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Convecting–radiating fins: Explicit solutions, efficiency and optimization

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ABSTRACT

We analyze the second order, non-linear, one dimensional differential equation describing the steady conduction of heat in convecting–radiating longitudinal fins. We introduce an auxiliary dependent variable, solving a first order differential equation and related to the thickness of the fin, giving the distribution of the heat along the fin. The purely convecting case, corresponding to a linear equation, and the convecting–radiating case, corresponding to a non-linear equation, are treated separately. For the linear case, different solutions, corresponding to different shapes of the fin, are analyzed. For the non-linear case, an explicit solution in terms of the auxiliary variable is obtained. The distribution of the temperature, the efficiency of the fin and the applicability of the results are discussed.

1. Introduction

Longitudinal fins are one of the most common tools used to enhance the heat transfer from a given device or surface. There is a broad amount of literature related to this subject (see e.g. [1,2] and references therein). Nevertheless, the analysis and design of convecting and convecting–radiating fins are still a topic of great interest in the research community, mainly due to two reasons: the first one is the very wide range of engineering applications of this type of devices, varying from heat sinks in electronic instruments to heat exchangers in power plants. The second one is the variety of physical mechanisms responsible for the thermal exchange: conduction, convection and radiation must be associated to the proper set of boundary conditions in order to fit the particular application and environment for which the fin is thought of. To different types of mechanisms and conditions there correspond different types of differential equations describing the distribution of temperature along the fin: if only conduction and convection are considered, assuming the thermal coefficients to be independent of the temperature, then the underlying equations are linear (see e.g. [2]), but if the radiation term is added, a non-linear term appears in the equations and the analysis becomes more involved. For these reasons, most of the investigations of the convecting–radiating case are non-exact, in the sense that approximations ([1,3]), numerical analysis ([2,4–6]), Adomian decomposition methods ([7,8]), differential algorithms and optimization techniques ([9–11]) or iterative methods ([12]) are applied to describe the properties of the model considered.

In this work we consider a longitudinal fin of arbitrary profile with radiation losses. The boundary conditions, given at the base and at the tip of the fin, embrace different physical scenarios: they are the sum of a convective and radiative part, with different Biot and radiation–conduction numbers. The cases of a fin with an isolated tip and/or with a base at a given

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Nomenclature

Roman Symbols

a	parameter in subsection (4.2)
$\hat{a}$	parameter in subsection (5)
b	parameter in Section 4.2
$\hat{b}$	parameter in Section 5
c	parameter in Eq. (10)
$Bi_{0,1}$	Biot numbers
f_0	fin half-thickness (m)
f	dimensionless fin half-thickness
h	convective-heat transfer coefficient ($\mathrm{Wm^{-2}K^{-1}}$)
k	ratio between absorptivity and emissivity of the fin
L	fin length (m)
ℓ	fin depth (m)
$N_{0,1}$	radiation conduction numbers
T	temperature (K)
T_0	temperature of the fluid (K)
T_1	effective radiation environment temperature (K)
T_b	temperature of the base (K)
$y(z)$	auxiliary function of z
u	parameter in subsection (4.3.1)
w	parameter in Section 5
x	distance along the depth of the fin measured from the base (m)
z	dimensionless distance ($= x/\ell$)

Greek Symbols

α	dimensionless convective coefficient
β	dimensionless radiative coefficient
ε	emissivity
η	efficiency
σ	Stefan-Boltzmann constant ($\mathrm{Wm^{-2}K^{-4}}$)
κ	thermal conductivity ($\mathrm{Wm^{-1}K^{-1}}$)
θ	dimensionless temperature

temperature (that of the hot plate at the base) are seen as limiting cases of our results and, in this sense, they provide a relaxation of the Murray-Gardner limiting assumptions (see e.g. [1]). We analyze the behavior of temperature at steady state: it is clear that, in all heat transfer processes, transient phenomena always precede the steady state and in many applications the transient behavior may be relevant (see e.g. [13–15]). On the other hand, in particular applications when the time interval of the transient state is small with respect to the operational time interval of the device, a steady state analysis can give interesting insights on the design of such devices. The transient and steady states are two aspects that we are taking separate and in a future work we are planning to extend the discussion here given to the transient behavior of the fin.

What we are going to discuss here is the optimization of the profile of the fin in order to maximize its efficiency to dissipate heat by convection and radiation, the efficiency being defined as the ratio of the actual heat transfer to the ideal heat transfer for a fin of infinite thermal conductivity [1,5]. Clearly, in suitable applications other constraints, like weight, total volume or cost may be relevant. For example, the optimization of the fin profile by minimizing the total volume for a given value of the transferred heat flux is treated in [16].

The paper is organized as follows: in Section 2 the main equation, the boundary conditions and an auxiliary variable giving an implicit definition of the temperature are introduced. In Section 3 the general formula for the efficiency of the fin is presented, both in terms of the auxiliary variables and of the temperatures at the ends of the fin. In Section 4 the linear case (pure convection) is analyzed and the change of variable is used to (re-)obtain the classical results for the rectangular profile, for a square root profile in terms of the Airy functions and for an arbitrary profile, with an explicit application in terms of the Mathieu functions: as far as we know, these last results are new and the square root case can be seen as a generalization of the rectangular profile. A general formula for the efficiency for the arbitrary profile is given. In Section 5 we present the non-linear case: an explicit solution in terms of the auxiliary variable is given, depending also on the radiative and convective parameters appearing in the differential equation. The formula for the efficiency is given and analyzed also for this case and numerical plots are presented. In Section 6 we discuss the results, highlighting their applicability and clarifying in which physical conditions the radiative effects become relevant. Finally, in the conclusion, the results are summarized along with possible future extensions, especially for what concerns the conducting–radiating case.

2. The main equation and the auxiliary variable

We consider a longitudinal fin of arbitrary profile attached to a base at a temperature T_b. The fin length is L, whereas the fin thickness at a distance x from the base is $2f_0(x)$. The half-thickness at the base is $f_b = f_0(x=0)$ and the fin tip, located at $x = \ell$, has a half-thickness equal to $f_t = f_0(\ell)$. We assume the Fourier law to holds inside the fin, so that the quantity of heat flowing between x and $x + dx$ is [1,2]

$$dq = \kappa \frac{d}{dx}\left(Lf_0(x)\frac{dT}{dx}\right)dx, \tag{1}$$

where $\kappa > 0$ is the thermal conductivity of the fin. At steady state, this heat is equal to the heat dissipated by the surface. We assume the heat to be dissipated by convection and radiation. The energy balance gives the following equation for the steady state temperature along the fin profile [1,2]:

$$\kappa\frac{d}{dx}\left(f_0(x)\frac{dT}{dx}\right) = 2\mathrm{h}\left(1+2\frac{f_0}{L}\right)(T(x)-T_0) + 2\sigma\varepsilon\left(1+2\frac{f_0}{L}\right)(T(x)^4 - T_1^4), \tag{2}$$

where h is the convective heat transfer coefficient, σ the Stefan–Boltzmann constant and ε the emissivity of the fin.

If the fin thickness is small compared to its length, then the term f_0/L can be ignored and we obtain

$$\kappa\frac{d}{dx}\left(f_0(x)\frac{dT}{dx}\right) = 2\mathrm{h}(T(x)-T_0) + 2\sigma\varepsilon(T(x)^4 - T_1^4). \tag{3}$$

In this equation T_0 is the temperature of the fluid adjacent to the fin and T_1 represents the temperature of the effective radiation environment (i.e. the radiant energy absorbed by the fin per unit of time and surface is $\varepsilon\sigma T_1^4$). In the following we assume the fin to be in general non-gray, with $T_1^4 = kT_0^4$, where k is the ratio between the absorptivity and the emissivity of the fin [2]. For a gray fin one has to set $k = 1$ [2]. Eq. (3) must be supplied with the boundary conditions: the usual boundary condition at the base of the fin is $T(x=0) = T_b$. However we make a more general assumption by considering the base of the fin to exchange heat both convectively and radiatively with the source, i.e.:

$$f_0(x)\left.\frac{dT}{dx}\right|_{x=0} - \eta_0(T-T_b)|_{x=0} - \xi_0\left.(T^4 - kT_b^4)\right|_{x=0} = 0. \tag{4}$$

A similar condition holds for the other end of the fin (see also [5] for an assumption of this type for the tip of the fin):

$$f_0(x)\left.\frac{dT}{dx}\right|_{x=\ell} + \eta_1(T-T_0)|_{x=\ell} + \xi_1\left.(T^4 - kT_0^4)\right|_{x=\ell} = 0. \tag{5}$$

In (4) and (5) η_i and ξ_i, $i = 0, 1$, are positive constants proportional, as we will see, to the Biot and radiation-conduction numbers of the ends of the fin.

By introducing the dimensionless variables $\theta(x) = T(x)/T_b$, $z = x/\ell$, $\alpha = 2\mathrm{h}\ell^2/(f_b\kappa)$, $\beta = 2\sigma\varepsilon\ell^2T_b^3/(f_b\kappa)$ and $f(z) = f_0(z\ell)/f_b$, Eq. (3) becomes

$$\frac{d}{dz}\left(f(z)\frac{d\theta}{dz}\right) = \alpha(\theta-\theta_0) + \beta(\theta^4 - k\theta_0^4), \tag{6}$$

with the boundary conditions

$$\begin{aligned} &f(z)\left.\frac{d\theta}{dz}\right|_{z=0} - Bi_0(\theta-1)|_{z=0} - N_0\left.(\theta^4 - k)\right|_{z=0} = 0,\\ &f(z)\left.\frac{d\theta}{dz}\right|_{z=1} + Bi_1(\theta-\theta_0)|_{z=1} + N_1\left.(\theta^4 - k\theta_0^4)\right|_{z=1} = 0, \end{aligned} \tag{7}$$

where we introduced the Biot numbers $Bi_j = \eta_j\ell/f_b$, $j = 0, 1$ and the radiation-conduction numbers $N_j = \xi_j\ell T_b^3/f_b$, $j = 0, 1$.

The main idea is to approach the previous problem by making a change of variables: we set $\theta(z) = F(y)$, where $y = y(z)$ is an "auxiliary dependent variable" satisfying $f(z)\frac{dy}{dz} = 1$, with $f(z)$ the thickness function. The corresponding equation for $F(y)$ is

$$\frac{d^2F}{dy^2} = f(z)\Big(\alpha(F-\theta_0) + \beta(F^4 - k\theta_0^4)\Big). \tag{8}$$

The function $f(z)$ may be equivalently defined by the previous Eq. (8). Let us make some observations about the r.h.s. of Eq. (8).

i. Eq. (8) is a *first order* equation for the function $y(z)$. Indeed, multiplying both sides of Eq. (8) by $\frac{dy}{dz}$, we get

$$\frac{dy}{dz}\frac{d^2F}{dy^2} = \Big(\alpha(F-\theta_0) + \beta(F^4 - k\theta_0^4)\Big) \tag{9}$$

or

$$\int^y \frac{F_{yy}}{\left(\alpha(F-\theta_0)+\beta(F^4-k\theta_0^4)\right)}dy = z + c\,, \tag{10}$$

where c is an arbitrary integration constant.

ii. With a suitable choice of the function $F(y)$, the explicit integration of Eq. (10) may give an implicit definition of $y(z)$ as a function of z. In turns, this function will define also the function $f(z)$ and the temperature $T(z)$ by virtue of $f(z)\frac{dy}{dz}=1$ and $T(z)=F(y(z))$.

Clearly the function $F(y)$ in point (ii) should be prescribed by taking into account the physical constrains on $T(z)$ and $f(z)$, since these two functions are positive definite. At this point it is not clear why there should be advantages in the change of variables described above: an easiest reasoning could be just to choose a trial function for $T(z)$ with some parameters, insert this function in Eq. (3) and integrate once to find $f(z)$. In the next section however we will see that the efficiency of the fin can be described by a very simple formula in terms of the function y alone.

3. Efficiency of the fin

The efficiency of the fin is defined as the ratio of the actual heat transfer to the ideal heat transfer for a fin of infinite thermal conductivity [1,5]. The heat transfer from the fin to the surround is given by [1,5]

$$\mathrm{q}_{tot} = \kappa L f_0(x)\left.\frac{dT}{dx}\right|_{x=\ell} - \kappa L f_0(x)\left.\frac{dT}{dx}\right|_{x=0}, \tag{11}$$

or, in terms of dimensionless variables

$$\frac{\mathrm{q}_{tot}}{\kappa L f_b T_b/\ell} = \left.\frac{dF}{dy}\right|_{z=1} - \left.\frac{dF}{dy}\right|_{z=0}. \tag{12}$$

With an infinite thermal conductivity the entire fin is at the base temperature and the ideal heat transfer is [1,5]

$$\mathrm{q}_{id} = 2\mathrm{h}L\ell(T_b-T_0) + 2\sigma\varepsilon L\ell(T_b^4-kT_0^4), \tag{13}$$

or, again, in terms of dimensionless variables

$$\frac{\mathrm{q}_{id}}{\kappa L f_b T_b/\ell} = \alpha(1-\theta_0)+\beta(1-k\theta_0^4), \tag{14}$$

giving an efficiency η equal to

$$\eta = \frac{\left.\frac{dF}{dy}\right|_{z=1} - \left.\frac{dF}{dy}\right|_{z=0}}{\alpha(1-\theta_0)+\beta(1-k\theta_0^4)}. \tag{15}$$

It is possible to get an equivalent expression of the efficiency η involving explicitly the Biot numbers Bi_j, $j=0,1$, and the radiation–conduction numbers N_j, $j=0,1$, by using the boundary conditions (7). Indeed we notice that, from the definition of the function $F(y)$ and $y(z)$, one has:

$$\frac{d\theta}{dz} = \frac{dy}{dz}\frac{dF}{dy} \quad \text{or} \quad f(z)\frac{d\theta}{dz} = \frac{dF}{dy}. \tag{16}$$

Eqs. (7), (15) and (16) then give

$$\eta = \frac{Bi_1(\theta_0-\theta(1))+Bi_0(1-\theta(0))+N_1(k\theta_0^4-\theta(1)^4)+N_0(k-\theta(0)^4)}{\alpha(1-\theta_0)+\beta(1-k\theta_0^4)}. \tag{17}$$

This equation shows that, in principle, it is not necessary to calculate the evolution of temperature for every point along the fin in order to get its efficiency, but it may be sufficient to have the values of the temperature at the boundaries, i.e. $\theta(0)$ and $\theta(1)$. We will take advantage of this observation in the next sections.

4. The pure convective case

It is interesting, also for comparison with the more general case, to look at what happens when the convection is the only mechanism of heat transfer for the fin. We will show how the application of the formulae obtained in the previous sections can give results well known in the literature but also new ones. We will consider different instances for the profile of the fin: a rectangular profile, a profile described by a square root and a profile involving an arbitrary function.

4.1. Rectangular profile.

The efficiency of a rectangular longitudinal fin is well-known in literature. In order to clarify the approach introduced in the previous sections, we will describe how to obtain this efficiency by employing this approach. As pointed out in the observation (ii) of Section 2, we need to pick a function $F(y)$ and integrate the following equation

$$\int^{y} \frac{F_{yy}}{(\alpha(F-\theta_0))} dy = z + c, \tag{18}$$

The simplest case is obtained when the function $F(y)$ is such that $F_{yy}/(F-\theta_0)$ is a positive constant, that we denote by m^2. The function $F(y)$ is then given by $F(y) = \theta_0 + c_1 e^{my} + c_2 e^{-my}$. The function $y(z)$ is defined by

$$m^2 y(z) = \alpha(z + c). \tag{19}$$

The corresponding function $f(z)$ is then $f(z) = 1/\frac{dy}{dz} = m^2/\alpha$: this is a constant, so the fin is rectangular. Notice that, since the a-dimensional thickness function $f(z)$ is given by $f(z) = f_0(z\ell)/f_b$ (see after Eq. (5)), one has $f(0) = 1$, implying $\alpha = m^2$. The application of the boundary conditions (7) gives the explicit evolution of the temperature as

$$\theta(z) = \theta_0 + (1-\theta_0)Bi_0 \frac{m\cosh(m(1-z)) + Bi_1 \sinh(m(1-z))}{m(Bi_0 + Bi_1)\cosh(m) + (m^2 + Bi_0 Bi_1)\sinh(m)}. \tag{20}$$

The efficiency η is explicitly given by:

$$\eta = Bi_0 \frac{Bi_1(\cosh(m) - 1) + m\sinh(m)}{m\big(m(Bi_0 + Bi_1)\cosh(m) + (m^2 + Bi_0 Bi_1)\sinh(m)\big)}. \tag{21}$$

Eq. (21), for $Bi_0 \to \infty$ and $Bi_1 = 0$ gives the well-known Gardner's formula [17], [1] for a rectangular fin with insulated tip and a base at $T = T_b$, i.e. $\eta = \tanh(m)/m$. We notice also that to any other type of boundary conditions there correspond a lower efficiency: indeed it is possible to show that the difference between the Gardner's efficiency and (21) is always non negative for any choice of Bi_1 and Bi_0.

4.2. Square root profile.

The next case that we consider is obtained by choosing a function $F(y)$ in Eq. (18) such that the ratio $F_{yy}/(F-\theta_0)$ is linear in y, i.e. $F_{yy} = (b^3 y + a^2)(F-\theta_0)$. As we will show, this example can be handled explicitly and can be considered a generalization of the rectangular case, with the additional parameter b. In the case $b = 0$ we obtain the case discussed in the previous subsection. The coefficient proportional to y is a cube for reasons that will be clear in the next lines. It is known that the general solution of the equation $F_{yy} = (b^3 y + a^2)(F-\theta_0)$ is defined in terms of the Airy functions Ai and Bi as [18,19]:

$$F = \theta_0 + c_1 \mathrm{Ai}\left(by + \frac{a^2}{b^2}\right) + c_2 \mathrm{Bi}\left(by + \frac{a^2}{b^2}\right), \tag{22}$$

where c_1 and c_2 are two arbitrary constants. The function $y(z)$ is then defined by

$$\frac{b^3}{2} y(z)^2 + a^2 y(z) = \alpha(z + c). \tag{23}$$

The corresponding function $f(z)$ is given by

$$f(z) = \frac{b^3 y(z) + a^2}{\alpha} = \frac{\sqrt{2\alpha b^3(z+c) + a^4}}{\alpha}, \tag{24}$$

where we chose the positive root of Eq. (23) in order to have a positive definite function $f(z)$. Again, we must satisfy the constraint $f(0) = 1$, giving a value of c equal to $c = \frac{\alpha^2 - a^4}{2\alpha b^3}$. With this constraint the functions $y(z)$ and $f(z)$ are:

$$y(z) = \frac{-a^2 + \alpha\sqrt{1 + \frac{2b^3 z}{\alpha}}}{b^3}, \qquad f(z) = \sqrt{1 + \frac{2b^3 z}{\alpha}}. \tag{25}$$

The evolution of the temperature, by taking into account the boundary conditions, results in a quite cumbersome formula. Let us introduce the following combinations of Airy functions:

$$\mathrm{Ai}\circ\mathrm{Bi}_{xy} = \mathrm{Ai}(x)\mathrm{Bi}(y) - \mathrm{Ai}(y)\mathrm{Bi}(x), \quad \mathrm{Ai}'\circ\mathrm{Bi}_{xy} = \mathrm{Ai}'(x)\mathrm{Bi}(y) - \mathrm{Ai}(y)\mathrm{Bi}'(x) \tag{26}$$

$$\mathrm{Ai}'\circ\mathrm{Bi}'_{xy} = \mathrm{Ai}'(x)\mathrm{Bi}'(y) - \mathrm{Ai}'(y)\mathrm{Bi}'(x) \tag{27}$$

and the following variables:

$$h = \frac{\sqrt{\alpha^2 + 2b^3\alpha z}}{b^2}, \quad h_0 = h|_{z=0} = \frac{\alpha}{b^2}, \quad h_1 = h|_{z=1} = \frac{\sqrt{\alpha^2 + 2b^3\alpha}}{b^2}. \tag{28}$$

In terms of the previous definitions, the temperature is written as

$$\theta(z) = \theta_0 + \frac{(1-\theta_0)Bi_0\big(b\mathrm{Ai}'\circ\mathrm{Bi}_{h_1h} + Bi_1\mathrm{Ai}\circ\mathrm{Bi}_{h_1h}\big)}{b^2\mathrm{Ai}'\circ\mathrm{Bi}'_{h_0h_1} + b\big(Bi_1\mathrm{Ai}'\circ\mathrm{Bi}_{h_0h_1} + Bi_0\mathrm{Ai}'\circ\mathrm{Bi}_{h_1h_0}\big) + Bi_0Bi_1\mathrm{Ai}\circ\mathrm{Bi}_{h_1h_0}}, \tag{29}$$

whereas the efficiency η of the fin as

$$\eta = \frac{Bi_0b\big(Bi_1(\frac{1}{\pi} + \mathrm{Ai}'\circ\mathrm{Bi}_{h_0h_1}) - b\mathrm{Ai}'\circ\mathrm{Bi}'_{h_1h_0}\big)}{\alpha\big(b^2\mathrm{Ai}'\circ\mathrm{Bi}'_{h_0h_1} + b\big(Bi_1\mathrm{Ai}'\circ\mathrm{Bi}_{h_0h_1} + Bi_0\mathrm{Ai}'\circ\mathrm{Bi}_{h_1h_0}\big) + Bi_0Bi_1\mathrm{Ai}\circ\mathrm{Bi}_{h_1h_0}\big)}. \tag{30}$$

It is not clear how this formulae are connected to the results of Subsection 4.1. Although the functions Ai and Bi are by far the most common basis for the solutions of the equation $F_{yy} = (b^3y + a^2)(F - \theta_0)$ (mainly due to their good behavior at infinity), it is possible to write the previous results by using other basis. Particularly appropriate in this case is the basis introduced in [18,19]: we introduce the two functions $C(y, b^3, a^2)$ and $S(y, b^3, a^2)$ as the solutions of $Y_{yy} = (b^3y + a^2)Y$ uniquely defined by the following behavior around $z = 0$[1]:

$$C(y, b^3, a^2) \sim 1 + O(y^2), \quad S(y, b^3, a^2) \sim y + O(y^3). \tag{31}$$

The two previous functions can be considered a non-autonomous version of the trigonometric functions $\cos(ay)$ and $\sin(ay)$ (or the corresponding hyperbolic functions) since many properties of these periodic functions can be extended in a natural way to the functions $C(y, b^3, a^2)$ and $S(y, b^3, a^2)$ (see [18,19] for more details). In the limit $b \to 0$ the functions $C(y, b^3, a^2)$ and $S(y, b^3, a^2)$ are

$$\lim_{b\to 0} C(y, b^3, a^2) = \cosh(ay), \quad \lim_{b\to 0} S(y, b^3, a^2) = \frac{\sinh(ay)}{a}. \tag{32}$$

In terms of these functions the efficiency η (30) can be written as

$$\eta = \frac{Bi_0}{\alpha}\frac{Bi_1\big(C(\frac{h_1-h_0}{b}, b^3, b^2h_0) - 1\big) + C'(\frac{h_1-h_0}{b}, b^3, b^2h_0)}{C'(\frac{h_1-h_0}{b}, b^3, b^2h_0) + \big(Bi_1C(\frac{h_1-h_0}{b}, b^3, b^2h_0) + Bi_0C(\frac{h_0-h_1}{b}, b^3, b^2h_1)\big) + Bi_0Bi_1S(\frac{h_1-h_0}{b}, b^3, b^2h_0)}. \tag{33}$$

By looking at the efficiency for the rectangular fin (21) it is clearer now the similarity between the formulae (21) and (33). It is possible to get a neater formula by using the homogeneity properties of the functions C, S, C' and S'. Indeed, for every $\lambda \neq 0$ it holds [18,19]:

$$\begin{aligned} &C\Big(\frac{y}{\lambda}, (\lambda b)^3, (\lambda a)^2\Big) = C(y, b^3, a^2), \quad C'\Big(\frac{y}{\lambda}, (\lambda b)^3, (\lambda a)^2\Big) = \lambda C(y, b^3, a^2),\\ &S\Big(\frac{y}{\lambda}, (\lambda b)^3, (\lambda a)^2\Big) = \frac{1}{\lambda}S(y, b^3, a^2), \quad S'\Big(\frac{y}{\lambda}, (\lambda b)^3, (\lambda a)^2\Big) = S(y, b^3, a^2). \end{aligned} \tag{34}$$

From the previous relations it follows that the efficiency (33) can be rewritten, for $b \neq 0$, as:

$$\eta = \frac{Bi_0b}{\alpha}\frac{Bi_1(C(h_1-h_0, 1, h_0) - 1) + bC'(h_1-h_0, 1, h_0)}{b^2C'(h_1-h_0, 1, h_0) + b(Bi_1C(h_1-h_0, 1, h_0) + Bi_0C(h_0-h_1, 1, h_1)) + Bi_0Bi_1S(h_1-h_0, 1, h_0)}. \tag{35}$$

The limiting case of a fin with the base temperature fixed and equal to T_b corresponds to take the limit $Bi_0 \to \infty$ in formula (35), giving

$$\eta = \frac{b}{\alpha}\frac{Bi_1(C(h_1-h_0, 1, h_0) - 1) + bC'(h_1-h_0, 1, h_0)}{bC(h_0-h_1, 1, h_1) + Bi_1S(h_1-h_0, 1, h_0)}, \tag{36}$$

and, for a fin with an insulated tip (i.e. $Bi_1 = 0$):

$$\eta = \frac{b}{\alpha}\frac{C'(h_1-h_0, 1, h_0)}{C(h_0-h_1, 1, h_1)}. \tag{37}$$

The free parameters in Eq. (35) can be assumed to be Bi_0, Bi_1, h_0 and h_1, or Bi_0, Bi_1, α and b, indeed from the definitions (28) it follows that $\alpha = \frac{(h_1^2-h_0^2)^2}{4h_0}$ and $b = \frac{(h_1^2-h_0^2)}{2h_0}$. The result (37) is reminiscent of the well known Gardner's formula [17] for the efficiency of the rectangular fin, with the base at $T = T_b$ and insulated tip, given by $\eta = \frac{\tanh(m)}{m}$ (see Eq. (21) in the limit $Bi_0 \to \infty$ and with $Bi_1 = 0$). Indeed, it can be shown that in the limit $b \to 0$ and identifying α with m^2, formula (37) gives exactly the result (21).

In the limit $h_1 \to h_0$, it can be shown that the efficiency (35) becomes (for any b):

$$\eta = \frac{Bi_0}{Bi_0 + Bi_1} \tag{38}$$

showing that for a fin with a base at $T = T_b$ (i.e. $Bi_0 \to \infty$) or a fin with an insulated tip (i.e. $Bi_1 = 0$) one has $\eta = 1$ in the limit $h_1 \to h_0$. On the left of Fig. 1 we report, in blue, the plot of the efficiency of the fin with the boundary conditions

[1] For completeness we report the expressions of $C(y, b^3, a^2)$ and $S(y, b^3, a^2)$ in terms of Ai and Bi: $C(y, b^3, a^2) = \pi\big(\mathrm{Ai}\big(by + \frac{a^2}{b^2}\big)\mathrm{Bi}'\big(\frac{a^2}{b^2}\big) - \mathrm{Bi}\big(by + \frac{a^2}{b^2}\big)\mathrm{Ai}'\big(\frac{a^2}{b^2}\big)\big)$ and $S(y, b^3, a^2) = \frac{\pi}{b}\big(\mathrm{Bi}\big(by + \frac{a^2}{b^2}\big)\mathrm{Ai}\big(\frac{a^2}{b^2}\big) - \mathrm{Ai}\big(by + \frac{a^2}{b^2}\big)\mathrm{Bi}\big(\frac{a^2}{b^2}\big)\big)$.

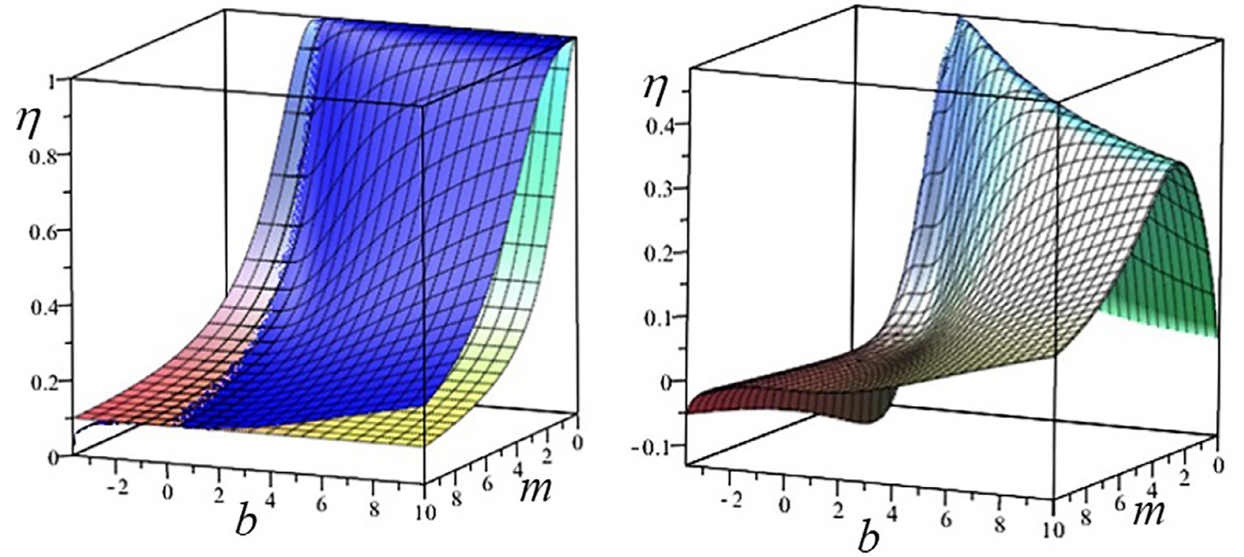

Fig. 1. On the left: the efficiency of the fin (37), corresponding to a base at $T = T_b$ and an insulated tip, as a function of m and b. On the right: the difference between the efficiency (37) and the efficiency of the fin (36) for $Bi_1 = 10$ as a function of m and b.

corresponding to a base at $T = T_b$ and an insulated tip as a function of $m = \sqrt{\alpha}$ and b (with $b^3 > -m^2/2$ as can be seen from Eq. (28)), as given by formula (37). We fixed $\alpha = m^2$ in formula (37) in order to make a direct comparison with the efficiency of the rectangular profile $\eta = \tanh(m)/m$ (see 21). The plot below the blue surface is the reference efficiency (21), obtained for $b = 0$: the figure shows that for $b > 0$ the efficiency of the square root profile is higher than the efficiency of the rectangular profile, whereas for $b < 0$ the rectangular profile as a higher efficiency. In order to compare, in an illustrative example, the efficiency of a fin with an insulated tip with the efficiency of a fin with convection on the tip, we report in the left of Fig. 1 the plot of the difference between two efficiencies, the efficiency (37) and the efficiency of the fin (36) for $Bi_1 = 10$ as functions of b and m: apart a narrow region for $b < 0$, where both the efficiencies are small, the difference is always positive, showing that the efficiency (37) is greater than the efficiency (36) for the given value of Bi_1.

4.3. Arbitrary profile.

Finally, we consider the case of a profile described by an arbitrary function p: to be consistent with the assumptions made in Section 2 we assume that p depends on z through the function $y(z)$, i.e. is a composed function of z, $p = p(y(z))$. The profile $f(z)$ is assumed to be equal to

$$f(z) \doteq \frac{1}{\alpha}\left(\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right)\Bigg|_{y=y(z)}. \tag{39}$$

It can be shown that the general solution of Eq. (8), in the case $\beta = 0$ and with $f(z)$ defined as above, can be explicitly written in terms of the function $p(y)$ as

$$F(y) = \theta_0 + c_1 e^{p(y)} + c_2 e^{p(y)} \int^y e^{-2p(x)dx} \doteq \theta_0 + c_1 e^{p(y)} + c_2 e^{s(y)}, \tag{40}$$

where we introduced the function $s(y) = p(y) + \ln\left(\int^y e^{-2p(x)dx}\right)$ for later convenience. The explicit dependence of the function y on z, according to the observation (i) made in Section 2, is given by

$$\int^{y(z)} \left(\frac{d^2p(u)}{du^2} + \left(\frac{dp(u)}{du}\right)^2\right) du = \alpha(z + c) \tag{41}$$

that can be written also in terms of the integral of the profile function $f(z)$ as

$$\int^{y(z)} f(y^{-1}(u))du = z + c. \tag{42}$$

We still need to apply the boundary conditions to obtain the particular solution we are interested in. In the following, as for the Airy's formulae above, in order to avoid quite cumbersome equations, let us introduce the following symbols for the functions $p(z)$ and $s(z)$ and their derivatives evaluated in the points $z=0$ and $z=1$:

$$\dot{p}_0 = \left.\frac{dp}{dy}\right|_{z=0}, \quad \dot{s}_0 = \left.\frac{ds}{dy}\right|_{z=0} \quad \dot{p}_1 = \left.\frac{dp}{dy}\right|_{z=1}, \quad \dot{s}_1 = \left.\frac{ds}{dy}\right|_{z=1} \tag{43}$$

$$p_0 = p(y(0))\ ,\ p_1 = p(y(1))\ ,\ s_0 = s(y(0))\ ,\ s_1 = s(y(1)). \tag{44}$$

The temperature along the fin is described by the following formula:

$$\theta(z) = \theta_0 + (1-\theta_0)Bi_0 \frac{e^{s(y)+p_1}(\dot{p}_1 + Bi_1) - e^{p(y)+s_1}(\dot{s}_1 + Bi_1)}{(\dot{s}_1 + Bi_1)(\dot{p}_0 - Bi_0)e^{p_0+s_1} - (\dot{p}_1 + Bi_1)(\dot{s}_0 - Bi_0)e^{s_0+p_1}}. \tag{45}$$

The expression for the efficiency of the fin is given by

$$\eta = \frac{Bi_0}{\alpha} \frac{\dot{p}_0 e^{p_0+s_1}(\dot{s}_1 + Bi_1) - \dot{s}_0 e^{s_0+p_1}(\dot{p}_1 + Bi_1) + Bi_1 e^{s_1+p_1}(\dot{s}_1 - \dot{p}_1)}{(\dot{s}_1 + Bi_1)(\dot{p}_0 - Bi_0)e^{p_0+s_1} - (\dot{p}_1 + Bi_1)(\dot{s}_0 - Bi_0)e^{s_0+p_1}}. \tag{46}$$

It is possible to consider the limiting case of the previous formulae, given by a fin with the base temperature fixed and equal to T_b, corresponding to the limit $Bi_0 \to \infty$ in (45) and (46), and the tip insulated, corresponding to $Bi_1 = 0$. Eqs. (45) and (46) then give

$$\theta(z) = \theta_0 + (1-\theta_0)\frac{\dot{p}_1 e^{s(y)+p_1} - \dot{s}_1 e^{p(y)+s_1}}{\dot{p}_1 e^{s_0+p_1} - \dot{s}_1 e^{p_0+s_1}} \tag{47}$$

and

$$\eta = \frac{\dot{p}_0 \dot{s}_1 e^{p_0+s_1} - \dot{s}_0 \dot{p}_1 e^{s_0+p_1}}{\alpha(\dot{p}_1 e^{s_0+p_1} - \dot{s}_1 e^{p_0+s_1})}. \tag{48}$$

4.3.1. An application

As an application of the previous general formulae, we consider a profile given by a trigonometric function in the variable y. More in particular, we set

$$f(z) \doteq \left.\frac{1}{\alpha}(2q\cos(2ry) - u)\right|_{y=y(z)}, \tag{49}$$

where q, r and u are free parameters. We get the following equation for $p(y)$ (see eq. (39)):

$$\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2 = 2q\cos(2ry) - u. \tag{50}$$

By the change of variable $p(y) = \ln(v(y))$, the previous equation reduces to the Mathieu differential equation (see e.g. [20] and [21]):

$$\frac{d^2v}{dy^2} = (2q\cos(2ry) - u)v. \tag{51}$$

The general solution of the previous equation can be written in terms of linear combinations of Mathieu functions Mc and Ms, that are defined by the following behavior at $y=0$:

$$\mathrm{Mc}(y,u,q) \sim 1 + O(y^2), \quad \mathrm{Ms}(y,u,q) \sim y + O(y^3). \tag{52}$$

The general solution of (51) is then given by

$$v(y) = c_1\mathrm{Mc}(ry, u/r^2, q/r^2) + c_2\mathrm{Ms}(ry, u/r^2, q/r^2). \tag{53}$$

The dependence of y on z is specified by Eq. (41), that in this case gives

$$\frac{q\sin(2ry)}{r} - uy = \alpha(z+c). \tag{54}$$

From Eqs. (53) and (54), due to the arbitrariness in the choice of the parameters, we see that it is not reductive to fix $r=1$. Further, to enhance the readability of the paper, we only consider in this subsection the limit $Bi_0 \to \infty$ and set $Bi_1 = 0$ (corresponding to the base of the fin at $T = T_b$ and the tip insulated). Eq. (54) is known as Kepler's equation [22,23]: fixed a value of α, z and c, and provided $r=1$, Eq. (54) is invertible if $|2q/u| \le 1$. Further, we need the function $f(z)$ defined in (57) to be positive and, since $|2q/u| \le 1$, this is possible only if $u < 0$: the parameters u and q are then limited to take values in the ranges $u \in (-\infty, 0)$ and $q \in (-|u|/2, |u|/2)$.

By applying the boundary conditions $\theta(0) = 1$ and $d\theta/dz|_{z=1} = 0$, from the explicit expression of the temperature it is possible to get the following equation for the efficiency of the fin

$$\eta = \frac{\mathrm{Mc}'(y(0),u,q)\mathrm{Ms}'(y(1),u,q) - \mathrm{Ms}'(y(0),u,q)\mathrm{Mc}'(y(1),u,q)}{\alpha\big(\mathrm{Mc}'(y(1),u,q)\mathrm{Ms}(y(0),u,q) - \mathrm{Ms}'(y(1),u,q)\mathrm{Mc}(y(0),u,q)\big)}, \tag{55}$$

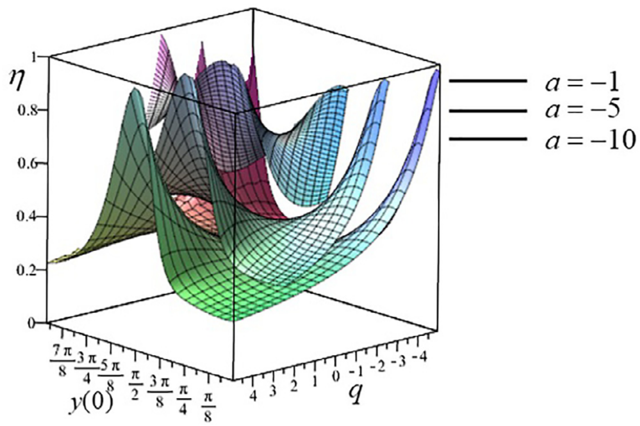

Fig. 2. The efficiency of the fin (55), corresponding to a base at $T = T_b$ and an insulated tip, as a function of $y(0)$ and q for $u = -1$ (top), $u = -5$ (middle) and $u = -10$ (bottom).

where Mc′ and Ms′ are respectively the derivatives of the functions Mc(y, u, q) and Ms(y, u, q) with respect to y. We must also to take into account the constraint $f(0) = 1$. From Eq. (57) we get:

$$f(0) = 1 \to \alpha = 2q\cos(2y(0)) - u. \tag{56}$$

In applications, the parameter α is given and the value of $y(0)$ must be chosen accordingly to Eq. (56). However it is easier, in what follows, to fix the value of α by Eq. (56), thinking to fix a suitable value of $y(0)$ corresponding to the given α. With this definition of the parameter α, the profile function $f(z)$ becomes

$$f(z) = \frac{2q\cos(2y(z)) - u}{2q\cos(2y(0)) - u}. \tag{57}$$

The constant $y(0)$ must be also a solution of the Kepler's Eq. (54): this equation fixes the value of the parameter c:

$$q\sin(2y(0)) - uy(0) = \alpha c \to c = \frac{q\sin(2y(0)) - uy(0)}{2q\cos(2y(0)) - u}. \tag{58}$$

The values of $y(1)$ is fixed by the solutions of the corresponding Kepler's Eq. (54):

$$q\sin(2y(1)) - uy(1) = \alpha(1 + c). \tag{59}$$

The free parameters left are then u, q and $y(0)$. As said, u and q are limited to the ranges $u \in (-\infty, 0)$ and $q \in (-|u|/2, |u|/2)$, whereas from (56) we see that $y(0)$ can be chosen to vary in $(0, \pi)$.

To plot the values of the efficiency as a function, for example, of u and q for fixed $y(0)$ or of $y(0)$ and q for fixed u, we need of the expression of $y(1)$ as functions of these parameters. Further, to plot the profile function $f(z)$ (57), we need of $y(z)$, solution of (54), for all $z \in (0, 1)$. There are various techniques to solve the Kepler's Eqs. (54)–(59) (see e.g. [22], [23] and references therein). We apply the following iterative method (see [23]): we define the sequences $y_n(1)$ and $y_n(z)$ by the recursions

$$y_{n+1}(1) = \frac{q}{u}\sin(2y_n(1)) - \frac{\alpha}{u}(1 + c), \quad y_{n+1}(z) = \frac{q}{u}\sin(2y_n(z)) - \frac{\alpha}{u}(z + c), \tag{60}$$

with the initial conditions

$$y_0(1) = -\frac{\alpha}{u}(1 + c), \quad y_0(z) = -\frac{\alpha}{u}(z + c). \tag{61}$$

The iterations always converge in our range of parameters and indeed it is possible to show that the convergence is linear in $2q/u$ [23] (i.e. if e_n is the nth error in y_n, then e_n is of order $(2q/u)^n e_0$). In Fig. 2 we report the plot of the efficiency (55) as a function of $y(0)$ and q for three different values of u, i.e. $u = -1$, $u = -5$ and $u = -10$. The values of the efficiency are decreasing with the increase of the modulus of u but maintain three maxima around $y(0) = \pi/2$ and $q = |u|/2$, $y(0) = 0$ and $q = -|u|/2$ and around $y(0) = \pi$ and $q = -|u|/2$. The plots of the profile function $f(z)$ (57) corresponding to $u = -1$ and $y(0) = \pi/2$ (left) and $y(0) = 0$ (right) as a function of z and q are given in Fig. 3: as it can be seen, to a larger tip of the fin it corresponds a larger value of the efficiency.

5. The radiative–convective case

The main difference, form the mathematical point of view, between the case of pure convection and the case of convection and radiation is the non-linear term in the differential equation arising from the radiative term. This reflects in the impossibility to write the general solution of Eq. (6) or (8) as we did, for example, in the Section 4.3 (see Eq. (40)).

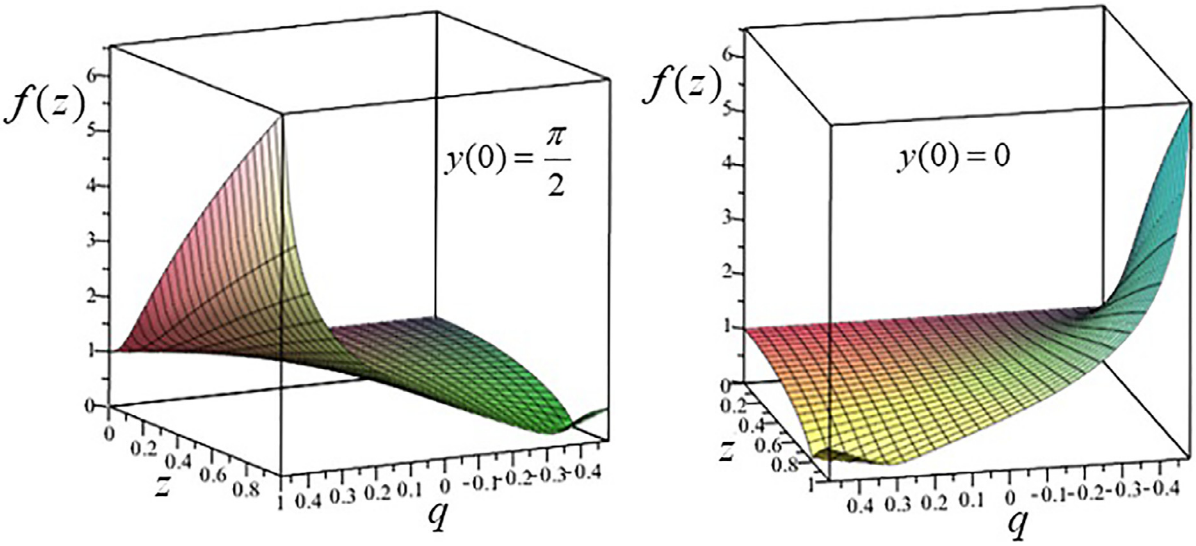

Fig. 3. The profile function $f(z)$ (57) for a fin with a base at $T = T_b$ and an insulated tip, as a function of q and z for $\alpha = -1$ and $u = -3$ (left) and $u = -1$ (right).

There are many numerical results about solutions of the nonlinear Eq. (6) and certain results about approximated or asymptotic analytical solutions. As far as we know, there are no examples of explicit solutions however. Here we consider the possibility to obtain explicit solutions thanks to the observations (i) and (ii) made in Section 2. Other observations, that will be useful in the next discussion, are the following:

iii. We notice that since α, β, θ_0 and k are all real and positive, the polynomial $F^4 - k\theta_0^4 + \alpha/\beta(F - \theta_0)$ has two real roots and two complex conjugated roots. The two real roots are one positive and one negative (the sum of the two is negative). So we can factorize the polynomial as

$$F^4 - k\theta_0^4 + \alpha/\beta(F - \theta_0) = (F - \hat{a})(F + \hat{a} + \hat{b})\left(\left(F - \frac{\hat{b}}{2}\right)^2 + \hat{a}(\hat{a} + \hat{b}) + \frac{3}{4}\hat{b}^2\right) \tag{62}$$

where $\hat{a}$ and $\hat{b}$ are two positive constants depending on θ_0, k and α/β. Clearly, for $k = 1$, $\hat{a} = \theta_0$.

iv. Physically, we expect that there exist solutions of Eq. (6), with boundary conditions given by (7), such that the dimensionless temperature $\theta(z)$ is bounded from below by the root $\hat{a}$ ($= \theta_0$ in the case $k = 1$) and from above by the base temperature (equal to 1 in dimensionless variables). From Eq. (8) it follows that $F(y)$ is a convex function of y. Further, $y(z)$ is an increasing function of z, since $\frac{dy}{dz} f(z) = 1$. The temperature is expected to be decreasing along the fin[2], so $F(y(z)) = \theta(z)$ is a decreasing function of z, implying that $F(y)$ is a decreasing function of y. In order to obtain physical relevant solutions, the function $F(y)$ must be chosen in the class of convex decreasing functions.

We look at Eq. (10) and assume $F(y)$ to be given by a polynomial in y. To fix the idea, we choose a polynomial of second degree. More specifically:

$$F(y) = wy^2 + \hat{a}, \tag{63}$$

where w is a positive constant and $\hat{a}$ is the positive root in the polynomial Eq. (62). Clearly, the function is convex. Also, we need the function $y(z)$ to be negative in order to have a decreasing $F(z)$. This is compatible with the boundary conditions (7). Indeed, with the help of (16) and (63), we can write:

$$f(z)\frac{d\theta}{dz}\bigg|_{z=0} = \frac{dF}{dy}\bigg|_{z=0} = 2wy(0) = Bi_0(F - 1)|_{z=0} + N_0(F^4 - k)\big|_{z=0}, \tag{64a}$$

$$f(z)\frac{d\theta}{dz}\bigg|_{z=1} = \frac{dF}{dy}\bigg|_{z=0} = 2wy(1) = -Bi_1(F - \theta_0)|_{z=1} - N_1(F^4 - k\theta_0^4)\big|_{z=1}. \tag{64b}$$

[2] Actually it may be possible, according to a suitable choice of the constant parameters appearing in the boundary conditions (7), that the temperature presents a minimum in the interval $z \in (0, 1)$. For simplicity we do not consider such possibility in this paper.

As we will show in the next subsection, both these equations always possess a real negative solution $y(0)$ and $y(1)$. To explicitly determine the function $y(z)$, we need to evaluate the integral (10), that in this case takes the form

$$\frac{2}{\beta}\int^{y}\frac{1}{u^2\left(wu^2+2\hat{a}+\hat{b}\right)\left(\left(wu^2+\hat{a}-\frac{\hat{b}}{2}\right)^2+\hat{a}(\hat{a}+\hat{b})+\frac{3}{4}\hat{b}^2\right)}du=z+c. \tag{65}$$

The polynomial appearing in the denominator of the integral has a double root in $u=0$ and six complex roots, paired by complex conjugation. We give the general formula for an integral of the previous type with $2n$ complex roots paired by complex conjugation. We have:

$$\int^{y}\frac{1}{u^2\prod_{i=1}^{n}(u-\alpha_i)(u-\bar{\alpha}_i)}du=-\frac{A}{y}+B\ln(|y|)+\sum_{i=1}^{n}E_i\arctan\left(\frac{y-a_i}{b_i}\right)+\sum_{i=1}^{n}F_i\ln\left((y-a_i)^2+b_i^2\right), \tag{66}$$

where a_i and b_i are the real and imaginary parts of the roots α_i, with $\alpha_i\neq\alpha_j$ for $i\neq j$, $b_i\neq 0\,\forall i$ and the real constants A, B, E_i and F_i are determined by

$$A=\prod_{i=1}^{n}\frac{1}{a_i^2+b_i^2},\quad B=2A\sum_{i=1}^{n}\frac{a_i}{a_i^2+b_i^2},\quad b_k(E_k+2\mathrm{i}F_k)=\frac{1}{\alpha_k^2}\prod_{k\neq i=1}^{n}\frac{1}{(\alpha_k-\alpha_i)(\alpha_k-\bar{\alpha}_i)}. \tag{67}$$

In the last equation the real and imaginary part should be compared in order to obtain the values of the constants E_k and F_k.

We apply the result (66) to our formula (65): in this case $n=3$, but we notice that there are further symmetries that can be used to simplify the result. Indeed the polynomial at the denominator of (65) depends on the square of u, so if u_0 is a root of the polynomial, $-u_0$ is another root. Also, two roots are purely imaginary. So the six roots are $\pm\mathrm{i}b_1$, $a_2\pm\mathrm{i}b_2$ and $-a_2\pm\mathrm{i}b_2$. It follows that the constants B and F_1 vanish, whereas $E_3=E_2$ and $F_3=-F_2$. The values of A, E_1 E_2 and F_2 are given by:

$$\begin{aligned}
E_1 &= -\frac{1}{b_1^3\left(a_2^2+(b_1-b_2)^2\right)\left(a_2^2+(b_1+b_2)^2\right)},\\
E_2 &= \frac{1}{4}\frac{\left((a_2^2-3b_2^2)(a_2^2+b_1^2-b_2^2)-2b_2^2(3a_2^2-b_2^2)\right)}{\left(a_2^2+(b_1-b_2)^2\right)\left(a_2^2+(b_1+b_2)^2\right)\left(a_2^2+b_2^2\right)^3 b_2},\\
A &= \frac{1}{b_1^2(a_2^2+b_2^2)^2},\quad F_2=\frac{1}{8}\frac{\left((b_2^2-3a_2^2)(a_2^2+b_1^2-b_2^2)+2a_2^2(3b_2^2-a_2^2)\right)}{\left(a_2^2+(b_1-b_2)^2\right)\left(a_2^2+(b_1+b_2)^2\right)\left(a_2^2+b_2^2\right)^3 a_2}.
\end{aligned}$$

The function $y(z)$ is specified by the implicit equation

$$-\frac{A}{y}+E_1\arctan\left(\frac{y}{b_1}\right)+E_2\left(\arctan\left(\frac{y+a_2}{b_2}\right)+\arctan\left(\frac{y-a_2}{b_2}\right)\right)+F_2\ln\left(\frac{(y+a_2)^2+b_2^2}{(y-a_2)^2+b_2^2}\right)=\frac{\beta w^3}{2}(z+c). \tag{68}$$

The constants b_1, a_2 and b_2 are respectively given, in terms of the constants $\hat{a}$, $\hat{b}$ and w appearing in (65), as

$$b_1=\sqrt{\frac{2\hat{a}+\hat{b}}{w}},\quad a_2^2=\frac{\hat{b}-2\hat{a}+2\sqrt{2\hat{a}^2+\hat{b}^2}}{4w},\quad \hat{b}_2^2=\frac{2\hat{a}-\hat{b}+2\sqrt{2\hat{a}^2+\hat{b}^2}}{4w}. \tag{69}$$

The function $f(z)$ is defined in terms of the function $y(z)$ (68) as:

$$f(z)=\frac{2}{\beta y^2\left(wy^2+2\hat{a}+\hat{b}\right)\left(\left(wy^2+\hat{a}-\frac{\hat{b}}{2}\right)^2+\hat{a}(\hat{a}+\hat{b})+\frac{3}{4}\hat{b}^2\right)}. \tag{70}$$

We must require that the function $f(z)$ complies with the constraint $f(0)=1$: once the value of $y(0)$ is specified by the boundary condition (64a), Eq. (70) evaluated in $z=0$ fixes the value of w that must be used. The real negative root $y(0)$ of (64a) (see the next subsection) fixes also the value of the arbitrary constant c appearing in the expression (68) of $y(z)$. Then, the function $y(z)$ is implicitly determined by Eq. (68) for any choice of the parameters β, $\hat{a}$ and $\hat{b}$ (i.e. of the parameters θ_0, α and β of the differential Eq. (6)), giving, through Eq. (63), the corresponding value of the temperature $\theta(z)=F(y(z))$. Finally, we notice that the value of $y(1)$ at this point is also fixed by Eq. (68) and this must be consistent with the root of Eq. (64b): this means that not both the values of the parameters Bi_1 and N_1 are arbitrary, but are constrained by Eq. (64b) itself.

5.1. Efficiency of the nonlinear fin.

The efficiency of the fin, given by formula (17), can be written as (see Eq. (63)):

$$\eta = 2w\frac{y(1)-y(0)}{\alpha(1-\theta_0)+\beta(1-k\theta_0^4)}. \tag{71}$$

As explained, the value of $y(0)$ is determined by Eq. (64a), whereas the value of $y(1)$ by Eq. (68): indeed the differential equation determining $y(z)$ is of first order, so once the value of $y(0)$ is fixed by the boundary conditions the value of $y(1)$ must be determined by the solution of the differential equation.

Now we will show that the equation determining $y(0)$, and for completeness also the equation determining $y(1)$, always possesses just one negative root. The equations are:

$$Bi_0(\hat{a}+wy(0)^2-1)+N_0\big((\hat{a}+wy(0)^2)^4-k\big)-2wy(0)=0,$$
$$Bi_1(\hat{a}+wy(1)^2-\theta_0)+N_1\big((\hat{a}+wy(1)^2)^4-k\theta_0^4\big)+2wy(1)=0. \tag{72}$$

For the sake of simplicity, let us consider the special case $k=1$ (giving also $\hat{a}=\theta_0$, see Eq. (62)). The equations become:

$$Bi_0(wy(0)^2-(1-\theta_0))+N_0\big((wy(0)^2+\theta_0)^4-1\big)-2wy(0)=0, \tag{73a}$$

$$Bi_1wy(1)^2+N_1\big((wy(1)^2+\theta_0)^4-\theta_0^4\big)+2wy(1)=0. \tag{73b}$$

Eq. (73a) has a first derivative, with respect to $y(0)$, proportional to

$$y(0)\left(4(wy(0)^2+\theta_0)^3+\frac{Bi_0}{N_0}\right)-\frac{1}{N_0}, \tag{74}$$

and this has clearly only one real root. It follows that Eq. (73a) has only a negative global minimum with respect to $y(0)$ (since for $y(0)\to\ \pm\infty$ the left hand side is positive, whereas for $y(0)=0$ is negative), giving two roots for Eq. (73a), one positive and one negative. As for Eq. (73b) determining $y(1)$, apart the trivial solution $y(1)=0$ that can be factored, the derivative with respect to $y(1)$ is always positive and so the function increases from $-\infty$ to $+\infty$, giving one real root, that must be negative since Eq. (73b) can be rewritten in the form

$$y(1)\left((wy(1)^2+2\theta_0)((wy(1)^2+\theta_0)^2+\theta_0^2)+\frac{Bi_1}{N_1}\right)+\frac{1}{N_1}=0. \tag{75}$$

In the next part of this subsection, we analyze the efficiency of the non-linear fin (71) by assuming, for simplicity, the fin to be gray, i.e. we set $k=1$ in Eq. (6). In this case the constant $\hat{a}$ in Eqs. (62), (69) and (70) is equal to θ_0, whereas the constant $\hat{b}$ appearing in the same equations is determined in terms of the ratio α/β by the unique real solution of the following equation:

$$\frac{\alpha}{\beta}=\hat{b}(\hat{b}^2+2\hat{b}\theta_0+2\theta_0^2). \tag{76}$$

This last equation comes from the comparison of the two sides of Eq. (62).

Further, we assume that the base of the fin is at $T=T_b$, i.e. $\theta(0)=1$, corresponding to $Bi_0>\ >0$ and/or $N_0>\ >0$. It follows that the value of $y(0)$ is fixed by

$$y(0)=-\sqrt{\frac{1-\theta_0}{w}}. \tag{77}$$

From Eq. (8) we see that the constraint $f(0)=1$ gives the following equation for w:

$$w=\frac{1}{2}\big(\alpha(1-\theta_0)+\beta(1-\theta_0^4)\big), \tag{78}$$

resulting in a simplification of the formula (71) for the efficiency:

$$\eta=y(1)-y(0)=y(1)+\sqrt{\frac{2}{\alpha+\beta(1+\theta_0)(1-\theta_0^2)}}. \tag{79}$$

The value of $y(1)$ is fixed by Eq. (68), as explained in the previous subsection. The plot of the efficiency (79) is reported in Fig. 4 as a function of the parameter β for various values of α and θ_0. The dependence of $y(0)$ on the parameters α, β and θ_0 is explicit and is given by minus the term in the square root in Eq. (79). The dependence of $y(1)$ on the same parameters, being fixed by Eq. (68), is quite involved. From Fig. 4 we can conjecture that the efficiency is a decreasing function of the parameter β. The analytical description on how the efficiency curves decrease is an interesting problem and deserves more research to be understood. It is clear however that the decreasing rate depends also on the values of α and θ_0, so at different values of α, for any fixed θ_0, there correspond different decreasing rate of the efficiency. This explains the possibility to have intersections between the curves, as shown in Fig. 4.

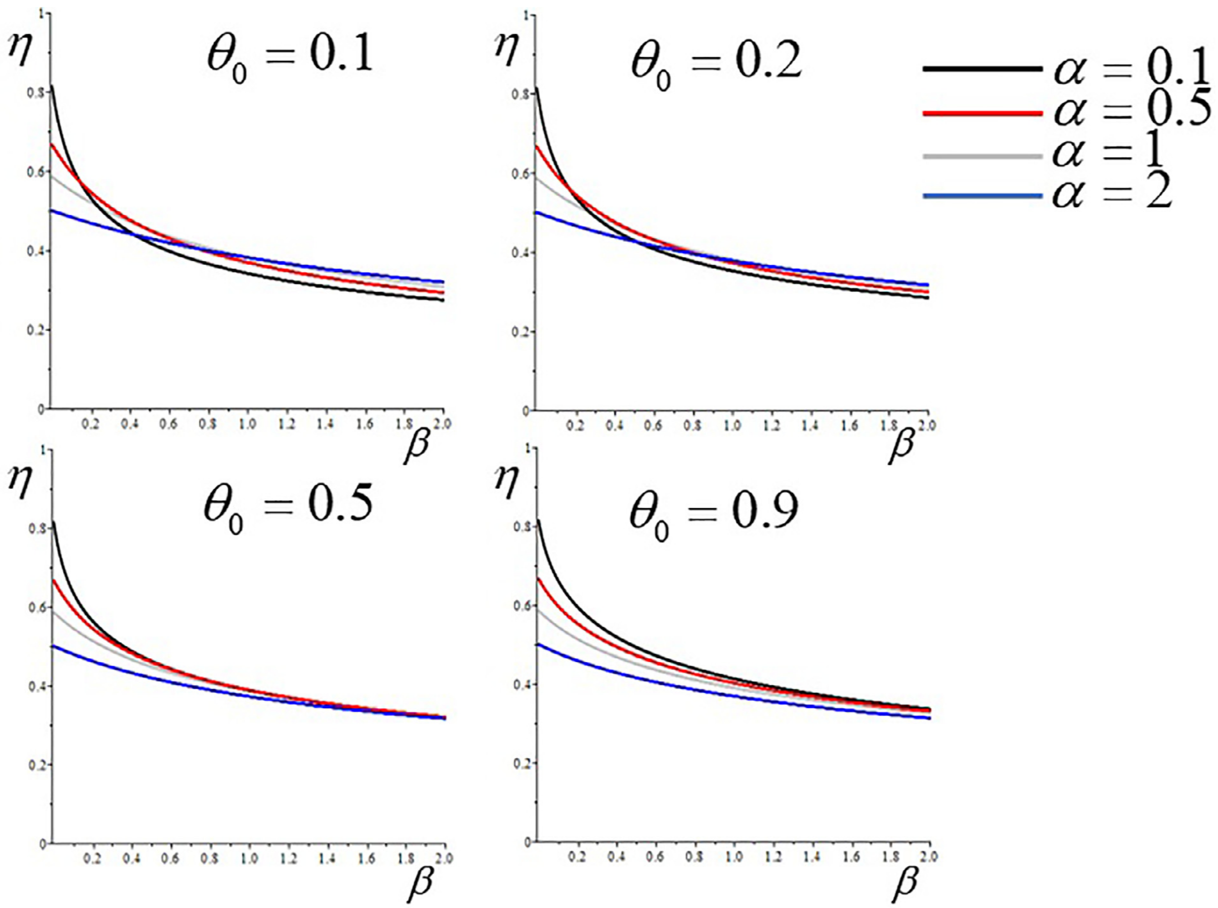

Fig. 4. The plot of the efficiency (79) as a function of the parameter $\beta \in (0, 2)$ for four different values of α and four different values of the parameter θ_0.

In Fig. 5 we report an example of temperature variation along the fin for a fixed value of θ_0 (=0.5) and a fixed value of β (=0.1) and four different values of α. The black curves have been obtained directly by Eq. (68). For comparison, we plotted also the temperatures obtained with a numerical integration of the differential Eq. (6) with a seventh-eighth order continuous Runge–Kutta method. They are given by the red circles. The corresponding profiles $f(z)$ for the same values of the parameters are shown in Fig. 6.

6. Discussion

The geometry of the fins is only a particular aspect of the problem of optimization of a fin heat sink. The material, its manufacturability and cost, its weight (for aerospace applications), the reference temperature of the environment are examples of the different aspects that may become relevant in specific applications. It is not our aim to enter in all these details, since the literature is abundant (see e.g. [1,2] and references therein). In the following we are going to discuss the results obtained focusing on their applicability.

When the parameter β is small compared to α in Eq. (6), the non-linear term may be seen as a perturbation to the linear equation. From the definitions of α and β we see that the condition $\beta < \ < \alpha$ entails

$$T_b << \left(\frac{\mathrm{h}}{\sigma \epsilon}\right)^{1/3}. \tag{80}$$

If T_b complies with the previous inequivalence it is possible to use perturbation techniques to approximate the solutions of Eq. (6) by a power series in β (see for example [3], where the approximated solutions have been compared also with the numerical ones).

Vice-versa, if $T_b \gtrapprox (\mathrm{h}/\sigma\epsilon)^{1/3}$, the radiative effects are relevant and perturbation techniques cannot be applied. To get an idea of the order of magnitude at which radiative effects become significant, we report in Fig. 7 the plot of $(\mathrm{h}/\sigma\epsilon)^{1/3}$ as a function of h for two fixed values of ϵ, $\epsilon = 1$ and $\epsilon = 0.5$. The plot shows the interval $\mathrm{h} \in [5, 10^4]\mathrm{Wm}^{-2}\mathrm{K}^{-1}$, range comparable to typical values of the heat transfer coefficient for a fluid adjacent to the fin given by gas or water (see e.g.

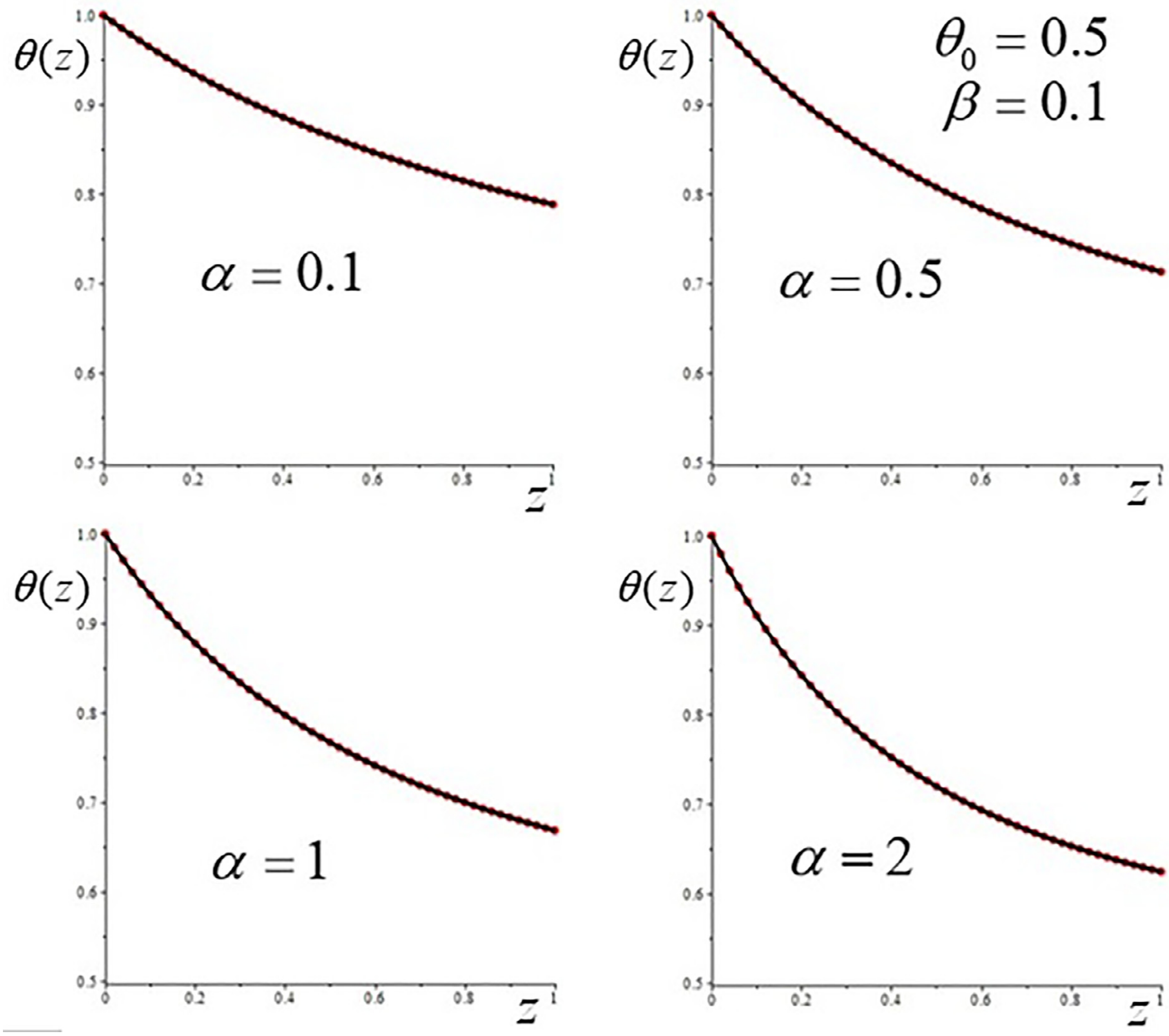

Fig. 5. The variation of temperature along the fin for four different values of α and with $\theta_0 = 0.5$ and $\beta = 0.1$. The black curves have been obtained directly by Eq. (68), whereas the red circles have been obtained by a numerical integration of the differential Eq. (6) with a seventh-eighth order continuous Runge–Kutta method. The corresponding profiles are given in Fig. 6.

[24]). If, for any given ϵ and h, the base of the fin has a temperature T_b greater or equal to the one on the corresponding curve, the non-linear term in Eq. (6) cannot be neglected.

Up to now we used dimensionless quantities since we assumed that all the thermal coefficients (thermal conductivity, convective heat transfer coefficient and emissivity) are constants. Indeed, in this case there is a scale invariant property for solutions of Eq. (3), in the sense that for any $C \in \mathbb{R}$, if $T \to CT$, $T_0 \to CT_0$, $T_1 \to CT_1$ and $\epsilon \to \epsilon/C^3$ (3) remains unchanged. Clearly, for any given material, the thermal coefficients can be considered constant only for finite ranges of temperature and this should be taken into account in the applications of the results of the above sections. To clarify this point, we make a more practical example. We consider a base at 800 K, a fluid at 400 K and an aluminum fin. The thermal conductivity for aluminum between 400 K and 800 K varies from 220 $\mathrm{Wm^{-1}K^{-1}}$ to 240 $\mathrm{Wm^{-1}K^{-1}}$ [25] and so can be considered constant with a good approximation. We assume that the heat transfer coefficient is 50 $\mathrm{Wm^{-2}K^{-1}}$. The ratio ℓ^2/f_b is assumed to be equal to 1m (i.e., if both ℓ and f_b are in cm, then $\ell = 10\sqrt{f_b}$). With these values and taking $\kappa = 230\,\mathrm{Wm^{-1}K^{-1}}$ for the thermal conductivity, we get $\alpha \sim 0.4348$ and $\beta \sim 0.2272$ (we took the reference value $\epsilon = 0.9$ for anodized aluminum). These two values are of the same order of magnitude (equivalently, the condition (80) is not satisfied) and the radiative effects should be taken into account. To make a comparison with a case when the radiative effects are negligible, we make a second example by increasing the value of the heat transfer coefficient to 250 $\mathrm{Wm^{-2}K^{-1}}$ and keeping all other constants the same as before. In this second example the value of α is $\alpha \sim 2,\ 1739$, whereas β remains unchanged, i.e. $\beta \sim 0.2272$. Now the two constants are of different orders of magnitude and the radiative term may be neglected. In Fig. 8 we plot the values of the dimensional temperature T as a function of the dimensionless fin length $z = x/\ell$ for both the examples. The

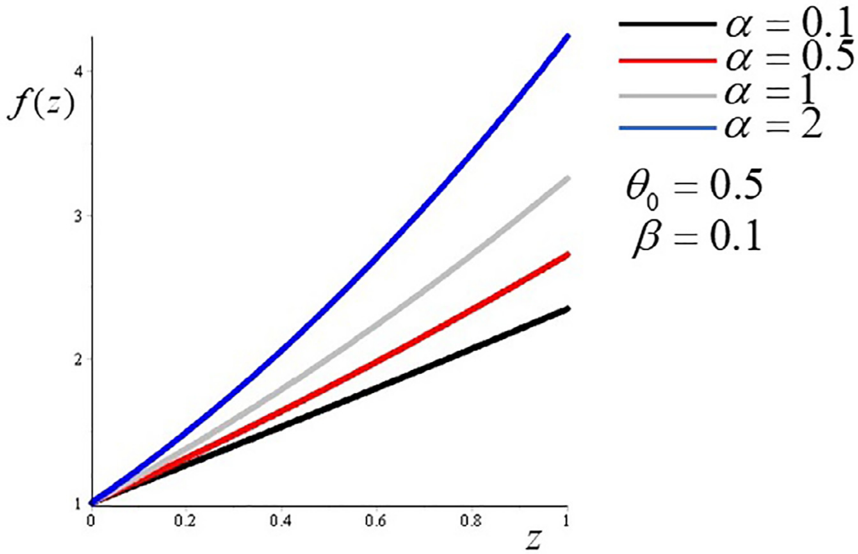

Fig. 6. The plot of the profile function $f(z)$ for four different values of α and with $\theta_0 = 0.5$ and $\beta = 0.1$. The corresponding variations of temperature along the fin are given in Fig. 5.

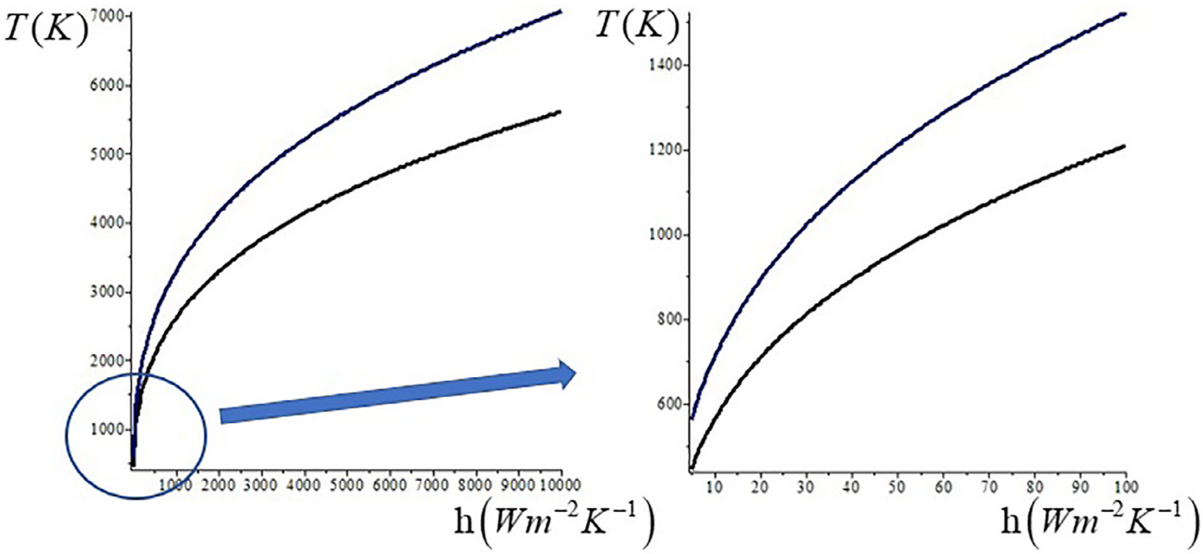

Fig. 7. Plot of the function $(h/\sigma\epsilon)^{1/3}$ as a function of h for $\epsilon = 1$ (lower curve) and $\epsilon = 0.5$ (upper curve). A magnification of the interval $h \in [5, 10^2]\mathrm{Wm}^{-2}\mathrm{K}^{-1}$ has been added for readableness.

black curves are the functions $T(z)$, solutions of Eq. (3) obtained with the ansatz (63) and the explicit expression (68). The boundary conditions are assumed to be the same as those at the end of the previous section (i.e. $T(z=0) = T_b$, fixing the value of $y(0)$ and $y(1)$ given by the solution of (68) in $z = 1$). The blue curves are the purely convective solutions, obtained with the same boundary conditions and the same set of thermal constants apart the value of β, the radiative coefficient, taken to be equal to zero. Explicitly, in this two examples, the purely convective solutions are given by

$$T(z) = T_0 + \frac{T_b - T_0}{\left(1 + \sqrt{\frac{\alpha}{2}}z\right)^2}$$

with $T_b = 800\mathrm{K}$, $T_0 = 400\mathrm{K}$ and $\alpha = 0.4348$ (plot on the left) and $\alpha = 2.1739$ (plot on the right). The comparison between the black and blue curves confirms that in the second example (plot on the right) the radiative effects may be neglected. Also, in the first example, the combination of radiation and convection results in a reduction of the temperature along the fin with respect to the purely convective case: the relative difference between the two curves, rather small in this case, can be taken as a measure of the capacity of dissipation by radiation of the fin.

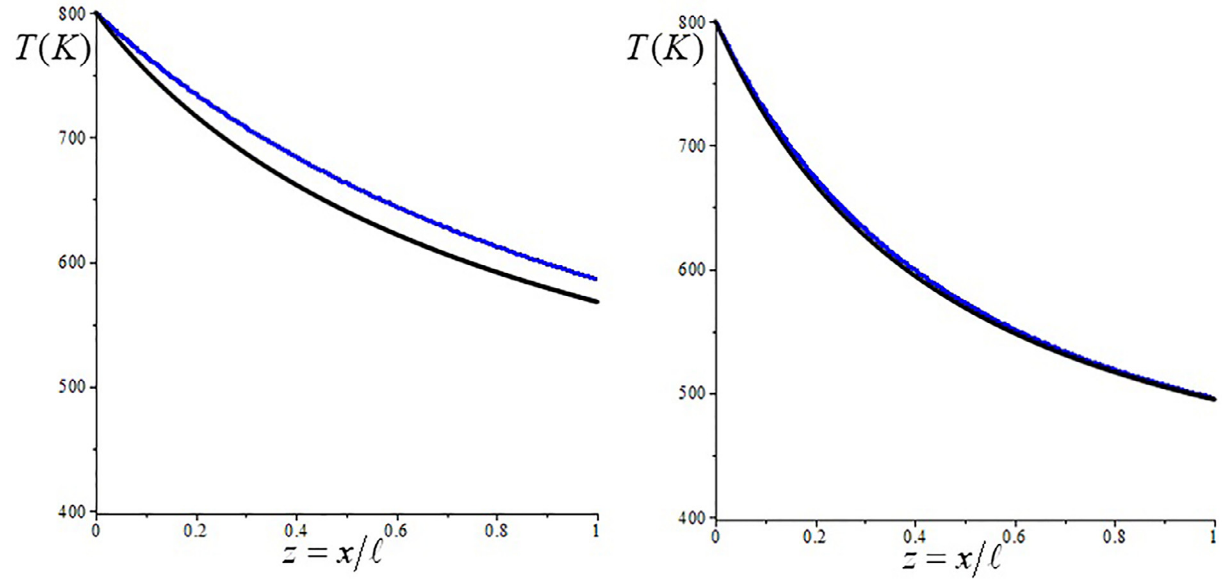

Fig. 8. Plot of the function $(h/\sigma\epsilon)^{1/3}$ as a function of h for $\epsilon = 1$ (lower curve) and $\epsilon = 0.5$ (upper curve). A magnification of the interval $h \in [5, 10^2]\mathrm{Wm}^{-2}\mathrm{K}^{-1}$ has been added for readableness.

7. Conclusions

We analyzed the non-linear equation describing the steady state temperature in convecting-radiating longitudinal fins. The introduction of a change of variable into the differential Eq. (6) gives the possibility to obtain new solutions both for the convective-linear case and for the convective-radiative non-linear case. The solution given in terms of the Airy functions (29) can be considered a generalization of the Gardner's classical result (20). The results about the efficiency of the fin with an arbitrary general profile and the application involving the Mathieu functions are interesting, since the comparison with the Airy case shows that the underlying mathematical model here considered provides higher efficiencies for profiles of the fin larger at the tip. The explicit solution in terms of the auxiliary variable for the non-linear case is a novel result and is worthy of further consideration since with different ansatz for the function $F(y)$ (63) it is possible to obtain other explicit solutions. Further, with an ansatz involving arbitrary parameters, it may be possible to get parametric solutions of the non-linear equation.

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