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The Follow-The-Leader model without a leader: An infinite-dimensional Cauchy problem

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A B S T R A C T

We introduce a *Follow-The-Leader* model where all drivers are indistinguishable, in the sense that each driver has a role and a behavior depending on the driver immediately in front, but none of them has the *a-priori* privilege to drive at maximal speed. We prove that the resulting Cauchy Problem with infinitely many differential equations admits a global unique solution. The total variation of the discrete density being uniformly bounded, as the proper length of each vehicle vanishes the infinite microscopic model converges to the macroscopic LWR model based on a first order PDE. Finally, the case of traffic flow with monotone density is discussed, with applications to real situations.

1. Introduction

The literature on the modeling of vehicular traffic offers a large variety of different approaches, typically they can be characterized as either macroscopic, microscopic, kinetic or based on cellular automata. In particular, *macroscopic* descriptions are usually based on partial differential equations describing traffic viewed as a compressible fluid flow, while *microscopic* descriptions consist of a finite set of ordinary differential equations, describing the motion of each vehicle. For a combination of the two approaches see [4–6,12,18,21].

The basic model for the macroscopic ones is the classical Lighthill-Whitham [20] and Richards [24] (LWR) traffic model, given by the following scalar conservation law

$$\partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0\,, \tag{1.1}$$

which expresses the conservation of the number of cars, where $\rho = \rho(t,x)$ is the traffic density at time t and position x, while $v : [0,1] \to [0, V_{\max}]$ is the speed law, for a suitable positive $V_{\max}$.

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On the other hand, the prototype of microscopic descriptions is the so called *Follow-The-Leader* (FTL) model, where the traffic dynamics is governed by the interaction between a vehicle and the vehicle immediately in front. More precisely, in its classical version, see [13], the Cauchy problem takes the form

$$\begin{cases} \dot{x}_i = v\left(\frac{\ell}{x_{i+1}-x_i}\right) & i = 1,\dots,n\,, \\ \dot{x}_{n+1} = V_{\max} & \\ x_i(0) = x_i^0 & i = 1,\dots,n+1\,, \end{cases} \tag{1.2}$$

where $x_i = x_i(t)$ is the position of the i-th driver at $t \geq 0$, x_i^0 is the initial position and ℓ is the vehicle's length. The function v, as in (1.1), is the velocity of drivers and $V_{\max} = \max v$ is the speed assigned to the *leader*, i.e., to the first driver; in the above model each driver adjusts his/her velocity according to the distance to the vehicle in front. We fix $L > 0$ and assume that at time $t = 0$ the $n+1$ drivers are distributed along $[-L, L]$, that is $-L = x_1^0 < x_2^0 < \dots < x_n^0 < x_{n+1}^0 = L - \ell$. In this model a crucial role is played by the *leader*, who in (1.2) is denoted by x_{n+1} and moves with fixed maximal speed $\dot{x}_{n+1} = V_{\max}$, as if the road in front were empty. A common slight extension leads to let the leader travel with an *a priori* assigned speed, i.e., $V_{\max} = V_{\max}(t)$.

In the applications to real traffic a condition of this type is often hardly acceptable as natural: the specific behavior assigned to the leader heavily influences the behavior of the whole population. As a consequence, for instance, optimal control problems based on (1.2) are in general depending on the particular route assigned to the leader.

In general, in presence of a leader, the resulting framework is not always realistic. The present approach allows to get rid of this somewhat artificial ingredient – the leader's movement: all drivers are here considered with equivalent characteristics.

In this paper we introduce a *Follow-The-Leader Model without a Leader*. We propose a model where all drivers are indistinguishable, in the sense that each driver x_i has a role and a behavior depending on x_{i+1}, but none of them has the *a-priori* privilege to drive at maximal speed. More precisely, a rigorous formulation of the model consists of a Cauchy Problem with infinitely many differential equations and initial conditions indexed by $i \in \mathbb{Z}$. That is

$$\begin{cases} \dot{x}_i = v\left(\frac{\ell}{x_{i+1}-x_i}\right) & i \in \mathbb{Z}\,, \\ x_i(0) = x_i^0 & i \in \mathbb{Z}\,, \end{cases} \tag{1.3}$$

where, as before, $x_i = x_i(t)$ is the position of the i-th driver, ℓ is the vehicle's length and v is the speed function which satisfies the condition

(V) $v \in \mathbf{C^{0,1}}([0,1]; [0, V_{\max}])$ is a decreasing function such that $v(\rho) = 0$ for all $\rho \geq 1$.

Again, each driver adjusts his/her velocity to the vehicle in front. The drivers at the initial position are ordered and distributed all along $(-\infty, +\infty)$, that is

$$\dots < x_{-i}^0 < x_{-i+1}^0 < \dots < x_{-1}^0 < x_0^0 < x_1^0 < \dots < x_i^0 < x_{i+1}^0 < \dots$$

and all remaining at least at ℓ distance from each other, i.e.,

$$x_{i+1}^0 - x_i^0 \geq \ell \quad \forall i \in \mathbb{Z}\,. \tag{1.4}$$

We prove below that the above inequality remains true along the solutions to (1.3). The maximal speed $V_{\max}$ steel plays a relevant role, although not explicitly imposed on any driver. The maximal speed is a bound for the velocity $v : [0, 1] \to [0, V_{\max}]$. In this paper, we do not assume that v is strictly decreasing for $\rho \in [0, 1]$; in fact, a strictly decreasing speed law implies that x_i drives at the maximal speed if and only if there are no drivers in front of x_i. A more realistic model is obtained introducing a threshold ρ_0 such that $V(\rho) = V_{\max}$ for all $\rho \in [0, \rho_0]$, which requires v not to be strictly decreasing.

We prove in Section 2 (see Theorem 2.2) that the *infinite Cauchy Problem* (1.3) admits a unique solution $(x_i(t))_{i\in\mathbb{Z}}$, globally defined in a given time interval $[0, T]$. We remark that this theorem does not require the space variable to be merely one dimensional. This feature, in particular, allows to apply the present result to the microscopic modeling of crowd movements where, typically, $x \in \mathbb{R}^2$, see § 2.2.

In the case of vehicular traffic, moreover, we also ensure that the solution satisfies

$$x_{i+1}(t) - x_i(t) \geq \ell, \quad \forall t \in [0, T],\ \forall i \in \mathbb{Z}\,.$$

We then use the solutions $(x_i(t))_{i\in\mathbb{Z}}$ to the Cauchy Problem (1.3) to define an approximate solution to the conservation law (1.1), as was already considered for instance in [1,2,8,11,14–16]. We prove that the solution to (1.1) is obtained as limit of the *Follow-The-Leader model without a Leader* (1.3). This is formalized in Sections 3, 4 and 5. More precisely, in Section 3 we consider the *discrete density* $\rho_{discr}(t, x)$, defined in (3.1), and we prove that it has *bounded total variation* locally in any compact set of $[0, +\infty[\times\mathbb{R}$, provided this property holds for the initial datum at $t = 0$. This is achieved assuming that a driver travels at a constant speed, not necessarily being the maximal one (Section 4). This step is obtained adapting an *a-priori* bound recently provided by Di Francesco and Rosini in [10], where the condition that the leader drives at the maximal velocity plays a relevant role.

In Section 5, following [7], we prove that the infinite microscopic model (1.3) yields the macroscopic one (1.1) as the vehicle length ℓ converges to zero. Finally, in Section 6, we study the case of traffic flow with monotone density, both decreasing and increasing, and we interpret it as an approach to the real situation of a traffic light.

2. The Cauchy theorem with infinitely many equations

With the aim to obtain the global existence of (1.3) we consider the following form of the Cauchy problem for systems of infinitely many differential equations of the form

$$\begin{cases} \dot{x}_i = f_i\left(t, ..., x_j(t), ...\right) & i, j \in \mathbb{Z} \\ x_i(0) = x_i^0 & i \in \mathbb{Z}\,. \end{cases} \tag{2.1}$$

In the mathematical literature we can find the problem (2.1) in the framework of general Banach spaces. See for instance Deimling [9]; in particular Theorem 6.4 and its Corollary 6.1 in [9], with f_i bounded and other additional conditions. See also Lakshmikantham-Leela [17] and Martin [23] under compactness conditions on $f = (f_i)_{i\in\mathbb{Z}}$. We give here a direct proof of the existence and uniqueness of the solution of the Cauchy problem (2.1) under natural conditions on f that can be easily tested in our context of the Follow-The-Leader (1.3). In order to treat the Cauchy problem (2.1) in its generality without further assumptions on $(f_i)_{i\in\mathbb{Z}}$, we introduce the notation $x = (x_i)_{i\in\mathbb{Z}}$, $f = (f_i)_{i\in\mathbb{Z}}$ and (2.1) in compact form

$$\begin{cases} \dot{x} = f\left(t, x\right) \\ x(0) = x^0\,. \end{cases} \tag{2.2}$$

Given positive numbers M, T, we denote by B_{MT} the functional set

$$B_{MT} = \left\{ x(t) = (x_i(t))_{i\in\mathbb{Z}} : \, x_i \in \mathbf{C^0}([0, T]),\ \left\| x_i(t) - x_i^0 \right\|_{L^\infty([0,T])} \leq MT,\, \forall i \in \mathbb{Z} \right\}. \tag{2.3}$$

As proved below, the set B_{MT} is a metric space under the distance $d(x^\alpha, x^\beta) = \left\|x^\alpha - x^\beta\right\|_{B_{MT}}$ with

$$\|x(t)\|_{B_{MT}} = \sum_{i=0}^{\infty} 2^{-i}\|x_i(t)\|_{\mathbf{L}^\infty([0,T])} + \sum_{i=-1}^{-\infty} 2^{i}\|x_i(t)\|_{\mathbf{L}^\infty([0,T])}\,. \tag{2.4}$$

Remark 2.1. For any fixed $t \geq 0$, the property that $x(t)$ is bounded in the norm of B_{MT} in (2.4) is not equivalent to be bounded neither in ℓ^∞, the space of $x = (x_i)_{i\in\mathbb{Z}}$ such that $|x_i| \leq M$ for every $i \in \mathbb{Z}$ and for some $M \geq 0$, nor in ℓ^p, $p \in [1, +\infty[$. In fact if $x = (..., 0, 0, x_i, 0, 0, ...)$ with $i > 0$, then $\|x\|_p = \|x\|_\infty = |x_i|$ and $\|x\|_{B_{MT}} = 2^{-i}|x_i| \leq M$ if and only if $|x_i| \leq M2^i$. However if $x(t) \in B_{MT}$ for some t, then as set inclusion $x(t) \in \ell^\infty$; i.e., B_{MT} is an ℓ^∞-ball of continuous functions in $[0, T]$ with center in $x^0 = (x_i^0)_{i\in\mathbb{Z}}$ and radius MT. In the proof of Theorem 2.2 we are concerned with $x = (x_i)_{i\in\mathbb{Z}}$ with equibounded components x_i because of the main assumption (2.8).

We now prove that (2.4) defines a norm and that B_{MT} is a complete metric space. The distance in (2.4) is finite since

$$\begin{aligned} d(x^\alpha, x^\beta) = \left\|x^\alpha - x^\beta\right\|_{B_{MT}} &\leq \left\|x^\alpha - x^0\right\|_{B_{MT}} + \left\|x^\beta - x^0\right\|_{B_{MT}} \\ &\leq 2MT\left(\sum_{i=0}^{\infty} 2^{-i} + \sum_{i=-1}^{-\infty} 2^{i}\right) = 6MT\,. \end{aligned}$$

B_{MT} is a complete metric space. In fact, if x^n is a Cauchy sequence in B_{MT}, then $\forall\varepsilon > 0$, there exist ν such that $d(x^n, x^m) < \varepsilon$ for every $n, m > \nu$. For every fixed $i \in \mathbb{Z}$, for instance when $i \geq 0$, we have

$$2^{-i}\|x_i^n - x_i^m\|_{\mathbf{C^0}([0,T])} \leq \|x^n - x^m\|_{B_{MT}} = d(x^n, x^m) < \varepsilon\,.$$

Therefore every component $x_i^n(t)$ of $x^n(t)$ is a Cauchy sequence in $\mathbf{C^0}([0, T])$ and uniformly converges in $[0, T]$ to a continuous function which is the i-component of the limit of x^n. We denote by $\bar{x}_i(t)$ this limit; i.e., $\lim_{n\to+\infty} \|x_i - \bar{x}_i\|_{\mathbf{C^0}([0,T])} = 0$. We also use the notation $\bar{x} = (\bar{x}_i)_{i\in\mathbb{Z}}$. We need to prove that

$$\lim_{n\to+\infty} \|x^n - \bar{x}\|_{B_{MT}} = 0\,. \tag{2.5}$$

To this aim we fix $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0}^{+\infty} 2^{-i} = 2^{-i_0}\sum_{i=0}^{+\infty} 2^{-i} = 2^{1-i_0} < \varepsilon\,. \tag{2.6}$$

Then, the set of indexes $\{0, \pm1, ..., \pm i_0\}$ is finite and we consider ν such that

$$\|x_i^n - \bar{x}_i\|_{\mathbf{C^0}([0,T])} < \varepsilon, \;\; \forall n > \nu, \;\; \forall\, |i| \leq i_0\,. \tag{2.7}$$

Since by (2.6), $\sum_{i=i_0-1}^{-\infty} 2^i = \sum_{i=i_0+1}^{+\infty} 2^{-i} < \varepsilon$, and since $\|x^n - \bar{x}\|_{\mathbf{C^0}([0,T])} \leq \left\|x^n - x^0\right\|_{\mathbf{C^0}([0,T])} + \left\|\bar{x} - x^0\right\|_{\mathbf{C^0}([0,T])} \leq 2MT$, we obtain

$$\begin{aligned} d(x^n, \bar{x}) &= \|x^n - \bar{x}\|_{B_{MT}} = \\ &= \sum_{i=0}^{i_0} 2^{-i}\|x_i^n - \bar{x}_i\|_{\mathbf{C^0}([0,T])} + \sum_{i=-1}^{-i_0} 2^{i}\|x_i^n - \bar{x}_i\|_{\mathbf{C^0}([0,T])} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=i_0+1}^{+\infty} 2^{-i}\|x_i^n - \bar{x}_i\|_{\mathbf{C^0}([0,T])} + \sum_{i=-i_0-1}^{-\infty} 2^{i}\|x_i^n - \bar{x}_i\|_{\mathbf{C^0}([0,T])} \\
&\leq \varepsilon \left(\sum_{i=0}^{+\infty} 2^{-i} + \sum_{i=-1}^{-\infty} 2^{i} \right) + 2MT \left(\sum_{i=i_0+1}^{+\infty} 2^{-i} + \sum_{i=-i_0-1}^{-\infty} 2^{i} \right) \\
&\leq 3\varepsilon + 4MT\varepsilon = (3 + 4MT)\varepsilon \,,
\end{aligned}
$$

for every $n > \nu$ where ν is defined in (2.7). Therefore x^n converges to $\bar{x}$ in the B_{MT}-norm defined in (2.4) and the proof of (2.5) is complete.

We are in the position to state the Cauchy Theorem in this context. Under the previous definition of the metric space B_{MT}, the proof follows the usual steps of the standard Cauchy Theorem.

Theorem 2.2. *Let $f(t,x) = (f_i(t,x))_{i\in\mathbb{Z}}$ be a function defined for $t \in [0,T]$ and for every $x = (x_i)_{i\in\mathbb{Z}}$. We assume that every component $f_i(t,x)$ is continuous and equibounded by a constant M, i.e.*

$$|f_i(t,x)| \leq M, \qquad \forall i \in \mathbb{Z}\,, t \in [0,T]\,, x = (x_i)_{i\in\mathbb{Z}} \tag{2.8}$$

and Lipschitz continuous with respect to x in B_{MT}, i.e.,

$$\left\|f(t,x^\alpha) - f(t,x^\beta)\right\|_{B_{MT}} \leq L\left\|x^\alpha - x^\beta\right\|_{B_{MT}}\,, \tag{2.9}$$

for some constant $L > 0$. Then, there exists a unique function $x(t) = (x_i(t))_{i\in\mathbb{Z}}$, with $x_i(t)$ in the class $\mathbf{C^1}[0,T]$ for all $i \in \mathbb{Z}$, which solves in $[0,T]$ the Cauchy problem (2.1).

Proof. We consider the map in integral form

$$F(x(t)) = x^0 + \int_0^t f(s,x(s))ds \tag{2.10}$$

which is an application from B_{MT} to B_{MT}; in fact $F(x(t))$ has components $F(x(t)) = (F_i(x(t)))_{i\in\mathbb{Z}}$ of the form

$$F_i(x(t)) = x_i^0 + \int_0^t f_i(s,x(s))ds \tag{2.11}$$

and thus, by (2.8), for every $i \in \mathbb{Z}$,

$$\left\|F_i(x(t)) - x_i^0\right\|_{L^\infty([0,T])} \leq \int_0^t |f_i(s,x(s))|ds \leq MT\,. \tag{2.12}$$

In order to verify that $F : B_{MT} \to B_{MT}$ is a contraction, we consider $t \in [0,\delta] \subset [0,T]$ for some δ to be fixed below. We obtain

$$\left\|F(x^\alpha) - F(x^\beta)\right\|_{B_{MT}} \leq \int_0^t \left\|f(s,x^\alpha(s)) - f(s,x^\beta(s))\right\|_{B_{MT}} ds \tag{2.13}$$

$$\leq L\int_0^t \left\|x^\alpha - x^\beta\right\|_{B_{MT}} ds \leq L\delta\left\|x^\alpha - x^\beta\right\|_{B_{MT}}. \tag{2.14}$$

If $\delta < 1/L$, then F is a contraction on B_{MT} and it has a unique fixed point $F(x(t)) = x(t)$ and we obtain the equivalent integral formulation of the Cauchy problem (2.1)

$$x(t) = x^0 + \int_0^t f(s, x(s))ds\,. \tag{2.15}$$

We found a local solution of the Cauchy problem (2.1) in the interval $[0, \delta]$. Since f is bounded by (2.8), with a standard argument we itarate the process starting from the initial time $t = \delta$; after a finite number of interations we obtain that $x(t)$ is a solution of the Cauchy problem (2.1) in the whole interval $[0, T]$. □

2.1. The case of traffic modeling

Finally we can prove that the stated problem (1.3) on the *Follow-The-Leader Model without a Leader* has a solution. To this aim we consider the function $f(x) = (f_i(x))_{i\in\mathbb{Z}}$ whose components, for every $i \in \mathbb{Z}$, are defined by

$$f_i(x) = \begin{cases} v\left(\frac{\ell}{x_{i+1}(t)-x_i(t)}\right) & \text{if } x_{i+1}(t) - x_i(t) \quad \geq \ell \\ 0 & \text{if } x_{i+1}(t) - x_i(t) \quad < \ell\,. \end{cases} \tag{2.16}$$

Theorem 2.3. *Let v satisfy* **(V)** *and (1.4) for the initial data holds. The Cauchy problem with infinitely many differential equations (1.3) admits a unique global solution $x(t) = (x_i(t))_{i\in\mathbb{Z}}$ with $x_i(t)$ in the class $\mathbf{C^1}[0, T]$ for all $i \in \mathbb{Z}$. Moreover*

$$x_{i+1}(t) - x_i(t) \geq \ell \quad \forall t \in [0, T],\ \forall i \in \mathbb{Z}\,. \tag{2.17}$$

Proof. The function $f(x) = (f_i(x))_{i\in\mathbb{Z}}$ defined in (2.16) is continuous since $f_i = 0$ when $x_{i+1}(t) - x_i(t) = \ell$. It is also Lipshitz continuous and bounded. In fact, if we limit to the case $x_{i+1}(t) - x_i(t) \geq \ell$,

$$\left\|f(x^\alpha(t)) - f(x^\beta(t))\right\|_{B_{MT}} = \sum_{i\in\mathbb{Z}} 2^{-i}\left\|f_i(x^\alpha(t)) - f_i(x^\beta(t))\right\|_{L^\infty[0,T]} \tag{2.18}$$
$$= \sum_{i\in\mathbb{Z}} 2^{-i}\left\|v\left(\frac{\ell}{x^\alpha_{i+1}(t) - x^\alpha_i(t)}\right) - v\left(\frac{\ell}{x^\beta_{i+1}(t) - x^\beta_i(t)}\right)\right\|_{L^\infty[0,T]}.$$

If we denote with L_V the Lipshitz constant of the speed v satisfying **(V)**,

$$\left|v\left(\frac{\ell}{x^\alpha_{i+1}(t) - x^\alpha_i(t)}\right) - v\left(\frac{\ell}{x^\beta_{i+1}(t) - x^\beta_i(t)}\right)\right| \leq \frac{L_V}{\ell}\left(\left|x^\alpha_i(t) - x^\beta_i(t)\right| + \left|x^\alpha_{i+1}(t) - x^\beta_{i+1}(t)\right|\right).$$

From (2.18), we get

$$\begin{aligned} &\left\|f(x^\alpha(t)) - f(x^\beta(t))\right\|_{B_{MT}} \\ &\leq \frac{L_V}{\ell}\sum_{i\in\mathbb{Z}} 2^{-i}\left\|x^\alpha_i - x^\beta_i\right\|_{L^\infty[0,T]} + \frac{L_V}{\ell}\sum_{i\in\mathbb{Z}} 2^{-i}\left\|x^\alpha_{i+1} - x^\beta_{i+1}\right\|_{L^\infty[0,T]} \\ &= \frac{L_V}{\ell}\sum_{i\in\mathbb{Z}} 2^{-i}\left\|x^\alpha_i - x^\beta_i\right\|_{L^\infty[0,T]} + 2\frac{L_V}{\ell}\sum_{i\in\mathbb{Z}} 2^{-(i+1)}\left\|x^\alpha_{i+1} - x^\beta_{i+1}\right\|_{L^\infty[0,T]} \\ &= 3\frac{L_V}{\ell}\left\|x^\alpha - x^\beta\right\|_{B_{MT}}. \end{aligned}$$

Therefore $f(x) = (f_i(x))_{i\in\mathbb{Z}}$ defined in (2.16) is Lipshitz continuous and satisfies the condition (2.9) with constant $L = 3\frac{L_V}{\ell}$. We can apply the global existence Theorem 2.2. With the argument of [7, Proposition 4.1], under the assumption (1.4) we can prove that (2.17) holds. Therefore the solution $x(t) = (x_i(t))_{i\in\mathbb{Z}}$ of the Cauchy problem with right hand side given by (2.16) is also a solution to (1.3). □

2.2. The case of crowd dynamics

In the microscopic modeling of crowd's movements we typically have that each individual is described through its time dependent position $x_i = x_i(t)$, with $x_i \in \mathbb{R}^2$. Then, a reasonable model reads

$$\begin{cases} \dot{x}_i = v\left(\min_{j\neq i}\|x_j - x_i\|\right) d_i(t) & i\in\mathbb{Z} \\ x_i(0) = x_i^o\,. & i\in\mathbb{Z} \end{cases} \tag{2.19}$$

Here the speed law v is a Lipschitz continuous real *increasing* function. Indeed, it describes the velocity at which an individual moves as a function of his/her distance from the nearest neighbor. The, possibly time dependent, vector d_i gives the direction of the i–th individual; precisely $d_i(t) = (\cos(\alpha_i(t), \sin(\alpha_i(t))$ and $\alpha_i(t)$ is the direction-angle of the individual x_i at the time t.

Theorem 2.2 ensures the existence and uniqueness of solutions to (2.19).

Theorem 2.4. *Let $v \in \mathbf{C}^{\mathbf{0},\mathbf{1}}(\mathbb{R};\mathbb{R})$ be a bounded increasing function. Let $e_i \in \mathbf{C}^{\mathbf{0}}([0,T];\mathbb{R}^2)$ be such that $\|d_i(t)\| = 1$. Then, for any initial datum $(x_i^o)_{i\in\mathbb{Z}}$, problem (2.19) admits a unique solution.*

Proof. The proof consists in showing that the map

$$(t,x) \to v(t,x) \quad \text{where} \quad v_i(t,x) = v\left(\min_{j\neq i}\|x_j - x_i\|\right) d_i(t) \quad \text{for} \quad i\in\mathbb{Z}$$

is continuous, bounded and Lipschitz continuous in x. The former two properties are immediate. Let L_p be a Lipschitz constant for v and consider the latter term:

$$\begin{aligned} \|v_i(t,x') - v(t,x'')\| &= \left| v\left(\min_{j\neq i}\|x_j - x_i'\|\right) - v\left(\min_{j\neq i}\|x_j - x_i'\|\right)\right| \|d_i(t)\| \\ &\le L_p \left|\min_{j\neq i}\|x_j - x_i'\| - \min_{j\neq i}\|x_j - x_i'\|\right|. \end{aligned}$$

Assume now that $\min_{j\neq i}\|x_j - x_i'\| = \|x_{j'} - x_i'\|$ and $\min_{j\neq i}\|x_j - x_i''\| = \|x_{j''} - x_i''\|$. We then have:

$$\begin{aligned} &\left|\min_{j\neq i}\|x_j - x_i'\| - \min_{j\neq i}\|x_j - x_i'\|\right| \\ &= \left|\min\left\{\|x_{j'} - x_i'\|, \|x_{j''} - x_i'\|\right\} - \min\left\{\|x_{j'} - x_{i\cdot}'\|, \|x_{j''} - x_i'\|\right\}\right| \end{aligned}$$

and the Lipschitz estimate follows, since the map $x \to \min\{\|a - x\|, \|b - x\|\}$ is Lipschitz continuous, as it immediately follows from the equality $\min\{\alpha,\beta\} = (\alpha + \beta - |\alpha - \beta|)/2$. □

The case considered in this section, i.e. the crowd's movements, is a typical example where the existence of a leader is not a realistic assumption. The number of individuals could be either finite or infinite. The Cauchy Theorem 2.2 can be applied to this framework.

3. A-priori local bound for the total variation

In order to prove that the infinite microscopic model (1.3) yields the macroscopic one (1.1), we consider the *discrete density* $\rho_{discr}(t,x)$ of the macroscopic variable ρ related to the Cauchy problem (1.3), defined for $t \geq 0$ and $x \in (-\infty, +\infty)$ by

$$\rho_{discr}(t,x) = \sum_{i\in\mathbb{Z}} \frac{\ell}{x_{i+1}(t) - x_i(t)} \, \chi_{[x_i(t), x_{i+1}(t)[}(x)\,, \tag{3.1}$$

where $x_i = x_i(t)$, for all $i \in \mathbb{Z}$ are the positions of the drivers. In this section we prove a *local* bound for the total variation of $\rho_{discr}(t,x)$. To this aim, for every compact interval $[a,b] \subset \mathbb{R}$ we consider the set of indexes $i \in \mathbb{Z}$ such that $x_i(0) = x_i^0 \in [a,b]$. We denote by i_{min} the minimum of the indices $i \in \mathbb{Z}$ such that $a < x_i^0$ and i_{max} the maximum of the indices $i \in \mathbb{Z}$ such that $x_i^0 < b$, i.e., we consider indexes $i \in \mathbb{Z}$ such that

$$\{i \in \mathbb{Z} : i_{min} \leq i \leq i_{max}, \quad a < x_i^0 < b\}\,. \tag{3.2}$$

Note that the above set is not empty if the interval $[a,b]$ is large enough; precisely if $x_i^0 \in (a,b)$ for some index $i \in \mathbb{Z}$, we assume that $x_i^0 \in (a,b)$ at least for two indexes, so that $i_{min} < i_{max}$. The set (3.2) is finite since $x_{i+1}(0) - x_i(0) \geq \ell$; thus i_{min} and i_{max} are (finite) integer numbers. In correspondence, we consider the *discrete density* $\rho_{[a,b]}(t,x)$, which is equal to the density defined in (3.1) when restricted to the interval $x \in [a,b]$

$$\rho_{[a,b]}(t,x) = \sum_{i=i_{min}}^{i_{max}-1} \frac{\ell}{x_{i+1}(t) - x_i(t)} \, \chi_{[x_i(t), x_{i+1}(t)[}(x)\,. \tag{3.3}$$

Remark 3.1. The interval $[a,b]$ contains the values $x_i(t)$ for $i = i_{min}, ..., i_{max}$ at $t = 0$. The functions $x_i(t)$ are increasing and can exit this interval at positive time. Thus, in the notation $\rho_{[a,b]}$, the interval $[a,b]$ refers to the initial positions. For positive times, $\rho_{[a,b]}(t)$ "follows" the drivers whose trajectories started from $[a,b]$.

We also consider the total variation $TV(\rho_{[a,b]})(t)$ of the discrete density $\rho_{[a,b]}(0,x)$, defined by

$$TV(\rho_{[a,b]})(t) = \sum_{i=i_{min}}^{i_{max}} \left| \frac{\ell}{x_{i+1}(t) - x_i(t)} - \frac{\ell}{x_i(t) - x_{i-1}(t)} \right|. \tag{3.4}$$

Theorem 3.2. *If the total variation of the discrete density $\rho_{[a,b]}(0,x)$, defined in (3.4), is bounded at $t = 0$, then it remains bounded in any compact set $[0,T] \times [a,b]$. Precisely the following estimates hold for every $t \geq 0$ and $x \in [a,b]$ by*

$$TV(\rho_{[a,b]})(t) \leq TV(\rho_{[a,b]})(0) + \frac{V_{\max}}{\ell} t\,. \tag{3.5}$$

Proof. We compute the partial derivative with respect to time of the $TV(\rho_{[a,b]})(t)$ defined in (3.4). In order to consider it, we observe that we need to take into account only the non-zero terms in the sum (3.4), so that also the absolute value does not give any problem in computing the derivative. We obtain

$$\frac{d}{dt} TV(\rho_{[a,b]})(t) =$$

$$
\begin{aligned}
&= -\sum_{i=i_{min}}^{i_{max}} \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \times \left(\frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) - \frac{\ell}{(x_i-x_{i-1})^2}(\dot{x}_i-\dot{x}_{i-1}) \right) \\
&= -\sum_{i=i_{min}}^{i_{max}} \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) \\
&+ \sum_{i=i_{min}}^{i_{max}} \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \frac{\ell}{(x_i-x_{i-1})^2}(\dot{x}_i-\dot{x}_{i-1}) \,.
\end{aligned}
$$

In the last sum we change i with $i+1$ and we obtain

$$
\begin{aligned}
&\frac{d}{dt}TV(\rho_{[a,b]})(t) \\
&= -\sum_{i=i_{min}}^{i_{max}} \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) \\
&+ \sum_{i=i_{min}-1}^{i_{max}-1} \frac{\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}}{\left|\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}\right|} \frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) \\
&= \sum_{i=i_{min}}^{i_{max}-1} \left(\frac{\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}}{\left|\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}\right|} - \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \right) \frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) \\
&+ \left. \frac{\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}}{\left|\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}\right|} \frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) \right]_{i=i_{min}-1} \\
&- \left. \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \frac{\ell}{(x_{i+1}-x_i)^2}(\dot{x}_{i+1}-\dot{x}_i) \right]_{i=i_{max}}
\end{aligned}
\tag{3.6}
$$

The absolute value of the last term is bounded by $\frac{1}{\ell}V_{\max}$, since (1.3). All the other addends in the above sum are less then or equal to zero. In fact, by limiting ourselves to the term with the sum (for the other the argument is identical), again by (1.3) we have

$$
\dot{x}_{i+1} - \dot{x}_i = v\left(\frac{\ell}{x_{i+2}-x_{i+1}}\right) - v\left(\frac{\ell}{x_{i+1}-x_i}\right) \,.
$$

We consider two cases. If $\frac{\ell}{x_{i+2}-x_{i+1}} > \frac{\ell}{x_{i+1}-x_i}$, since v is a decreasing function, then $\dot{x}_{i+1}-\dot{x}_i \le 0$. Therefore

$$
\begin{aligned}
&\left(\frac{\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}}{\left|\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}\right|} - \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \right)(\dot{x}_{i+1}-\dot{x}_i) \\
&= (+1 \pm 1)(\dot{x}_{i+1}-\dot{x}_i) \le 0 \,.
\end{aligned}
$$

If $\frac{\ell}{x_{i+2}-x_{i+1}} < \frac{\ell}{x_{i+1}-x_i}$, then $\dot{x}_{i+1}-\dot{x}_i \ge 0$ and

$$
\left(\frac{\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}}{\left|\frac{\ell}{x_{i+2}-x_{i+1}} - \frac{\ell}{x_{i+1}-x_i}\right|} - \frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i-x_{i-1}}\right|} \right)(\dot{x}_{i+1}-\dot{x}_i)
$$

$$= (-1 \pm 1)\,(\dot{x}_{i+1} - \dot{x}_i) \leq 0\,.$$

We finally obtain $\frac{d}{dt} TV(\rho_{[a,b]})(t) \leq \frac{1}{\ell} V_{\max}$ and thus the conclusion of the proof. □

Remark 3.3. In the above proof we adapted the Di Francesco and Rosini method in [10] used for the *Follow-The-Leader* model (1.2) to prove that the total variation of the density $\rho(t,x)$ is decreasing with respect to $t > 0$. Note however that there the condition that the leader drives at the maximal speed plays a relevant role. Under our assumptions, in Theorem 3.2 we proved that the total variation of the density remains locally bounded.

Remark 3.4. The bound in Theorem 3.2 of the derivative of the total variation of $\rho_{[a,b]}$ depends not only on T but also on ℓ and explodes to $+\infty$ if $\ell \to 0$. With the aim to pass from the microscopic description to the macroscopic one we need some more assumptions as considered in the next section.

4. A generalization of the follow-the-leader model: the case of one driver with constant speed

In this section, we consider the case that at least one driver travels at constant speed. Precisely, we assume that exist $i_0 \in \mathbb{Z}$, $t_0 \geq 0$ and $v_0 \in [0, V_{\max}[$ such that

$$\dot{x}_{i_0}(t) = v_0, \quad \forall t \geq t_0\,. \tag{4.1}$$

If $v_0 = V_{\max}$ we are in the case of the Follow-the-Leader model (1.2) and, being well known, we do not consider this case here. We introduce the following notations. Let $\rho_0 \in]0,1]$ such that $v(\rho_0) = v_0$. Moreover let $d = l/\rho_0$, so that

$$v\left(\frac{l}{d}\right) = v(\rho_0) = v_0\,. \tag{4.2}$$

Proposition 4.1. *Let $(x_i(t))_{i\in\mathbb{Z}}$ be a solution to the Cauchy problem (1.3), with x_{i_0} satisfying (4.1). We consider the auxiliary solution $(y_i(t))_{i\in\mathbb{Z}}$ to the Cauchy problem (1.3) for $t > t_0$ with initial data*

$$\begin{cases} y_i(t_0) = x_i(t_0) & \forall i \leq i_0 \\ y_{i_0+k}(t_0) = x_{i_0}(t_0) + kd & \forall k \in \mathbb{N}\,. \end{cases} \tag{4.3}$$

Then for $t > t_0$, $y_i(t) = x_i(t)$ for all $i \leq i_0$ and $\dot{y}_i(t) = v_0$ for all $i > i_0$.

Proof. The Cauchy problem (1.3) has a solution $(y_i(t))_{i\in\mathbb{Z}}$ for every $t > t_0$ with the initial data given in (4.3). We observe that for $i \geq i_0$ the solution is given by

$$y_i(t) = y_i(t_0) + v_0(t - t_0), \quad \forall t \geq t_0\,.$$

In fact, by (4.3),

$$y_{i+1}(t) - y_i(t) = y_{i+1}(t_0) - y_i(t_0) = d, \quad \forall t \geq t_0\,,$$

thus, by (4.2),

$$\dot{y}_i(t) = v_0 = v\left(\frac{l}{d}\right) = v\left(\frac{l}{y_{i+1}(t) - y_i(t)}\right), \qquad \forall i \geq i_0,\ \forall t \geq t_0\,.$$

Since by (4.1) $\dot{x}_{i_0}(t) = v_0 = \dot{y}_{i_0}(t)$, for all $t \geq t_0$, and since $x_{i_0}(t_0) = y_{i_0}(t_0)$, then $x_{i_0}(t) = y_{i_0}(t)$, for all $t \geq t_0$. Since each driver is influenced only by the previous one, also $x_{i_0-1}(t) = y_{i_0-1}(t)$. By iterating $x_i(t) = y_i(t)$ for all $t \geq t_0$, also for every $i \leq i_0$. □

Remark 4.2. The function $v : [0,1] \to [0, V_{\max}]$ is decreasing for $\rho \in [0,1]$. If we assume that v is *strictly* decreasing in $[0,1]$, then

$$v_0 = \dot{x}_{i_0}(t) = v\left(\frac{\ell}{x_{i_0+1} - x_{i_0}}\right) \tag{4.4}$$

and thus $x_{i_0+1}(t) - x_{i_0}(t) = \ell/\rho_0 = d$ is constant with respect to t. Therefore $\dot{x}_{i_0+1}(t) - \dot{x}_{i_0}(t) = 0$ and $\dot{x}_{i_0+1}(t) = v_0$ for all $t \geq t_0$. By induction this is not only true for $i = i_0+1$ and we obtain $x_{i_0+k}(t) - x_{i_0}(t) = d$ and $\dot{x}_{i_0+k}(t) = v_0$ for all $t > t_0$ and $k \in \mathbb{N}$. In this case, in Proposition 4.1, $y_i(t) = x_i(t)$ for all $t > t_0$ and $i \in \mathbb{Z}$.

As in the previous section, under condition (4.1), we consider the interval $[a,b] \subset \mathbb{R}$ and the set I of indexes $i \in \mathbb{Z}$ such that $x_i(0) = x_i^0 \in [a,b]$. In particular $i_0 = i_{max}$ is the maximum of the indices $i \in \mathbb{Z}$ such that $x_i^0 < b$, i.e.,

$$I = \{i \in \mathbb{Z} : i_{min} \leq i \leq i_0, \quad a < x_i^0 < b\}. \tag{4.5}$$

We emphasize that i_0 is the maximum index in I, while the index i_{min} can be chosen arbitrarily.

Theorem 4.3. *Fix $[a,b]$ such that $a < x_{i_0}(0) < b \leq x_{i_0+1}(0)$. Under assumption (4.1) the total variation $TV(\rho_{[a,b]})(t)$ defined in (3.4) of the discrete density $\rho_{[a,b]}(t,x)$ is decreasing with respect to $t \geq t_0$. If the speed v is strictly decreasing in $[0,1]$, then we can fix any interval $[a,b]$ such that $x_{i_0}(0) \in]a,b[$.*

Proof. We prove that the derivative of the total variation $TV(\rho_{[a,b]})(t)$ is less then or equal to zero. By assumption (4.1) and Proposition 4.1, for every $i \geq i_0$ we replace $x_i(t)$ with $y_i(t)$. We repeat the computation in the proof of Theorem 4.3 and we arrive to formula (3.6). All the addends are less then or equal to zero but the last one, with $i = i_{max}$ replaced by $i = i_0$, is given by

$$-\left.\frac{\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i - x_{i-1}}}{\left|\frac{\ell}{x_{i+1}-x_i} - \frac{\ell}{x_i - x_{i-1}}\right|} \frac{\ell}{(x_{i+1}-x_i)^2} (\dot{x}_{i+1} - \dot{x}_i)\right]_{i=i_0} \tag{4.6}$$

Since $x_{i_0+1} = y_{i_0+1}$ we obtain

$$-\left.\frac{\frac{\ell}{y_{i+1}-x_i} - \frac{\ell}{x_i - x_{i-1}}}{\left|\frac{\ell}{y_{i+1}-x_i} - \frac{\ell}{x_i - x_{i-1}}\right|} \frac{\ell}{(y_{i+1}-x_i)^2} (\dot{y}_{i+1} - \dot{x}_i)\right]_{i=i_0} \tag{4.7}$$

and this quantity is equal to zero for $t \geq t_0$, since $\dot{y}_{i+1} = \dot{x}_i = v_0$.

Finally, if the speed v is strictly decreasing in $[0,1]$, by Remark 4.2, $y_i(t) = x_i(t)$ for every $t \geq t_0$ and for every $i \in \mathbb{Z}$. Therefore the role of index i_0 in (4.1) is played by any index $i \geq i_0$. □

5. The micro-macro limit

Let $(x_i(t))_{i\in\mathbb{Z}}$ be a solution to the Cauchy problem (1.3), with x_{i_0} satisfying (4.1). Without loss of generality we assume $t_0 = 0$. Fix $[a,b]$ such that $a < x_{i_0-1}(0) < x_{i_0}(0) < b \leq x_{i_0+1}(0)$. If the speed v

is strictly decreasing in $[0,1]$, then we can fix any interval $[a,b]$ such that $x_{i_0}(0) \in]a,b[$. The interval $[a,b]$ contains at least two drivers at $t=0$; in a general case $n+1$ drivers are in $[a,b]$ at $t=0$, with $n \geq 1$,

$$a < x_{i_{min}}(0),\, x_{i_{min}+1}(0),\, \ldots, x_{i_0-1}(0),\, x_{i_0}(0) < b\,.$$

We consider the discrete density of the traffic flow $\rho_{[a,b]}(t,x)$ defined in (3.3). In this section we explicitly denote the n dependence, therefore formula (3.3) takes the form of n addenda related to the n intervals $[x_i^n(t), x_{i+1}^n(t)[$, with indices i from i_{min} to i_0-1,

$$\rho^n_{[a,b]}(t,x) = \sum_{i=i_{min}}^{i_0-1} \frac{\ell^n}{x^n_{i+1}(t) - x^n_i(t)}\, \chi_{[x_i^n(t), x_{i+1}^n(t)[}(x)\,. \tag{5.1}$$

For every $t \geq 0$ the function $\rho^n_{[a,b]}(t,x)$ is positive when $x \in [x^n_{i_{min}}(t), x^n_{i_0}(t)[$ and it is equal to zero outside this interval. In the following, for every $t \geq 0$, we consider the integral of $\rho^n_{[a,b]}(t,x)$ with respect to $x \in]-\infty,+\infty[$, which in fact coincides with the integral of $\rho^n_{[a,b]}(t,x)$ on the interval $[x^n_{i_{min}}(t), x^n_{i_0}(t)[$ of $\rho^n_{[a,b]}(t,x)$. Fix T in $\mathbb{R}^+$, for every i and $t \in [0,T]$, we have $x_i(0) \leq x_i(t) \leq x_i(0) + v_0 T$. Therefore, for every $x \in \mathbb{R}$ and $t \in [0,T]$

$$supp\ \rho^n_{[a,b]}(t,x) \subset [x^n_{i_{min}}(t), x^n_{i_0}(t)[\subset [a, b+v_0T]\,.$$

Since the number of addenda in the sum is equal to n, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \rho^n_{[a,b]}(t,x)\,dx \\ &= \sum_{i=i_{min}}^{i_0-1} \frac{\ell^n}{x^n_{i+1}(t) - x^n_i(t)} \int_{-\infty}^{+\infty} \chi_{[x_i^n(t), x_{i+1}^n(t)[}(x)\,dx \\ &= n\,\ell^n \end{aligned}$$

and we obtain the estimate $\ell^n \leq (b-a+v_0T)/n$, in fact

$$0 < n\,\ell^n = \int_{-\infty}^{+\infty} \rho^n_{[a,b]}(t,x)\,dx \leq x^n_{i_0}(t) - x^n_{i_{min}}(t) \leq b-a+v_0T\,. \tag{5.2}$$

With the method in [7], we prove that the infinite microscopic model in (1.3) yields the macroscopic one in (1.1), as the proper length of each vehicle vanishes. To this aim, we recall the notion of weak solution to the macroscopic system (1.1), that we consider for $x \in \mathbb{R}$ and $t \in [0,T]$. See also [7,22].

Definition 5.1. The function $\rho(t,x)$ is a weak solution of (1.1) if

$$\int_0^T \int_{-\infty}^{+\infty} (\rho\,\partial_t\varphi + \rho v\,\partial_x\varphi)\ dtdx + \int_{-\infty}^{+\infty} \rho(0,x)\varphi(0,x)\,dx = 0,$$

for every $\varphi(t,x) \in \mathbf{C}^\infty$ with compact support in $\mathbb{R}$ and $\varphi(T,x) = 0$.

Theorem 5.2. *Let v satisfy **(V)** and (4.1) holds. Assume that the initial total variation of $\rho^n_{[a,b]}(0,x)$ is bounded for $n \in \mathbb{N}$. Then, there exists a subsequence of $\rho^n_{[a,b]}(t,x)$ which as $n \to +\infty$ converges in $\mathbf{L^1}([0,T] \times \mathbb{R})$ to a map ρ^∞ which is a weak solution to system (1.1).*

Before proving the above theorem we need the following results.

Lemma 5.3. *Let v satisfy **(V)** and (4.1) holds. The following limit relation holds*

$$\int_0^T \int_{-\infty}^{+\infty} \left(\rho^n_{[a,b]} \partial_t \varphi(t,x) + \rho^n_{[a,b]} v(\rho^n_{[a,b]}) \partial_x \varphi(t,x)\right) dtdx = -\ell^n \sum_{i=i_{min}}^{i_0-1} \varphi(0, x_i^n(0)) + I^n$$

for very $\varphi(t,x) \in \mathbf{C^2}$ with compact support with respect to $x \in [a,b]$ and $\varphi(T,x) = 0$, where $I^n \to 0$ as $\ell^n \to 0$.

Proof. Since $v(\rho^n_{[a,b]}) = \dot{x}_i(t)$ for $x \in \left[x_i^n(t), x_{i+1}^n(t)\right[$, we have

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} \left(\rho^n_{[a,b]} \partial_t \varphi(t,x) + \rho^n_{[a,b]} v(\rho^n_{[a,b]}) \partial_x \varphi(t,x)\right) dtdx \\
&= \int_0^T dt \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(t)}^{x_{i+1}^n(t)} \frac{\ell^n}{x_{i+1}^n(t) - x_i^n(t)} \left(\partial_t \varphi(t,x) + \dot{x}_i(t) \partial_x \varphi(t,x)\right) dx \\
&= \int_0^T dt \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(t)}^{x_{i+1}^n(t)} \frac{\ell^n}{x_{i+1}^n(t) - x_i^n(t)} \left(\partial_t \varphi(t,x_i^n(t)) + \dot{x}_i(t) \partial_x \varphi(t,x_i^n(t))\right) dx \\
&\quad + \int_0^T dt \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(t)}^{x_{i+1}^n(t)} \frac{\ell^n}{x_{i+1}^n(t) - x_i^n(t)} \left[\left(\partial_t \varphi(t,x) - \partial_t \varphi(t,x_i^n(t))\right)\right. \\
&\quad \left. + \dot{x}_i(t) \left(\partial_x \varphi(t,x) - \partial_x \varphi(t,x_i^n(t))\right)\right] dx \\
&= \int_0^T dt \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(t)}^{x_{i+1}^n(t)} \frac{\ell^n}{x_{i+1}^n(t) - x_i^n(t)} \frac{d}{dt} \varphi(t, x_i^n(t))\, dx + I^n.
\end{aligned}$$

We compute the first integral and we estimate the second one. For the first integral we get

$$\begin{aligned}
&\int_0^T dt \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(t)}^{x_{i+1}^n(t)} \frac{\ell^n}{x_{i+1}^n(t) - x_i^n(t)} \frac{d}{dt} \varphi(t, x_i^n(t))\, dx \\
&= \ell^n \sum_{i=i_{min}}^{i_0-1} \int_0^T \frac{d}{dt} \varphi(t, x_i^n(t))\, dt = \ell^n \sum_{i=i_{min}}^{i_0-1} \left[\varphi(T, x_i^n(T)) - \varphi(0, x_i^n(0))\right] \\
&= -\ell^n \sum_{i=i_{min}}^{i_0-1} \varphi(0, x_i^n(0)),
\end{aligned}$$

since $\varphi = 0$ for $t = T$. For the integral I^n, we use the Lipschitz estimate valid for $x \in \left[x_i^n(t), x_{i+1}^n(t)\right[$

$$|\partial_t \varphi(t,x) - \partial_t \varphi(t, x_i^n(t))| \leq \max |\partial_x \partial_t \varphi| (x - x_i^n) ;$$
$$|\partial_x \varphi(t,x) - \partial_x \varphi(t, x_i^n(t))| \leq \max |\partial_x \partial_x \varphi| (x - x_i^n) .$$

We get the estimate

$$\begin{aligned} |I^n| &\leq \ell^n \|\varphi\|_{\mathbf{C^2}} (1 + V_{\max}) \int_0^T \sum_{i=i_{min}}^{i_0-1} \left(x_{i+1}^n(t) - x_i^n(t)\right) dt \\ &\leq \ell^n \|\varphi\|_{\mathbf{C^2}} (1 + V_{\max}) \int_0^T \left(x_{i_0}^n(t) - x_{i_{min}}^n(t)\right) dt . \end{aligned}$$

Since $x_{i_{min}}^n(t) \geq x_1^n(0) = a$ and $x_{i_0}^n(t) \leq b + v_0 T$ we obtain

$$|I^n| \leq \ell^n \|\varphi\|_{\mathbf{C^2}} (1 + V_{\max}) T (b - a + v_0 T) ,$$

which converges to zero as $\ell^n \to 0$, see (5.2). □

Lemma 5.4. *Let v satisfy **(V)** and (4.1) holds. The following limit relation holds*

$$\int_{-\infty}^{+\infty} \rho_{[a,b]}^n(0,x)\varphi(0,x)\, dx = \ell^n \sum_{i=i_{min}}^{i_0-1} \varphi(0, x_i^n(0)) + J^n$$

for very $\varphi(t,x) \in \mathbf{C^1}$ with compact support with respect to $x \in [a,b]$ and $\varphi(T,x) = 0$, where $J^n \to 0$ as $\ell^n \to 0$.

Proof. We have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \rho_{[a,b]}^n(0,x)\varphi(0,x)\, dx \\ &= \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(0)}^{x_{i+1}^n(0)} \frac{\ell^n}{x_{i+1}^n(0) - x_i^n(0)} \varphi(0,x)\, dx \\ &= \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(0)}^{x_{i+1}^n(0)} \frac{\ell^n}{x_{i+1}^n(0) - x_i^n(0)} \varphi(0,x_i^n(0))\, dx \\ &\quad + \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(0)}^{x_{i+1}^n(0)} \frac{\ell^n}{x_{i+1}^n(0) - x_i^n(0)} [\varphi(0,x) - \varphi(0,x_i^n(0))]\, dx \\ &= \ell^n \sum_{i=i_{min}}^{i_0-1} \varphi(0, x_i^n(0)) + J^n \end{aligned}$$

where $J^n \to 0$ as $\ell^n \to 0$. In fact

$$
\begin{aligned}
|J^n| &= \sum_{i=i_{min}}^{i_0-1} \int_{x_i^n(0)}^{x_{i+1}^n(0)} \frac{\ell^n}{x_{i+1}^n(0)-x_i^n(0)} \left[\varphi(0,x)-\varphi(0,x_i^n(0))\right] dx \\
&\le \ell^n \|\varphi\|_{\mathbf{C^1}} \sum_{i=i_{min}}^{i_0-1} \left(x_{i+1}^n(0)-x_i^n(0)\right) \\
&\le \ell^n \|\varphi\|_{\mathbf{C^1}} \left(x_{i_0}^n(0)-x_{i_{min}}^n(0)\right) \\
&\le (b-a)\ell^n \|\varphi\|_{\mathbf{C^1}},
\end{aligned}
$$

which converges to zero as $\ell^n \to 0$, see (5.2). □

By Lemma 5.3 and Lemma 5.4, we can now prove Theorem 5.2.

Proof of Theorem 5.2. By assumption, the total variation of $\rho^n_{[a,b]}(0,x)$ is bounded. Then, the total variation of $\rho^n_{[a,b]}(t,x)$ remains bounded also for any $t>0$, since by Theorem 4.3 it is a decreasing equibounded nonnegative function. By Helly's Theorem (see [3, Theorem 2.3]) there exists a subsequence, which we relabel again by $\rho^n_{[a,b]}$, which as $n\to+\infty$ converges in $\mathbf{L^1}((0,T)\times(a,b))$ to a function $\rho^\infty \in \mathbf{BV}((0,T)\times(a,b))$.

In particular, by Lemma 5.3 and Lemma 5.4 we obtain as $\ell^n \to 0$, equivalently by (5.2) as $n\to+\infty$,

$$
\lim_{n\to+\infty}\left(\int_0^T\int_{-\infty}^{+\infty}\left(\rho^n_{[a,b]}\partial_t\varphi(t,x)+\rho^n_{[a,b]}v(\rho^n_{[a,b]})\partial_x\varphi(t,x)\right)dtdx + \int_{-\infty}^{+\infty}\rho^n(0,x)\varphi(0,x)\,dx\right)=0\,,
$$

for every $\varphi(t,x)\in\mathbf{C^2}$ with compact support with respect to $x\in\mathbb{R}$ and $\varphi(T,x)=0$.

Now we expand the left hand side and we pass to the limit as $n\to+\infty$ inside the integral, by the Lebesgue dominated convergence theorem. At this stage we extract, if necessary, a further subsequence from $\rho^n(t,x)$ which converges almost everywhere to $\rho^\infty(t,x)$ for $(t,x)\in[0,T]\times[a,b]$. As $n\to+\infty$ we obtain

$$
\left(\int_0^T\int_{-\infty}^{+\infty}\left(\rho^\infty_{[a,b]}\partial_t\varphi(t,x)+\rho^\infty_{[a,b]}v(\rho^\infty_{[a,b]})\partial_x\varphi(t,x)\right)dtdx + \int_{-\infty}^{+\infty}\rho^\infty(0,x)\varphi(0,x)\,dx\right)=0\,,
$$

for very $\varphi(t,x)\in\mathbf{C^2}$ with compact support with respect to $x\in\mathbb{R}$ and $\varphi(T,x)=0$. This proves that ρ^∞ is a weak solution to the LWR equation in (1.1) in the sense of Definition (5.1). □

Remark 5.5. The microscopic model (1.3) yields the macroscopic one (1.1), as the proper length of each vehicle vanishes. As a byproduct of the limit procedure of Theorem 5.2, it seems that in the macroscopic traffic description governed by the conservation law (1.1), the presence/absence of a possibly fictitious leader in the original microscopic model, related to the Caucy Problem (1.3), is uninfluential.

6. Traffic flow with monotone density

A relevant particular case of Theorem 5.2 is a monotone traffic density. We continue to use the previous notations, in particular we denote by i_{min} the minimum of the indices $i \in \mathbb{Z}$ such that $a < x_i^0$ and i_{max} the maximum of the indices $i \in \mathbb{Z}$ such that $x_i^0 < b$, i.e.,

$$\{i \in \mathbb{Z} : i_{min} \le i \le i_{max}, \quad a < x_i^0 < b\}. \tag{6.1}$$

Let $x_i = x_i(t)$ is the position of the i-th driver, solution of the differential Cauchy problem in (1.3). As before we consider the case when at least one driver travels at constant speed, that is we assume that exist $i_0 \in \mathbb{Z}$ and $v_0 \in [0, V_{\max}[$ such that $\dot{x}_{i_0}(t) = v_0$, for all $t \ge 0$, see (4.1). Fix the interval $[a, b]$ such that $a < x_{i_0-1}(0) < x_{i_0}(0) < b \le x_{i_0+1}(0)$. If the speed v is strictly decreasing in $[0, 1]$, then we can fix any interval $[a, b]$ such that $x_{i_0}(0) \in]a, b[$.

In this section we assume that the initial density is monotone in $[a, b]$. By the definition of the density $\rho_{[a,b]}(t, x)$ in (3.3), an equivalent condition is to assume that the initial distance of the drivers is monotone in $[a, b]$. Precisely we assume that the initial distances $x_i(0) = x_i^0$ are either monotone decreasing, i.e.,

$$x_{i+1}(0) - x_i(0) \ge x_i(0) - x_{i-1}(0), \quad \forall i \in [i_{min}, i_{max}], \tag{6.2}$$

or monotone increasing, i.e.,

$$x_{i+1}(0) - x_i(0) \le x_i(0) - x_{i-1}(0), \quad \forall i \in [i_{min}, i_{max}]. \tag{6.3}$$

In the next theorem we prove that the above condition remains valid also for $t > 0$.

Theorem 6.1. *Let v satisfy* **(V)** *and (4.1) holds. Let $x_i = x_i(t)$, for $i \in \mathbb{Z}$ be the solution of the differential Cauchy problem (1.3), with initial condition satisfying either (6.2) or (6.3). Then, for every $i \in [i_{min}, i_{max}]$, respectively either*

$$x_{i+1}(t) - x_i(t) \ge x_i(t) - x_{i-1}(t), \quad \forall t \ge 0, \tag{6.4}$$

or

$$x_{i+1}(t) - x_i(t) \le x_i(t) - x_{i-1}(t), \quad \forall t \ge 0. \tag{6.5}$$

Proof. We consider the assumption (6.2) and the conclusion (6.4), the other being similar. By assumption (4.1) and Proposition 4.1, for every $i \ge i_0$ we replace $x_i(t)$ with $y_i(t)$. As in Remark 4.2, we recall that $y_i = x_i$ for every $i \in \mathbb{Z}$ if the speed v is strictly decreasing. In any case, under this replacement (we continue to denote y_i by x_i also when $i > i_0$), $\dot{x}_i = v_0$ for every $i \ge i_0$ and $x_i(t) = x_i(0) + v_0 t$. By assumption (6.2), immediately we get

$$x_{i+1}(t) - x_i(t) \ge x_i(t) - x_{i-1}(t), \quad \forall i \ge i_0 + 1, \quad t \ge 0. \tag{6.6}$$

By contradiction we assume that the conclusion (6.4) is not satisfied. Thus there exists $\bar{t} > 0$ and at least an index in $[i_{min}, i_0]$ such that in (6.4) the reverse inequality holds for $t = \bar{t}$. For every i we define

$$t_0^i = \inf\{t \ge 0 : x_{i+1}(t) - x_i(t) < x_i(t) - x_{i-1}(t)\}, \tag{6.7}$$

and also

$$t_0 = \min\left\{t_0^i : i \in [i_{min}, i_o]\right\}\,, \tag{6.8}$$

where the set in (6.8) is not empty by the contradiction assumption. Therefore t_0 is the smallest of the t_0^i for $i \in [i_{min}, i_0]$. Then the minimum is realized for some index $i = i_*$, that is $t_0 = t_0^{i_*}$ for some $i_* \in [i_{min}, i_0]$ and by (6.7)

$$x_{i_*+1}(t_0^{i_*}) - x_{i_*}(t_0^{i_*}) = x_{i_*}(t_0^{i_*}) - x_{i_*-1}(t_0^{i_*})\,. \tag{6.9}$$

If $t_0 = t_0^{i_*}$ for more than one index i_* then we choose the largest possible $i_* \in [i_{min}, i_0]$. The case $i_* = i_0$ will be considered in a moment. If $i_* < i_0$, since i_* is the largest index such that $t_0 = t_0^{i_*}$, then $t_0 = t_0^{i_*} < t_0^{i_*+1}$

$$x_{i_*+2}(t) - x_{i_*+1}(t) \geq x_{i_*+1}(t) - x_{i_*}(t)\,, \quad \forall t \in \left[t_0, t_0^{i_*+1}\right[\,. \tag{6.10}$$

By the differential equations in (1.3), since the speed $v = v(\rho)$ satisfies **(V)** and it is decreasing as a function of $\rho \in [0,1]$, for every $t \in \left[t_0, t_0^{i_*+1}\right[$

$$\begin{aligned} &\dot{x}_{i_*+1}(t) - \dot{x}_{i_*}(t) \\ &= v\left(\frac{\ell}{x_{i_*+2}(t) - x_{i_*+1}(t)}\right) - v\left(\frac{\ell}{x_{i_*+1}(t) - x_{i_*}(t)}\right) \\ &\geq 0\,. \end{aligned}$$

Therefore if $i_* < i_0$,

$$\dot{x}_{i_*+1}(t) - \dot{x}_{i_*}(t) \geq 0\,, \quad \forall t \in \left[t_0, t_0^{i_*+1}\right[\,. \tag{6.11}$$

Otherwise, if $i_* = i_0$, then $\dot{x}_{i_*+1} = \dot{x}_{i_*} = v_0$ and (6.11) holds too. By the definitions of t_0 in (6.7), (6.8) we recall (6.9) and there exists $\delta > 0$ such that

$$\begin{cases} x_{i_*+1}(t_0) - x_{i_*}(t_0) = x_{i_*}(t_0) - x_{i_*-1}(t_0) \\ x_{i_*+1}(t) - x_{i_*}(t) < x_{i_*}(t) - x_{i_*-1}(t)\,, \quad \forall t \in \left]t_0, t_0+\delta\right[\,. \end{cases} \tag{6.12}$$

Again, by the differential equations in (1.3), since the speed v satisfies **(V)**, for every $t \in [t_0, t_0+\delta[$ we have

$$\begin{aligned} &\dot{x}_{i_*}(t) - \dot{x}_{i_*-1}(t) \\ &= v\left(\frac{\ell}{x_{i_*+1}(t) - x_{i_*}(t)}\right) - v\left(\frac{\ell}{x_{i_*}(t) - x_{i_*-1}(t)}\right) \\ &\leq 0\,. \end{aligned}$$

We denote by $\varphi_{i_*}(t)$ the following function

$$\varphi_{i_*}(t) = \left(x_{i_*+1}(t) - x_{i_*}(t)\right) - \left(x_{i_*}(t) - x_{i_*-1}(t)\right)\,. \tag{6.13}$$

By the above inequality and (6.11), if $\delta_1 = \min\{\delta, t_0^{k+1} - t_0\}$ we have

$$\dot{\varphi}_{i_*}(t_0) \geq 0\,, \quad \forall t \in [t_0, t_0+\delta_1[\tag{6.14}$$

and thus $\varphi_{i_*}(t) \geq 0$ for every $t \in [t_0, t_0+\delta_1[$. This yields a contradiction with the assumption $\varphi_{i_*}(t) < 0$ for every $t \in]t_0, t_0+\delta_1[$. The proof is complete. □

Remark 6.2. Note that, by Theorem 4.3, assumption (4.1) implies that the total variation $TV(\rho_{[a,b]})(t)$ defined in (3.4) of the discrete density $\rho_{[a,b]}(t,x)$ in (3.3) remains bounded if it is bounded for $t = 0$. However, under assumption (6.2) (the other (6.3 being similar), we can show more directly that the total variation $TV(\rho_{[a,b]})(t)$ is bounded. In fact,

$$\sum_{i=i_{min}}^{i=i_{max}} \left| \frac{\ell}{x_{i+1}(t) - x_i(t)} - \frac{\ell}{x_i(t) - x_{i-1}(t)} \right| = \sum_{i=i_{min}}^{i=i_{max}} \left(\frac{\ell}{x_i(t) - x_{i-1}(t)} - \frac{\ell}{x_{i+1}(t) - x_i(t)} \right)$$
$$= \left(\frac{\ell}{x_{min+1}(t) - x_{min}(t)} - \frac{\ell}{x_{max+1}(t) - x_{max}(t)} \right) \leq \frac{\ell}{x_{min+1}(t) - x_{min}(t)} \leq 1 \,.$$

Therefore we can apply directly Theorem 5.2 to get the convergence of the microscopic model (1.3) to the macroscopic one (1.1) without using Theorem 4.3.

The case of a traffic light We recall that a well known situation where a monotone traffic density well describes the real behavior of vehicular traffic is in correspondence of a traffic light. Indeed, when a traffic light turns red, the braking of the first driver approaching it results in a (locally) increasing traffic density. On the contrary, when a traffic light turns green, the first vehicle accelerates and moves faster than the second and so on, resulting in a decreasing traffic density.

More precisely, an ordered approach of drivers to a traffic light can be described using the construction of this section. We consider initially smoothly ordered drivers, for instance with equal distance between each other, that is

$$x^0_{i-1} - x^0_{i-2} = x^0_i - x^0_{i-1} \,, \quad \forall i \,.$$

If one of the drivers approaches a red light, then he/she stops and all the following drivers necessarily start to decelerate arriving all to stop smoothly. We can modelize this case with the first driver x_{i_0} finding a red light at time $t = t_0$; so that, $\dot{x}_{i_0}(t_0) = 0$. We can consider a disposition of the drivers x_i at $t = t_0$ of the type

$$... = x_i - x_{i-1} = ... = x_{i_0-1} - x_{i_0-2} > x_{i_0} - x_{i_0-1} \,.$$

The drivers x_i with $i < i_0$, originally equidistributed, start to decrease their speed for $t > t_0$ and they arrive to stop following an ordered decreasing distance in the sense of Theorem 4.3 (assumption (6.2), with increasing density and decreasing reciprocal distances for $t > t_0$.

When the green light turns on at a time $t_1 > t_0$, the first driver x_{i_0} starts moving with a positive velocity ($\dot{x}_{i_0}(t) = v_0 > 0$) for $t \geq t_1$. All the following drivers x_i with $i < i_0$, initially at equal distance ℓ each other at time $t = t_1$, start to move at a time $t > t_1$

$$... = x_i - x_{i-1} = ... = x_{i_0-1} - x_{i_0-2} < x_{i_0} - x_{i_0-1} \,.$$

Then, they continue to move with increasing velocity again according to Theorem 4.3 (Assumption 6.3), resulting in a decreasing density and increasing reciprocal distance.

7. Appendix

We may ask if a genuine infinitely dimensional Cauchy Problem as considered in Section 2 can naturally appear in different contexts away from the traffic modeling. These kinds of Cauchy Problems enter in the general class of *Cauchy Problems in Banach spaces* widely studied in the literature; see for instance the quoted books by Deimling [9], Lakshmikantham-Leela [17] and Martin [23]. Here we propose an infinite-dimensional example related to the classical approach in the resolution by the Fourier methods of the Heat Equation in one space variable. Precisely, we consider the parabolic second order linear equation

$$\partial_t u = \partial_{xx} u, \quad x \in [a,b],\, t \geq 0, \tag{7.1}$$

under the initial and boundary conditions for the unknown functions $u = u(t,x)$, $u : [0 \times +\infty) \times [a,b] \to \mathbb{R}$

$$\begin{cases} u(0,x) = u^0(x) & \forall\ x\ \in [a,b]\,, \\ u(t,a) = u(t,b) = 0 & \forall\ t\ \in [0,+\infty)\,, \end{cases} \tag{7.2}$$

under the compatibility condition $u^0(a) = u^0(b) = 0$. Classically we assume that the initial datum $u^0(x)$ has the Fourier representation

$$u^0(x) = \sum_{k=1}^{\infty} a_k \sin\left(k\pi \frac{x-a}{b-a}\right)\,, \tag{7.3}$$

for some coefficients $a_k \in \mathbb{R}$, $k \in \mathbb{N}$. Of course $u^0(a) = u^0(b) = 0$. Then, it is natural to look for a solution $u(t,x)$ to the Initial Boundary Value Problem (7.1)–(7.2) of the form

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) \sin\left(k\pi \frac{x-a}{b-a}\right)\,. \tag{7.4}$$

In fact the function $u(t,x)$ in (7.4) satisfies the conditions $u(t,a) = u(t,b) = 0$ and the initial condition

$$u(0,x) = \sum_{k=1}^{\infty} u_k(0) \sin\left(k\pi \frac{x-a}{b-a}\right) = u^0(x)$$

provided that $u_k(0) = a_k$, for all $k \in \mathbb{N}$. We also have

$$\partial_t u(t,x) = \sum_{k=1}^{\infty} \dot{u}_k(t) \sin\left(k\pi \frac{x-a}{b-a}\right)\,,$$

$$\partial_{xx} u(t,x) = -\sum_{k=1}^{\infty} u_k(t) \sin\left(k\pi \frac{x-a}{b-a}\right)\left(\frac{k\pi}{b-a}\right)^2\,.$$

To check if $u(t,x)$ satisfies the heat equation (7.1), we impose

$$\sum_{k=1}^{\infty} \dot{u}_k(t) \sin\left(k\pi \frac{x-a}{b-a}\right) = -\sum_{k=1}^{\infty} u_k(t) \sin\left(k\pi \frac{x-a}{b-a}\right)\left(\frac{k\pi}{b-a}\right)^2\,.$$

We multiply both sides of the above equation by $\sin\left(h\pi\frac{x-a}{b-a}\right)$ and we integrate for $x \in [a,b]$. Since, when $h \neq k$,

$$\int_a^b \sin\left(h\pi \frac{x-a}{b-a}\right) \cdot \sin\left(k\pi \frac{x-a}{b-a}\right)\, dx = 0\,,$$

then we obtain

$$\dot{u}_k(t) = -u_k(t) \left(\frac{k\pi}{b-a} \right)^2 , \quad \forall k \in \mathbb{N} \, .$$

This procedure gives rise to an infinite dimensional Cauchy Problem which, in this simple linear context, it can be solved elementary equation by equation; i.e., for every fixed $k \in \mathbb{N}$. This *linear* case is elementary. However the Heat Equation may present a source nonlinear term depending also on u, of the form

$$\partial_t u = \partial_{xx} u + f(t, x, u, \partial_x u), \quad x \in [a, b], \, t \geq 0. \tag{7.5}$$

With the same procedure described above, we arrive to a nonlinear infinite-dimensional Cauchy Problem of the type

$$\begin{cases} \dot{u}_k(t) = -u_k(t) \left(\frac{k\pi}{b-a} \right)^2 + F_k(t, u_1, u_2, ...), & \\ u_k(0) = a_k & \forall \; k \in \mathbb{N} \, , \end{cases} \tag{7.6}$$

where for some constant c

$$F_k(t, u_1, u_2, ...) = \\ c \int_a^b f \left(t, x, \sum_{i=1}^{\infty} u_i(t) \sin \left(i\pi \frac{x-a}{b-a} \right), \sum_{i=1}^{\infty} \left(\frac{i\pi}{b-a} \right) u_i(t) \cos \left(i\pi \frac{x-a}{b-a} \right) \right) \sin \left(k\pi \frac{x-a}{b-a} \right) dx.$$

In this case we have a genuine infinite-dimensional Cauchy system and it can not be solved separately equation by equation, but it needs a global approach. For more details we refer to Lewis in [19], who solved a particular infinite system of ordinary differential equations under assumptions satisfied for system (7.6). Theorem 2.2 can not be applied to (7.6), but it is modelized to be applied specifically to traffic modeling as in this manuscript and in particular in § 2.1.

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