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GLOBAL WEAK SOLUTIONS TO THE CAUCHY PROBLEM FOR A TWO-PHASE MODEL AT A NODE*

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Abstract. This paper considers a system of two conservation laws with a Lipschitz continuous flow and deals with solutions to the Cauchy problems at a node. The system considered is the two-phase traffic model, proposed in [R. M. Colombo, F. Marcellini, and M. Rascle, *SIAM J. Appl. Math.*, 70 (2010), pp. 2652–2666]. We are able to provide global in time existence of solutions at a node with a single incoming road and m outgoing roads. Neither assumptions on the smallness of the total variation of initial data are required, as usual in the case of systems, nor on the range of the initial data.

Key words. hyperbolic systems of conservation laws, continuum traffic models, Cauchy problem, networks

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1. Introduction.

We prove global in time existence of solutions to the Cauchy problem at a node for a possibly degenerate system of two conservation laws with Lipschitz continuous flow. This work fits in the field of conservation laws on networks, which has attracted a lot of interest in recent years especially due to a wide variety of applications that range from traffic to gas dynamics, supply chains, telecommunications, and biology; see [6] and the references therein. In general, existence of solutions to Cauchy problems for a hyperbolic system of conservation laws holds provided the initial data have small total variation. Here we are able to treat initial conditions with finite total variation, not necessarily small.

Motivated by applications to traffic flow, the PDE system considered in this paper is the two-phase traffic model proposed in 2010 in [12]. There are two key ideas in the development of this traffic model: each driver has his/her own preferred maximal speed and there exists a maximal velocity that all the drivers respect. The resulting macroscopic model displays two traffic regimes or phases: the *free phase* F and the *congested phase* C. In the *free* region all drivers travel at the same maximal speed, the system is degenerate, and it reduces to a single conservation law describing the evolution of cars' density, while in the *congested* region the model is a 2×2 strictly hyperbolic system of conservation laws.

In the literature there are various macroscopic models describing traffic evolution; see [3, 16, 23, 30, 34, 36] for first order models and their extensions in [2, 14, 15, 29, 35, 37] for second order ones; see [4, 9, 21] for descriptions with phase transitions, and see [25] for a third order model. The prototype for macroscopic traffic models is the well known Lighthill–Whitham–Richards, proposed independently in [30, 34],

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and consisting of a single equation based on the conservation of the total number of vehicles. The two-phase traffic model (also called the phase transition model), considered in this paper, belongs to the class of *second order models*, i.e., models based on systems of two equations. The most famous in this context is the Aw–Rascle–Zhang introduced in [2, 37], then extended to the general Aw–Rascle–Zhang in [14]; see also the collapsed general Aw–Rascle–Zhang in [15]. The present two-phase traffic model belongs also to the class of models with phase transitions. They are mainly characterized by two regimes of traffic, which correspond to different regions possibly disconnected, in the fundamental diagram. In the *free* region usually traffic is described by a single conservation law, whereas in the *congested* region traffic is described by a system of two conservation laws. The first model with phase transition was proposed in 2002 by Colombo [9]; other examples can be found in [4, 21, 28, 31].

This paper is mainly devoted to the study of the Cauchy problem for the extension of the two-phase traffic model to a simple network, consisting of a node with a single incoming road and m outgoing roads. The extension of the two-phase model at simple junctions is proposed in [17], where the Riemann problem both for the case of a node with one incoming road and m outgoing ones and the case of a node with two incoming roads and a single outgoing one are considered. The motivation for treating these two special kinds of junctions in [17] relies on the rule for the distribution of the maximal speed in outgoing roads, which is very simple in the case of a single incoming road, but it becomes more intricate as the number of incoming roads increases; see [18] for a justification of this rule. For the previous reasons and to avoid additional technicalities, here we decide to study the existence of solutions for the Cauchy problem only in the case of a junction with a single incoming road.

The study of hyperbolic conservation laws on networks originated in the work by Holden and Risebro [26] in 1995, which opened the way to a vast literature considering this topic under a variety of points of view that range from the theoretical aspects to modeling and numerics.

The analysis in this paper is also preliminary to deal with control problems for traffic in a general network using the phase transition model. In recent years, this topic has raised relevant interest, clearly motivated by real applications. In this direction, a possibility is to control traffic through time varying parameters corresponding, for example, to junction distribution coefficients; see [7, 8, 20, 24, 33] and the references therein. Another possibility consists of controlling traffic evolution by regulating the inflows; see [1, 11, 22].

As underlined above, the main result here is the global in time existence of solutions to the Cauchy problem at the node, provided the initial conditions are bounded variation functions. We remark that neither assumptions on the smallness of the total variation on initial data are required nor any constraint on the fundamental diagram, as in [10], is imposed. The proof is based on the wave-front tracking technique, which is one of the natural methods in the conservation laws setting. Among other techniques we recall the vanishing viscosity method, the compensated compactness argument, and the random choice method; see [13] and the references therein. The wave-front tracking technique consists of constructing a piecewise constant approximate solution and proving that, up to a subsequence, it converges to a solution of the problem. A key role is played by our introduction of specific functionals enjoying two properties. First, they provide a uniform control on the total variation and, second, their variation along approximate solutions can be contolled. Moreover, bounds on the number of waves and interaction esimates on approximate solutions are obtained.

In the present case, the functionals here defined are not always decreasing in time, since several wave interactions with the junction cause a strict increment of their values. The present deeper analysis allows us to prove the existence of a uniform bound, which depends on the initial data. The bound on the number of waves and of the interaction estimates is obtained extending the techniques in [19] to the case of systems, by using a contradiction argument and the quasi-periodic behavior of approximate solutions.

The paper is organized as follows. In the next section we briefly recall the two-phase model introduced in [12]. In section 3 we state and prove the main result concerning the existence of solutions to the Cauchy problem at a node with an incoming road and m outgoing roads; the proof is divided into three different subsections. Finally Appendix A contains some technical results.

2. Notation: The two-phase model.

The two-phase model, introduced in [12], is derived as an extension of the Lighthill–Whitham–Richards model [30, 34], by assuming that each driver has his/her own maximal speed wR, where $w \in [\check{W}, \widehat{W}]$ plays the role of a marker and R is maximal possible car density. The two-phase model is given by the system of two conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, v(\rho, \eta)) = 0 \\ \partial_t \eta + \partial_x (\eta\, v(\rho, \eta)) = 0 \end{cases} \quad \text{with } \; v(\rho, \eta) = \min \{V_{\max}, w(R - \rho)\}, \tag{2.1}$$

where t is the time, x is the space, ρ is the traffic density, $\eta = \rho w$ is the second conserved variable, $v \in [0, V_{\max}]$ is the speed of cars, and $V_{\max}$ is a uniform constant bound of the cars' speed.

The model (2.1) is characterized by two different phases, the free one and the congested one, which are described by the sets

$$F = \left\{(\rho, \eta) \in [0, R] \times [0, \widehat{W} R] \colon \check{W} \rho \le \eta \le \widehat{W} \rho,\, v(\rho, \eta) = V_{\max}\right\},$$
$$C = \left\{(\rho, \eta) \in [0, R] \times [0, \widehat{W} R] \colon \check{W} \rho \le \eta \le \widehat{W} \rho,\, v(\rho, \eta) = w(R - \rho)\right\};$$

see Figure 1. As in [12, 17], we assume the following assumptions:

(H-1) $R, \check{W}, \widehat{W}, V_{\max}$ are positive constants, with $V_{\max} < \check{W} < \widehat{W}$.

(H-2) Waves of the first family in the congested phase C have negative speed; more precisely, we assume that there exists a positive constant λ_0 such that

$$\lambda_1(\rho, \eta) \le -\lambda_0 < 0,$$

where $\lambda_1 = w(R - 2\rho)$ is the first eigenvalue of the Jacobian matrix of the flux.

For the Riemann problem along a single road see [12]. Here, we recall all the possible waves in the solution.

- *A linear wave* is a wave connecting two states in the free phase.
- *A phase transition wave* is a wave connecting a left state $(\rho_l, \eta_l) \in F$ with a right state $(\rho_r, \eta_r) \in C$ satisfying $\frac{\eta_l}{\rho_l} = \frac{\eta_r}{\rho_r}$.
- *A wave of the first family* is a wave connecting a left state $(\rho_l, \eta_l) \in C$ with a right state $(\rho_r, \eta_r) \in C$ such that $\frac{\eta_l}{\rho_l} = \frac{\eta_r}{\rho_r}$.
- *A wave of the second family* is a wave connecting a left state $(\rho_l, \eta_l) \in C$ with a right state $(\rho_r, \eta_r) \in C$ such that $v(\rho_l, \eta_l) = v(\rho_r, \eta_r)$.

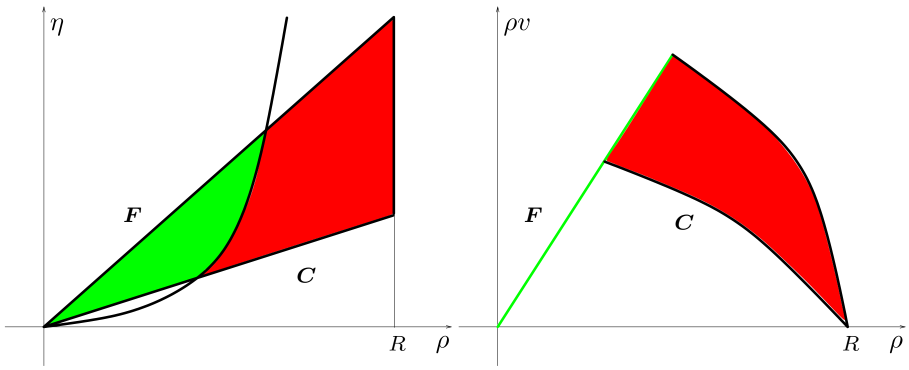

FIG. 1. *The free phase F in green and the congested phase C in red, resulting from* (2.1) *in the coordinates* (ρ, η)*, left, where the two phases are both two-dimensional, and in the coordinates* $(\rho, \rho v)$*, right, where F is one-dimensional and C is two-dimensional.*

3. The Cauchy problem at a $1 \times m$ node. In this section, we deal with the Cauchy problem at a node with a single incoming road I_1, modeled by the real interval $(-\infty, 0)$, and m outgoing ones $I_2, \ldots, I_{1+m}$, modeled by $(0, +\infty)$. More precisely, we consider the following Cauchy problem:

$$\left\{\begin{array}{ll}\left\{\begin{array}{l}\partial_t \rho_1 + \partial_x \left(\rho_1 \, v(\rho_1, \eta_1)\right) = 0,\\ \partial_t \eta_1 + \partial_x \left(\eta_1 \, v(\rho_1, \eta_1)\right) = 0,\\ \left(\rho_1, \eta_1\right)(0, x) = \left(\bar\rho_1, \bar\eta_1\right)(x),\end{array}\right. & x < 0,\\ \quad\vdots & \\ \left\{\begin{array}{l}\partial_t \rho_{1+m} + \partial_x \left(\rho_{1+m} \, v(\rho_{1+m}, \eta_{1+m})\right) = 0,\\ \partial_t \eta_{1+m} + \partial_x \left(\eta_{1+m} \, v(\rho_{1+m}, \eta_{1+m})\right) = 0,\\ \left(\rho_{1+m}, \eta_{1+m}\right)(0, x) = \left(\bar\rho_{1+m}, \bar\eta_{1+m}\right)(x),\end{array}\right. & x > 0,\end{array}\right. \tag{3.1}$$

where $(\bar\rho_1, \bar\eta_1) \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)((-\infty, 0); F \cup C)$ and, for $j \in \{2, \ldots, m+1\}$, $(\bar\rho_j, \bar\eta_j) \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)((0, +\infty); F \cup C)$.

Before stating the definition of solution for (3.1) and the main result of this paper, we introduce the concept of a Riemann solver and of equilibrium for the Riemann problem at the node. We recall that a Riemann problem at a node is the Cauchy problem (3.1), where the initial conditions are constant in each road. It is well known that the solution to a Riemann problem can be given through a function, called the Riemann solver, which associates to each initial condition the traces at the node of the solution. In this paper, we consider the Riemann solver, denoted by $\mathcal{RS}_J$, proposed in [17], which produces a solution conserving both the number of cars and the maximal speed of each single driver at J; such a Riemann solver has been obtained imposing the so-called Rankine–Hugoniot conditions at the junction. For a detailed description see [17, section 4].

DEFINITION 3.1. *We say that* $\left((\bar\rho_1, \bar\eta_1), \ldots, (\bar\rho_{1+m}, \bar\eta_{1+m})\right)$ *is an equilibrium for the Riemann solver* $\mathcal{RS}_J$ *if*

$$\mathcal{RS}_J\left((\bar\rho_1, \bar\eta_1), \ldots, (\bar\rho_{1+m}, \bar\eta_{1+m})\right) = \left((\bar\rho_1, \bar\eta_1), \ldots, (\bar\rho_{1+m}, \bar\eta_{1+m})\right),$$

i.e., $\left((\bar\rho_1, \bar\eta_1), \ldots, (\bar\rho_{1+m}, \bar\eta_{1+m})\right)$ *is a fixed point of* $\mathcal{RS}_J$*.*

DEFINITION 3.2. *An equilibrium* $((\bar\rho_1,\bar\eta_1),\dots,(\bar\rho_{1+m},\bar\eta_{1+m}))$ *for the Riemann solver* $\mathcal{RS}_J$ *is said to be generic if either*

$$(\bar\rho_1,\bar\eta_1)\in F,\qquad (\bar\rho_j,\bar\eta_j)\in F\setminus C\qquad \forall j\in\{2,\dots,1+m\}$$

or exists $\bar{\jmath}\in\{2,\dots,1+m\}$ *such that*

$$(\bar\rho_1,\bar\eta_1)\in C\setminus F,\quad (\bar\rho_{\bar{\jmath}},\bar\eta_{\bar{\jmath}})\in C,\quad (\bar\rho_j,\bar\eta_j)\in F\setminus C\quad \forall j\in\{2,\dots,1+m\}\setminus\{\bar{\jmath}\}.$$

In the former case, we say that the state $(\bar\rho_1,\bar\eta_1)$ *is the active constraint for the equilibrium. In the latter case, we say that the state* $(\bar\rho_{\bar{\jmath}},\bar\eta_{\bar{\jmath}})$ *is the active constraint for the equilibrium.*

We are now ready to give a constructive definition of the solution to the Cauchy problem (3.1) and to state the main result of this paper.

DEFINITION 3.3. *A tuple* $\left((\rho_1^*,\eta_1^*),\dots,(\rho_{1+m}^*,\eta_{1+m}^*)\right)$ *is a solution to* (3.1) *if*

1. $(\rho_1^*,\eta_1^*)\in \mathbf{C^0}\left([0,+\infty[;\mathbf{L^1}(I_1;F\cup C)\right)$ *is a weak solution to* (2.1) *for* $x<0$ *such that* $(\rho_1^*(0,x),\eta_1^*(0,x))=(\bar\rho_1(x),\bar\eta_1(x))$ *for a.e.* $x<0$*;*
2. *for every* $j\in\{2,\dots,1+m\}$, $\left(\rho_j^*,\eta_j^*\right)\in\mathbf{C^0}\left([0,+\infty[;\mathbf{L^1}(I_j;F\cup C)\right)$ *is a weak solution to* (2.1) *for* $x>0$ *such that* $\left(\rho_j^*(0,x),\eta_j^*(0,x)\right)=(\bar\rho_j(x),\bar\eta_j(x))$ *for a.e.* $x>0$*;*
3. *for a.e.* $t>0$ *and for* $i\in\{1,\dots,1+m\}$*, the function* $x\mapsto(\rho_i^*(t,x),\eta_i^*(t,x))$ *has bounded total variation;*
4. *the couple* $\left((\rho_1(t,0-),\eta_1(t,0-)),\dots,(\rho_{1+m}(t,0+),\eta_{1+m}(t,0+))\right)$*, for a.e.* $t>0$*, provides an equilibrium for the Riemann solver* $\mathcal{RS}_J$ *in the sense of Definition* 3.1.

THEOREM 3.4. *Under assumptions* (H-1) *and* (H-2)*, fix initial data* $(\bar\rho_1,\bar\eta_1)\in\left(\mathbf{L^1}\cap\mathbf{BV}\right)((-\infty,0);F\cup C)$ *and* $(\bar\rho_j,\bar\eta_j)\in\left(\mathbf{L^1}\cap\mathbf{BV}\right)((0,+\infty);F\cup C)$ *for every* $j\in\{2,\dots,1+m\}$*. Then there exists a solution to the Cauchy problem* (3.1) *in the sense of Definition* 3.3*.*

The proof is based on the wave-front tracking technique and it is contained in the following subsections.

3.1. Wave-front tracking approximate solution. We construct piecewise constant approximations via the wave-front tracking technique; see [5, 13, 27] for the general theory.

DEFINITION 3.5. *Given* $\varepsilon>0$*, we say that the map* $u_\varepsilon=(u_{1,\varepsilon},\dots,u_{1+m,\varepsilon})=((\rho_{1,\varepsilon},\eta_{1,\varepsilon}),\dots,(\rho_{1+m,\varepsilon},\eta_{1+m,\varepsilon}))$ *is an* ε*-approximate wave-front tracking solution to* (3.1) *if the following conditions hold:*

1. $u_{i,\varepsilon}\in\mathbf{C^0}((0,+\infty);\mathbf{L^1}(I_i;F\cup C))$ *for every* $i\in\{1,\dots,1+m\}$*.*
2. *For every* $i\in\{1,\dots,1+m\}$*,* $u_{i,\varepsilon}$ *is piecewise constant, with discontinuities occurring along finitely many straight lines in* $(0,+\infty)\times I_i$*. Moreover the jumps can be of the first family, of the second family, linear waves, or phase-transition waves. They are indexed by* $\mathcal{J}(t)=\mathit{1}(t)\cup\mathit{2}(t)\cup\mathcal{LW}(t)\cup\mathcal{PT}(t)$*.*
3. *For every* $i\in\{1,\dots,1+m\}$*, it holds that*

$$\begin{cases}\left\|(\rho_{i,\varepsilon}(0,\cdot),\eta_{i,\varepsilon}(0,\cdot))-(\bar\rho_i(\cdot),\bar\eta_i(\cdot))\right\|_{\mathbf{L^1}(I_i;F\cup C)}<\varepsilon,\\ \mathrm{TV}\,(\rho_{i,\varepsilon}(0,\cdot),\eta_{i,\varepsilon}(0,\cdot))\le\mathrm{TV}\,(\bar\rho_i(\cdot),\bar\eta_i(\cdot)).\end{cases}$$

4. *For a.e.* $t>0$*, it holds that*

$$\mathcal{RS}_J\left(u_{1,\varepsilon}(t,0-),\dots,u_{1+m,\varepsilon}(t,0+)\right)=\left(u_{1,\varepsilon}(t,0-),\dots,u_{1+m,\varepsilon}(t,0+)\right).$$

Consider, for every $i \in \{1, \ldots, 1+m\}$, a sequence $(\bar\rho_{i,\nu}, \bar\eta_{i,\nu})$ of piecewise constant functions with a finite number of discontinuities such that

1. $(\bar\rho_{i,\nu}, \bar\eta_{i,\nu}) : I_i \to F \cup C$ for every $i \in \{1, \ldots, 1+m\}$;
2. for every $i \in \{1, \ldots, 1+m\}$, the following limit holds:
$$\lim_{\nu \to +\infty} (\bar\rho_{i,\nu}, \bar\eta_{i,\nu}) = (\bar\rho_i, \bar\eta_i) \qquad \text{in } \mathbf{L^1}(I_i; F \cup C);$$
3. for every $i \in \{1, \ldots, 1+m\}$, the following inequality holds:
$$\mathrm{TV}(\bar\rho_{i,\nu}, \bar\eta_{i,\nu}) \le \mathrm{TV}(\bar\rho_i, \bar\eta_i).$$

For every $\nu \in \mathbb{N} \setminus \{0\}$, we apply the following procedure. At time $t = 0$, we solve the Riemann problem at the junction and all the Riemann problems inside the roads. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ traveling with the Rankine–Hugoniot speed. At every interaction between two waves, we solve the corresponding Riemann problem. Finally, when a wave interacts with the junction, we solve the corresponding Riemann problem.

Remark 3.6. As usual, by slightly modifying the speed of waves or the position of the discontinuities for the boundary values, we may assume that, at every positive time t, at most one of the following possibilities happens:

1. two waves interact together inside a road;
2. a wave interacts with the junction;
3. for a.e. $t > 0$, the equilibrium at J is generic, in the sense of Definition 3.2.

Remark 3.7. For interactions inside a road, we split rarefaction waves into rarefaction fans only at time $t = 0$. At the junction, on the contrary, we allow the formation of rarefaction fans at every positive time.

Fix a constant $K_w > 0$. Given an ε-approximate wave-front tracking solution, define, for a.e. $t > 0$, the following functionals:

$$\begin{aligned} \mathcal{F}_w(t) = {} & K_w \sum_{x \in I_1} \left| w\left(u_{1,\varepsilon}(t, x+)\right) - w\left(u_{1,\varepsilon}(t, x-)\right) \right| \\ & + \sum_{i=2}^{1+m} \sum_{x \in I_i} \left| w\left(u_{i,\varepsilon}(t, x+)\right) - w\left(u_{i,\varepsilon}(t, x-)\right) \right|, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathcal{F}_{\tilde v}(t) = {} & \sum_{i=1}^{1+m} \sum_{x \in I_i} \left| \tilde v\left(\bar u_{i,\varepsilon}(t, x+)\right) - \tilde v\left(\bar u_{i,\varepsilon}(t, x-)\right) \right| \\ & + \tilde v\left(\bar u_{1,\varepsilon}(t, 0-)\right) - \sum_{i=2}^{1+m} \tilde v\left(\bar u_{i,\varepsilon}(t, 0+)\right), \end{aligned} \tag{3.3}$$

$$\mathcal{N}(t) = \sum_{i=1}^{1+m} \# \left\{ x \in I_i : \bar u_{i,\varepsilon}(t, x-) \neq \bar u_{i,\varepsilon}(t, x+) \right\}, \tag{3.4}$$

where the symbol # stands for cardinality of a set and $\tilde v(\rho, \eta) = w(R - \rho)$. Note that the previous functionals may vary only at times $\bar t$ when two waves interact or a wave hits the junction.

3.2. Interaction estimates. We consider here estimates for wave interactions. In the following, we distinguish wave interactions by the nature of the involved waves: for example, if a wave of the second family hits a wave of the first family producing a phase transition wave, we write 2-1/$\mathcal{PT}$. Here the symbol "/" divides the waves

TABLE 1

The possible cases when there is an interaction of the wave $\left(\left(\rho^l,\eta^l\right),\left(\hat\rho_1,\hat\eta_1\right)\right)$ *with the junction from the incoming road* I_1*. Lemmas* 3.9, 3.10, *and* 3.11 *describe respectively the cases of interacting wave of type* 2, $\mathcal{LW}$, *and* $\mathcal{PT}$.

Wave	$\Delta\mathcal{N}$	$\Delta\mathcal{F}_w$	$\Delta\mathcal{F}_{\tilde v}$
2	$\le m+1$	$\le (m-K_w)\left\|w^l-\hat w_1\right\|$	$\le K\left\|w^l-\hat w_1\right\|$
$\mathcal{LW}$	$\le m$	$\le (m-K_w)\left\|w^l-\hat w_1\right\|$	$\le K\left[\left\|w^l-\hat w_1\right\|+\left\|v^l-\hat v_1\right\|\right]$
$\mathcal{PT}$	$\le m-1$	$=0$	$=0$

TABLE 2

The possible cases when there is an interaction of the wave $\left(\left(\hat\rho_{\bar{\jmath}},\hat\eta_{\bar{\jmath}}\right),\left(\rho^r,\eta^r\right)\right)$ *with the junction from the outgoing road* $I_{\bar{\jmath}}$*. Lemmas* 3.12 *and* 3.13 *describe respectively the cases of interacting wave of type* 1 *and* $\mathcal{PT}$.

Wave	$\Delta\mathcal{N}$	$\Delta\mathcal{F}_w$	$\Delta\mathcal{F}_{\tilde v}$
1	$\le m$	$=0$	$\le K\|v^r-\hat v_{\bar{\jmath}}\|$
$\mathcal{PT}$	$\le m-1$	$=0$	$=0$

before and after the interaction. Moreover, with the notation $\Delta\mathcal{F}_w(t)$, $\Delta\mathcal{F}_{\tilde v}(t)$, and $\Delta\mathcal{N}(t)$, we intend respectively $\mathcal{F}_w(t+)-\mathcal{F}_w(t-)$, $\mathcal{F}_{\tilde v}(t+)-\mathcal{F}_{\tilde v}(t-)$, and $\mathcal{N}(t+)-\mathcal{N}(t-)$. Table 1 contains all the possible interactions between a wave, coming from the incoming road, and the junction. Table 2 contains all the possible interactions between a wave, coming from an outgoing road, and the junction. For later use, define the constant

$$\begin{aligned}K = {}& 2\frac{V_{\max}^2}{R\widecheck{W}^2}+4Rm+2\frac{\widehat{W}}{\widecheck{W}}+\frac{2\widehat{W}V_{\max}}{\widecheck{W}^2\min\{\alpha_{1j}:j\in\{2,\dots,m+1\}\}}\\ &+\frac{2R\widehat{W}}{\min\{\alpha_{1j}:j\in\{2,\dots,m+1\}\}}\left(\frac{1}{\lambda_0}+\frac{1}{V_{\max}}\right).\end{aligned} \tag{3.5}$$

3.2.1. Interactions along a road. In the case of an interaction between two waves inside a road the functionals (3.2), (3.3), and (3.4) do not increase. Indeed the following lemma holds.

LEMMA 3.8. *Assume that the wave* $((\rho^l,\eta^l),(\rho^m,\eta^m))$ *interacts with the wave* $((\rho^m,\eta^m),(\rho^r,\eta^r))$ *in a road* I_i*, for* $i=1,\dots,1+m$*, at the point* $(\bar t,\bar x)$ *with* $\bar t>0$ *and* $\bar x\in I_i$*. Then*

$$\Delta F_w(\bar t)=0 \qquad \textit{and} \qquad \Delta F_{\tilde v}(\bar t)\le 0\,.$$

Moreover the possible interactions are 2-1/1-2, $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2, 1-1/1, $\mathcal{PT}$-1/$\mathcal{PT}$*; hence* $\Delta\mathcal{N}(\bar t)\le 0$.

The proof is completely identical to [32, Lemma 3.4]; hence we omit it.

3.2.2. Wave hitting the junction from the incoming road. We consider the case of a wave $\left(\rho^l,\eta^l\right)$, coming from I_1 and interacting with the junction. For simplicity we denote by

$$(\hat\rho_1,\hat\eta_1)\,,\quad(\hat\rho_2,\hat\eta_2)\,,\quad\dots,\quad(\hat\rho_{1+m},\hat\eta_{1+m})$$

the states at the junction before the interaction. By Remark 3.6, without loss of generality, we may assume that $((\hat\rho_1,\hat\eta_1),\dots,(\hat\rho_{1+m},\hat\eta_{1+m}))$ is a generic equilibrium for the Riemann solver $\mathcal{RS}_J$. Moreover denote

$$\left((\rho_1^*,\eta_1^*),\dots,(\rho_{1+m}^*,\eta_{1+m}^*)\right)=\mathcal{RS}_J\left(\left(\rho^l,\eta^l\right),(\hat\rho_2,\hat\eta_2),\dots,(\hat\rho_{1+m},\hat\eta_{1+m})\right).$$

LEMMA 3.9. *Assume that the wave of the second family* $((\rho^l, \eta^l), (\hat\rho_1, \hat\eta_1))$ *interacts with the junction for the incoming road* I_1 *at time* $\bar t$. *Then*

$$\Delta \mathcal{F}_w(\bar t) \le (m - K_w)\left|w^l - \hat w_1\right|, \tag{3.6}$$

$$\Delta F_{\tilde v}(\bar t) \le K\left|w^l - \hat w_1\right|, \tag{3.7}$$

where K *is given by* (3.5). *Moreover* $\Delta \mathcal{N}(\bar t) \le m+1$.

Proof. For simplicity, for $i = 1, \dots 1+m$, we define

$$\begin{aligned} v^l &= \tilde v(\rho^l, \eta^l), \quad \hat v_i = \tilde v(\hat\rho_i, \hat\eta_i), \quad v_i^* = \tilde v(\rho_i^*, \eta_i^*), \\ w^l &= w(\rho^l, \eta^l), \quad \hat w_i = w(\hat\rho_i, \hat\eta_i), \quad w_i^* = w(\rho_i^*, \eta_i^*). \end{aligned} \tag{3.8}$$

The interacting wave is a wave of the second family; then both the states (ρ^l, η^l) and $(\hat\rho_1, \hat\eta_1)$ belong to $C \setminus F$. Since the equilibrium is generic, there exists $\bar\jmath \in \{2, \dots, 1+m\}$ such that

$$(\hat\rho_{\bar\jmath}, \hat\eta_{\bar\jmath}) \in C, \qquad (\hat\rho_j, \hat\eta_j) \in F \quad \forall j \in \{2, \dots, 1+m\} \setminus \{\bar\jmath\}.$$

We have two possibilities: either $\rho^l > \hat\rho_1$ or $\rho^l < \hat\rho_1$.

Assume first $\rho^l > \hat\rho_1$. In this case Lemma A.1 implies that the state $(\rho^*_{\bar\jmath}, \eta^*_{\bar\jmath}) \in C$ is the active constraint after the interaction and is determined uniquely by the conditions

$$\begin{cases} (\rho^*_{\bar\jmath}, \eta^*_{\bar\jmath}) \in C, \\ v(\rho^*_{\bar\jmath}, \eta^*_{\bar\jmath}) = v(\hat\rho_{\bar\jmath}, \hat\eta_{\bar\jmath}), \\ w^*_{\bar\jmath} = w^l. \end{cases}$$

Therefore in the road I_1 a wave of the first family, connecting (ρ_1^*, η_1^*) with (ρ^l, η^l), is produced; in the road $I_{\bar\jmath}$ a wave of the second family, connecting $(\rho^*_{\bar\jmath}, \eta^*_{\bar\jmath})$ with $(\hat\rho_{\bar\jmath}, \hat\eta_{\bar\jmath})$, is produced; finally in each of the remaining outgoing roads a single linear wave is produced. For the functional (3.2) we have

$$\begin{aligned} \Delta \mathcal{F}_w(\bar t) &= K_w\left|w_1^* - w^l\right| + \sum_{j=2}^{1+m}\left|w_j^* - \hat w_j\right| - K_w\left|w^l - \hat w_1\right| \\ &= \left[\sum_{j=2}^{1+m}\left|w^l - \hat w_1\right|\right] - K_w\left|w^l - \hat w_1\right| = (m - K_w)\left|w^l - \hat w_1\right|, \end{aligned} \tag{3.9}$$

since $w^l = w_1^* = \cdots = w_{1+m}^*$ and $\hat w_1 = \cdots = \hat w_{1+m}$; this proves (3.6). For the functional (3.3) we deduce that

$$\begin{aligned} \Delta F_{\tilde v}(\bar t) &= \left|v_1^* - v^l\right| + \sum_{j=2}^{1+m}\left|v_j^* - \hat v_j\right| - \left|v^l - \hat v_1\right| + (v_1^* - \hat v_1) - \sum_{j=2}^{1+m}(v_j^* - \hat v_j) \\ &= \sum_{j \ne \bar\jmath, j=2}^{1+m}\left|v_j^* - \hat v_j\right| - \sum_{j=2}^{1+m}(v_j^* - \hat v_j) \le 2\sum_{j \ne \bar\jmath, j=2}^{1+m}\left|v_j^* - \hat v_j\right|, \end{aligned}$$

since $\hat v_1 = v^l > v_1^*$. For every $j \in \{2, \dots, m+1\}$, $v_j^* = w_1^*(R - \rho_j^*)$ and $\hat v_j = \hat w_1(R - \hat\rho_j)$; thus

$$\Delta F_{\tilde v}(\bar t) \le 2R \sum_{j\neq \bar{\jmath}, j=2}^{1+m} |w_1^* - \hat w_1| + 2 \sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|w_1^*\rho_j^* - \hat w_1 \hat\rho_j\right|$$

$$\le 4R \sum_{j\neq \bar{\jmath}, j=2}^{1+m} |w_1^* - \hat w_1| + 2\widehat{W} \sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* - \hat\rho_j\right|.$$

Since $(\hat\rho_j, \hat w_j) \in F$ and $\left(\rho_j^*, w_j^*\right) \in F$ for every $j \in \{2, \ldots, m+1\} \setminus \{\bar{\jmath}\}$,

$$\sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* - \hat\rho_j\right| = \frac{1}{V_{\max}} \sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* V_{\max} - \hat\rho_j V_{\max}\right|$$

$$= \frac{|\rho_1^* v_1^* - \hat\rho_1 \hat v_1|}{V_{\max}} \sum_{j\neq \bar{\jmath}, j=2}^{1+m} \alpha_{1,j} \le \frac{|\rho_1^* v_1^* - \hat\rho_1 \hat v_1|}{V_{\max}}$$

$$= \frac{\left|\rho_{\bar{\jmath}}^* - \hat\rho_{\bar{\jmath}}\right| v_{\bar{\jmath}}^*}{\alpha_{1,\bar{\jmath}} V_{\max}}.$$

Moreover we have that $\rho_{\bar{\jmath}}^* = R - \frac{v_{\bar{\jmath}}^*}{w_1^*}$ and $\hat\rho_{\bar{\jmath}} = R - \frac{v_{\bar{\jmath}}^*}{\hat w_1}$; therefore

$$\sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* - \hat\rho_j\right| \le \frac{\left|\rho_{\bar{\jmath}}^* - \hat\rho_{\bar{\jmath}}\right| v_{\bar{\jmath}}^*}{\alpha_{1,\bar{\jmath}} V_{\max}} = \frac{\left(v_{\bar{\jmath}}^*\right)^2}{\alpha_{1,\bar{\jmath}} V_{\max}} \frac{|\hat w_1 - w_1^*|}{\hat w_1 w_1^*}.$$

Hence we get that

$$\Delta F_{\tilde v}(\bar t) \le 4R \sum_{j\neq \bar{\jmath}, j=2}^{1+m} |w_1^* - \hat w_1| + 2\widehat{W} \sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* - \hat\rho_j\right|$$

$$\le \left[4R\,(m-1) + 2\widehat{W} \frac{V_{\max}}{\alpha_{1,\bar{\jmath}}} \frac{1}{\widetilde{W}^2}\right] |\hat w_1 - w_1^*| \le K |\hat w_1 - w_1^*|,$$

proving (3.7).

Assume now $\rho^l < \hat\rho_1$. Define $v_i^\circ = \tilde v(\rho_i^\circ, \eta_i^\circ)$ and $w_i^\circ = w(\rho_i^\circ, \eta_i^\circ)$, where the state $\left(\rho_{\bar{\jmath}}^\circ, \eta_{\bar{\jmath}}^\circ\right) \in C$ is uniquely determined by

$$\begin{cases} v\left(\rho_{\bar{\jmath}}^\circ, \eta_{\bar{\jmath}}^\circ\right) = v\left(\hat\rho_{\bar{\jmath}}, \hat\eta_{\bar{\jmath}}\right), \\ w_{\bar{\jmath}}^\circ = w^l. \end{cases}$$

We have two possibilities.

1. The state $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) \in C$ is the active constraint for the new equilibrium. Therefore in the road I_1 a wave of the first family, connecting (ρ_1^*, η_1^*) with $\left(\rho^l, \eta^l\right)$, is produced; in the road $I_{\bar{\jmath}}$ a wave of the second family, connecting $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) = \left(\rho_{\bar{\jmath}}^\circ, \eta_{\bar{\jmath}}^\circ\right)$ with $(\hat\rho_{\bar{\jmath}}, \hat\eta_{\bar{\jmath}})$, is produced; finally in each of the remaining outgoing roads a single linear wave is produced. For the functional (3.2), the estimate (3.9) holds, since $w^l = w_1^* = \cdots = w_{1+m}^*$ and $\hat w_1 = \cdots = \hat w_{1+m}$. For the functional (3.3) we deduce that

$$\Delta F_{\tilde v}(\bar t) = \left|v_1^* - v^l\right| + \sum_{j=2}^{1+m} \left|v_j^* - \hat v_j\right| - \left|v^l - \hat v_1\right| + (v_1^* - \hat v_1)$$

$$- \sum_{j=2}^{1+m} (v_j^* - \hat v_j) \le 2\,(v_1^* - \hat v_1) + 2 \sum_{j\neq \bar{\jmath}, j=2}^{1+m} \left|v_j^* - \hat v_j\right|$$

since $\hat{v}_1 = v^l < v_1^*$. The same computations as before imply that

$$\Delta F_{\tilde{v}}(\bar{t}) \le 2\left(v_1^* - \hat{v}_1\right) + \left[4R\left(m-1\right) + 2\widehat{W}\frac{V_{\max}}{\alpha_{1,\bar{\jmath}}}\frac{1}{\widecheck{W}^2}\right]\left|\hat{w}_1 - w_1^*\right|.$$

We now estimate the term $v_1^* - \hat{v}_1$, which is based on the fact that in this situation the flux inequalities $\rho^l v^l < \rho_1^* v_1^* \le \hat{\rho}_1 \hat{v}_1$ hold. Since $v^l = \hat{v}_1$,

$$\hat{\rho}_1\hat{v}_1 - \rho^l v^l = \frac{\hat{v}_1^2}{\hat{w}_1 w_1^*}\left(\hat{w}_1 - w_1^*\right).$$

Moreover, by $w^l = w_1^*$,

$$\begin{aligned} \rho_1^* v_1^* - \rho^l v^l &= v_1^*\left(R - \frac{v_1^*}{w_1^*}\right) - \hat{v}_1\left(R - \frac{\hat{v}_1}{w_1^*}\right) \\ &= \left[R + \frac{1}{w_1^*}\left(\hat{v}_1 + v_1^*\right)\right]\left(v_1^* - \hat{v}_1\right) \ge R\left(v_1^* - \hat{v}_1\right). \end{aligned}$$

The inequality $\rho_1^* v_1^* - \rho^l v^l \le \hat{\rho}_1\hat{v}_1 - \rho^l v^l$ now implies that

$$v_1^* - \hat{v}_1 \le \frac{\hat{v}_1^2}{R\hat{w}_1 w_1^*}\left(\hat{w}_1 - w_1^*\right) \le \frac{V_{\max}^2}{R\widecheck{W}^2}\left(\hat{w}_1 - w_1^*\right)$$

and so

$$\begin{aligned} \Delta F_{\tilde{v}}(\bar{t}) &\le \left[2\frac{V_{\max}^2}{R\widecheck{W}^2} + 4R\left(m-1\right) + 2\widehat{W}\frac{V_{\max}}{\alpha_{1,\bar{\jmath}}}\frac{1}{\widecheck{W}^2}\right]\left|\hat{w}_1 - w_1^*\right| \\ &\le K\left|\hat{w}_1 - w^l\right|, \end{aligned}$$

proving (3.7).

2. The state $(\rho_1^*, \eta_1^*) \in F \cap C$ is the active constraint for the new equilibrium. Therefore in the road I_1 a wave of the first family, connecting (ρ_1^*, η_1^*) with $\left(\rho^l, \eta^l\right)$, is produced; in the road $I_{\bar{\jmath}}$ a phase transition wave, connecting $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right)$ with $\left(\rho_{\bar{\jmath}}^\circ, \eta_{\bar{\jmath}}^\circ\right)$, and a wave of the second family, connecting $\left(\rho_{\bar{\jmath}}^\circ, \eta_{\bar{\jmath}}^\circ\right)$ with $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}})$, are produced; finally in each of the remaining outgoing roads a single linear wave is produced. Thus (3.9) holds, since $w^l = w_1^* = \cdots = w_{1+m}^* = w_{\bar{\jmath}}^\circ$ and $\hat{w}_1 = \cdots = \hat{w}_{1+m}$. Moreover $v_{\bar{\jmath}}^* > v_{\bar{\jmath}}^\circ = \hat{v}_{\bar{\jmath}}$ and the functional (3.3) can be estimated by

$$\begin{aligned} \Delta F_{\tilde{v}}(\bar{t}) &= \left|v_1^* - v^l\right| + \sum_{j=2}^{1+m}\left|v_j^* - \hat{v}_j\right| - \left|v^l - \hat{v}_1\right| + \left(v_1^* - \hat{v}_1\right) \\ &\quad - \sum_{j=2}^{1+m}\left(v_j^* - \hat{v}_j\right) \le 2\left(v_1^* - \hat{v}_1\right) + 2\sum_{j=2, j\neq\bar{\jmath}}^{1+m}\left|v_j^* - \hat{v}_j\right|. \end{aligned}$$

The estimate on the term $v_1^* - \hat{v}_1$ is completely identical to the previous case, since $\rho^l v^l < \rho_1^* v_1^* \le \hat{\rho}_1\hat{v}_1$; thus

$$v_1^* - \hat{v}_1 \le \frac{V_{\max}^2}{R\widecheck{W}^2}\left(\hat{w}_1 - w_1^*\right).$$

For every $j \in \{2, \dots, m+1\}$, $v_j^* = w_1^* \left(R - \rho_j^*\right)$ and $\hat{v}_j = \hat{w}_1 \left(R - \hat{\rho}_j\right)$; thus

$$\begin{aligned}\left|v_j^* - \hat{v}_j\right| &\le \left(R + \hat{\rho}_j\right)\left|w_1^* - \hat{w}_1\right| + w_1^*\left|\rho_j^* - \hat{\rho}_j\right| \\ &\le 2R\left|w_1^* - \hat{w}_1\right| + \widehat{W}\left|\rho_j^* - \hat{\rho}_j\right|.\end{aligned}$$

Since $(\hat{\rho}_j, \hat{w}_j) \in F$ and $\left(\rho_j^*, w_j^*\right) \in F$ for every $j \in \{2, \dots, m+1\} \setminus \{\bar{\jmath}\}$,

$$\begin{aligned}\sum_{j \ne \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* - \hat{\rho}_j\right| &= \frac{1}{V_{\max}} \sum_{j \ne \bar{\jmath}, j=2}^{1+m} \left|\rho_j^* V_{\max} - \hat{\rho}_j V_{\max}\right| \\ &= \frac{\left|\rho_1^* v_1^* - \hat{\rho}_1 \hat{v}_1\right|}{V_{\max}} \sum_{j \ne \bar{\jmath}, j=2}^{1+m} \alpha_{1,j} \le \frac{\left|\rho_1^* v_1^* - \hat{\rho}_1 \hat{v}_1\right|}{V_{\max}}.\end{aligned}$$

Moreover

$$\begin{aligned}\left|\rho_1^* v_1^* - \hat{\rho}_1 \hat{v}_1\right| \le \left|\hat{\rho}_1 \hat{v}_1 - \rho^l v^l\right| = \left|\hat{\rho}_1 \hat{v}_1 - \rho^l \hat{v}_1\right| = \hat{v}_1 \left|\frac{\hat{v}_1}{w_1^*} - \frac{\hat{v}_1}{\hat{w}_1}\right| \\ \le \frac{V_{\max}^2}{\widecheck{W}^2}\left|\hat{w}_1 - w_1^*\right|.\end{aligned}$$

Hence we deduce that

$$\begin{aligned}\Delta F_{\tilde{v}}(\bar{t}) &\le 2\left(v_1^* - \hat{v}_1\right) + 2\sum_{j=2, j \ne \bar{\jmath}}^{1+m} \left|v_j^* - \hat{v}_j\right| \\ &\le \left[2\frac{V_{\max}^2}{R\widecheck{W}^2} + 4R(m-1)\right]\left(\hat{w}_1 - w_1^*\right) + 2\widehat{W} \sum_{j=2, j \ne \bar{\jmath}}^{m+1} \left|\rho_j^* - \hat{\rho}_j\right| \\ &\le \left[2\frac{V_{\max}^2}{R\widecheck{W}^2} + 4R(m-1) + 2\widehat{W}\frac{V_{\max}}{\widecheck{W}^2}\right]\left(\hat{w}_1 - w_1^*\right) \\ &\le K\left(\hat{w}_1 - w_1^*\right),\end{aligned}$$

proving (3.7).

The proof is thus completed, since in all the previous cases the inequality $\Delta\mathcal{N}(\bar{t}) \le m+1$ holds. ∎

LEMMA 3.10. *Assume that the linear wave $((\rho^l, \eta^l), (\hat{\rho}_1, \hat{\eta}_1))$ interacts with the junction for the incoming road I_1 at time $\bar{t}$. Then*

$$\Delta\mathcal{F}_w(\bar{t}) \le (m - K_w)\left|w^l - \hat{w}_1\right|, \tag{3.10}$$

$$\Delta F_{\tilde{v}}(\bar{t}) \le K\left[\left|w^l - \hat{w}_1\right| + \left|v^l - \hat{v}_1\right|\right], \tag{3.11}$$

where K is given by (3.5). *Moreover $\Delta\mathcal{N}(\bar{t}) \le m$.*

Proof. For simplicity we use the notation in (3.8). The interacting wave is a linear wave; then both the states $\left(\rho^l, \eta^l\right)$ and $(\hat{\rho}_1, \hat{\eta}_1)$ belong to F. Since the equilibrium is generic, then

$$(\hat{\rho}_j, \hat{\eta}_j) \in F \setminus C \quad \forall j \in \{2, \dots, 1+m\}.$$

Note that either $(\rho_1^*, \eta_1^*) = \left(\rho^l, \eta^l\right)$ or the states (ρ_1^*, η_1^*) and $\left(\rho^l, \eta^l\right)$ are connected by a phase transition wave.

In the former case, the state $(\rho_1^*, \eta_1^*) = \left(\rho^l, \eta^l\right) \in F$ is the active constraint for the equilibrium after the interaction and $\left(\rho_j^*, \eta_j^*\right) \in F$ for all $j \in \{2, \dots, 1+m\}$. Therefore no wave is produced in the incoming road, while in each of the outgoing roads a single linear wave is produced.

In the latter case, there exists $\bar{\jmath} \in \{2, \dots, 1+m\}$ such that the state $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) \in F \cap C$, uniquely determined by the conditions

$$\begin{cases} w_{\bar{\jmath}}^* = w^l, \\ v\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) = \tilde{v}\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) = V_{\max}, \end{cases}$$

becomes the active constraint for the equilibrium after the interaction. Therefore in I_1 is produced a phase transition wave with negative speed connecting $(\rho_1^*, \eta_1^*) \in C$ with $\left(\rho^l, \eta^l\right) \in F$; in the road $I_{\bar{\jmath}}$ a linear wave, connecting $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) \in F \cap C$ with $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}}) \in F$, is produced; in each of the remaining outgoing roads a single linear wave is produced.

In all the cases, $\hat{w}_1 = \cdots = \hat{w}_{1+m}$ and $w^l = w_1^* = \cdots = w_{1+m}^*$. Thus, for the functional (3.2), we deduce that

$$\begin{aligned} \Delta\mathcal{F}_w(\bar{t}) &= K_w\left|w_1^* - w^l\right| + \sum_{j=2}^{1+m}\left|w_j^* - \hat{w}_j\right| - K_w\left|w^l - \hat{w}_1\right| \\ &= \left[\sum_{j=2}^{1+m}\left|w^l - \hat{w}_1\right|\right] - K_w\left|w^l - \hat{w}_1\right| = (m - K_w)\left|w^l - \hat{w}_1\right|, \end{aligned}$$

proving (3.10). We consider now the functional $\mathcal{F}_{\tilde{v}}$. We have

$$\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) = \left|v_1^* - v^l\right| + \sum_{j=2}^{1+m}\left|v_j^* - \hat{v}_j\right| - \left|v^l - \hat{v}_1\right| + (v_1^* - \hat{v}_1) - \sum_{j=2}^{1+m}\left(v_j^* - \hat{v}_j\right).$$

If in the first road no wave is produced, then $v_1^* = v^l$ and so

$$\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) \le 2\sum_{j=2}^{1+m}\left|v_j^* - \hat{v}_j\right|. \tag{3.12}$$

In the other case in I_1 a phase transition wave is produced and $v^l > v_1^*$; so (3.12) holds. Similarly to the proof of Lemma 3.9, for every $j \in \{2, \dots, m+1\}$, $v_j^* = w_1^*\left(R - \rho_j^*\right)$ and $\hat{v}_j = \hat{w}_1\left(R - \hat{\rho}_j\right)$ and so

$$\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) \le 4R\,m\left|w_1^* - \hat{w}_1\right| + 2\widehat{W}\sum_{j=2}^{1+m}\left|\rho_j^* - \hat{\rho}_j\right|.$$

Since $(\hat{\rho}_j, \hat{w}_j) \in F$ and $\left(\rho_j^*, w_j^*\right) \in F$ for every $j \in \{2, \dots, m+1\}$,

$$\begin{aligned} \sum_{j=2}^{1+m}\left|\rho_j^* - \hat{\rho}_j\right| &= \frac{1}{V_{\max}}\sum_{j=2}^{1+m}\left|\rho_j^* V_{\max} - \hat{\rho}_j V_{\max}\right| \\ &= \frac{\left|\gamma^* - \hat{\rho}_1 V_{\max}\right|}{V_{\max}}\sum_{j=2}^{1+m}\alpha_{1,j} = \frac{\left|\gamma^* - \hat{\rho}_1 V_{\max}\right|}{V_{\max}}, \end{aligned}$$

where either $\gamma^* = \rho_1^* v_1^*$ if a phase transition wave in I_1 is produced or $\gamma^* = \rho^l V_{\max}$. In the latter case we have

$$\sum_{j=2}^{1+m} \left|\rho_j^* - \hat\rho_j\right| = \left|\rho^l - \hat\rho_1\right| = \left|\frac{\hat v_1}{\hat w_1} - \frac{v^l}{w^l}\right|$$

$$\leq \frac{1}{\widecheck{W}}\left|\hat v_1 - v^l\right| + \frac{V_{\max}}{\widecheck{W}^2}\left|\hat w_1 - w^l\right| \tag{3.13}$$

and so

$$\Delta\mathcal{F}_{\tilde v}(\bar t) \leq 4R\,m\,|w_1^* - \hat w_1| + 2\widehat{W}\sum_{j=2}^{1+m}\left|\rho_j^* - \hat\rho_j\right|.$$

$$\leq \left(4R\,m + \frac{2\widehat{W}V_{\max}}{\widecheck{W}^2}\right)\left|w^l - \hat w_1\right| + \frac{2\widehat{W}}{\widecheck{W}}\left|\hat v_1 - v^l\right|, \tag{3.14}$$

proving (3.11). In the former case, i.e., when a phase transition wave is produced in I_1, we have two different cases:

1. $w^l \geq \hat w_1$. In this case the maximum possible flux for the outgoing roads increases. The fact that, after the interaction, the road I_1 is not an active constraint implies that $\hat\rho_1 V_{\max} < \rho_1^* v_1^* < \rho^l V_{\max}$. Consequently

$$\sum_{j=2}^{1+m}\left|\rho_j^* - \hat\rho_j\right| < \rho^l - \hat\rho_1$$

and so, by (3.13) and (3.14), the estimate (3.11) easily follows.

2. $w^l < \hat w_1$. In this case the maximum possible flux for the outgoing roads decreases. The fact that, after the interaction, the road I_1 is not an active constraint implies that $\hat\rho_1 V_{\max} \geq \rho_1^* v_1^*$. Consequently

$$\sum_{j=2}^{1+m}\left|\rho_j^* - \hat\rho_j\right| = \frac{|\rho_1^* v_1^* - \hat\rho_1 V_{\max}|}{V_{\max}} = \frac{\hat\rho_1 V_{\max} - \rho_1^* v_1^*}{V_{\max}}$$

$$= \frac{1}{V_{\max}\,\alpha_{1\bar{\jmath}}}\left(\hat\rho_{\bar{\jmath}} V_{\max} - \rho_{\bar{\jmath}}^* V_{\max}\right).$$

Before the interaction the road $I_{\bar{\jmath}}$ is not an active constraint; thus the flux $\hat\rho_{\bar{\jmath}} V_{\max}$ is lower than the maximum possible flux $RV_{\max} - \frac{V_{\max}^2}{\hat w_1}$. After the interaction the road $I_{\bar{\jmath}}$ is an active constraint; thus the flux $\rho_{\bar{\jmath}}^* V_{\max} = RV_{\max} - \frac{V_{\max}^2}{w_l}$. Therefore

$$\sum_{j=2}^{1+m}\left|\rho_j^* - \hat\rho_j\right| \leq \frac{1}{V_{\max}\,\alpha_{1\bar{\jmath}}}\left(\frac{V_{\max}^2}{w_l} - \frac{V_{\max}^2}{\hat w_1}\right) \leq \frac{V_{\max}}{\alpha_{1,\bar{\jmath}}\widecheck{W}^2}\left(\hat w_1 - w_l\right)$$

and so

$$\Delta\mathcal{F}_{\tilde v}(\bar t) \leq \left[4Rm + 2\frac{V_{\max}\widehat{W}}{\alpha_{1\bar{\jmath}}\widecheck{W}^2}\right]|\hat w_1 - w_l|,$$

proving (3.11).

This concludes the proof, since in all the previous cases the inequality $\Delta\mathcal{N}(\bar{t}) \le m$ holds. ∎

LEMMA 3.11. *Assume that the phase transition wave* $((\rho^l, \eta^l), (\hat{\rho}_1, \hat{\eta}_1))$ *with positive speed interacts with the junction for the incoming road* I_1 *at time* $\hat{t}$. *Then*

$$\Delta F_w(\bar{t}) = 0\,, \qquad \Delta F_{\tilde{v}}(\bar{t}) = 0\,, \qquad \Delta\mathcal{N}(\bar{t}) \le m - 1.$$

Proof. For simplicity we use the notation in (3.8). The interacting wave is a phase transition with positive speed; then the state $\left(\rho^l, \eta^l\right)$ is in the free phase while the state $(\hat{\rho}_1, \hat{\eta}_1)$ belongs to the congested phase and $\rho^l V_{\max} < \hat{\rho}_1 \hat{v}_1$. Note that $\rho^l < \hat{\rho}_1$. Since the equilibrium is generic, then there exists $\bar{\jmath} \in \{2, \dots, 1+m\}$ such that

$$(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}}) \in C, \qquad (\hat{\rho}_j, \hat{\eta}_j) \in F \quad \forall j \in \{2, \dots, 1+m\} \setminus \{\bar{\jmath}\}\ ,$$

with $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}})$ active constraint.

After the interaction, the state $(\rho_1^*, \eta_1^*) = (\rho^l, \eta^l) \in F$ becomes the active constraint for the equilibrium and $(\rho_j^*, \eta_j^*) \in F$ for all $j \in \{2, \dots, 1+m\}$. No wave is produced in the incoming road; in the road $I_{\bar{\jmath}}$ a phase transition wave with positive speed, connecting $(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*) \in F$ with $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}}) \in C$, is produced; finally in each of the remaining outgoing roads a single linear wave is produced. Note also that the flux through the junction decreases after the interaction; thus $v_{\bar{\jmath}}^* > \hat{v}_j$, for every $j \in \{2, \dots, 1+m\}$. Regarding the functional (3.2), we deduce that

$$\Delta\mathcal{F}_w(\bar{t}) = \sum_{j=2}^{1+m} \left|w_j^* - \hat{w}_j\right| - K_w\left|w^l - \hat{w}_1\right| = 0\,,$$

since $w^l = w_1^* = \dots = w_{1+m}^* = \hat{w}_1 = \dots = \hat{w}_{1+m}$. For the functional (3.3),

$$\begin{aligned}
\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) &= \sum_{j=2}^{1+m} \left|v_j^* - \hat{v}_j\right| - \left|v^l - \hat{v}_1\right| + \left(v^l - \hat{v}_1\right) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) \\
&= \sum_{j=2}^{1+m} (v_j^* - \hat{v}_j) - \left(v^l - \hat{v}_1\right) + \left(v^l - \hat{v}_1\right) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) = 0
\end{aligned}$$

since $\hat{v}_1 < v^l$ and $v_{\bar{\jmath}}^* > \hat{v}_j$, for every $j \in \{2, \dots, 1+m\}$. The proof is completed, since clearly $\Delta\mathcal{N}(\bar{t}) \le m - 1$. ∎

3.2.3. Wave hitting the junction from an outgoing road. We consider the case of a wave (ρ^r, η^r), coming from an outgoing road $I_{\bar{\jmath}}$ and interacting with the junction. As in the previous subsection, we denote

$$(\hat{\rho}_1, \hat{\eta}_1)\,, \quad (\hat{\rho}_2, \hat{\eta}_2)\,, \quad \dots, \quad (\hat{\rho}_{1+m}, \hat{\eta}_{1+m})$$

and

$$(\rho_1^*, \eta_1^*)\,, \quad (\rho_2^*, \eta_2^*)\,, \quad \dots, \quad (\rho_{1+m}^*, \eta_{1+m}^*)$$

the states at the junction respectively before and after the interaction. By Remark 3.6, we may assume that $((\hat{\rho}_1, \hat{\eta}_1), \dots, (\hat{\rho}_{1+m}, \hat{\eta}_{1+m}))$ is a generic equilibrium for the Riemann solver $\mathcal{RS}_J$.

LEMMA 3.12. *Assume that there exists* $\bar{\jmath} \in \{2, \dots, 1+m\}$ *such that the wave of the first family* $((\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}}), (\rho^r, \eta^r))$ *interacts with the junction for the outgoing road* $I_{\bar{\jmath}}$. *Then*

$$\Delta F_w(\bar{t}) = 0\,, \tag{3.15}$$

$$\Delta F_{\tilde{v}}(\bar{t}) \leq K|v^r - \hat{v}_{\bar{\jmath}}|, \tag{3.16}$$

where K *is given by* (3.5). *Moreover* $\Delta \mathcal{N}(\bar{t}) \leq m$.

Proof. For simplicity, for $i = 1, \dots 1+m$, we define

$$\begin{aligned} v^r &= \tilde{v}(\rho^r, \eta^r), \quad \hat{v}_i = \tilde{v}(\hat{\rho}_i, \hat{\eta}_i), \quad v_i^* = \tilde{v}(\rho_i^*, \eta_i^*), \\ w^r &= w(\rho^r, \eta^r), \quad \hat{w}_i = w(\hat{\rho}_i, \hat{\eta}_i), \quad w_i^* = w(\rho_i^*, \eta_i^*). \end{aligned} \tag{3.17}$$

The interacting wave is a wave of the first family; then both the states (ρ^r, η^r) and $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}})$ belong to C. Since the equilibrium is generic, then

$$(\hat{\rho}_1, \hat{\eta}_1) \in C \setminus F, \qquad (\hat{\rho}_j, \hat{\eta}_j) \in F \setminus C \quad \forall j \in \{2, \dots, 1+m\} \setminus \{\bar{\jmath}\}$$

and $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}})$ is the active constraint for the equilibrium. Note that there exists $\bar{w}$ such that

$$\bar{w} = \hat{w}_1 = \dots = \hat{w}_{1+m} = w_1^* = \dots = w_{1+m}^* = w^r.$$

Therefore, for the functional (3.2), we deduce that

$$\Delta \mathcal{F}_w(\bar{t}) = K_w |w_1^* - \hat{w}_1| + \sum_{j=2}^{1+m} \left|w_j^* - \hat{w}_j\right| - |w^r - \hat{w}_{\bar{\jmath}}| = 0\,,$$

proving (3.15).

We consider now the functional (3.3). We have two possibilities: either $\rho^r > \hat{\rho}_{\bar{\jmath}}$ or $\rho^r < \hat{\rho}_{\bar{\jmath}}$.

Assume first $\rho^r > \hat{\rho}_{\bar{\jmath}}$. After the interaction, the state $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) = (\rho^r, \eta^r) \in C$ is the active constraint for the new equilibrium, $(\rho_1^*, \eta_1^*) \in C$ and $\left(\rho_j^*, \eta_j^*\right) \in F$ for all $j \in \{2, \dots, 1+m\} \setminus \bar{\jmath}$. Therefore, no wave is produced in $I_{\bar{\jmath}}$; in the incoming road a first family wave, connecting $(\rho_1^*, \eta_1^*) \in C$ with $(\hat{\rho}_1, \hat{\eta}_1) \in C$, is produced; finally in each of the remaining outgoing roads a single linear wave is produced. We deduce that $\hat{v}_1 > v_1^*$, $\hat{v}_{\bar{\jmath}} > v^r = v_{\bar{\jmath}}^*$, and $\hat{v}_j < v_j^*$ for every $j \in \{2, \dots, 1+m\} \setminus \bar{\jmath}$; hence

$$\begin{aligned} \Delta \mathcal{F}_{\tilde{v}}(\bar{t}) &= |v_1^* - \hat{v}_1| + \sum_{j \neq \bar{\jmath}, j=2}^{1+m} \left|\hat{v}_j - v_j^*\right| - |v^r - \hat{v}_{\bar{\jmath}}| + (v_1^* - \hat{v}_1) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) \\ &= \sum_{j \neq \bar{\jmath}, j=2}^{1+m} \left(\hat{v}_j - v_j^*\right) - (\hat{v}_{\bar{\jmath}} - v^r) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) = 0, \end{aligned}$$

proving (3.16) in the case $\rho^r > \bar{\rho}_j$.

Next, assume $\rho^r < \hat{\rho}_{\bar{\jmath}}$. In this case, we have three possibilities.

1. After the interaction, the state $(\rho_1^*, \eta_1^*) \in F \cap C$ is the active constraint for the new equilibrium and $\left(\rho_j^*, \eta_j^*\right) \in F$ for all $j \in \{2, \dots, 1+m\}$. Therefore, a phase transition wave with positive speed, connecting $\left(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*\right) \in F$ with $(\rho^r, \eta^r) \in C$, is produced in $I_{\bar{\jmath}}$; in the incoming road a first family wave,

connecting $(\rho_1^*, \eta_1^*) \in F \cap C$ with $(\hat{\rho}_1, \hat{\eta}_1) \in C$, is produced; finally in each of the remaining outgoing roads a single linear wave is produced.
We deduce that $v_1^* > \hat{v}_1$, $v_{\bar{\jmath}}^* > v^r > \hat{v}_{\bar{\jmath}}$, and $\hat{v}_j > v_j^*$ for every $j \neq \bar{\jmath}$; hence

$$\begin{aligned}\Delta \mathcal{F}_{\tilde{v}}(\bar{t}) &= |v_1^* - \hat{v}_1| + \sum_{j=2, j\neq \bar{\jmath}}^{1+m} \left|v_j^* - \hat{v}_j\right| + \left|v_{\bar{\jmath}}^* - v^r\right| - |v^r - \hat{v}_{\bar{\jmath}}| \\ &\quad + (v_1^* - \hat{v}_1) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) \\ &= 2\left(v_1^* - \hat{v}_1\right) + 2\sum_{j=2, j\neq \bar{\jmath}}^{1+m} \left(\hat{v}_j - v_j^*\right) - 2v^r + 2\hat{v}_{\bar{\jmath}}.\end{aligned}$$

Moreover, since $\hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}} < \rho_{\bar{\jmath}}^* V_{\max} < \rho^r v^r$ and $\rho^r < \hat{\rho}_{\bar{\jmath}}$,

$$\begin{aligned}\sum_{j=2, j\neq \bar{\jmath}}^{1+m} \left(\hat{v}_j - v_j^*\right) &= \bar{w} \sum_{j=2, j\neq \bar{\jmath}}^{1+m} \left(\rho_j^* - \hat{\rho}_j\right) \\ &= \frac{\bar{w}}{V_{\max}} \sum_{j=2, j\neq \bar{\jmath}}^{1+m} V_{\max}\left(\rho_j^* - \hat{\rho}_j\right) \\ &= \frac{\bar{w}}{V_{\max}} \sum_{j=2, j\neq \bar{\jmath}}^{1+m} \alpha_{1,j}\left(\rho_1^* v_1^* - \hat{\rho}_1 \hat{v}_1\right) \\ &= \frac{\bar{w}}{V_{\max}\,\alpha_{1,\bar{\jmath}}} \sum_{j=2, j\neq \bar{\jmath}}^{1+m} \alpha_{1,j}\left(\rho_{\bar{\jmath}}^* V_{\max} - \hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}}\right) \\ &\leq \frac{\bar{w}}{V_{\max}\,\alpha_{1,\bar{\jmath}}}\left(\rho^r v^r - \hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}}\right) \\ &\leq \frac{R\widehat{W}}{V_{\max}\,\alpha_{1,\bar{\jmath}}}\left(v^r - \hat{v}_{\bar{\jmath}}\right).\end{aligned} \tag{3.18}$$

Finally, by Lemma A.2 and since $\hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}} < \rho_{\bar{\jmath}}^* V_{\max} < \rho^r v^r$, $\rho^r < \hat{\rho}_{\bar{\jmath}}$,

$$\begin{aligned}(v_1^* - \hat{v}_1) &\leq \frac{\bar{w}}{\lambda_0}\left(\rho_1^* v_1^* - \hat{\rho}_1\hat{v}_1\right) = \frac{\bar{w}}{\lambda_0\,\alpha_{1,\bar{\jmath}}}\left(\rho_{\bar{\jmath}}^* V_{\max} - \hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}}\right) \\ &\leq \frac{\bar{w}}{\lambda_0\,\alpha_{1,\bar{\jmath}}}\left(\rho^r v^r - \hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}}\right) \leq \frac{R\widehat{W}}{\lambda_0\,\alpha_{1,\bar{\jmath}}}\left(v^r - \hat{v}_{\bar{\jmath}}\right)\end{aligned} \tag{3.19}$$

proving that

$$\Delta \mathcal{F}_{\tilde{v}}(\bar{t}) \leq 2\left[\frac{R\widehat{W}}{\lambda_0 \alpha_{1,\bar{\jmath}}} + \frac{R\widehat{W}}{V_{\max}\alpha_{1,\bar{\jmath}}} - 1\right]\left(v^r - \hat{v}_{\bar{\jmath}}\right) \tag{3.20}$$

and, consequently, (3.16).

2. After the interaction the equilibrium does not change, in the sense that $\left(\rho_{\bar{\jmath}}^*, v_{\bar{\jmath}}^*\right) = (\rho^r, v^r) \in C$ is the active constraint, the state $(\rho_1^*, \eta_1^*) \in C \setminus F$, and $\left(\rho_j^*, \eta_j^*\right) \in F$ for all $j \in \{2, \ldots, 1+m\}$.

Therefore, no wave is produced in $I_{\bar{\jmath}}$; in the incoming road a first family wave, connecting $(\rho_1^*, \eta_1^*) \in C$ with $(\hat{\rho}_1, \hat{\eta}_1) \in C$, is produced; finally in each of the remaining outgoing roads a single linear wave is produced.
We deduce that $v_1^* > \hat{v}_1$, $v_{\bar{\jmath}}^* = v^r > \hat{v}_{\bar{\jmath}}$, and $\hat{v}_j > v_j^*$ for every $j \neq \bar{\jmath}$; hence

$$\begin{aligned}\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) &= |v_1^* - \hat{v}_1| + \sum_{j=2, j\neq\bar{\jmath}}^{1+m} \left|v_j^* - \hat{v}_j\right| - |v^r - \hat{v}_{\bar{\jmath}}| \\ &\quad + (v_1^* - \hat{v}_1) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) \\ &= 2\left(v_1^* - \hat{v}_1\right) + 2\sum_{j=2, j\neq\bar{\jmath}}^{1+m} \left(\hat{v}_j - v_j^*\right) - 2v^r + 2\hat{v}_{\bar{\jmath}}.\end{aligned}$$

Since $\rho^r < \hat{\rho}_{\bar{\jmath}}$, we deduce that (3.18) holds. Finally, by Lemma A.2, $\rho^r < \hat{\rho}_{\bar{\jmath}}$ and $\hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}} < \rho_{\bar{\jmath}}^* v_{\bar{\jmath}}^* = \rho_{\bar{\jmath}}^* V_{\max} = \rho^r v^r$, we obtain as before (3.19)–(3.20) and so (3.16).

3. After the interaction there exists $\bar{\bar{\jmath}} \in \{2, \dots, 1+m\} \setminus \bar{\jmath}$, such that the state $(\rho_{\bar{\bar{\jmath}}}^*, \eta_{\bar{\bar{\jmath}}}^*) \in F \cap C$ is the active constraint for the new equilibrium, $(\rho_1^*, \eta_1^*) \in C$, and $\left(\rho_j^*, \eta_j^*\right) \in F$ for all $j \in \{2, \dots, 1+m\} \setminus \bar{\bar{\jmath}}$.
Thus, a phase transition wave with positive speed, connecting $(\rho_{\bar{\jmath}}^*, \eta_{\bar{\jmath}}^*) \in F$ with $(\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}}) \in C$ is produced in $I_{\bar{\jmath}}$; a linear wave, connecting $(\rho_{\bar{\bar{\jmath}}}^*, \eta_{\bar{\bar{\jmath}}}^*) \in F \cap C$ with $(\hat{\rho}_{\bar{\bar{\jmath}}}, \hat{\eta}_{\bar{\bar{\jmath}}}) \in F$ is produced in $I_{\bar{\bar{\jmath}}}$; a first family wave, connecting $(\rho_1^*, \eta_1^*) \in C$ with $(\hat{\rho}_1, \hat{\eta}_1) \in C$, is produced in the incoming road; finally in each of the remaining outgoing roads a single linear wave is produced. We deduce that $v_1^* > \hat{v}_1$, $v_{\bar{\jmath}}^* > v^r > \hat{v}_{\bar{\jmath}}$, $\hat{v}_{\bar{\bar{\jmath}}} > v_{\bar{\bar{\jmath}}}^*$, and $\hat{v}_j > v_j^*$ for every $j \neq \bar{\jmath}$ and $j \neq \bar{\bar{\jmath}}$; hence

$$\begin{aligned}\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) &= |v_1^* - \hat{v}_1| + \sum_{j=2, j\neq\bar{\jmath}}^{1+m} \left|v_j^* - \hat{v}_j\right| + \left|v_{\bar{\jmath}}^* - v^r\right| - |v^r - \hat{v}_{\bar{\jmath}}| \\ &\quad + (v_1^* - \hat{v}_1) - \sum_{j=2}^{1+m} \left(v_j^* - \hat{v}_j\right) \\ &\leq 2(v_1^* - \hat{v}_1) + 2\sum_{j\neq\bar{\jmath}, j=2}^{1+m} \left(\hat{v}_j - v_j^*\right) + 2\left(\hat{v}_{\bar{\jmath}} - v^r\right).\end{aligned}$$

Moreover, since $\rho^r < \hat{\rho}_{\bar{\jmath}}$ and $\hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}} < \rho_{\bar{\jmath}}^* V_{\max} < \rho^r v^r$, we obtain as before (3.18). Finally, by Lemma A.2, $\hat{\rho}_{\bar{\jmath}}\hat{v}_{\bar{\jmath}} < \rho_{\bar{\jmath}}^* V_{\max} < \rho^r v^r$, and $\rho^r < \hat{\rho}_{\bar{\jmath}}$, we deduce (3.19)–(3.20) and so (3.16).

The proof is concluded, since in the previous cases, the inequality $\Delta\mathcal{N}(\bar{t}) \leq m$ holds. ∎

LEMMA 3.13. *Assume that there exists* $\bar{\jmath} \in \{2, \dots, 1+m\}$ *such that the phase transition wave* $((\hat{\rho}_{\bar{\jmath}}, \hat{\eta}_{\bar{\jmath}}), (\rho^r, \eta^r))$ *with negative speed interacts with the junction for the outgoing road* $I_{\bar{\jmath}}$. *Then*

$$\Delta F_w(\bar{t}) = 0\,, \qquad \Delta F_{\tilde{v}}(\bar{t}) = 0, \qquad \Delta\mathcal{N}(\bar{t}) \leq m - 1.$$

Proof. For simplicity we use the notation in (3.17). Note that there exists $\bar{w}$ such that

$$\bar{w} = \hat{w}_1 = \cdots = \hat{w}_{1+m} = w_1^* = \cdots = w_{1+m}^* = w^r.$$

The interacting wave is a phase transition with negative speed; then the state $(\hat\rho_{\bar{\jmath}}, \hat\eta_{\bar{\jmath}})$ is in the free phase F and the state (ρ^r, η^r) is in the congested phase C. Therefore $\rho^r > \hat\rho_{\bar{\jmath}}$ and, after the interaction, the state $\left(\rho^*_{\bar{\jmath}}, \eta^*_{\bar{\jmath}}\right) = (\rho^r, \eta^r) \in C$ becomes the active constraint for the equilibrium. This also implies that $(\rho^*_1, \eta^*_1) \in C$ and $\left(\rho^*_j, \eta^*_j\right) \in F$ for all $j \in \{2, \dots, 1+m\} \setminus \bar{\jmath}$. Moreover no wave is produced in $I_{\bar{\jmath}}$. For the functional (3.2), we deduce that

$$\Delta \mathcal{F}_w(\bar t) = K_w |w^*_1 - \hat w_1| + \sum_{j \neq \bar{\jmath}, j=2}^{1+m} \left|w^*_j - \hat w_j\right| - |w^r - \hat w_{\bar{\jmath}}| = 0\,.$$

Consider now the functional (3.3). Since the equilibrium, before the interaction, is generic, we have two possibilities.

1. Before the interaction an outgoing road is the active constraint, i.e., there exists $\bar{\bar{\jmath}} \in \{2, \dots, 1+m\} \setminus \{\bar{\jmath}\}$ such that

   $$(\hat\rho_1, \hat\eta_1) \in C, \qquad (\hat\rho_j, \hat\eta_j) \in F \quad \forall j \in \{2, \dots, 1+m\} \setminus \{\bar{\bar{\jmath}}\}$$

   and $(\hat\rho_{\bar{\bar{\jmath}}}, \hat\eta_{\bar{\bar{\jmath}}}) \in C$ is the active constraint for the equilibrium.
   Therefore, in I_1 a first family wave, connecting $(\rho^*_1, \eta^*_1) \in C$ with $(\hat\rho_1, \hat\eta_1) \in C$, is produced, a phase transition wave with positive speed, connecting $(\rho^*_{\bar{\bar{\jmath}}}, \eta^*_{\bar{\bar{\jmath}}}) \in F$ with $(\hat\rho_{\bar{\bar{\jmath}}}, \hat\eta_{\bar{\bar{\jmath}}}) \in C$ is produced in $I_{\bar{\bar{\jmath}}}$, and linear waves are produced in every I_j with $j \neq \bar{\jmath}$ and $j \neq \bar{\bar{\jmath}}$. Thus $v^*_1 < \hat v_1$, $v^*_{\bar{\jmath}} = v^r < \hat v_{\bar{\jmath}}$, and $\hat v_j < v^*_j$ for every $j \neq \bar{\jmath}$, and

   $$\begin{aligned}\Delta \mathcal{F}_{\tilde v}(\bar t) &= |v^*_1 - \hat v_1| + \sum_{j=2, j\neq\bar{\jmath}}^{1+m} \left|v^*_j - \hat v_j\right| - |\hat v_{\bar{\jmath}} - v^r| + (v^*_1 - \hat v_1) \\ &\quad - \sum_{j=2}^{1+m} \left(v^*_j - \hat v_j\right) = (v^r - \hat v_{\bar{\jmath}}) - \left(v^*_{\bar{\jmath}} - \hat v_{\bar{\jmath}}\right) = 0.\end{aligned}$$

2. Before the interaction the incoming road I_1 is the active constraint, i.e.,

   $$(\hat\rho_1, \hat\eta_1) \in F, \qquad (\hat\rho_j, \hat\eta_j) \in F \quad \forall j \in \{2, \dots, 1+m\}$$

   and $(\hat\rho_1, \hat\eta_1) \in F$ is the active constraint for the equilibrium. Therefore, in I_1 a phase transition wave with negative speed, connecting $(\hat\rho_1, \hat\eta_1) \in F$ with $(\rho^*_1, \eta^*_1) \in C$, is produced, and a single linear wave is produced in every I_j with $j \neq \bar{\jmath}$. Thus $\hat v_1 > v^*_1$, $v^*_{\bar{\jmath}} = v^r < \hat v_{\bar{\jmath}}$, and $\hat v_j < v^*_j$ for every $j \neq \bar{\jmath}$; hence

   $$\begin{aligned}\Delta \mathcal{F}_{\tilde v}(\bar t) &= |v^*_1 - \hat v_1| + \sum_{j=2, j\neq\bar{\jmath}}^{1+m} \left|v^*_j - \hat v_j\right| - |v^r - \hat v_{\bar{\jmath}}| + (v^*_1 - \hat v_1) \\ &\quad - \sum_{j=2}^{1+m} \left(v^*_j - \hat v_j\right) = 0\,.\end{aligned}$$

The proof is so completed, since $\Delta\mathcal{N}(\bar t) \leq m-1$. ∎

3.3. Existence of wave-front tracking solutions.

PROPOSITION 3.14. *Assume $K_w \geq m$, where K_w is defined in* (3.2). *Then, given K by* (3.5), *we deduce that, for every $t > 0$,*

$$\mathcal{F}_w(t) \leq \mathcal{F}_w(0+) \qquad \textit{and} \qquad \mathcal{F}_{\tilde v}(t) \leq 2K\left[\mathcal{F}_w(0+) + \mathcal{F}_{\tilde v}(0+)\right]. \tag{3.21}$$

Proof. By the interaction estimates in subsection 3.2, the functional $\mathcal{F}_w$ does not increase after wave interactions. Hence $\mathcal{F}_w(t) \leq \mathcal{F}_w(0+)$ for every $t > 0$, proving the first inequality in (3.21).

Consider now the functional $\mathcal{F}_{\tilde{v}}$. If two waves interact in a road or a phase transition wave hits J, then $\mathcal{F}_{\tilde{v}}$ does not increase; see Lemmas 3.8, 3.11, and 3.13. Instead $\mathcal{F}_{\tilde{v}}$ can increase if there is an interaction with J due to a wave of the first family, or of the second family, or due to a linear wave.

Waves of the first family have negative speed; hence they can interact with the junction J only from an outgoing road. Lemma 3.8 implies that waves of the first family can be generated by interactions of two waves inside a road (precisely by interactions 2-1/1-2 and 1-1/1), and in this case $\mathcal{F}_{\tilde{v}}$ does not increase. Moreover waves of the second family do not contribute to $\mathcal{F}_{\tilde{v}}$. Therefore the maximum increment of $\mathcal{F}_{\tilde{v}}$ due to the interaction of waves of the first family with J is $K\mathcal{F}_{\tilde{v}}(0+)$.

Waves of the second family have positive speed; hence they can interact with the junction J only from I_1. Lemma 3.8 implies that waves of the second family can be generated by interactions of two waves inside a road (precisely by interactions 2-1/1-2 and $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2), and in this case $\mathcal{F}_{\tilde{v}}$ does not increase. Moreover waves of the first family or phase transition waves do not contribute to $\mathcal{F}_w$. Since linear waves are not generated in a road (see Lemma 3.8), the maximum increment of $\mathcal{F}_{\tilde{v}}$ due to the interaction of waves of the second family with J is $K\mathcal{F}_w(0+)$.

Finally linear waves have positive speed; hence they can interact with the junction J only from I_1. Lemma 3.8 implies that linear waves are not generated by interactions of two waves inside a road. Then the maximum increment of $\mathcal{F}_{\tilde{v}}$ due to the interaction of linear waves with J is $K\mathcal{F}_w(0+) + K\mathcal{F}_{\tilde{v}}(0+)$. Consequently, we deduce that for every $t > 0$ $\mathcal{F}_{\tilde{v}}(t) \leq 2K\left[\mathcal{F}_w(0+) + \mathcal{F}_{\tilde{v}}(0+)\right]$, proving (3.21). ∎

PROPOSITION 3.15. *The construction, described in subsection* 3.1*, can be done for every positive time and, for every $\nu \in \mathbb{N} \setminus \{0\}$, it produces a $\frac{1}{\nu}$-approximate wave-front tracking solution to* (3.1)*, in the sense of Definition* 3.5*.*

Proof. We consider the construction in subsection 3.1 and the function $u_\nu = (u_{1,\nu}, \ldots, u_{1+m,\nu})$ built there, for $\nu \in \mathbb{N} \setminus \{0\}$. It is sufficient to prove that the number of waves and interactions generated is finite. Note that the functional $\mathcal{N}(t)$, corresponding to u_ν, is piecewise constant and can vary at interaction times in the following way:

1. If at time $\bar{t} > 0$ two waves interact inside a road, then $\Delta\mathcal{N}(\bar{t}) \leq 0$. More precisely, $\Delta\mathcal{N}(\bar{t}) = 0$ if and only if the interaction is either 2-1/1-2 or $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2; see Lemma 3.8.
2. If at time $\bar{t} > 0$ a wave of the second family interacts with J, then $\Delta\mathcal{N}(\bar{t}) \leq m + 1$; see Table 1.
3. If at time $\bar{t} > 0$ a linear wave interacts with J, then $\Delta\mathcal{N}(\bar{t}) \leq m$; see Table 1.
4. If at time $\bar{t} > 0$ a phase transition wave interacts with J, then $\Delta\mathcal{N}(\bar{t}) \leq m-1$; see Tables 1 and 2.
5. If at time $\bar{t} > 0$ a first family wave interacts with J, then $\Delta\mathcal{N}(\bar{t}) \leq m$; see Table 2.

The number of waves may increase in cases 2, 3, 4, and 5. Note that linear waves have positive speed; hence they can interact with the junction J only from the incoming road I_1. Lemma 3.8 implies that linear waves are not generated by interactions of two waves inside a road. Consequently the linear waves responsible for the increment of $\mathcal{N}$ of the point 3 are those generated in I_1 at time $t = 0+$. Thus point 3 can happen at most a finite number of times.

Waves of the first family have negative speed; hence they can interact with the junction J only from an outgoing road. Lemma 3.8 implies that waves of the first family can be generated by interactions of two waves inside a road, but only if an interacting wave is of the first family. Consequently the waves of the first family responsible for the increment of $\mathcal{N}$ of point 5 are those generated in the outgoing roads at time $t = 0+$. Thus point 5 can happen at most a finite number of times.

Waves of the second family have positive speed; hence they can interact with the junction J only from the incoming road I_1. Lemma 3.8 implies that waves of the second family can be generated by interactions of two waves inside a road. More precisely, this happens if an interacting wave is either a wave of the second family or a linear wave. Since linear waves are generated only at time $t = 0+$, then the number of waves of second family interacting with J is bounded by the number of waves in I_1 at $t = 0+$. Thus point 2 can happen at most a finite number of times.

Consider now the phase transition waves. Inside the roads, by Lemma 3.8, a phase transition wave can emerge from an interaction of two waves only if an interacting wave is a phase transition. Therefore the only possibility for having an infinite number of waves is that a phase transition wave hits J, producing a new phase transition, which comes back to J after interacting with another wave. Since linear waves have positive speed and waves of the first family have negative speed, the previous situation can happen if the phase transition wave interacts in I_1 with the wave of the first family or in an outgoing road with a linear wave. More precisely, in the first case we have the interaction $\mathcal{PT}$-1/$\mathcal{PT}$ (the interacting $\mathcal{PT}$ is produced at J and the generated $\mathcal{PT}$ has positive speed and comes back to J) and in the second case we have the interaction $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2 (the interacting $\mathcal{PT}$ is produced at J and the generated $\mathcal{PT}$ has negative speed and comes back to J).

Assume, by contradiction, that there exist $T > 0$ and $\bar{\nu} \in \mathbb{N} \setminus \{0\}$ such that

$$\mathcal{N}(t) < +\infty$$

for $0 < t < T$ and

$$\limsup_{t \to T^-} \mathcal{N}(t) = +\infty.$$

The previous considerations imply that there exists $\xi > 0$ such that in the time interval $(T - \xi, T)$ there are infinitely many interactions of phase transition waves with J. Assume that there exist $t_1, t_2 \in (T - \xi, T)$ with $t_1 < t_2$ such that a phase transition wave is originated at J at time t_1 in I_i ($i \in \{1, \dots, 1 + m\}$) and comes back to J at time t_2 after interacting with a wave of the first family if $i = 1$ or after interacting with a linear wave if $i \neq 1$. Obviously in I_i the datum before t_1 is the same as that after t_2 and such datum is in $F \setminus C$ if $i = 1$ and belongs to $C \setminus F$ if $i \neq 1$. Thus, similarly to [19, Proposition 10], there are finitely many possible combinations of data at the node J. Consequently we deduce that, for $t \in (T - \xi, T)$, $u_{\bar{\nu}}$ at J may take only a finite number of values, thus waves produced by J have a finite set of possible velocities. Therefore, as in the proof of [19, Proposition 10], if the number of discontinuities cannot be bounded by a constant, then also the functionals $\mathcal{F}_w$ and $\mathcal{F}_{\bar{v}}$ cannot be bounded, contradicting Proposition 3.14.

In this way we have also proved that the number of interactions at J is finite. Moreover the number of interactions inside a road is finite; see [32, Proposition 2]. The proof is thus completed. ∎

3.4. Existence of solutions. This part contains the proof of the main theorem.

Proof of Theorem 3.4. Fix an ε-approximate wave-front tracking solution $u_{i,\varepsilon}$ to (3.1), in the sense of Definition 3.5. By Proposition 3.14, we deduce that there exists a constant $M > 0$, depending on the $\mathcal{F}_w(0+)$ and on $\mathcal{F}_{\tilde{v}}(0+)$, such that

$$\mathcal{F}_w(t) \le M \qquad \text{and} \qquad \mathcal{F}_{\tilde{v}}(t) \le M$$

for every $t > 0$. Therefore, for every $i \in \{1, \ldots, 1+m\}$,

$$\mathrm{TV}\left(w\left(u_{i,\varepsilon}\left(t,\cdot\right)\right)\right) \le M \qquad \text{and} \qquad \mathrm{TV}\left(\tilde{v}\left(u_{i,\varepsilon}\left(t,\cdot\right)\right)\right) \le M + m\widehat{W}R,$$

since $0 \le \tilde{v} < \widehat{W}R$. Hence, at least passing to a subsequence, there exist w^* and $\tilde{v}^*$ such that

$$w\left(u_{i,\varepsilon}\left(t,\cdot\right)\right) \to w^*(t,\cdot) \qquad \text{and} \qquad \tilde{v}\left(u_{i,\varepsilon}\left(t,\cdot\right)\right) \to \tilde{v}^*(t,\cdot)$$

pointwise for a.e. $t > 0$. Finally, since the transformation $(\rho, \eta) \to (w, \tilde{v})$ is invertible, then we deduce the existence of ρ^* and η^* such that

$$u_{i,\varepsilon}(t,\cdot) \to (\rho_i^*(t,\cdot), \eta_i^*(t,\cdot))$$

pointwise for a.e. $t > 0$. The latter convergence holds also in $\mathbf{L}^1_{\mathbf{loc}}\left([0,+\infty[\times\mathbb{R}; F \cup C\right)$. Therefore $(\rho_i^*(t,\cdot), \eta_i^*(t,\cdot))$ provides a solution to (3.1). ∎

Appendix A. Technical lemmas.

LEMMA A.1. *Fix $\bar{w}, \tilde{w} \in [\check{w}, \hat{w}]$, $\bar{w} < \tilde{w}$, and two elements $(\bar{\rho}_1, \bar{\eta}_1)$ and $(\bar{\rho}_2, \bar{\eta}_2)$ in the congested region C. Assume that*

$$\bar{\rho}_1 < \bar{\rho}_2 \qquad \textit{and} \qquad \frac{\bar{\eta}_1}{\bar{\rho}_1} = \frac{\bar{\eta}_2}{\bar{\rho}_2} = \bar{w}.$$

Denote with $(\tilde{\rho}_1, \tilde{\eta}_1)$ and with $(\tilde{\rho}_2, \tilde{\eta}_2)$ the unique elements in C satisfying

$$\begin{aligned} v\left(\tilde{\rho}_1, \tilde{\eta}_1\right) &= v\left(\bar{\rho}_1, \bar{\eta}_1\right), \qquad \frac{\tilde{\eta}_1}{\tilde{\rho}_1} = \tilde{w}, \\ v\left(\tilde{\rho}_2, \tilde{\eta}_2\right) &= v\left(\bar{\rho}_2, \bar{\eta}_2\right), \qquad \frac{\tilde{\eta}_2}{\tilde{\rho}_2} = \tilde{w}. \end{aligned} \tag{A.1}$$

Then

$$\frac{\bar{\rho}_1 v\left(\bar{\rho}_1, \bar{\eta}_1\right)}{\bar{\rho}_2 v\left(\bar{\rho}_2, \bar{\eta}_2\right)} < \frac{\tilde{\rho}_1 v\left(\tilde{\rho}_1, \tilde{\eta}_1\right)}{\tilde{\rho}_2 v\left(\tilde{\rho}_2, \tilde{\eta}_2\right)}. \tag{A.2}$$

Proof. Given V and w, the point $(\rho, \eta) \in C$ satisfying $v\left(\rho, \eta\right) = V$ and $\frac{\eta}{\rho} = w$ is given by

$$\rho = R - \frac{V}{w} \qquad \text{and} \qquad \eta = Rw - V.$$

Since (A.1) holds, in order to prove (A.2), it is sufficient to show that the function

$$\begin{aligned} [\bar{w}, \tilde{w}] &\longrightarrow \mathbb{R}, \\ w &\longmapsto \frac{\left(R - \frac{V_1}{w}\right) V_1}{\left(R - \frac{V_2}{w}\right) V_2} \end{aligned}$$

is strictly increasing, where $V_1 = v(\bar\rho_1, \bar\eta_1)$ and $V_2 = v(\bar\rho_2, \bar\eta_2)$. Its derivative is given by

$$\frac{RV_1V_2}{\left(RV_1w - V_1^2\right)^2}\,(V_1 - V_2) > 0.$$

This concludes the proof. ∎

LEMMA A.2. *Fix* $\bar w \in [\widecheck{W}, \widehat{W}]$. *Consider the function* $v_{\bar w} : [0, \Gamma_1] \to \mathbb{R}^+$, *which satisfies*

$$\left(R - \frac{v_{\bar w}(\Gamma)}{\bar w}\right) v_{\bar w}(\Gamma) = \Gamma \qquad \textit{and} \qquad \left(\left(R - \frac{v_{\bar w}(\Gamma)}{\bar w}\right), \bar w \left(R - \frac{v_{\bar w}(\Gamma)}{\bar w}\right)\right) \in C$$

for every $\Gamma \in [0, \Gamma_1]$, *where* $\Gamma_1 = RV_{\max} - \frac{V_{\max}^2}{\bar w}$.

Then the function $v_{\bar w}$ *is Lipschitz continuous with Lipschitz constant* $\frac{\bar w}{\lambda_0}$, *where* λ_0 *is defined in* (H-2).

Proof. The function $v_{\bar w}$ returns the velocity of the point $(\rho, \eta) \in C$ belonging to the line $\eta = \bar w \rho$ and whose flux $\rho v(\rho, \eta)$ is equal to Γ. Therefore its analytic expression is

$$v_{\bar w}(\Gamma) = \frac{R\bar w - \sqrt{R^2\bar w^2 - 4\Gamma \bar w}}{2}.$$

Its derivative $\partial_\Gamma v_{\bar w}(\Gamma)$ is equal to $\frac{\bar w}{\sqrt{R^2\bar w^2 - 4\Gamma\bar w}}$ and so

$$\left|\partial_\Gamma v_{\bar w}(\Gamma)\right| \le \frac{\bar w}{|R\bar w - 2V_{\max}|} \le \frac{\bar w}{\lambda_0},$$

concluding the proof. ∎

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