  $w'_i(t, x) = w''_i(t, x) = 0$  for all  $x \in [0, \varepsilon]$ , as in (4.36). Denoting with  $u'$  and  $u''$  the corresponding solutions, we have, for all  $i \in \{1, \dots, n\}$ ,

$$\|u'_i(t) - u''_i(t)\|_{L^1(\mathbb{R}_+; \mathbb{R})} = \int_0^{\gamma_i(t)} |u'_i(t, x) - u''_i(t, x)| dx + \int_{\gamma_i(t)}^{+\infty} |u'_i(t, x) - u''_i(t, x)| dx. \quad (4.42)$$

Consider the second term in the right-hand side of (4.42). If  $x > \gamma_i(t)$ , then the construction in the first part of the proof implies that

$$\begin{aligned} u'_i(t, x) &= u'_{0,i}(X_i(0; t, x)) \mathcal{E}_i(0, t, x) + \int_0^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau, \\ u''_i(t, x) &= u''_{0,i}(X_i(0; t, x)) \mathcal{E}_i(0, t, x) + \int_0^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau. \end{aligned} \quad (4.43)$$

Therefore,

$$\begin{aligned}
& \int_{\gamma_i(t)}^{+\infty} |u'_i(t, x) - u''_i(t, x)| dx && \text{[Use (4.43)]} \\
&= \int_{\gamma_i(t)}^{+\infty} \mathcal{E}_i(0, t, x) |u'_{o,i}(X_i(0; t, x)) - u''_{o,i}(X_i(0; t, x))| dx && \text{[Set } X_i(0; t, x) = \xi] \\
&= \int_0^{+\infty} \mathcal{E}_i(0, t, X(t; 0, \xi)) |u'_{o,i}(\xi) - u''_{o,i}(\xi)| && \text{[By (4.10)]} \\
&\quad \times \exp\left(\int_0^t \partial_x g_i(s; X(s; 0, \xi)) ds\right) d\xi && \text{[Use (4.13)]} \\
&= \int_0^{+\infty} \exp\left(\int_0^t m_i(s; X(s; 0, \xi)) ds\right) |u'_{o,i}(\xi) - u''_{o,i}(\xi)| d\xi && \text{[Use (m)]} \\
&\leq e^{Mt} \|u'_{o,i} - u''_{o,i}\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})}.
\end{aligned}$$

Consider the first term in the right-hand side of (4.42). If  $x < \gamma_i(t)$ , then the construction in the first part of the proof implies that

$$\begin{aligned}
u'_i(t, x) &= \frac{b'_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}_i(T_i(0; t, x), t, x) + \int_{T_i(0; t, x)}^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau, \\
u''_i(t, x) &= \frac{b''_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}_i(T_i(0; t, x), t, x) + \int_{T_i(0; t, x)}^t f_i(\tau, X_i(\tau; t, x)) \mathcal{E}_i(\tau, t, x) d\tau,
\end{aligned} \tag{4.44}$$

where, for  $\tau \in I$ ,

$$\begin{aligned}
b'_i(\tau) &= \alpha'_i(\tau, u'_1(\tau, \bar{x}_1-), \dots, u'_n(\tau, \bar{x}_n-)) \\
&\quad + \beta'_i\left(\tau, \int_{\mathbb{R}_+} w'_1(\tau, \xi) u'_1(\tau, \xi) d\xi, \dots, \int_{\mathbb{R}_+} w'_n(\tau, \xi) u'_n(\tau, \xi) d\xi\right), \\
b''_i(\tau) &= \alpha''_i(\tau, u''_1(\tau, \bar{x}_1-), \dots, u''_n(\tau, \bar{x}_n-)) \\
&\quad + \beta''_i\left(\tau, \int_{\mathbb{R}_+} w''_1(\tau, \xi) u''_1(\tau, \xi) d\xi, \dots, \int_{\mathbb{R}_+} w''_n(\tau, \xi) u''_n(\tau, \xi) d\xi\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_0^{\gamma_i(t)} |u'_i(t, x) - u''_i(t, x)| dx && \text{[Use (4.44)]} \\
&= \int_0^{\gamma_i(t)} \left| \frac{\mathcal{E}_i(T_i(0; t, x), t, x)}{g_i(T_i(0; t, x), 0)} \right| |b'_i(T_i(0; t, x)) - b''_i(T_i(0; t, x))| dx && \text{[Set } T_i(0; t, x) = \tau] \\
&= \int_0^t \frac{|\mathcal{E}_i(\tau, t, X(t; \tau, 0))|}{|g_i(\tau, 0)|} |b'_i(\tau) - b''_i(\tau)| |g_i(\tau, 0)| && \text{[By (4.10)]} \\
&\quad \times \exp\left(\int_\tau^t \partial_x g_i(s, X(s; \tau, 0)) ds\right) d\tau && \text{[Use (4.13)]} \\
&= \int_0^t \exp\left(-\int_\tau^t m_i(s, X(s; \tau, 0)) ds\right) |b'_i(\tau) - b''_i(\tau)| d\tau && \text{[Use (m)]} \\
&\leq e^{Mt} \|b'_i - b''_i\|_{\mathbf{L}^1([0, t]; \mathbb{R})}.
\end{aligned}$$

We now estimate the latter right-hand side, using **(b)**, (4.32), and  $t \leq T_1$ :

$$\begin{aligned}
& \|b'_i - b''_i\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\
& \leq \int_0^t \left| \alpha'_i(\tau, u'_1(\tau, \bar{x}_1-), \dots, u'_n(\tau, \bar{x}_n-)) - \alpha''_i(\tau, u''_1(\tau, \bar{x}_1-), \dots, u''_n(\tau, \bar{x}_n-)) \right| d\tau \\
& \quad + \int_0^t \left| \beta'_i \left( \tau, \int_{\mathbb{R}_+} w'_1(\tau, x) u'_1(\tau, x) dx, \dots \right) - \beta''_i \left( \tau, \int_{\mathbb{R}_+} w''_1(\tau, x) u''_1(\tau, x) dx, \dots \right) \right| d\tau \\
& \leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t \left| u'_j(\tau, \bar{x}_j-) - u''_j(\tau, \bar{x}_j-) \right| d\tau + t \|\alpha' - \alpha''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} \\
& \quad + \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \int_{\varepsilon}^{+\infty} |w'_j(\tau, x)| |u'_j(\tau, x) - u''_j(\tau, x)| dx d\tau \\
& \quad + \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \int_{\varepsilon}^{+\infty} |u''_j(\tau, x)| |w'_j(\tau, x) - w''_j(\tau, x)| dx d\tau + t \|\beta' - \beta''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} \\
& \leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t \left| u'_{o,j}(X_j(0; \tau, \bar{x}_j)) - u''_{o,j}(X_j(0; \tau, \bar{x}_j)) \right| \mathcal{E}_j(0, \tau, \bar{x}_j) d\tau \\
& \quad + t \|\alpha' - \alpha''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} + t \|\beta' - \beta''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} \\
& \quad + \mathbf{Lip}(\beta) W \sum_{j=1}^n \int_0^t \|u'_j(\tau) - u''_j(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} d\tau \\
& \quad + \mathcal{L} \mathbf{Lip}(\beta) \left( 1 + \|u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right) \sum_{j=1}^n \int_0^t \|w'_j(\tau) - w''_j(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} d\tau \\
& \leq \mathbf{Lip}(\alpha) \frac{e^{Mt}}{\mathfrak{g}} \sum_{j=1}^n \|u'_{o,j} - u''_{o,j}\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} + t \|\alpha' - \alpha''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} + t \|\beta' - \beta''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} \\
& \quad + \mathbf{Lip}(\beta) W \sum_{j=1}^n \int_0^t \|u'_j(\tau) - u''_j(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} d\tau \\
& \quad + \mathcal{L} \mathbf{Lip}(\beta) \left( 1 + \|u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right) \sum_{j=1}^n \int_0^t \|w'_j(\tau) - w''_j(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} d\tau,
\end{aligned}$$

where  $\mathcal{L}$ , defined in Lemma 4.7, does not depend on  $\varepsilon$ . Using (4.42), we deduce that

$$\begin{aligned}
\sum_{i=1}^n \|u'_i(t) - u''_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} & \leq e^{Mt} \sum_{i=1}^n \|b'_i - b''_i\|_{\mathbf{L}^1([0,t]; \mathbb{R})} + e^{Mt} \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \\
& \leq \left( 1 + n \mathbf{Lip}(\alpha) \frac{e^{Mt}}{\mathfrak{g}} \right) e^{Mt} \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \\
& \quad + n t e^{Mt} \left( \|\alpha' - \alpha''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} + \|\beta' - \beta''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} \right) \\
& \quad + n \mathcal{L} \mathbf{Lip}(\beta) t e^{Mt} \|w' - w''\|_{\mathbf{C}^0([0,t]; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))} \\
& \quad \times \left( 1 + \|u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right) \\
& \quad + n e^{Mt} \mathbf{Lip}(\beta) W \int_0^t \sum_{i=1}^n \|u'_i(\tau) - u''_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} d\tau.
\end{aligned}$$

By Grönwall lemma, we obtain the existence of a positive constant  $\mathcal{M}$ , depending only on  $T_1$  and on the constants defined in **(b)**, **(f)**, **(g)**, and **(m)**, such that

$$\begin{aligned} \sum_{i=1}^n \|u'_i(t) - u''_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} &\leq \mathcal{M}e^{Mt} \left[ \|\alpha' - \alpha''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} + \|\beta' - \beta''\|_{\mathbf{C}^0(I \times \mathbb{R}^n; \mathbb{R}^n)} \right] \\ &\quad + \mathcal{M}e^{Mt} \left[ \|w' - w''\|_{\mathbf{C}^0([0,t]; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))} + \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right]. \end{aligned}$$

Iterating on the intervals  $[0, T_1]$ ,  $[T_1, 2T_1]$ , and so on, we deduce that the  $\mathbf{L}^1$  well posedness estimate (2.1) and the stability estimate (2.4) hold under assumption (4.36) and in the case  $m' = m''$  and  $f' = f''$ . Remark that  $\mathcal{M}$  does not depend on  $\varepsilon$ .

**Stability estimates w.r.t.  $m$  and  $f$ .** Fix  $t \in I$ ,  $t \leq T_1$  as defined in (4.37). Consider two different sets of data, namely,  $m', f'$  and  $m'', f''$ . Denoting with  $u'$  and  $u''$  the corresponding solutions, we have, for all  $i \in \{1, \dots, n\}$ ,

$$\|u'_i(t) - u''_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} = \int_0^{\gamma_i(t)} |u'_i(t, x) - u''_i(t, x)| dx + \int_{\gamma_i(t)}^{+\infty} |u'_i(t, x) - u''_i(t, x)| dx. \quad (4.45)$$

Consider the second term in the right-hand side of (4.45). If  $x > \gamma_i(t)$ , then the construction in the first part of the proof implies that

$$\begin{aligned} u'_i(t, x) &= u_{o,i}(X_i(0; t, x)) \mathcal{E}'_i(0, t, x) + \int_0^t f'_i(\tau, X_i(\tau; t, x)) \mathcal{E}'_i(\tau, t, x) d\tau, \\ u''_i(t, x) &= u_{o,i}(X_i(0; t, x)) \mathcal{E}''_i(0, t, x) + \int_0^t f''_i(\tau, X_i(\tau; t, x)) \mathcal{E}''_i(\tau, t, x) d\tau. \end{aligned} \quad (4.46)$$

Therefore, using (4.46), the change of coordinate  $X_i(0; t, x) = \xi$ ,  $(\mathbf{f})$  and  $(\mathbf{m})$ , we get

$$\begin{aligned} &\int_{\gamma_i(t)}^{+\infty} |u'_i(t, x) - u''_i(t, x)| dx \\ &\leq \int_{\gamma_i(t)}^{+\infty} |u_{o,i}(X_i(0; t, x))| |\mathcal{E}'_i(0, t, x) - \mathcal{E}''_i(0, t, x)| dx \\ &\quad + \int_{\gamma_i(t)}^{+\infty} \int_0^t |f'_i(\tau, X_i(\tau; t, x)) \mathcal{E}'_i(\tau, t, x) - f''_i(\tau, X_i(\tau; t, x)) \mathcal{E}''_i(\tau, t, x)| d\tau dx \\ &\leq \|u_{o,i}\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}_+; \mathbb{R})} \\ &\quad \times \int_{\gamma_i(t)}^{+\infty} \left| \exp\left(-\int_0^t m'_i(s, X(s; t, x)) ds\right) - \exp\left(-\int_0^t m''_i(s, X(s; t, x)) ds\right) \right| \\ &\quad \times \exp\left(-\int_0^t \partial_x g(s, X(s; t, x)) ds\right) dx \\ &\quad + \int_{\gamma_i(t)}^{+\infty} \int_0^t |f'_i(\tau, X_i(\tau; t, x))| |\mathcal{E}'_i(\tau, t, x) - \mathcal{E}''_i(\tau, t, x)| d\tau dx \\ &\quad + \int_{\gamma_i(t)}^{+\infty} \int_0^t |f'_i(\tau, X_i(\tau; t, x)) - f''_i(\tau, X_i(\tau; t, x))| \mathcal{E}''_i(\tau, t, x) d\tau dx \\ &\leq \|u_{o,i}\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}_+; \mathbb{R})} e^{Mt} \\ &\quad \times \int_{\gamma_i(t)}^{+\infty} \int_0^t |m'_i(s, X(s; t, x)) - m''_i(s, X(s; t, x))| ds \exp\left(-\int_0^t \partial_x g(s, X(s; t, x)) ds\right) dx \\ &\quad + F_\infty \int_{\gamma_i(t)}^{+\infty} \int_0^t \left| \exp\left(-\int_\tau^t m'_i(s, X(s; t, x)) ds\right) - \exp\left(-\int_\tau^t m''_i(s, X(s; t, x)) ds\right) \right| \end{aligned}$$

$$\begin{aligned} & \times \exp \left( - \int_{\tau}^t \partial_x g(s, X(s; t, x)) ds \right) d\tau dx \\ & + e^{Mt} \|f'_i - f''_i\|_{L^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\ \leq & \left( \|u_{o,i}\|_{L^\infty([0,t] \times \mathbb{R}_+; \mathbb{R})} + F_\infty t \right) e^{Mt} \|m'_i - m''_i\|_{L^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + e^{Mt} \|f'_i - f''_i\|_{L^1([0,t] \times \mathbb{R}_+; \mathbb{R})}. \end{aligned}$$

Consider the first term in the right-hand side of (4.45). If  $x < \gamma_i(t)$ , then the construction in the first part of the proof implies that

$$\begin{aligned} u'_i(t, x) &= \frac{b'_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}'_i(T_i(0; t, x), t, x) + \int_{T_i(0; t, x)}^t f'_i(\tau, X_i(\tau; t, x)) \mathcal{E}'_i(\tau, t, x) d\tau, \\ u''_i(t, x) &= \frac{b''_i(T_i(0; t, x))}{g_i(T_i(0; t, x), 0)} \mathcal{E}''_i(T_i(0; t, x), t, x) + \int_{T_i(0; t, x)}^t f''_i(\tau, X_i(\tau; t, x)) \mathcal{E}''_i(\tau, t, x) d\tau, \end{aligned} \tag{4.47}$$

where, for  $\tau \in I$ ,

$$\begin{aligned} b'_i(\tau) &= \alpha_i(\tau, u'_1(\tau, \bar{x}_1-), \dots, u'_n(\tau, \bar{x}_n-)) \\ &+ \beta_i \left( \tau, \int_{\mathbb{R}_+} w_1^i(\tau, \xi) u'_1(\tau, \xi) d\xi, \dots, \int_{\mathbb{R}_+} w_n^i(\tau, \xi) u'_n(\tau, \xi) d\xi \right), \\ b''_i(\tau) &= \alpha_i(\tau, u''_1(\tau, \bar{x}_1-), \dots, u''_n(\tau, \bar{x}_n-)) \\ &+ \beta_i \left( \tau, \int_{\mathbb{R}_+} w_1^i(\tau, \xi) u''_1(\tau, \xi) d\xi, \dots, \int_{\mathbb{R}_+} w_n^i(\tau, \xi) u''_n(\tau, \xi) d\xi \right). \end{aligned} \tag{4.48}$$

Preliminary, we claim that, for  $t \in I, t \leq T_1$ ,

$$\begin{aligned} \|b'_i - b''_i\|_{L^1([0,t]; \mathbb{R})} &\leq e^{Mt} \mathbf{Lip}(\alpha) \left( \|u_o\|_{L^\infty(\mathbb{R}_+; \mathbb{R}^n)} + tF_\infty \right) \sum_{j=1}^n \|m'_j - m''_j\|_{L^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\ &+ e^{Mt} \mathbf{Lip}(\alpha) \sum_{j=1}^n \|f'_j - f''_j\|_{L^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\ &+ W \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \|u'_j(\tau) - u''_j(\tau)\|_{L^1(\mathbb{R}_+; \mathbb{R})} d\tau \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} \|b''_i\|_{L^\infty([0,t]; \mathbb{R})} &\leq n \mathcal{L} \mathbf{Lip}(\alpha) \left( 1 + \|u_o\|_{L^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u_o\|_{L^1(\mathbb{R}_+; \mathbb{R}^n)} \right) \\ &+ n \mathbf{Lip}(\beta) \left( \mathcal{L}_1(t) + \mathcal{L}_2(t) \|u_o\|_{L^1(\mathbb{R}_+; \mathbb{R}^n)} \right). \end{aligned} \tag{4.50}$$

Using (4.48), **(b)**, **(f)**, and **(m)**, we deduce that

$$\begin{aligned} \|b'_i - b''_i\|_{L^1([0,t]; \mathbb{R})} &\leq \int_0^t \left| \alpha_i(\tau, u'_1(\tau, \bar{x}_1-), \dots, u'_n(\tau, \bar{x}_n-)) - \alpha_i(\tau, u''_1(\tau, \bar{x}_1-), \dots, u''_n(\tau, \bar{x}_n-)) \right| d\tau \\ &+ \int_0^t \left| \beta_i \left( \tau, \int_{\mathbb{R}_+} w_1^i(\tau, x) u'_1(\tau, x) dx, \dots \right) - \beta_i \left( \tau, \int_{\mathbb{R}_+} w_1^i(\tau, x) u''_1(\tau, x) dx, \dots \right) \right| d\tau \\ &\leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t \left| u'_j(\tau, \bar{x}_j-) - u''_j(\tau, \bar{x}_j-) \right| d\tau \\ &+ W \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \int_{\varepsilon}^{+\infty} \left| u'_j(\tau, x) - u''_j(\tau, x) \right| dx d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t \left| u_{o,j} (X_j (0; \tau, \bar{x}_j) -) \right| \left| \mathcal{E}'_j (0, \tau, \bar{x}_j) - \mathcal{E}''_j (0, \tau, \bar{x}_j) \right| d\tau \\
&\quad + \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t \int_0^\tau \left| f'_j (s, X_j (s; \tau, \bar{x}_j)) - f''_j (s, X_j (s; \tau, \bar{x}_j)) \right| \mathcal{E}'_j (s, \tau, \bar{x}_j) ds d\tau \\
&\quad + \mathbf{Lip}(\alpha) \sum_{j=1}^n \int_0^t \int_0^\tau \left| f''_j (s, X_j (s; \tau, \bar{x}_j)) \right| \left| \mathcal{E}'_j (s, \tau, \bar{x}_j) - \mathcal{E}''_j (s, \tau, \bar{x}_j) \right| ds d\tau \\
&\quad + W \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \left\| u'_j(\tau) - u''_j(\tau) \right\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \\
&\leq e^{Mt} \mathbf{Lip}(\alpha) \sum_{j=1}^n \left\| u_{o,j} \right\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} \left\| m'_j - m''_j \right\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\
&\quad + e^{Mt} \mathbf{Lip}(\alpha) \sum_{j=1}^n \left\| f'_j - f''_j \right\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + F_\infty e^{Mt} \mathbf{Lip}(\alpha) t \sum_{j=1}^n \left\| m'_j - m''_j \right\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\
&\quad + W \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \left\| u'_j(\tau) - u''_j(\tau) \right\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})},
\end{aligned}$$

so that (4.49) holds. Using **(b)** and the estimates (4.31) and (4.32) of Lemma 4.7 we have that, for  $\tau \leq t$ ,  $t \in I$ ,  $t \leq T_1$ ,

$$\begin{aligned}
|b'_i(\tau)| &\leq \mathbf{Lip}(\alpha) \sum_{j=1}^n \left| u'_j(\tau, \bar{x}_j -) \right| + \mathbf{Lip}(\beta) \sum_{j=1}^n \left| \int_{\mathbb{R}_+} w(\tau, \xi) u'_j(\tau, \xi) d\xi \right| \\
&\leq \mathbf{Lip}(\alpha) n \mathcal{L} (1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}) \\
&\quad + \mathbf{Lip}(\beta) n (\mathcal{L}_1(t) + \mathcal{L}_2(t) \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}),
\end{aligned}$$

where  $\mathcal{L}$ ,  $\mathcal{L}_1$ , and  $\mathcal{L}_2$  are defined in Lemma 4.7. This proves (4.50).

Using (4.47), (4.49), (4.50), **(m)**, and **(f)**, we bound the first term in the right-hand side of (4.45):

$$\begin{aligned}
&\int_0^{\gamma_i(t)} |u'_i(t, x) - u''_i(t, x)| dx \\
&\leq \int_0^{\gamma_i(t)} \frac{|b'_i(T_i(0; t, x)) - b''_i(T_i(0; t, x))|}{g_i(T_i(0; t, x), 0)} \mathcal{E}'_i(T_i(0; t, x), t, x) dx \\
&\quad + \int_0^{\gamma_i(t)} \frac{|b''_i(T_i(0; t, x))|}{g_i(T_i(0; t, x), 0)} \left| \mathcal{E}'_i(T_i(0; t, x), t, x) - \mathcal{E}''_i(T_i(0; t, x), t, x) \right| dx \\
&\quad + \int_0^{\gamma_i(t)} \int_{T_i(0; t, x)}^t |f'_i(\tau, X_i(\tau; t, x)) - f''_i(\tau, X_i(\tau; t, x))| \mathcal{E}'_i(\tau, t, x) d\tau dx \\
&\quad + \int_0^{\gamma_i(t)} \int_{T_i(0; t, x)}^t |f''_i(\tau, X_i(\tau; t, x))| \left| \mathcal{E}'_i(\tau, t, x) - \mathcal{E}''_i(\tau, t, x) \right| d\tau dx \\
&\leq e^{Mt} \|b'_i - b''_i\|_{\mathbf{L}^1([0,t]; \mathbb{R})} + \int_0^t |b''_i(\tau)| \left| \mathcal{E}'_i(\tau, t, X_i(t; \tau, 0)) - \mathcal{E}''_i(\tau, t, X_i(t; \tau, 0)) \right| d\tau \\
&\quad + e^{Mt} \|f'_i - f''_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + F_\infty e^{Mt} t \|m'_i - m''_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})}
\end{aligned}$$

$$\begin{aligned}
&\leq e^{2Mt} \mathbf{Lip}(\alpha) \left( \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + tF_\infty \right) \sum_{j=1}^n \|m'_j - m''_j\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\
&\quad + e^{2Mt} \mathbf{Lip}(\alpha) \sum_{j=1}^n \|f'_j - f''_j\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + e^{Mt} W \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \|u'_j(\tau) - u''_j(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \\
&\quad + n \mathcal{L} \mathbf{Lip}(\alpha) \left( 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right) e^{Mt} \|m'_i - m''_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\
&\quad + n \mathbf{Lip}(\beta) \left( \mathcal{L}_1(t) + \mathcal{L}_2(t) \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right) e^{Mt} \|m'_i - m''_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} \\
&\quad + e^{Mt} \|f'_i - f''_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})} + F_\infty e^{Mt} t \|m'_i - m''_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R})}.
\end{aligned}$$

Finally, by (4.45), we have that

$$\begin{aligned}
&\sum_{i=1}^n \|u'_i(t) - u''_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \\
&\leq \left[ (\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + 2F_\infty t) e^{Mt} + e^{2Mt} \mathbf{Lip}(\alpha) (\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + tF_\infty) \right. \\
&\quad \left. + n \mathcal{L} \mathbf{Lip}(\alpha) (1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}) e^{Mt} \right. \\
&\quad \left. + n \mathbf{Lip}(\beta) (\mathcal{L}_1(t) + \mathcal{L}_2(t) \|u_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}) e^{Mt} \right] \|m' - m''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R}^n)} \\
&\quad + [2e^{Mt} + e^{2Mt} \mathbf{Lip}(\alpha)] \|f' - f''\|_{\mathbf{L}^1([0,t] \times \mathbb{R}_+; \mathbb{R}^n)} \\
&\quad + e^{Mt} W \mathbf{Lip}(\beta) \int_0^t \sum_{i=1}^n \|u'_i(\tau) - u''_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})}.
\end{aligned}$$

By Grönwall lemma, we obtain the existence of a positive constant  $\mathcal{M}$ , depending only on  $T_1$  and on the constants defined in **(b)**, **(f)**, **(g)**, and **(m)**, such that

$$\sum_{i=1}^n \|u'_i(t) - u''_i(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R})} \leq \mathcal{M} e^{Mt} \left[ \|m' - m''\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} + \|f' - f''\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \right].$$

Iterating on  $[0, T_1]$ ,  $[T_1, 2T_1]$ , and so on, we deduce that the stability estimate (2.4) holds under assumption (4.36), if  $\alpha' = \alpha''$ ,  $\beta' = \beta''$ , and  $w' = w''$ . Remark that  $\mathcal{M}$  does not depend on  $\varepsilon$ .

### $\mathbf{L}^\infty$ estimate

Consider two initial conditions, namely,  $u'_o$  and  $u''_o$ , and fix  $t \in [0, T_1]$ . Denoting with  $u'$  and  $u''$  the solutions corresponding to  $u'_o$  and  $u''_o$ , we have, for all  $i \in \{1, \dots, n\}$ ,

$$\|u'_i(t) - u''_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} \leq \max \left\{ \|u'_i(t) - u''_i(t)\|_{\mathbf{L}^\infty([0, \gamma_i(t)]; \mathbb{R})}, \|u'_i(t) - u''_i(t)\|_{\mathbf{L}^\infty([\gamma_i(t), +\infty[; \mathbb{R})} \right\} \quad (4.51)$$

and bound both terms in the right-hand side above. Consider first the  $i$ th component of the second one. Use (4.38) and (4.14):

$$\begin{aligned}
\|u'_i(t) - u''_i(t)\|_{\mathbf{L}^\infty([\gamma_i(t), +\infty[; \mathbb{R})} &\leq \|\mathcal{E}_i(0, t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} \\
&\leq e^{(G_1+M)t} \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)}.
\end{aligned}$$

Consider now the first term in the right-hand side of (4.51), using (4.14) and (4.40),

$$\begin{aligned}
\|u'_i(t) - u''_i(t)\|_{\mathbf{L}^\infty([0, \gamma_i(t)]; \mathbb{R})} &\leq \frac{1}{g} \|\mathcal{E}_i(T_i(0; t, \cdot), t, \cdot)\|_{\mathbf{L}^\infty([0, \gamma_i(t)]; \mathbb{R})} \|b'_i(T_i(0; t, \cdot)) - b''_i(T_i(0; t, \cdot))\|_{\mathbf{L}^\infty([0, \gamma_i(t)]; \mathbb{R})} \\
&\leq \frac{1}{g} e^{(G_1+M)t} \|b'_i - b''_i\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})}.
\end{aligned}$$

For all  $\tau \in [0, t]$ , we have

$$\begin{aligned}
& |b'_i(\tau) - b''_i(\tau)| && \text{[Use (4.39)]} \\
& \leq \left| \alpha_i(\tau, u'_1(\tau, \bar{x}_1-), \dots, u'_n(\tau, \bar{x}_n-)) - \alpha_i(\tau, u''_1(\tau, \bar{x}_1-), \dots, u''_n(\tau, \bar{x}_n-)) \right| && \text{[Use (b)]} \\
& \quad + \left| \beta_i \left( \tau, \int_{\mathbb{R}_+} w_1^i(\tau, x) u'_1(\tau, x) dx, \dots \right) - \beta_i \left( \tau, \int_{\mathbb{R}_+} w_1^i(\tau, x) u''_1(\tau, x) dx, \dots \right) \right| \\
& \leq \mathbf{Lip}(\alpha_i) \sum_{j=1}^n \left| u'_j(\tau, \bar{x}_j-) - u''_j(\tau, \bar{x}_j-) \right| \\
& \quad + \mathbf{Lip}(\beta_i) \sum_{j=1}^n \int_{\varepsilon}^{+\infty} |w_j^i(\tau, x)| \left| u'_j(\tau, x) - u''_j(\tau, x) \right| dx && \text{[By (4.36)]} \\
& \leq n \mathbf{Lip}(\alpha) e^{(G_1+M)t} \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} + n \mathbf{Lip}(\beta) W e^{(G_1+M)t} \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u'_i(t) - u''_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} & \leq \left( \frac{n \mathbf{Lip}(\alpha) e^{(G_1+M)t}}{\check{g}} + 1 \right) e^{(G_1+M)t} \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)} \\
& \quad + \frac{n \mathbf{Lip}(\beta) W e^{2(G_1+M)t}}{\check{g}} \|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)}.
\end{aligned}$$

Iterating on the time intervals  $[0, T_1]$ ,  $[T_1, 2T_1]$ , and so on permits to conclude the proof of (2.2).

### Positivity

This point directly follows from the explicit expressions (4.38), (4.39), and (4.40).

### Assumption (4.36) does not hold

Fix an initial condition  $u_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}_+; \mathbb{R}^n)$  and data  $\alpha, \beta$ , and  $w$  satisfying (b). Consider, for every  $k \in \mathbb{N} \setminus \{0\}$ , the sequence  $w_k$  where

$$(w^k)_j^i(t, x) = \begin{cases} w_j^i(t, x), & \text{if } x > \frac{1}{k}, \\ 0, & \text{if } x \leq \frac{1}{k}. \end{cases} \quad (4.52)$$

for all  $i, j \in \{1, \dots, n\}$ . Denote by  $u^k$  the solution corresponding to  $w^k$ , which exists since  $w^k(t, x) = 0$  for  $t \in I$  and  $x \in [0, \frac{1}{k}]$ . We prove that  $u^k$  is a Cauchy sequence in  $\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))$ .

By (2.4), applicable since  $w^{k'}$  and  $w^{k''}$  satisfy (4.36), there exists  $\mathcal{H} \in \mathbf{C}^0(I; \mathbb{R}_+)$  such that

$$\|u^{k'}(t) - u^{k''}(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n)} \leq \mathcal{H}(t) \|w^{k'} - w^{k''}\|_{\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^{n \times n}))} \quad (4.53)$$

for every  $k', k'' \in \mathbb{N} \setminus \{0\}$  with  $k' < k''$ . Moreover, by (4.52) and (b), for every  $t \in I$  and  $k', k'' \in \mathbb{N} \setminus \{0\}$  with  $k' < k''$ , we have

$$\|w^{k'}(t) - w^{k''}(t)\|_{\mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^{n \times n})} \leq \sum_{i=1}^n \int_{\frac{1}{k''}}^{\frac{1}{k'}} |w_i(t, x)| dx \leq \frac{nW}{k'}.$$

Thus, (4.53) implies that  $u^k$  is a Cauchy sequence in  $\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))$ ; hence, there exists  $u^* \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))$  such that  $u^k$  converges to  $u^*$  in  $\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}_+; \mathbb{R}^n))$ . For every  $k \in \mathbb{N} \setminus \{0\}$ , each component of  $u^k$  is a regular entropy solution to an IBVP similar to (4.41). Therefore, each component of  $u^*$  is also a regular entropy solution; see Rossi<sup>16</sup>, Remark 3.6 and Málek et al.<sup>29</sup>, Chapter 2, Remark 7.33 Hence,  $u^*$  solves (1.1) with initial datum  $u_o$  and data  $\alpha, \beta$ , and  $w$ .  $\square$

*Proof of Corollary 2.4.* Let  $\vartheta_k$  be a sequence of positive mollifiers in  $\mathbf{C}^\infty$  converging to the Dirac delta centered at the origin. Assume that  $f$  and  $m$  satisfy (F) and (M). Define  $f_k = f * \vartheta_k$  and  $m_k = m * \vartheta_k$ . Then, the computations related to  $f$  being entirely similar,  $m_k \in \mathbf{C}^0(I \times \mathbb{R}_+; \mathbb{R})$  by the standard properties of the convolution. Moreover, for  $t \in I$ ,

$$\|m_k(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbb{R})} \leq \|m_k\|_{\mathbf{L}^\infty(I \times \mathbb{R}_+; \mathbb{R})} \leq \|m\|_{\mathbf{L}^\infty(I \times \mathbb{R}_+; \mathbb{R})} \|\vartheta_k\|_{\mathbf{L}^1(I \times \mathbb{R}_+; \mathbb{R})} = \|m\|_{\mathbf{L}^\infty(I \times \mathbb{R}_+; \mathbb{R})}$$

and similar, elementary but lengthier, computations ensure that an analogous inequality holds between the total variations. For every  $k \in \mathbb{N}$ , Theorem 2.2 can be applied with the source term  $f_k$  and the mortality function  $m_k$ ,

producing a solution  $u^k$ . Finally, Theorem 2.3 allows to pass to the limit to the sequence  $u^k$  as  $k \rightarrow +\infty$ , concluding the proof.  $\square$

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## CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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