Line Hermitian Grassmann Codes and their Parameters

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Abstract

In this paper we introduce and study line Hermitian Grassmann codes as those subcodes of the Grassmann codes associated to the 2-Grassmannian of a Hermitian polar space defined over a finite field. In particular, we determine the parameters and characterize the words of minimum weight.

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1. Introduction

Let $V := V(K, q)$ be a vector space of dimension K over a finite field $\mathbb{F}_q$ and Ω a projective system of $\mathrm{PG}(V)$, i.e. a set of N distinct points in $\mathrm{PG}(V)$ such that $\dim\langle\Omega\rangle = \dim(V)$. A projective code $\mathcal{C}(\Omega)$ induced by Ω is a $[N, K]$-linear code admitting a generator matrix G whose columns are vector representatives of the points of Ω; see [24]. There is a well-known relationship between the maximum number of points of Ω lying in a hyperplane of $\mathrm{PG}(V)$ and the minimum Hamming distance $d_{\min}$ of $\mathcal{C}(\Omega)$, namely

$$d_{\min} = N - \max_{\substack{\Pi \leq \mathrm{PG}(V) \\ \mathrm{codim}(\Pi)=1}} |\Pi \cap \Omega| \,.$$

Interesting cases arise when Ω is the point-set of a Grassmann variety. The associated codes $\mathcal{C}(\Omega)$ are called *Grassman codes* and have been extensively studied, see e.g. [20, 21, 22, 19, 15, 14, 18].

In [3], we started investigating some projective codes arising from subgeometries of the Grassmann variety associated to orthogonal and symplectic k-Grassmannians. We called such codes respectively *orthogonal* [3, 5, 6, 7] and *symplectic Grassman codes* [4, 6]. In the cases of line orthogonal and symplectic Grassmann codes, i.e. for $k = 2$, we determined all the parameters; see [5], [7]

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and [4]. For both these families we also proposed in [6] an efficient encoding algorithm, based on the techniques of enumerative coding introduced in [12].

In this paper we define *line Hermitian Grassmann codes* as the projective codes defined by the projective system of the points of the image under the Plücker embeddings of line Hermitian Grassmannians and determine their parameters. We refer the reader to Section 2.1 for the definition and properties of Hermitian Grassmannians.

Main Theorem. *A line Hermitian Grassmann code defined by a non–degenerate Hermitian form on a vector space $V(m, q^2)$ is a $[N, K, d_{\min}]$-linear code where*

$$N = \frac{(q^m + (-1)^{m-1})(q^{m-1} - (-1)^{m-1})(q^{m-2} + (-1)^{m-3})(q^{m-3} - (-1)^{m-3})}{(q^2-1)^2(q^2+1)};$$

$$K = \binom{m}{2};$$

$$d_{\min} = \begin{cases} q^{4m-12} - q^{2m-6} & \text{if } m = 4, 6 \\ q^{4m-12} & \text{if } m \geq 8 \text{ is even} \\ q^{4m-12} - q^{3m-9} & \text{if } m \text{ is odd.} \end{cases}$$

As a byproduct of the proof of the Main Theorem, we obtain a characterization of the words of minimum weight for any m and q, except for $(m, q) = (5, 2)$, see Corollaries 3.11 and 3.15.

In a forthcoming paper [8] we plan to describe and discuss algorithms for implementing encoding, decoding and error correction for line Hermitian Grassmann codes in the same spirit of [6].

1.1. Organization of the paper

In Section 2 we recall some preliminaries and set our notation. In particular, in Section 2.1 some basic notions about projective codes, Hermitian Grassmannians and their Plücker embeddings are recalled, while in Section 2.2 we recall a formula for estimating the weight of codewords for Grassmann codes. The same formula appears also in [5], but we now offer a much simplified and shorter proof. Section 3 is dedicated to the proof of our main result, by determining the minimum weight of line Hermitian Grassmann codes and, contextually, obtaining a description of the words of minimum weight in geometric terms. In particular, in Section 3.1 we provide bounds on the values of the weights given by the formula of Section 2.2 and in Sections 3.2 and 3.3 we investigate in detail the minimum weight in the cases where the hosting space has odd or even dimension.

2. Preliminaries

2.1. Hermitian Grassmannians and their embeddings

There is an extensive literature on the properties of Hermitian varieties over finite fields; for the basic notions as well as proofs for the counting formulas we use, we refer to the monograph [23] as well as well as to the survey [2]; see also [17, Chapter 2]. We warn the reader that we choose to uniformly use vector dimension in all statements throughout this paper.

Given any m-dimensional vector space $V := V(m, \mathbb{K})$ over a field $\mathbb{K}$ and $k \in \{1, \ldots, m-1\}$, let $\mathcal{G}_{m,k}$ be the k-Grassmannian of the projective space $\mathrm{PG}(V)$, that is the point–line geometry whose points are the k-dimensional subspaces of V and whose lines are the sets

$$\ell_{W,T} := \{X : W \leq X \leq T, \dim X = k\}$$

with $\dim W = k-1$ and $\dim T = k+1$.

When we want to stress on the role of the vector space V rather than its dimension m, we shall write $\mathcal{G}_k(V)$ instead of $\mathcal{G}_{m,k}$. In general, the points of a projective space $\mathrm{PG}(V)$ will be denoted by $[u]$, where $u \in V$ is a non-zero vector. For any $X \subseteq V$, we shall also write $[X] := \{[x] \colon x \in \langle X \rangle\}$.

Let $e_k : \mathcal{G}_{m,k} \to \mathrm{PG}(\bigwedge^k V)$ be the Plücker (or Grassmann) embedding of $\mathcal{G}_{m,k}$, which maps an arbitrary k–dimensional subspace $X = \langle v_1, v_2, \ldots, v_k \rangle$ of V to the point $e_k(X) := [v_1 \wedge v_2 \wedge \cdots \wedge v_k]$ of $\mathrm{PG}(\bigwedge^k V)$. Note that lines of $\mathcal{G}_{m,k}$ are mapped onto (projective) lines of $\mathrm{PG}(\bigwedge^k V)$. The dimension $\dim(e_k)$ of the embedding is defined to be the vector dimension of the subspace spanned by its image. It is well known that $\dim(e_k) = \binom{m}{k}$.

The image $e_k(\mathcal{G}_{m,k})$ of the Plücker embedding is a projective variety of $\mathrm{PG}(\bigwedge^k V)$, called *Grassmann variety* and denoted by $\mathbb{G}(m, k)$.

By Chow's theorem [11], the semilinear automorphism group stabilizing the variety $\mathbb{G}(m, k)$ is the projective general semilinear group $\mathrm{P\Gamma L}(m, \mathbb{K})$ unless $k = m/2$, in which case it is $\mathrm{P\Gamma L}(m, \mathbb{K}) \rtimes \mathbb{Z}_2$. This is also the permutation automorphism group of the induced code; see [14].

In [9] we introduced the notion of *transparent embedding* e of a point-line geometry Δ, as a way to clarify the relationship between the automorphisms of Δ and the automorphisms of its image $\Omega := e(\Delta)$ (and, consequently, also the automorphisms of the codes $\mathcal{C}(\Omega)$). A projective embedding $e : \Delta \to \mathrm{PG}(W)$ where $W = \langle \Omega \rangle$, is called *(fully) transparent* when the pre-image of every line contained (as a point-set) in Ω is actually a line of Δ. When an embedding is homogeneous and transparent, the collineations of $\mathrm{PG}(W)$ stabilizing Ω lift to automorphisms of Δ and, conversely, every automorphism of Δ corresponds to a collineation of $\mathrm{PG}(W)$ stabilizing Ω. So, under this assumption, it is possible to easily describe the relationship between the groups which are involved. In particular, the Grassmann embedding $e_k : \mathcal{G}_{m,k} \to \mathbb{G}(m, k)$ is transparent.

Assume henceforth $\mathbb{K} = \mathbb{F}_{q^2}$, so V is an m-dimensional vector space defined over a finite field of order q^2. Suppose that V is equipped with a non-degenerate Hermitian form η of Witt index n (hence $m = 2n+1$ or $m = 2n$).

The Hermitian k-Grassmannian $\mathcal{H}_{n,k}$ induced by η is defined for $k = 1, \ldots, n$ as the subgeometry of $\mathcal{G}_{m,k}$ having as points the totally η–isotropic subspaces of V of dimension k and as lines

- for $k < n$, the sets of the form

$$\ell_{W,T} := \{X : W \leq X \leq T, \dim X = k\}$$

with T totally η–isotropic and $\dim W = k-1$, $\dim T = k+1$.

- for $k = n$, the sets of the form

$$\ell_W := \{X : W \leq X, \dim X = n, X \text{ totally } \eta\text{–isotropic}\}$$

with $\dim W = n-1$, W totally η–isotropic.

If $k = 1$, $\mathcal{H}_{n,1}$ indicates a Hermitian polar space of rank n and if $k = n$, $\mathcal{H}_{n,n}$ is usually called *Hermitian dual polar space of rank n.*

Let $\varepsilon_{n,k} := e_k|_{\mathcal{H}_{n,k}}$ be the restriction of the Plücker embedding e_k to the Hermitian k-Grassmannian $\mathcal{H}_{n,k}$. The map $\varepsilon_{n,k}$ is an embedding of $\mathcal{H}_{n,k}$ called *Plücker (or Grassmann) embedding* of $\mathcal{H}_{n,k}$; its dimension is proved to be $\dim(\varepsilon_{n,k}) = \binom{\dim(V)}{k}$ if $\dim(V)$ is even and k arbitrary by Blok and Cooperstein [1] and for $\dim(V)$ arbitrary and $k = 2$ by Cardinali and Pasini [10].

Put $\mathbb{H}_{n,k} := \varepsilon_{n,k}(\mathcal{H}_{n,k}) = \{\varepsilon_{n,k}(X) \colon X \text{ point of } \mathcal{H}_{n,k}\}$. Then $\mathbb{H}_{n,k}$ is a projective system of $\mathrm{PG}(\bigwedge^k V)$.

Note that if $k = 2$ and $n > 2$ then $\varepsilon_{n,2}$ maps lines of $\mathcal{H}_{n,2}$ onto projective lines of $\mathrm{PG}(\bigwedge^2 V)$, independently from the parity of $\dim(V)$, i.e. the embedding is projective. Otherwise, if $n = k = 2$ and $m = \dim(V) = 5$ then the lines of $\mathcal{H}_{2,2}$ are mapped onto Hermitian curves, while if $m = \dim(V) = 4$ then lines of $\mathcal{H}_{2,2}$ are mapped onto Baer sublines of $\mathrm{PG}(\bigwedge^2 V)$. In the latter case $\mathbb{H}_{2,2} \cong Q^-(5,q)$ is contained in a proper subgeometry of $\mathrm{PG}(\bigwedge^2 V)$ defined over $\mathbb{F}_q$. We observe that for $k = 2$, $\dim(V) = 4$ or $\dim(V) > 5$ the embeddings $\varepsilon_{n,2}$ are always transparent; see [9].

We will denote by $\mathcal{C}(\mathbb{H}_{n,k})$ the projective code arising from vector representatives of the elements of $\mathbb{H}_{n,k}$, as explained at the beginning of the Introduction.

The following theorem is a consequence of the transparency of the embedding $\varepsilon_{n,2}$ and the description of the monomial automorphism group of projective codes; see [14].

Theorem 2.1. *The monomial automorphism group of the codes $\mathcal{C}(\mathbb{H}_{n,2})$ is $\mathrm{P\Gamma U}(m,q)$ for $m > 5$. For $m = 4$, we have $\mathbb{H}_{2,2} \cong Q^-(5,q)$; so the monomial automorphism group of the code is isomorphic to $GO^-(5,q)$.*

Clearly, the length of $\mathcal{C}(\mathbb{H}_{n,k})$ is the number of points of a Hermitian k-Grassmannian $\mathcal{H}_{n,k}$ and the dimension of $\mathcal{C}(\mathbb{H}_{n,k})$ is the dimension of the embedding ε_k.

From here on we shall focus on the minimum distance $d_{\min}$ of a line Hermitian code, i.e. $k = 2$. There is a geometrical way to read the minimum distance of $\mathcal{C}(\mathbb{H}_{n,2})$: since any codeword of $\mathcal{C}(\mathbb{H}_{n,2})$ corresponds to a bilinear alternating form on V, it can be easily seen that the minimum distance of $\mathcal{C}(\mathbb{H}_{n,2})$ is precisely the length of $\mathcal{C}(\mathbb{H}_{n,2})$ minus the maximum number of lines which are simultaneously totally η-isotropic for the given Hermitian form η defining $\mathcal{H}_{n,2}$ and totally φ-isotropic for a (possibly degenerate) bilinear alternating form φ on V.

2.1.1. Notation

Since the cases $\dim(V)$ even and $\dim(V)$ odd behave differently, it will be sometimes useful to adopt the following notation. We will write $\mathcal{H}_{n,k}^{odd}$ for a Hermitian k-Grassmannian in the case $\dim(V) = 2n+1$ and $\mathcal{H}_{n,k}^{even}$ for a Hermitian k-Grassmannian in the case $\dim(V) = 2n$. Accordingly, μ_n^{odd} is the number of points of $\mathcal{H}_{n,1}^{odd}$ and μ_n^{even} is the number of points of $\mathcal{H}_{n,1}^{even}$.

For $k = 2$, the number of points of $\mathcal{H}_{n,2}^{odd}$ is the length N^{odd} of $\mathcal{C}(\mathbb{H}_{n,2}^{odd})$:

$$N^{odd} = \frac{\mu_{n-1}^{odd} \cdot \mu_n^{odd}}{q^2+1} \quad \text{where} \quad \mu_n^{odd} := \frac{(q^{2n+1}+1)(q^{2n}-1)}{(q^2-1)}. \tag{1}$$

Analogously, the number of points of $\mathcal{H}_{n,2}^{even}$ is the length N^{even} of $\mathcal{C}(\mathbb{H}_{n,2}^{even})$:

$$N^{even} = \frac{\mu_{n-1}^{even} \cdot \mu_n^{even}}{q^2+1} \quad \text{where} \quad \mu_n^{even} := \frac{(q^{2n-1}+1)(q^{2n}-1)}{(q^2-1)}. \tag{2}$$

Equations (1) and (2) together with the results from [1, 10] on the dimension of the Grassmann embedding of a line Hermitian Grassmannian prove the first claims of the Main Theorem about the length and the dimension of the code.

When we do not want to explicitly focus on the Witt index of η but we prefer to stress on $\dim(V) = m$ regardless of its parity, we write $\mathcal{H}_{m,k}$ for the Hermitian k-Grassmannian defined by η and $\varepsilon_{m,k}$ for its Plücker embedding; we also put $\varepsilon_{m,k}(\mathcal{H}_{m,k}) = \mathbb{H}_{m,k}$. Clearly, if m is odd (i.e. $m = 2n+1$) then the symbols $\mathcal{H}_{m,k}$ and $\mathcal{H}_{n,k}^{odd}$ have the same meaning and analogously, if m is even (i.e. $m = 2n$), the symbols $\mathcal{H}_{m,k}$ and $\mathcal{H}_{n,k}^{even}$. Accordingly,

$$\mu_m = \begin{cases} \mu_{m/2}^{even} & \text{if } m \text{ is even} \\ \mu_{(m-1)/2}^{odd} & \text{if } m \text{ is odd.} \end{cases}$$

For simplicity of notation, we shall always write $\mathcal{H}_m$ for the point-set of $\mathcal{H}_{m,1}$.

2.2. A recursive weight formula for Grassmann and polar Grassmann codes

Denote by V^* the dual of a vector space V. It is well known that $(\bigwedge^k V)^* \cong \bigwedge^k V^*$ and that the linear functionals belonging to $(\bigwedge^k V)^*$ correspond exactly to k-linear alternating forms defined on V. More in detail, given $\varphi \in \bigwedge^k V^*$, we have that

$$\varphi^*(v_1, \dots, v_k) := \varphi(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$$

is a k-linear alternating form on V. Conversely, given any k-linear alternating form $\varphi^* \colon V^k \to \mathbb{F}_q$, there is a unique element $\varphi \in (\bigwedge^k V)^*$ such that

$$\varphi(v_1 \wedge \ldots \wedge v_k) := \varphi^*(v_1, \dots, v_k)$$

for any $v_1, \dots, v_k \in V$. Observe that, given a point $[u] = [v_1 \wedge v_2 \wedge \cdots \wedge v_k] \in \mathrm{PG}(\bigwedge^k V)$, we have $\varphi(u) = 0$ if and only if all the k-tuples of elements of the vector space $U := \langle v_1, \dots, v_k \rangle$ are killed by φ^*. With a slight abuse of notation, in the remainder of this paper, we shall use the same symbol φ for both the linear functional and the related k-alternating form.

Suppose $\{X_1, \ldots, X_N\}$ is a set of k-spaces of V and consider the projective system $\Omega = \{[\omega_1], \ldots, [\omega_N]\}$ of $\mathrm{PG}(\bigwedge^k V)$ with $[\omega_i] := e_k(X_i)$, $1 \leq i \leq N$, where e_k is the Plücker embedding of $\mathcal{G}_k(V)$. Put $W := \langle \Omega \rangle$ and let

$$\mathcal{N}(\Omega) := \{\varphi \in \bigwedge^k V^* \colon \varphi|_\Omega \equiv 0\}$$

be the annihilator of the set Ω; clearly $\mathcal{N}(\Omega) = \mathcal{N}(W)$. There exists a correspondence between the elements of $(\bigwedge^k V^*)/\mathcal{N}(\Omega) \cong W^*$ and the codewords of $\mathcal{C}(\Omega)$, where $\mathcal{C}(\Omega)$ is the linear code associated to Ω. Indeed, given any $\varphi \in W^*$, the codeword c_φ corresponding to φ is

$$c_\varphi := (\varphi(\omega_1), \ldots, \varphi(\omega_N)).$$

As Ω spans W it is immediate to see that $c_\varphi = c_\psi$ if and only if $\varphi - \psi \in \mathcal{N}(\Omega)$, that is $\varphi = \psi$ as elements of W^*.

We define the weight $\mathrm{wt}\,(\varphi)$ of φ as the weight of the codeword c_φ

$$\mathrm{wt}\,(\varphi) := \mathrm{wt}\,(c_\varphi) = |\{[\omega] \in \Omega \colon \varphi(\omega) \neq 0\}|. \tag{3}$$

When φ is a non-null linear functional in $\bigwedge^k V^*$, its kernel determines a hyperplane $[\Pi_\varphi]$ of $\mathrm{PG}(\bigwedge^k V)$; hence Equation (3) says that the weight of a non-zero codeword c_φ is the number of points of the projective system Ω not lying on the hyperplane $[\Pi_\varphi]$.

For linear codes the minimum distance $d_{\min}$ is the minimum of the weights of the non-zero codewords; so, in order to obtain $d_{\min}$ for $\mathcal{C}(\Omega)$ we need to determine the maximum number of k–spaces of V mapped by the Plücker embedding to Ω that are also φ–totally isotropic, as φ is an arbitrary k–linear alternating form which is not identically null on the elements of Ω.

In [5, Lemma 2.2] we proved a condition relating the weight of a codeword φ of a k-(polar) Grassmann code with the weight of codewords of suitable $(k-1)$-(polar) Grassmann codes. Since that result will be useful also for the present paper, we recall it in Lemma 2.2, providing a much shorter and easier proof.

In order to properly state Lemma 2.2 in its more general form, so that it can be applied to obtain the weights of codes associated to arbitrary subsets of the Grassmann variety (and not only to polar Grassmann varieties), we need to set some further notation; as a consequence, the remainder of this section is unavoidably quite technical. In any case, we warn the reader that to obtain the weights of a line Hermitian code one can just use the arguments of Remark 2.3 in order to derive Equation (9) directly.

Given any vector $u \in V$ and a k-linear alternating form φ, define $u \bigwedge^{k-2} V := \{u \wedge y : y \in \bigwedge^{k-2} V\} \subseteq \bigwedge^{k-1} V$. Put $V_u := V/\langle u \rangle$. Clearly, for any $y \in u \bigwedge^{k-2} V$

we have $\varphi(u \wedge y) = 0$. We can now define the functional $\bar{\varphi}_u \in (\bigwedge^{k-1} V_u)^* \cong ((\bigwedge^{k-1} V)/(u \bigwedge^{k-2} V))^*$ by the clause

$$\bar{\varphi}_u : \begin{cases} \bigwedge^{k-1} V_u \to \mathbb{K} \\ x + (u \bigwedge^{k-2} V) \mapsto \varphi(u \wedge x) \end{cases}$$

where $x \in \bigwedge^{k-1} V$. The functional $\bar{\varphi}_u$ is well defined and it can naturally be regarded as a $(k-1)$-linear alternating form on the quotient V_u of V. Also observe that $\mathrm{wt}\,(\bar{\varphi}_u) = \mathrm{wt}\,(\bar{\varphi}_{\alpha u})$ for any non-zero scalar α, so the expression $\mathrm{wt}\,(\bar{\varphi}_{[u]}) := \mathrm{wt}\,(\bar{\varphi}_u)$ is well defined. Let

$$\Delta := \{X_i := e_k^{-1}([\omega_i])\colon [\omega_i] \in \Omega\}$$

be the set of k-spaces of V mapped by the Plücker embedding to Ω and let

$$\Delta_u := \{X/\langle u\rangle : u \in X, X \in \Delta\} \text{ and } u^{\Delta} := \langle X \in \Delta : u \in X\rangle.$$

Since $V_u^{\Delta} \leq V_u$ with $V_u^{\Delta} := u^{\Delta}/\langle u\rangle$, we can consider the restriction

$$\varphi_u := \bar{\varphi}_u|_{\bigwedge^{k-1} V_u^{\Delta}} \tag{4}$$

of the functional $\bar{\varphi}_u$ to the space $\bigwedge^{k-1} V_u^{\Delta}$.

Note that when writing φ_u, we are implicitly assuming that u belongs to one of the elements in Δ. We have $\mathrm{wt}\,(\bar{\varphi}_u) = \mathrm{wt}\,(\varphi_u)$ because all points of Δ_u are, by construction, contained in V_u^{Δ}. Hence, $\Omega_u := e_{k-1}(\Delta_u) = \{e_{k-1}(X/\langle u\rangle)\colon X/\langle u\rangle \in \Delta_u\}$ is a projective system of $\mathrm{PG}(\bigwedge^{k-1} V_u^{\Delta})$. The form φ_u can be regarded as a codeword of the $(k-1)$-Grassmann code $\mathcal{C}(\Omega_u)$ defined by the image Ω_u of Δ_u under the Plücker embedding e_{k-1} of $\mathcal{G}_{k-1}(V_u^{\Delta})$.

Lemma 2.2. *Let V be a vector space over $\mathbb{F}_q$, $\Omega = \{[\omega_i]\}_{i=1}^N$ a projective system of $\mathrm{PG}(\bigwedge^k V)$ and $\Delta = \{e_k^{-1}([\omega])\colon [\omega] \in \Omega\}$ where e_k is the Plücker embedding of V. Suppose $\varphi : \bigwedge^k V \to \mathbb{K}$. Then*

$$\mathrm{wt}\,(\varphi) = \frac{q-1}{q^k-1} \sum_{\substack{[u]\in \mathrm{PG}(V)\colon \\ [u]\in[X], X\in\Delta}} \mathrm{wt}\,(\varphi_{[u]}). \tag{5}$$

Proof. By Equation (3), $\mathrm{wt}\,(\varphi)$ is the number of k-spaces of V mapped to Ω and not killed by φ. For any point $[u]$ of $\mathrm{PG}(V)$ such that u is a vector in $X_i := e_k^{-1}([\omega_i])$ with $[\omega_i] \in \Omega$, the number of k-spaces through $[u]$ not killed by φ is $\mathrm{wt}\,(\varphi_{[u]}) := \mathrm{wt}\,(\varphi_u)$ (see Equations (3) and (4)). Since any projective space $[X_i]$ with $X_i \in \Delta$ contains $(q^k-1)/(q-1)$ points, the formula follows. $\square$

Since each projective point corresponds to $q-1$ non-zero vectors, when we sum over the *vectors* contained in $X_i \in \Delta$ rather than over projective points $[u] \in [X_i]$, Formula (5) reads as

$$\mathrm{wt}\,(\varphi) = \frac{1}{q^k-1} \sum_{u\in X_i\in\Delta} \mathrm{wt}\,(\varphi_u). \tag{6}$$

Even if Equations (5) and (6) are equivalent, in this paper we find it convenient to use more often Equation (6) then (5).

Remark 2.3. For the purposes of the present paper we observe that $\mathrm{wt}\,(\varphi_u)$ could also be defined just as the number of $(k-1)$-subspaces of $V_u = V/\langle u\rangle$ which are both isotropic with respect to both the polarities $\perp_\eta^u$ and $\perp_\varphi^u$ induced by respectively $\perp_\eta$ and $\perp_\varphi$ in V_u by the clauses

$$(x+\langle u\rangle) \perp_\eta^u (y+\langle u\rangle) \Leftrightarrow x \perp_\eta y;$$

$$(x+\langle u\rangle) \perp_\varphi^u (y+\langle u\rangle) \Leftrightarrow x \perp_\varphi y.$$

Later on, in Equation (9) we will rewrite Equation (6) for the special case of line Hermitian Grassmannians.

3. Weights for Hermitian Line Grassmann codes

In this section we shall always assume $V := V(m, q^2)$ to be an m-dimensional vector space over the finite field $\mathbb{F}_{q^2}$, regardless the parity of m, η to be a non-degenerate Hermitian form on V with $\mathcal{H}_m$ the (non-degenerate) Hermitian polar space associated to η and $\Delta := \mathcal{H}_{m,2}$ to be the set of lines of $\mathcal{H}_m$, i.e. the set of totally η-isotropic lines of $\mathrm{PG}(V)$. Since we clearly consider only the cases for which Δ is non-empty, we have $m \geq 4$.

3.1. Estimates

We start by explicitly rewriting Equation (6) for the case $k = 2$, i.e. for line Hermitian Grassmannian codes. According to the notation introduced above we have $\varphi \in \bigwedge^2 V^*$ and $\Omega := \{\varepsilon_{m,2}(\ell)\colon \ell \in \Delta\}$. For any $\varphi \in \bigwedge^2 V^*$ and for $u \in V$, put $u^{\perp_\eta} = \{y\colon \eta(x,y) = 0\}$ and $u^{\perp_\varphi} = \{y\colon \varphi(x,y) = 0\}$. Observe that $[u] \in \mathcal{H}_m$, is equivalent to $u \in u^{\perp_\eta}$; thus, $u^{\perp_\eta}$ corresponds precisely to the set u^Δ defined in Section 2.2. Explicitly, Equation (4) can be written as:

$$\varphi_u : \begin{cases} u^{\perp_\eta}/\langle u\rangle \to \mathbb{F}_{q^2} \\ \varphi_u(x+\langle u\rangle) = \varphi(u\wedge x)(=\varphi(u,x)). \end{cases} \tag{7}$$

The function φ_u can be regarded as a linear functional on $u^{\perp_\eta}/\langle u\rangle$. Its kernel $\ker(\varphi_u) = (u^{\perp_\varphi}/\langle u\rangle) \cap (u^{\perp_\eta}/\langle u\rangle)$ either is the whole $u^{\perp_\eta}/\langle u\rangle$ or it is a subspace Π_u inducing a hyperplane $[\Pi_u]$ of $\mathrm{PG}(u^{\perp_\eta}/\langle u\rangle)$.

Note that since $\eta(u,x) = 0$ for all $x \in u^{\perp_\eta}$, the vector space $u^{\perp_\eta}/\langle u\rangle$ is naturally endowed with the Hermitian form $\eta_u : (x+\langle u\rangle, y+\langle u\rangle) \to \eta(x,y)$ and $\dim(u^{\perp_\eta}/\langle u\rangle) = \dim(V) - 2$. It is well known that the set of all totally singular vectors for η_u defines (the point set of) a non-degenerate Hermitian polar space $\mathcal{H}_{m-2}$ embedded in $\mathrm{PG}(u^{\perp_\eta}/\langle u\rangle)$.

We shall now apply Equation (5) to the codewords of the line Hermitian Grassmann code $\mathcal{C}(\mathbb{H}_{m,2})$. To this aim, we rewrite it as

$$\mathrm{wt}\,(\varphi) = \frac{q-1}{q^4-1} \sum_{[u]\in\ell\in\mathcal{H}_{m,2}} \mathrm{wt}\,(\varphi_u) = \frac{q-1}{q^4-1} \sum_{[u]\in\mathcal{H}_m} \mathrm{wt}\,(\varphi_u). \tag{8}$$

Similarly, when considering vectors, Equation (6) can be rewritten as

$$\mathrm{wt}\,(\varphi) = \frac{1}{q^4-1} \sum_{u \in V : [u] \in \mathcal{H}_m} \mathrm{wt}\,(\varphi_u). \tag{9}$$

Let u be a vector such that $[u] \in \mathcal{H}_m$. By Equation (3) in Section 2.2, $\mathrm{wt}\,(\varphi_u)$ is the number of η-isotropic lines $\ell = [v_1, v_2]$ of $\mathrm{PG}(V)$ through $[u]$ such that $\varphi(v_1, v_2) \neq 0$ or, equivalently, working in the setting $u^{\perp_\eta}/\langle u \rangle$, $\mathrm{wt}\,(\varphi_u)$ is the number of points contained in the hyperplane $[\Pi_u]$ not lying on $\mathcal{H}_{m-2}$. The hyperplane $[\Pi_u]$ can be either secant (i.e. meeting in a non-degenerate variety) or tangent to $\mathcal{H}_{m-2}$. Recall that all secant sections of $\mathcal{H}_{m-2}$ are projectively equivalent to a Hermitian polar space $\mathcal{H}_{m-3}$ embedded in $\mathrm{PG}(m-3, q^2)$. So, we have the following three possibilities:

a) $[\Pi_u] \cap \mathcal{H}_{m-2} = \mathcal{H}_{m-2}$ if $\ker(\varphi_u) \cong u^{\perp_\eta}/\langle u \rangle$;

b) $[\Pi_u] \cap \mathcal{H}_{m-2} = \mathcal{H}_{m-3}$ if $[\Pi_u]$ is a secant hyperplane to $\mathcal{H}_{m-2}$;

c) $[\Pi_u] \cap \mathcal{H}_{m-2} = [u]\mathcal{H}_{m-4}$ if $[\Pi_u]$ is a hyperplane tangent to $\mathcal{H}_{m-2}$, where $[u]\mathcal{H}_{m-4}$ is a cone with vertex the point $[u]$ and basis a non-degenerate Hermitian polar space $\mathcal{H}_{m-4}$.

Put

$$\mu_m := |\mathcal{H}_m| = \frac{(q^m + (-1)^{m-1})(q^{m-1} - (-1)^{m-1})}{(q^2-1)} \tag{10}$$

for the number of points of $\mathcal{H}_m$. By convention, we put $\mu_0 = 0$. Three possibilities can occur for the weights of φ_u, namely

$$\mathrm{wt}\,(\varphi_u) = \begin{cases} 0 & \text{in case a)} \\ \mu_{m-2} - \mu_{m-3} = q^{2m-7} + (-1)^{m-4} q^{m-4} & \text{in case b)} \\ \mu_{m-2} - q^2 \mu_{m-4} - 1 = q^{2m-7} & \text{in case c).} \end{cases}$$

For any given form $\varphi \in (\bigwedge^2 V)^*$ write,

$$\begin{array}{lll} \mathfrak{A}_\varphi & := \{u : [u] \in \mathcal{H}_m, \mathrm{wt}\,(\varphi_u) = 0, u \neq \mathbf{0}\} & A := |\mathfrak{A}_\varphi| \\ \mathfrak{B}_\varphi & := \{u : [u] \in \mathcal{H}_m, \mathrm{wt}\,(\varphi_u) = q^{2m-7} + (-1)^m q^{m-4}\} & B := |\mathfrak{B}_\varphi| \\ \mathfrak{C}_\varphi & := \{u : [u] \in \mathcal{H}_m, \mathrm{wt}\,(\varphi_u) = q^{2m-7}\} & C := |\mathfrak{C}_\varphi|. \end{array} \tag{11}$$

Since u varies among all (totally η-singular) vectors such that $[u] \in \mathcal{H}_m$, we clearly have $A + B + C = (q^2-1)\mu_m$, and Equation (9) can be rewritten as

$$\mathrm{wt}\,(\varphi) = \frac{q^{2m-7}(B+C) + (-1)^m q^{m-4} B}{q^4-1} = \frac{(q^{2m-7} + (-1)^m q^{m-4})(\mu_m(q^2-1) - A) - (-1)^m q^{m-4} C}{q^4-1}; \tag{12}$$

thus, we can express $\mathrm{wt}\,(\varphi)$ either as a function depending on A and B or as a function depending on A and C as

$$\mathrm{wt}\,(\varphi) = \frac{q^{2m-7}}{q^2+1}\mu_m - \frac{q^{2m-7}}{q^4-1}A + (-1)^m \frac{q^{m-4}}{q^4-1}B =$$
$$\frac{(q^{2m-7}+(-1)^m q^{m-4})}{q^2+1}\mu_m - \frac{(q^{2m-7}+(-1)^m q^{m-4})}{q^4-1}A - (-1)^m \frac{q^{m-4}}{q^4-1}C. \quad (13)$$

Denote by $A_{\max}$ the maximum value A might assume as φ varies among all non-trivial bilinear alternating forms defined on V. Then, by the first Equation of (13) with $B = 0$ and by the second Equation of (13) with $C = 0$ we have the following lower bounds for the minimum distance of $\mathcal{C}(\mathbb{H}_{m,2})$:

$$d_{\min} \geq \begin{cases} \frac{q^{2m-7}}{q^2+1}\left(\mu_m - \frac{1}{q^2-1}A_{\max}\right) & \text{if } m \text{ is even} \\ \\ \frac{q^{2m-7}-q^{m-4}}{q^2+1}\left(\mu_m - \frac{1}{q^2-1}A_{\max}\right) & \text{if } m \text{ is odd.} \end{cases} \quad (14)$$

We shall determine the actual values of $d_{\min}$ and see that the bound in (14) is not sharp unless $m = 4, 6$. More in detail, in the remainder of this paper we shall determine the possible values of the parameter A appearing in Equation (14) as a function depending on the dimension $\dim(\mathrm{Rad}\,(\varphi))$ of the radical of the form φ and show that in all cases the minimum weight codewords occur for $A = A_{\max}$ (but, in general, $B, C \neq 0$). We will also characterize the minimal weight codewords.

Given a (possibly degenerate) alternating bilinear form φ on V, denote by $\mathrm{Rad}\,(\varphi)$ the radical of φ, i.e. $\mathrm{Rad}\,(\varphi) = \{x \in V : \varphi(x,y) = 0\,\forall y \in V\}$. Define also $f_\varphi : \mathrm{PG}(m-1, q^2) \to \mathrm{PG}(m-1, q^2)$ as the semilinear transformation given by

$$f_\varphi([x]) := [x]^{\perp_\varphi \perp_\eta}. \quad (15)$$

It is straightforward to see that $\ker(f_\varphi) = [\mathrm{Rad}\,(\varphi)]$.

Lemma 3.1. *Let $[u] \in \mathcal{H}_m$. Then $\varphi_u = 0 \Leftrightarrow u^{\perp_\eta} \subseteq u^{\perp_\varphi}$.*

Proof. Take $x \in u^{\perp_\eta}$ and suppose $u^{\perp_\eta} \subseteq u^{\perp_\varphi}$. Then $\varphi(u,x) = 0$, so $\varphi_u(x + \langle u \rangle) = 0\ \forall x \in u^{\perp_\eta}$. Conversely, suppose φ_u is identically null. Then $\varphi_u(x + \langle u \rangle) = \varphi(u,x) = 0\ \forall x \in u^{\perp_\eta}$. Hence $u^{\perp_\eta} \subseteq u^{\perp_\varphi}$. □

By Lemma 3.1, $\mathfrak{A}_\varphi = \{u : [u] \in \mathcal{H}_m, u^{\perp_\eta} \subseteq u^{\perp_\varphi}\} = \mathfrak{A}_\varphi^{(1)} \cup \mathfrak{A}_\varphi^{(2)}$ where

$$\mathfrak{A}_\varphi^{(1)} := \{u : [u] \in \mathcal{H}_m, u^{\perp_\eta} \subset u^{\perp_\varphi}\} \quad \text{and} \quad \mathfrak{A}_\varphi^{(2)} := \{u : [u] \in \mathcal{H}_m, u^{\perp_\eta} = u^{\perp_\varphi}\}. \quad (16)$$

The vectors u such that $u^{\perp_\eta} \subset u^{\perp_\varphi}$ are precisely those vectors for which $u^{\perp_\varphi} = V$, hence $\mathfrak{A}_\varphi^{(1)} = \{u : [u] \in [\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m\}$.

Let us focus now on the set $\mathfrak{A}_\varphi^{(2)}$. Note that $u^{\perp_\eta} = u^{\perp_\varphi}$ is equivalent to $\alpha u = u^{\perp_\varphi \perp_\eta}$ for some $0 \neq \alpha \in \mathbb{F}_{q^2}$, or, in terms of projective points, $[u] =$

$[u]^{\perp_\varphi\perp_\eta} = f_\varphi([u])$. Hence, $\mathfrak{A}_\varphi^{(2)} = \{u : [u] \in \mathrm{Fix}\,(f_\varphi) \cap \mathcal{H}_m\}$ where $\mathrm{Fix}\,(f_\varphi) = \{[u] : f_\varphi([u]) = [u]\} \leq \mathrm{PG}(t,q)$, $0 \leq t \leq m-1$, is contained in a subgeometry over $\mathbb{F}_q$ of $\mathrm{PG}(m-1,q^2)$.

Lemma 3.2. *Let $[u]$ be a point of $\mathcal{H}_m$. The following hold.*

a) $u \in \mathfrak{A}_\varphi \Leftrightarrow f_\varphi([u]) = [u]$ or $u \in \mathrm{Rad}\,(\varphi)$.

b) $u \in \mathfrak{B}_\varphi \Leftrightarrow f_\varphi([u]) \neq [u]$ and $f_\varphi([u])$ is a non-singular point for η.

c) $u \in \mathfrak{C}_\varphi \Leftrightarrow f_\varphi([u]) \neq [u]$ and $f_\varphi([u])$ is a singular point for η.

Proof. By Equations (15) and (16) we have $u \in \mathfrak{A}_\varphi^{(2)}$ if and only if $[u] \in \mathcal{H}_m$ and $[u]$ is a fixed point of f_φ. Also, $u \in \mathfrak{A}_\varphi^{(1)}$ if and only if $[u] \in [\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m$.

Suppose $u \notin \mathfrak{A}_\varphi$. Then $f_\varphi([u]) \neq [u]$; hence $[u, f_\varphi([u])]$ is a line. Since $[u] \in [u]^{\perp_\varphi}$, we always have $[u]^{\perp_\varphi\perp_\eta} = f_\varphi([u]) \in [u]^{\perp_\eta}$.

Suppose that the point $f_\varphi([u])$ is non-singular with respect to η; then $[u]^{\perp_\varphi} = f_\varphi([u])^{\perp_\eta}$ meets $[u]^{\perp_\eta} \cap \mathcal{H}_m$ in a non-degenerate polar space not containing $f_\varphi([u])$. This is equivalent to saying $u \in \mathfrak{B}_\varphi$.

In case $f_\varphi([u])$ is singular with respect to η we have that $[u]^{\perp_\eta} \cap \mathcal{H}_m \cap [u]^{\perp_\varphi}$ is a degenerate polar space with radical of dimension 2, i.e. with radical the line $[u, f_\varphi([u])]$. This is equivalent to saying $u \in \mathfrak{C}_\varphi$. $\square$

Lemma 3.3. *Suppose φ is a non-singular alternating form. If $A = (q^m-1)(q+1)$ then $B = 0$.*

Proof. By hypothesis, m is necessarily even because φ is non-singular. Since $A = (q^m-1)(q+1)$ and φ is non-singular, $A = |\mathfrak{A}_\varphi^{(2)}|$, see (16), and the semilinear transformation f_φ fixes a subgeometry $\mathrm{Fix}\,(f_\varphi) = \mathrm{Fix}\,(f_\varphi) \cap \mathcal{H}_m \cong \mathrm{PG}(m-1,q)$ of $\mathrm{PG}(V)$ of maximal dimension; so f_φ^2 is the identity (as it is a linear transformation fixing a frame), that is to say $f_\varphi = f_\varphi^{-1}$ is involutory. Thus, for each point $[p]$ we have $[p]^{\perp_\varphi\perp_\eta} = f_\varphi([p]) = f_\varphi^{-1}([p]) = [p]^{\perp_\eta\perp_\varphi}$, i.e. the polarities $\perp_\eta$ and $\perp_\varphi$ *commute.* The transformation f_φ stabilizes $\mathcal{H}_m$; indeed, we have $f_\varphi([x]) \in \mathcal{H}_m$ if and only if

$$[x]^{\perp_\varphi\perp_\eta} = f_\varphi([x]) \in f_\varphi([x])^{\perp_\eta} = [x]^{\perp_\varphi\perp_\eta\perp_\eta} = [x]^{\perp_\varphi},$$

whence, applying $\perp_\varphi$ once more, we obtain

$$f_\varphi([x]) \in \mathcal{H}_m \Leftrightarrow x \in x^{\perp_\eta} \Leftrightarrow [x] \in \mathcal{H}_m.$$

In particular, $\forall [p] \in \mathcal{H}_m$, $f_\varphi([p]) \in \mathcal{H}_m$. So, by Lemma 3.2, $p \in \mathfrak{C}_\varphi \cup \mathfrak{A}_\varphi$ and, in particular, $\mathfrak{B}_\varphi = \emptyset$, i.e. $B = 0$. $\square$

Fix now a basis E of V. Without loss of generality, we can assume that the matrix H representing the Hermitian form η with respect to E is the identity matrix. Denote by S the antisymmetric matrix representing the (possibly degenerate) alternating form φ with respect to E; recall that $\mathrm{Rad}\,(\varphi)$ is precisely the kernel $\ker(S)$ of the matrix S.

Under these assumptions, the transformation f_φ can be represented as $f_\varphi([x]) := [S^q x^q]$, $\forall x \in V$. Since the fixed points of a semilinear transformation of $\mathrm{PG}(m-1,q^2)$ are contained in a subgeometry $[\Sigma_\varphi] \cong \mathrm{PG}(t,q)$ with $0 \le t \le m-1$,

$$\mathfrak{A}_\varphi \subseteq \{u : [u] \in ([\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m) \cup ([\Sigma_\varphi] \cap \mathcal{H}_m)\}. \tag{17}$$

Put $\widetilde{\Sigma}_\varphi := \mathbb{F}_{q^2} \otimes \Sigma_\varphi$; $\dim(\Sigma_\varphi) = \dim(\widetilde{\Sigma}_\varphi)$ where Σ_φ, respectively $\widetilde{\Sigma}_\varphi$, is regarded as a vector space over $\mathbb{F}_q$, respectively over $\mathbb{F}_{q^2}$. It is easy to see that $\mathrm{Rad}\,(\varphi)$ and $\widetilde{\Sigma}_\varphi$ are subspaces of $V(m,q^2)$ intersecting trivially. Since $\mathrm{Rad}\,(\varphi) = \ker(S)$, we have $\dim(\ker(S)) + \dim(\widetilde{\Sigma}_\varphi) = \dim(\ker(S)) + \dim(\Sigma_\varphi) \le m$. Clearly, $\mathrm{rank}\,(S) = m - \dim(\ker(S))$, so $\dim(\widetilde{\Sigma}_\varphi) = \dim(\Sigma_\varphi) \le \mathrm{rank}\,(S)$, where $\mathrm{rank}\,(S)$ is the rank of the matrix S.

Put $2i := \mathrm{rank}\,(S)$. Hence $\dim(\mathrm{Rad}\,(\varphi)) = m - 2i$ and $0 < 2i \le m$. Define

$$A_i := \max\{|\mathfrak{A}_\varphi| : \ \dim(\mathrm{Rad}\,(\varphi)) = m-2i\}. \tag{18}$$

Note that if $i = 0$, φ is identically null and this gives the $\mathbf{0}$ codeword. Clearly, by (17),

$$\begin{aligned} A_i \le \quad & (|\Sigma_\varphi| + |([\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m)|)(q^2-1) = \\ & (q^2-1)\left(\tfrac{(q^{2i}-1)}{(q-1)} + |([\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m)|\right) = \\ & (q^{2i}-1)(q+1) + |([\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m)|(q^2-1). \end{aligned} \tag{19}$$

We shall need the following elementary technical lemma.

Lemma 3.4. *Let H be a non-singular matrix of order m and let $t \le m$. Then for an $(m-t)\times(m-t)$ submatrix M of H, we have $m - 2t \le \mathrm{rank}\,(M) \le (m-t)$.*

Proof. The submatrix M is obtained from H by deleting t rows and t columns. First delete t rows. Then the rank of the $(m-t)\times m$ matrix M' so obtained is $m-t$. If we now delete t columns from M' as to obtain M, the rank of M' decreases by at most t. So, $\mathrm{rank}\,(M') - t \le \mathrm{rank}\,(M) \le \mathrm{rank}\,(M')$. □

We want to explicitly determine the cardinality of $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m$. The $(m-2i)$-dimensional space $[\mathrm{Rad}\,(\varphi)]$ intersects $\mathcal{H}_m$ in a (possibly) degenerate Hermitian variety. Write $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m = [\Pi_t]\mathcal{H}_{m-2i-t}$ where $[\Pi_t]\mathcal{H}_{m-2i-t}$ is a degenerate Hermitian variety contained in $[\mathrm{Rad}\,(\varphi)]$ with radical $[\Pi_t]$ of dimension t.

By Lemma 3.4, $0 \le t \le 2i$. Moreover, since $\Pi_t \subseteq \mathrm{Rad}\,(\varphi)$, $0 \le t \le m-2i$. (Recall also that $2i \le m$.) If $t = m-2i$ then $\mathrm{Rad}\,(\varphi) = \Pi_{m-2i}$ and in this case we put $[\Pi_{m-2i}]\mathcal{H}_0 := [\Pi_{m-2i}]$.

The following function provides the number of points of $[\Pi_t]\mathcal{H}_{m-2i-t}$ in dependence of t.

$$\begin{aligned}
&\mu_{m-2i}\colon \{0,\dots,\min\{2i,m-2i\}\}\to\mathbb{N}\\
&\mu_{m-2i}(t) = \quad q^{2t}\mu_{m-2i-t}+\frac{q^{2t}-1}{q^2-1} =\\
&\qquad\qquad \frac{q^{2t}(q^{m-2i-t}+(-1)^{m-2i-t-1})(q^{m-2i-t-1}-(-1)^{m-2i-t-1})+q^{2t}-1}{q^2-1},
\end{aligned} \tag{20}$$

where, by convention, $\mu_0(0)=0$ and μ_{m-2i-t} (for $t<m-2i$) is the number of points of a non-degenerate Hermitian polar space $\mathcal{H}_{m-2i-t}$ (see Equation (10)).

Using Equation (20), we can rewrite Equation (19) as

$$A_i \le (q^{2i}-1)(q+1)+(q^2-1)\mu_{m-2i}(t). \tag{21}$$

Lemma 3.5. *For any $t\in\{0,\dots,\min\{2i,m-2i\}\}$, we have $\mu_{m-2i}(t)\le\mu^{\max}_{m-2i}$, where*

$$\mu^{\max}_{m-2i} := \begin{cases} \mu_{m-2i}(m-2i)=\frac{q^{2m-4i}-1}{q^2-1} & \text{if } i\ge m/4;\\ \mu_{m-2i}(2i) & \text{if } i<m/4 \text{ and } m \text{ is even};\\ \mu_{m-2i}(2i-1) & \text{if } i<m/4 \text{ and } m \text{ is odd.}\end{cases}$$

Proof. Let H' be the matrix representing the restriction $\eta' := \eta|_{\mathrm{Rad}\,(\varphi)}$ of the Hermitian form η to $\mathrm{Rad}\,(\varphi)$. Then, by Lemma 3.4, $m-4i\le\operatorname{rank}H'\le m-2i$.

When $i\ge m/4$, i.e. $m-2i\le 2i$ (and hence $0\le t\le m-2i$), the maximum number of points for $[\Pi_t]\mathcal{H}_{m-2i-t}$ is attained for $t=m-2i$, i.e. $[\Pi_{m-2i}]\mathcal{H}_0 = [\mathrm{Rad}\,(\varphi)]=[\Pi_{m-2i}]$. Indeed, if $m\le 4i$ and $t=m-2i$, it is always possible to construct an antisymmetric form φ such that η' is the null form. This implies that $\mu_{m-2i}(t)<\mu_{m-2i}(m-2i)=\frac{q^{2m-4i}-1}{q^2-1}$ for any $t\in\{0,\dots,\min\{m-2i\}\}$, since $[\Pi_t]\mathcal{H}_{m-2i-t}\subseteq[\mathrm{Rad}\,(\varphi)]$. Hence, in this case, $\mu^{\max}_{m-2i}:=\mu_{m-2i}(m-2i)$.

Suppose now $i<m/4$. Then, by a direct computation

$$\mu_{m-2i}(t)-\mu_{m-2i}(t+1)=(-1)^{m-2i-t-2}q^{m-2i+t-1}. \tag{22}$$

So,

$$\mu_{m-2i}(t+2)-\mu_{m-2i}(t)=(-1)^{m-t}q^{m-2i+t-1}(q-1). \tag{23}$$

Assume m even; then, by (23), if t is even, $\mu_{m-2i}(t+2)>\mu_{m-2i}(t)$, i.e. $\mu_{m-2i}(t)$ is a monotone increasing function in t even. If t is odd, then by (23), $\mu_{m-2i}(t+2)<\mu_{m-2i}(t)$, i.e. $\mu_{m-2i}(t)$ is a monotone decreasing function in t odd. By (22), $\mu_{m-2i}(0)>\mu_{m-2i}(1)$. Recall that $0\le t\le 2i$; so we have

$$\begin{aligned}\mu_{m-2i}(2i-1)<\mu_{m-2i}(2i-3)<\cdots<\mu_{m-2i}(1)<\\ \mu_{m-2i}(0)<\mu_{m-2i}(2)<\mu_{m-2i}(4)<\cdots<\mu_{m-2i}(2i).\end{aligned}$$

In particular, the maximum value of $\mu_{m-2i}(t)$ for $i<m/4$ and m even is assumed for $t=2i$, i.e. $\mu^{\max}_{m-2i}=\mu_{m-2i}(2i)$.

Assume m odd; then, by (23), if t is even, $\mu_{m-2i}(t+2) < \mu_{m-2i}(t)$, i.e. $\mu_{m-2i}(t)$ is a monotone decreasing function in t even. If t is odd, then by (23), $\mu_{m-2i}(t+2) > \mu_{m-2i}(t)$, i.e. $\mu_{m-2i}(t)$ is a monotone increasing function in t odd. By (22), $\mu_{m-2i}(0) < \mu_{m-2i}(1)$. Recall that $0 \leq t \leq 2i$; so we have

$$\mu_{m-2i}(2i) < \mu_{m-2i}(2i-2) < \cdots < \mu_{m-2i}(2) < \\ \mu_{m-2i}(0) < \mu_{m-2i}(1) < \mu_{m-2i}(3) < \cdots < \mu_{m-2i}(2i-1).$$

In particular, the maximum value of $\mu_{m-2i}(t)$ for $i < m/4$ and m odd is assumed for $t = 2i-1$, i.e. $\mu_{m-2i}^{\max} = \mu_{m-2i}(2i-1)$. □

Define the function $\xi_m \colon \{1, \ldots, \lfloor m/2 \rfloor\} \to \mathbb{N}$

$$\xi_m(i) := (q^{2i}-1)(q+1) + (q^2-1)\mu_{m-2i}^{\max}, \tag{24}$$

where $\mu_{m-2i}^{\max}$, introduced in Lemma 3.5, is regarded as a function in i.

Corollary 3.6. *The following hold.*

a) $A_i \leq \xi_m(i)$*;*

b)

$$d_i \geq \begin{cases} \frac{q^{2m-7}}{q^2+1}\left(\mu_m - \frac{1}{q^2-1}\xi_m(i)\right) & \text{if } m \text{ is even} \\ \\ \frac{q^{2m-7}-q^{m-4}}{q^2+1}\left(\mu_m - \frac{1}{q^2-1}\xi_m(i)\right) & \text{if } m \text{ is odd,} \end{cases} \tag{25}$$

where d_i is the minimum weight of the words corresponding to bilinear alternating forms φ with $\dim(\mathrm{Rad}\,(\varphi)) = m-2i$.

Proof. Case a) follows from Lemma 3.5 and Equation (21). Case b) follows from Equation (13), the definition (18) of A_i and case a). □

Note that the function $\xi_m(i)$ (see (24)) is not monotone in i. The following lemma provides its largest and second largest values.

Lemma 3.7. *If $m \neq 4, 6$ the maximum value assumed by the function $\xi_m(i)$ is attained for $i = 1$. If $m = 4, 6$, then the maximum of $\xi_m(i)$ is attained for $i = m/2$. If $m = 5$ then $\xi_5(1) = \xi_5(2)$.*

The second largest value of $\xi_m(i)$ is attained for

$$\begin{cases} i = 1 & \text{if } m = 6; \\ i = 3 & \text{if } m = 7; \\ i = 4 & \text{if } m = 8, 9; \\ i = 5 & \text{if } m = 10; \\ i = 2 & \text{if } m > 10. \end{cases}$$

Proof. Recall that the polar line Grassmannian $\mathcal{H}_{m,2}$ is non-empty only for $m \geq 4$.

- If $m = 4$ then the possible values that the function $\xi_4(i)$ can assume are $\xi_4(1)$ and $\xi_4(2)$. Precisely,

$$\xi_4(1) = q^4 + q^3 + q^2 - q - 2 < \xi_4(2) = q^5 + q^4 - q - 1.$$

- If $m = 5$ then $\xi_5(1) = q^5 + q^4 + q^2 - q - 2 = \xi_5(2)$.

- If $m = 6$ then the possible values that the function $\xi_6(i)$ can assume are the following

$$\begin{aligned} \xi_6(1) &= q^7 + q^6 - q^5 + q^3 + q^2 - q - 2; \\ \xi_6(2) &= q^5 + 2q^4 - q - 2; \\ \xi_6(3) &= q^7 + q^6 - q - 1. \end{aligned}$$

  Hence $\xi_6(3) > \xi_6(1) > \xi_6(2)$. So, for both $m = 4$ and $m = 6$, $\xi_m(1) < \xi_m(m/2)$ and the maximum value of $\xi_m(i)$ is attained for $i = m/2$.

We assume henceforth $m > 6$. For any two functions $f(x)$ and $g(x)$ we shall write $f(x) = O_q(g(x))$ if

$$\frac{1}{q} g(x) < f(x) < q \cdot g(x).$$

If m is even and $i = 1$ then

$$\xi_m(1) = (q^2 - 1)(q + 1) + (q^2 - 1)\mu_{m-2}^{\max} = (q^2 - 1)(q + 1 + \mu_{m-2}(2)) = \\ q^{2m-5} + q^m - q^{m-1} + q^3 + q^2 - q - 2 = O_q(q^{2m-5}). \quad (26)$$

If m is odd and $i = 1$ then

$$\xi_m(1) = (q^2 - 1)(q + 1) + (q^2 - 1)\mu_{m-2}^{\max} = (q^2 - 1)(q + 1 + \mu_{m-2}(1)) = \\ q^{2m-5} + q^{m-1} - q^{m-2} + q^3 + q^2 - q - 2 = O_q(q^{2m-5}). \quad (27)$$

If $i \geq m/4$ then $\xi_m(i) := (q^{2i} - 1)(q + 1) + (q^{2m-4i} - 1)$. Since we always have $2i \leq m$ (so $2i \leq m \leq 4i \leq 2m$, and clearly $i > 1$), we have

$$\xi_m(i) = (q^{2i} - 1)(q + 1) + (q^{2m-4i} - 1) \leq \\ \leq (q^m - 1)(q + 1) + (q^m - 1) = O_q(q^{m+1}). \quad (28)$$

Hence, for $m > 6$, by Equations (26), (27), (28), we have $\xi_m(1) > \xi_m(i)$ for any $i \geq m/4 (\geq 2)$.

We prove that also for any $1 < i < \lfloor m/4 \rfloor$ we have $\xi_m(1) > \xi_m(i)$. Note that now m can be either even or odd. By Equations (10) and (20), we have that

$$\mu_x(t) = O_q(q^{2t}\mu_{x-t} + q^{2t-2}) = O_q(q^{2t}q^{2x-3-2t} + q^{2t-2});$$

so $\mu_x(t) = O_q(q^{2x-3})$ for all $x \leq m$ and $t < x \leq m$, while $\mu_x(x) = O_q(q^{2x-2})$. Hence, for $x = m - 2i$,

$$O_q(q^{2m-4i-3}) < \mu_{m-2i}^{\max} < O_q(q^{2m-4i-2}),$$

since $\mu_{m-2i}(t) > O_q(q^{2m-4i-3})$ and $\mu_{m-2i}(t) < O_q(q^{2m-4i-2})$ $\forall t \leq m - 2i$. By Equation (24), we obtain

$$\xi_m(i) = O_q(q^{2i+1} + q^2 \mu_{m-2i}^{\max}) \leq O_q(q^{2i+1} + q^{2m-4i}).$$

However, for $2 \leq i \leq \lfloor m/2 \rfloor - 1$ (henceforth also for $1 < i < \lfloor m/4 \rfloor$),

$$q^{2i+1} + q^{2m-4i} < q^m + q^{2m-8},$$

so

$$\xi_m(i) < O_q(q^m + q^{2m-8}).$$

This latter value is smaller than $\xi_m(1) = O_q(q^{2m-5})$ (see Equations (26) and (27)). It follows that the maximum of $\xi_m(i)$ is attained for $i = 1$ for all cases $m > 6$.

Assume now $i > 2$. Since $O_q(q^{4m-11}) < \mu_{m-4}^{\max} < O_q(q^{2m-10})$ and $\mu_{m-2i}^{\max} < O_q(q^{2m-4i-2})$ we have $\mu_{m-4}^{\max} - \mu_{m-2i}^{\max} > O_q(q^{4m-11} - q^{2m-4i-2})$. Hence

$$\begin{aligned}\xi_m(2) - \xi_m(i) = (q^4 - q^{2i})(q+1) + (q^2-1)(\mu_{m-4}^{\max} - \mu_{m-2i}^{\max}) \geq \\ \geq O_q(q^5 + q^4 - q^{2i+1} - q^{2i} + q^{2m-9} - q^{2m-4i}).\end{aligned}$$

For $m/2 \geq i \geq 3$, we have

$$q^5+q^4-q^{2i+1}-q^{2i}+q^{2m-9}-q^{2m-4i} > q^5+q^4-q^{m+1}-q^m+q^{2m-9}-q^{2m-12} > 0,$$

so, for $m > 10$,

$$\xi_m(2) - \xi_m(i) > O_q(q^{2m-9}) > 0.$$

A direct computation gives the following:

$$\xi_7(1) > \xi_7(3) > \xi_7(2); \;\; \xi_8(1) > \xi_8(4) > \xi_8(2) > \xi_8(3);$$

$$\xi_9(1) > \xi_9(4) > \xi_9(2) > \xi_9(3); \;\; \xi_{10}(1) > \xi_{10}(5) > \xi_{10}(4) > \xi_{10}(3).$$

This completes the proof. □

By Corollary 3.6 and Lemma 3.7 we have:

Corollary 3.8. *Let φ be a form with* $\dim(\mathrm{Rad}\,(\varphi)) = m - 2i$. *Then,*

- *for $m > 6$, we have $|\mathfrak{A}_\varphi| \leq A_i \leq \xi_m(1)$;*
- *for $m = 4, 6$ we have $|\mathfrak{A}_\varphi| \leq A_i \leq \xi_m(m/2)$.*

3.2. Minimum distance of $\mathcal{H}_{m,2}$ with m odd

In this section we assume m to be odd. Then the Witt index of the Hermitian form η is $n = (m-1)/2$. Let φ be an alternating form on V. Recall from Equation (13) that

$$\mathrm{wt}\,(\varphi) = \frac{(q^{2m-7} - q^{m-4})(\mu_m(q^2-1) - A)}{q^4-1} + \frac{q^{m-4}}{q^4-1}C.$$

Proposition 3.9. *There exists a bilinear alternating form φ with $\dim(\mathrm{Rad}\,(\varphi)) = m-2$ such that $\mathrm{wt}\,(\varphi) = q^{4m-12} - q^{3m-9}$ and $\mathrm{wt}\,(\varphi') \geq q^{4m-12} - q^{3m-9}$ for any other form φ' with $\dim(\mathrm{Rad}\,(\varphi')) = m-2$.*

Proof. In order to determine the weight of the word of $\mathcal{H}_{m,2}$ induced by the form φ, we need to determine the number of lines of $\mathcal{H}_{m,2}$ which are not totally isotropic for φ.

Take φ with $\dim(\mathrm{Rad}\,(\varphi)) = m-2$. Then a line ℓ is totally isotropic for φ if and only if $\ell \cap [\mathrm{Rad}\,(\varphi)] \neq \emptyset$. If S denotes the matrix representing φ, we have $\mathrm{rank}\,(S) = 2$. According to the notation of Section 3.1, $\mathrm{rank}\,(S) = 2$ is equivalent to $i = 1$ and $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m = [\Pi_t]\mathcal{H}_{m-2-t}$ is a degenerate Hermitian variety with radical $[\Pi_t]$ of dimension t. By Equation (21) and Lemma 3.5, since $m > 4$ is odd, the maximum number of points $\mu_{m-2}^{\max}$ of $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m$ is attained for $t = 2i-1 = 1$, i.e. $\mu_{m-2}^{\max} = \mu_{m-2}(1) = q^2\mu_{m-3} + 1$ (last equality comes from Equation (20)). Hence the number of points of $\mathcal{H}_m \setminus [\mathrm{Rad}\,(\varphi)]$ (see Equation (20)) is at least

$$\mu_m - \mu_{m-2}^{\max} = \mu_m - q^2\mu_{m-3} - 1 = q^{m-2}(q^{m-1} + q^{m-3} - 1) = q^{2m-3} + q^{2m-5} - q^{m-2}.$$

Assume $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m = [\Pi_1]\mathcal{H}_{m-3}$ and consider a point $[p] \in \mathcal{H}_m \setminus [\mathrm{Rad}\,(\varphi)]$.

- $\boxed{\text{Case } [p] \in [\Pi_1]^{\perp_\eta}}$. Then $[p]^{\perp_\eta} \cap [\mathrm{Rad}\,(\varphi)]$ is a degenerate Hermitian polar space $[\Pi_1]\mathcal{H}_{m-4}$ with radical $[\Pi_1]$ of dimension 1; so there are $(\mu_{m-2} - q^2\mu_{m-4} - 1) = q^{2m-7}$ lines through $[p]$ disjoint from $[\mathrm{Rad}\,(\varphi)]$. The number of points $[p]$ collinear with the point $[\Pi_1]$ in $\mathcal{H}_m$ but not contained in $\mathcal{H}_m \cap [\mathrm{Rad}\,(\varphi)] = [\Pi_1]\mathcal{H}_{m-3}$ is $q^2(\mu_{m-2} - \mu_{m-3}) = q^2(q^{2m-7} - q^{m-4})$.

- $\boxed{\text{Case } [p] \notin [\Pi_1]^{\perp_\eta}}$. Then $[p]^{\perp_\eta} \cap [\mathrm{Rad}\,(\varphi)] = \mathcal{H}_{m-3}$, so, there are $(\mu_{m-2} - \mu_{m-3}) = (q^{2m-7} - q^{m-4})$ lines through $[p]$ which are not totally isotropic. The number of points not collinear with $[\Pi_1]$ in $\mathcal{H}_m$ and not in $\mathcal{H}_m \cap [\mathrm{Rad}\,(\varphi)]$ is $(\mu_m - q^2\mu_{m-3} - 1) - q^2(\mu_{m-2} - \mu_{m-3}) = (\mu_m - q^2\mu_{m-2} - 1) = q^{2m-3}$.

So, we have that the total number of lines disjoint from $[\mathrm{Rad}\,(\varphi)]$ is

$$\frac{1}{q^2+1}\left(q^2(q^{m-7} - q^{m-4}) \cdot q^{2m-7} + (q^{2m-7} - q^{m-4})q^{2m-3}\right),$$

i.e.

$$\mathrm{wt}\,(\varphi) = \frac{1}{q^2+1}\left(q^{4m-12} - q^{3m-9} + q^{4m-10} - q^{3m-7}\right) = q^{4m-12} - q^{3m-9}.$$

In case $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m \neq [\Pi_1]\mathcal{H}_{m-3}$, the number of totally η-isotropic lines incident (either in a point or contained in) $[\mathrm{Rad}\,(\varphi)]$ is smaller; thus the weight of the word induced by φ is larger than the value obtained above. □

We claim that $q^{4m-12} - q^{3m-9}$ is actually the minimum distance for m odd.
As before, let $i = (\mathrm{rank}\,(S))/2$. When $i = 1$, then by Proposition 3.9, the minimum weight of the codewords induced by S is $d_1 = q^{4m-12} - q^{3m-9}$. Suppose now $i > 1$; we need to distinguish several cases according to the value of m.

- $\boxed{m \geq 11}$. Then by Corollary 3.6 and Lemma 3.7 $A_i \leq \xi_m(i) \leq \xi_m(2) \leq \xi_m(1)$. By Case b) of Corollary 3.6,

$$d_i \geq \frac{q^{2m-7} - q^{m-4}}{q^4 - 1}((q^2-1)\mu_m - \xi_m(i)) \geq \frac{q^{2m-7} - q^{m-4}}{q^4 - 1}((q^2-1)\mu_m - \xi_m(2)).$$

  We will show that

$$\frac{q^{2m-7} - q^{m-4}}{q^4 - 1}((q^2 - 1)\mu_m - \xi_m(2)) > q^{4m-12} - q^{3m-9}.$$

  Actually, by straightforward computations, this becomes

$$q^{m-4}(q^{m-3} - 1)(q^{2m-9} - q - 1 - \frac{q^{m-2}}{q^2 + 1}) > 0$$

  which is true for all values of q; so $d_{\min} = d_1 = q^{4m-12} - q^{3m-9}$.

- $\boxed{m = 9}$. By Lemma 3.7 we have that the second largest value of $\xi_m(i)$ is for $i = 4$ and $\xi_9(4) = q^9 + q^8 + q^2 - q - 2$. This corresponds to the following bound on the minimum weight d_i of codewords associated with matrices S with $2i = \mathrm{rank}\,(S) > 2$ (see Case b) of Corollary 3.6):

$$d_i > \frac{q^{11} - q^5}{q^4 - 1}(q^{17} - 2q^9 - q^2 + q + 1) > q^{24} - q^{18}.$$

  So, the minimum distance is attained by codewords corresponding to matrices S of rank 2 and, by Proposition 3.9, $d_{\min} = d_1 = q^{4m-12} - q^{3m-9}$.

- $\boxed{m = 7}$. By Lemma 3.7 we have that the second largest value of $\xi_m(i)$ is for $i = 3$ and $\xi_7(3) = q^7 + q^6 + q^2 - q - 2$. This corresponds to the following bound on the minimum weight of codewords associated with matrices S with $2i = \mathrm{rank}\,(S) > 2$:

$$d_i > q^{16} - 2q^{10} - q^5 + q^4 + q^3 > q^{16} - q^{12}.$$

  So the minimum distance is attained by codewords corresponding to matrices S of rank 2 and, by Proposition 3.9, $d_{\min} = d_1 = q^{4m-12} - q^{3m-9}$.

- $\boxed{m=5}$. By Lemma 3.7, we have $\xi_5(1) = \xi_5(2) = q^5 + q^4 + q^2 - q - 2$. We will prove that the minimum distance of $\mathcal{H}_{m,2}$ is $q^8 - q^6$.

  Let φ be a (non-null) alternating bilinear form of $V(5, q^2)$ represented by a matrix S. The radical of φ can have dimension 1 or 3, hence $\mathrm{rank}\,(S)$ is either 2 or 4, i.e. $i = 1$ or 2. By Proposition 3.9, there exists an alternating bilinear form φ with $\dim(\mathrm{Rad}\,(\varphi)) = 3$ such that $\mathrm{wt}\,(\varphi) = q^8 - q^6$ and any other form φ with $\dim(\mathrm{Rad}\,(\varphi)) = 3$ has weight greater then $q^8 - q^6$. So, we need to show that there are no alternating bilinear forms with $\dim(\mathrm{Rad}\,(\varphi)) = 1$ inducing words of weight less than $q^8 - q^6$. Assume henceforth that φ is an alternating bilinear form with $\dim(\mathrm{Rad}\,(\varphi)) = 1$ (hence $i = 2$). We shall determine a lower bound d_2 for the weights $\mathrm{wt}\,(\varphi)$ and prove $\mathrm{wt}\,(\varphi) \geq d_2 \geq q^8 - q^6$.

  By Lemma 3.2, $p \in \mathfrak{C}_\varphi$ if and only if $f_\varphi([p]) \neq [p]$ and $f_\varphi([p]) \in \mathcal{H}_5$. If $[p] = [\mathrm{Rad}\,(\varphi)]$, then $p \in \mathfrak{A}_\varphi$. So, suppose $[p] \neq [\mathrm{Rad}\,(\varphi)]$. Since $[\mathrm{Rad}\,(\varphi)] = \ker(f_\varphi)$, if $[p] \in \mathcal{H}_5$ and $f_\varphi([p]) = [x] \in \mathcal{H}_5$, then $f_\varphi([p + \alpha\mathrm{Rad}\,(\varphi)]) = [x] \in \mathcal{H}_5$ for any $\alpha \in \mathbb{F}_{q^2}$. On the other hand, for any given point $[p] \in \mathcal{H}_5$, the line $[p, \mathrm{Rad}\,(\varphi)]$ meets $\mathcal{H}_5$ in either $(q+1)$ or (q^2+1) points; this yields that any point in $\mathcal{H}_5$ belonging to the image $f_\varphi(\mathcal{H}_5)$ of f_φ restricted to $\mathcal{H}_5$ admits at least $q - 1$ preimages in $\mathcal{H}_5$ distinct from itself. Since $\dim(\mathrm{Rad}\,(\varphi)) = 1$, the set $\mathrm{Fix}\,(f_\varphi)$ is contained in a $\mathrm{PG}(V')$ with $\dim V' = 4$. We need a preliminary lemma.

  Lemma 3.10. *Let $f_\varphi : \mathrm{PG}(V') \to \mathrm{PG}(V')$ be a semilinear collineation with $\dim V' = 4$. Then, either $\mathrm{Fix}\,(f_\varphi) \cong \mathrm{PG}(3, q)$ or $|\mathrm{Fix}\,(f_\varphi)| \leq q^2+q+2$.*

  Proof. In general, $\mathrm{Fix}\,(f_\varphi)$ is contained in a subgeometry $\mathrm{PG}(V'')$ with V'' a vector space over $\mathbb{F}_q$ and $\dim V'' = 4$.

  Assume $\mathrm{Fix}\,(f_\varphi) \neq \mathrm{PG}(V'') \cong \mathrm{PG}(3, q)$. Suppose first that there is a vector space W over $\mathbb{F}_q$ with $\dim W = 3$ such that $\mathrm{PG}(W) \subseteq \mathrm{Fix}\,(f_\varphi)$. If $\mathrm{Fix}\,(f_\varphi) = \mathrm{PG}(W)$, then we are done. Otherwise, let $[p] \in \mathrm{Fix}\,(f_\varphi) \backslash \mathrm{PG}(W)$ and define $W' := W + \langle p \rangle_q$. Since $\dim_q W' = 4$, we have that $\mathrm{Fix}\,(f_\varphi)$ is contained in the subgeometry $\mathrm{PG}(W')$. If there is $[r] \in \mathrm{Fix}\,(f_\varphi) \setminus \mathrm{PG}(W)$ with $[r] \neq [p]$, then the subline $\ell_q := [p, r]_q$ spanned by $[p]$ and $[r]$ meets the subplane $\mathrm{PG}(W)$ in a point $[s] \neq [r]$ which is also fixed. So ℓ_q is fixed pointwise. Take $[t] \in \mathrm{PG}(W)$ with $[t] \neq [s]$. The subline $[t, s]_q$ is also fixed pointwise, so the subplane $[p, s, t]_q \neq \mathrm{PG}(W)$ is also fixed pointwise. As f_φ fixes two (hyper)planes pointwise in $\mathrm{PG}(W') \cong \mathrm{PG}(3, q)$ we have that f_φ fixes $\mathrm{PG}(W')$ pointwise — a contradiction. Thus, in this case $|\mathrm{Fix}\,(f_\varphi)| \leq q^2 + q + 2$.

  Suppose now that $\mathrm{Fix}\,(f_\varphi)$ does not contain a subplane isomorphic to $\mathrm{PG}(2, q)$ and that there is a subline $\ell \subseteq \mathrm{Fix}\,(f_\varphi)$. If there were a subplane π_q through ℓ such that $\mathrm{Fix}\,(f_\varphi) \cap \pi_q$ contains two points not on ℓ, then (by the same argument we used in the case above), this subplane would have to be fixed pointwise by f_φ. This is a contradiction; so if $\mathrm{Fix}\,(f_\varphi)$ contains a subline ℓ, then there is at most one fixed point $[x] \notin \ell$ on any subplane through ℓ contained in $\mathrm{PG}(V'')$. So, $|\mathrm{Fix}\,(f_\varphi)| \leq 2q+2$.

Finally, if $\mathrm{Fix}\,(f_\varphi)$ does not contain sublines, then $\mathrm{Fix}\,(f_\varphi)$ cannot contain either frames or more than 3 points on a plane or more than 2 points on a line. It follows that $|\mathrm{Fix}\,(f_\varphi)| \leq 4$. This completes the proof. □

In light of Lemma 3.10 we now distinguish two subcases:

a) Suppose f_φ fixes a subgeometry $[\Sigma_\varphi] \cong \mathrm{PG}(3,q)$ of (vector) dimension 4. Clearly, $[\mathrm{Rad}\,(\varphi)] \notin [\Sigma_\varphi]$. By Lemma 3.3, f_φ restricted to $\mathcal{H}_4 = [\Sigma_\varphi] \cap \mathcal{H}_5$, bijectively maps points of $\mathcal{H}_4$ into points of $\mathcal{H}_4$ and fixes $(q^4-1)/(q-1)$ of them. Since every point in the image of f_φ admits at least $q-1$ preimages in $\mathcal{H}_5$ distinct from itself we get

$$C \geq (q^2-1)(q-1)\mu_4.$$

Plugging this in Equation (13) and using Corollary 3.6 and Lemma 3.7, we obtain that for any q:

$$\begin{aligned} \mathrm{wt}\,(\varphi) \geq \frac{(q^3-q)}{(q^2+1)}\mu_5 - \frac{(q^3-q)}{(q^4-1)}A + \frac{q}{(q^4-1)}C \geq \\ \frac{q(q^2-1)}{(q^2+1)}\mu_5 - \frac{(q^3-q)}{(q^4-1)}\xi_5(2) + \frac{q}{(q^4-1)}(q^2-1)(q-1)\mu_4 = \\ = \frac{q^{10}-2q^6+q^5-q^4-q^3+2q^2}{q+1} > q^8 - q^6. \quad (29) \end{aligned}$$

b) Suppose now that f_φ does not fix a subgeometry isomorphic to $\mathrm{PG}(3,q)$. Then, by Lemma 3.10, $|\mathrm{Fix}\,(f_\varphi)| \leq q^2+q+2$. By Lemma 3.2, $p \in \mathfrak{A}_\varphi$ if and only if $f_\varphi([p]) = [p]$ or $f_\varphi([p]) = 0$.

By Equation (19), we have

$$\begin{aligned} A \leq |\mathrm{Fix}\,(f_\varphi)| + (q^2-1)|([\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_5)| \leq \\ (q^2+q+2)(q^2-1) + (q^2-1) = q^4+q^3+2q^2-q-3. \end{aligned}$$

We now need to compute a lower bound for C. Since $[\mathrm{Rad}\,(\varphi)] = \ker(f_\varphi)$ and we are assuming $\dim(\mathrm{Rad}\,(\varphi)) = 1$, the image $\mathrm{Im}(f_\varphi)$ of the semilinear function f_φ is a subspace of $\mathrm{PG}(4,q^2)$ of (vector) dimension 4. In particular, the image $f_\varphi(\mathcal{H}_5)$ of its restriction to $\mathcal{H}_5$ is a (possibly degenerate) Hermitian surface contained in a projective space $\mathrm{PG}(3,q^2)$.

Let $\mathcal{H}' := f_\varphi(\mathcal{H}_5) \cap \mathcal{H}_5$. By Lemma 3.2, $p \in \mathfrak{C}_\varphi$ if and only if $f_\varphi([p]) \neq [p]$ and $f_\varphi([p]) \in \mathcal{H}'$. Using the descriptions of intersection of Hermitian varieties in [16, 13] we see that $|\mathcal{H}'| \geq q^3+1$. Thus we get

$$C \geq (q-1)(q^2-1)(q^3+1).$$

Plugging this in Equation (13), we obtain that for any q,

$$\mathrm{wt}\,(\varphi) \geq \frac{(q^3-q)}{(q^2+1)}\mu_5 - \frac{(q^3-q)}{(q^4-1)}A + \frac{q}{(q^4-1)}C \geq \frac{q^{10}-q^6+q^5-2q^4-2q^3+2q^2+q}{q^2+1} \tag{30}$$

For $q > 2$, (30) gives $\mathrm{wt}\,(\varphi) \geq q^8 - q^6$. And this completes the argument. For $q = 2$ a direct computer search proves that the minimum weight of the code is once more $192 = q^8 - q^6$.

The above proof directly implies the following characterization of the minimum weight codewords for m odd.

Corollary 3.11. *If either*

- $m > 5$ *is odd or*
- $m = 5$ *and* $q \neq 2$,

then the minimum weight codewords of $\mathcal{C}(\mathbb{H}_{m,2})$ *correspond to bilinear alternating forms* φ *with* $\dim(\mathrm{Rad}\,(\varphi)) = m-2$ *and such that* $[\mathrm{Rad}\,(\varphi)]$ *meets* $\mathcal{H}_m$ *in a Hermitian cone of the form* $[\Pi_1]\mathcal{H}_{m-3}$.

Remark 3.12. For $q = 2$ and $m = 5$, an exhaustive computer search shows that the characterization of Corollary 3.11 does not hold, as there are 24948 codewords of minimum weight 192; 5940 of these are associated with bilinear forms with radical of dimension 1 while the remaining 19008 are associated with forms with radical of dimension 3. The forms with 3-dimensional radical are as those described in Corollary 3.11. Incidentally, the full list of weights for this code is $0, 192, 216, 224, 232, 256$.

3.3. Minimum distance of $\mathcal{H}_{m,2}$ with m even

In this section we assume m to be even. Then the Witt index of the Hermitian form η is $n = m/2$. Let φ be an alternating form on V. Recall from Equation (13) that

$$\mathrm{wt}\,(\varphi) = \frac{q^{2m-7}}{(q^4-1)}((q^2-1)\mu_m - A) + \frac{q^{m-4}}{q^4-1}B.$$

Proposition 3.13. *There exists a bilinear alternating form* φ *with* $\dim(\mathrm{Rad}\,(\varphi)) = m-2$ *such that* $\mathrm{wt}\,(\varphi) = q^{4m-12}$ *and* $\mathrm{wt}\,(\varphi') \geq q^{4m-12}$ *for any other form* φ' *with* $\dim(\mathrm{Rad}\,(\varphi')) = m-2$.

Proof. Let φ be a bilinear alternating form with radical of dimension $m-2$. In order to determine the weight of the word of $\mathcal{H}_{m,2}$ induced by the form φ, we need to determine the number of lines of $\mathcal{H}_{m,2}$ which are not totally isotropic for φ.

Since, by hypothesis, the radical of φ has dimension $\dim(\mathrm{Rad}\,(\varphi)) = m-2$, a line ℓ is totally isotropic for φ if and only if $\ell \cap [\mathrm{Rad}\,(\varphi)] \neq \emptyset$. If S denotes the matrix representing φ, we have rank $(S) = 2$; so, according to the results obtained in Section 3.1, we have that for words associated to the value $A_{\max}$ it must be $i = 1$ and $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m = [\Pi_t]\mathcal{H}_{m-2-t}$ is a degenerate Hermitian variety with radical $[\Pi_t]$ of dimension t. By Equation (21) and Lemma 3.5, since m is even, the maximum number of points $\mu_{m-2}^{\max}$ of $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m$ is attained for $t = 2i = 2$ (in this case $[\Pi_2]$ is a line), i.e. $\mu_{m-2}^{\max} = \mu_{m-2}(2) = q^4\mu_{m-4} + q^2 + 1$ (last equality comes from Equation (20)). Hence the number of points of $\mathcal{H}_m \setminus [\mathrm{Rad}\,(\varphi)]$ (see Equations (10) and (20)) is at least

$$\mu_m - \mu_{m-2}^{\max} = \mu_m - q^4\mu_{m-4} - q^2 - 1 = q^{2m-3} + q^{2m-5}.$$

Assume $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m = [\Pi_2]\mathcal{H}_{m-4}$ and consider a point $[p] \in \mathcal{H}_m \setminus [\mathrm{Rad}\,(\varphi)]$; we study $[p]^{\perp_\eta} \cap \mathcal{H}_m \cap [\mathrm{Rad}\,(\varphi)]$.

- $\boxed{\text{Case } [\Pi_2] \subseteq [p]^{\perp_\eta}}$. Note that $[\mathrm{Rad}\,(\varphi)] = [\Pi_2]^{\perp_\eta}$ because $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m \subseteq [\Pi_2]^{\perp_\eta}$ and $\dim(\Pi_2^{\perp_\eta}) = \dim(\mathrm{Rad}\,(\varphi)) = m - 2$ (indeed, if $[x] \in [\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m$, then $[x] \in [\Pi_2]\mathcal{H}_{m-4}$; hence, $[x] \in [\Pi_2]^{\perp_\eta}$). This implies that every point $[p]$ such that $[\Pi_2] \subseteq [p]^{\perp_\eta}$, i.e. $[p] \in [\Pi_2]^{\perp_\eta}$ is also in $[\mathrm{Rad}\,(\varphi)]$ while we were assuming $[p] \in \mathcal{H}_m \setminus [\mathrm{Rad}\,(\varphi)]$. Thus, this case can not happen.

- $\boxed{\text{Case } [\Pi_2] \not\subseteq [p]^{\perp_\eta}}$. Then $[p]^{\perp_\eta} \cap (\mathcal{H}_m \cap [\mathrm{Rad}\,(\varphi)]) \cong [\Pi_1]\mathcal{H}_{m-4}$. In this case there are $(\mu_{m-2} - q^2\mu_{m-4} - 1) = q^{2m-7}$ lines through $[p]$ disjoint from $[\mathrm{Rad}\,(\varphi)]$.

Since all points of $\mathcal{H}_m$ not in $[\mathrm{Rad}\,(\varphi)]$ are such that $[\Pi_2] \not\subseteq [p]^{\perp_\eta}$, the total number of lines disjoint from $[\mathrm{Rad}\,(\varphi)]$ is $\frac{(q^{2m-3}+q^{2m-5})q^{2m-7}}{q^2+1} = q^{4m-12}$, i.e. it is always possible to find a bilinear alternating form φ with $\dim(\mathrm{Rad}\,(\varphi)) = m-2$ such that $\mathrm{wt}\,(\varphi) = q^{4m-12}$. Observe that for any form with $\dim(\mathrm{Rad}\,(\varphi)) = m-2$ such that $[\mathrm{Rad}\,(\varphi)] \cap \mathcal{H}_m \neq [\Pi_2]\mathcal{H}_{m-4}$, the number of totally η-isotropic lines disjoint from $[\mathrm{Rad}\,(\varphi)]$ is larger than the value obtained in the case considered above, so $\mathrm{wt}\,(\varphi) > q^{4m-12}$. □

We claim that q^{4m-12} is actually the minimum weight for m even unless $m = 4, 6$. We also compute the minimum weight for $m = 4, 6$.

As before, let $i = (\mathrm{rank}\,(S))/2$. When $i = 1$, then by Proposition 3.13, the minimum weight of the codewords is $d_1 = q^{4m-12}$. Assume $i > 1$; we need to distinguish several cases according to the value of m.

- $\boxed{m \geq 12}$. By Corollary 3.6 and Lemma 3.7 $A_i \leq \xi_m(i) \leq \xi_m(2) \leq \xi_m(1)$. From Case b) of Corollary 3.6 we get

$$d_i \geq \frac{q^{2m-7}(\mu_m(q^2-1) - \xi_m(i))}{q^4-1} \geq \frac{q^{2m-7}(\mu_m(q^2-1) - \xi_m(2))}{q^4-1}.$$

  We will show that

$$\frac{q^{2m-7}(\mu_m(q^2-1) - \xi_m(2))}{q^4-1} > q^{4m-12}.$$

Actually, by straightforward computations, the above condition becomes

$$q^{4m-12} + q^{4m-16} - q^{2m-6} - q^{2m-7} > q^{4m-12}$$

which is true for all values of q.

- $\boxed{m = 10}$. By Lemma 3.7 we have that the second largest value of $\xi_{10}(i)$ is for $i = 5$: $\xi_{10}(5) = q^{11} + q^{10} - q - 1$. By Case b) of Corollary 3.6 we have

$$d_i \geq \frac{q^{13}}{q^2+1}\left(\mu_{10} - \frac{1}{q^2-1}\xi_{10}(5)\right) = q^{28} + q^{24} - q^{18} - q^{14} > q^{28}.$$

  So, the minimum distance is attained by codewords corresponding to matrices S of rank 2 and, consequently, $d_{\min} = q^{4m-12}$.

- $\boxed{m = 8}$. By Lemma 3.7 we have that the second largest value of $\xi_8(i)$ is for $i = 4$: $\xi_8(4) = q^9 + q^8 - q - 1$. By Case b) of Corollary 3.6 we have

$$d_i \geq \frac{q^9}{q^2+1}\left(\mu_8 - \frac{1}{q^2-1}\xi_8(4)\right) = q^{20} + q^{16} - q^{14} - q^{10} > q^{20}.$$

  So the minimum distance is attained by codewords corresponding to matrices S of rank 2 and, consequently, $d_{\min} = q^{4m-12} = q^{20}$.

- $\boxed{m = 6, 4}$. By Lemma 3.7 we have $\xi_4(1) < \xi_4(2)$ and $\xi_6(2) < \xi_6(1) < \xi_6(3)$, hence the maximum value of $\xi_m(i)$ is for $i = m/2$, i.e. we have that the matrix S has maximum rank m and so it is non-singular.

  For $i \neq m/2$, by case b) of Corollary 3.6 we have

$$d_i \geq d_1 > \frac{q^{2m-7}}{q^2+1}\left(\mu_m - \frac{1}{q^2-1}\xi_m(m/2)\right) = q^{4m-12} - q^{2m-6} = d_{m/2}.$$

  We shall show that $q^{4m-12} - q^{2m-6}$ is the actual minimum distance.

Lemma 3.14. *If $m = 4, 6$ then there exists a non-singular alternating form φ of $V(m, q^2)$ such that $|\mathfrak{A}_\varphi| = A_{m/2} = (q^m - 1)(q + 1)$.*

Proof. For any non-singular bilinear alternating form φ we have $|\mathfrak{A}_\varphi| = |\mathrm{Fix}\,(f_\varphi) \cap \mathcal{H}_m|(q^2-1)$, see (16). Choose φ to be a symplectic polarity which permutes with η (i.e. $[u]^{\perp_\varphi \perp_\eta} = [u]^{\perp_\eta \perp_\varphi}$ for all $[u] \in \mathrm{PG}(m-1, q^2)$). Then $f_\varphi(\mathcal{H}_m) = \mathcal{H}_m$ and by [23, §74], $\mathrm{Fix}\,(f_\varphi) \cong \mathrm{PG}(m-1, q)$ is a subgeometry over $\mathbb{F}_q$ fully contained in $\mathcal{H}_m$. Hence,

$$|\mathfrak{A}_\varphi| = |\mathrm{Fix}\,(f_\varphi) \cap \mathcal{H}_m|(q^2-1) = |\mathrm{Fix}\,(f_\varphi)|(q^2-1) = \frac{q^m-1}{q-1}(q^2-1) = (q^m-1)(q+1).$$

$\square$

By Lemma 3.3, the bilinear alternating form φ given by Lemma 3.14 is such that $|\mathfrak{B}_\varphi| = 0$. Hence, since $A_{m/2} > A_i$ for all $i \neq m/2$ and $A_{m/2} = (q^m - 1)(q + 1)$, we have $\mathrm{wt}\,(\varphi) = q^{4m-12} - q^{2m-6}$. By Equation (14), as $\mathrm{wt}\,(\varphi) = q^{4m-12} - q^{2m-6}$, it follows that $d_{\min} = q^{4m-12} - q^{2m-6}$.

By the arguments presented before we have the following characterization of the minimum weight codewords for m even.

Corollary 3.15. *If $m = 4$ or $m = 6$, then the minimum weight codewords of $\mathcal{C}(\mathbb{H}_{m,2})$ correspond to bilinear alternating forms φ which are permutable with the given Hermitian form η. If $m > 6$ is even, then the minimum weight codewords correspond to bilinear alternating forms φ with $\dim(\mathrm{Rad}\,(\varphi)) = m - 2$ and such that $[\mathrm{Rad}\,(\varphi)]$ meets $\mathcal{H}_m$ in a Hermitian cone of the form $[\Pi_2]\mathcal{H}_{m-4}$.*

Sections 3.2 and 3.3 complete the proof of the Main Theorem.

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References

[1] R. J. Blok and B. N. Cooperstein. The generating rank of the unitary and symplectic Grassmannians. *J. Combin. Theory Ser. A*, 119(1):1–13, 2012.

[2] R. C. Bose and I. M. Chakravarti. Hermitian varieties in a finite projective space PG(N, q^2). *Canad. J. Math.*, 18:1161–1182, 1966.

[3] I. Cardinali and L. Giuzzi. Codes and caps from orthogonal Grassmannians. *Finite Fields Appl.*, 24:148–169, 2013.

[4] I. Cardinali and L. Giuzzi. Minimum distance of symplectic Grassmann codes. *Linear Algebra Appl.*, 488:124–134, 2016.

[5] I. Cardinali, L. Giuzzi, K. V. Kaipa, and A. Pasini. Line polar Grassmann codes of orthogonal type. *J. Pure Appl. Algebra*, 220(5):1924–1934, 2016.

[6] I. Cardinali and L. Giuzzi. Enumerative coding for line polar Grassmannians with applications to codes. *Finite Fields Appl.*, 46:107–138, 2017.

[7] I. Cardinali and L. Giuzzi. Minimum distance of line orthogonal Grassmann codes in even characteristic. *J. Pure Appl. Algebra*, to appear, doi:10.1016/j.jpaa.2017.11.009

[8] I. Cardinali and L. Giuzzi. Implementing line Hermitian Grassmann codes. *preprint*, 2017.

[9] I. Cardinali, L. Giuzzi, and A. Pasini. On transparent embeddings of point-line geometries. *J. Combin. Theory Series A*, 155:190–224, 2018.

[10] I. Cardinali and A. Pasini. Embeddings of line-Grassmannians of polar spaces in Grassmann varieties. In *Groups of exceptional type, Coxeter groups and related geometries*, volume 82 of *Springer Proc. Math. Stat.*, pages 75–109. Springer, New Delhi, 2014.

[11] W. L. Chow. On the geometry of algebraic homogeneous spaces. *Ann. of Math. (2)*, 50:32–67, 1949.

[12] T. M. Cover. Enumerative source encoding. *IEEE Trans. Information Theory*, IT-19(1):73–77, 1973.

[13] G. Donati and N. Durante. On the intersection of Hermitian curves and of Hermitian surfaces. *Discrete Math.*, 308(22):5196–5203, 2008.

[14] S. R. Ghorpade and K. V. Kaipa. Automorphism groups of Grassmann codes. *Finite Fields Appl.*, 23:80–102, 2013.

[15] S. R. Ghorpade, A. R. Patil, and H. K. Pillai. Decomposable subspaces, linear sections of Grassmann varieties, and higher weights of Grassmann codes. *Finite Fields Appl.*, 15(1):54–68, 2009.

[16] L. Giuzzi. On the intersection of Hermitian surfaces. *J. Geom.*, 85(1-2):49–60, 2006.

[17] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries*. Springer Monographs in Mathematics. Springer, London, 2016.

[18] K. V. Kaipa and H. K. Pillai. Weight spectrum of codes associated with the Grassmannian $G(3,7)$. *IEEE Trans. Inform. Theory*, 59(2):986–993, 2013.

[19] D. Y. Nogin. Codes associated to Grassmannians. In *Arithmetic, geometry and coding theory (Luminy, 1993)*, pages 145–154. de Gruyter, Berlin, 1996.

[20] C. T. Ryan. An application of Grassmannian varieties to coding theory. *Congr. Numer.*, 57:257–271, 1987. Sixteenth Manitoba conference on numerical mathematics and computing (Winnipeg, Man., 1986).

[21] C. T. Ryan. Projective codes based on Grassmann varieties. *Congr. Numer.*, 57:273–279, 1987. Sixteenth Manitoba conference on numerical mathematics and computing (Winnipeg, Man., 1986).

[22] C. T. Ryan and K. M. Ryan. The minimum weight of the Grassmann codes $C(k,n)$. *Discrete Appl. Math.*, 28(2):149–156, 1990.

[23] B. Segre. Forme e geometrie hermitiane, con particolare riguardo al caso finito. *Ann. Mat. Pura Appl. (4)*, 70:1–201, 1965.

[24] M. Tsfasman, S. Vlăduţ, and D. Nogin. *Algebraic geometric codes: basic notions*, volume 139 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.