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A New Approach to Equations with Memory

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Abstract

We discuss a novel approach to the mathematical analysis of equations with memory, based on a new notion of *state*. This is the initial configuration of the system at time $t = 0$ which can be unambiguously determined by the knowledge of the dynamics for positive times. As a model, for a nonincreasing convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$G(0) = \lim_{s\to 0} G(s) > \lim_{s\to\infty} G(s) > 0$$

we consider an abstract version of the evolution equation

$$\partial_{tt}\boldsymbol{u}(\boldsymbol{x}, t) - \Delta\left[G(0)\boldsymbol{u}(\boldsymbol{x}, t) + \int_0^\infty G'(s)\boldsymbol{u}(\boldsymbol{x}, t - s)\mathrm{d}s\right] = 0$$

arising from linear viscoelasticity.

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1. Preamble

1.1. A general introduction to equations with memory

Many interesting physical phenomena (such as viscoelasticity, hereditary polarization in dielectrics, population dynamics or heat flow in real conductors, to name some) are modelled by differential equations which are influenced by the past values of one or more variables in play: so-called *equations with memory*. The main problem in the analysis of equations of this kind lies in their nonlocal character, due to the presence of the memory term (in general, the time convolution of the unknown function against a suitable memory kernel). Loosely speaking, an evolution equation with memory has the following formal structure:

$$\partial_t w(t) = \mathcal{F}(w(t), w^t(\cdot)), \quad t > 0, \tag{1}$$

where

$$w^t(s) = w(t-s), \quad s > 0,$$

and $\mathcal{F}$ is some operator acting on $w(t)$, as well as on the *past values* of w up to the actual time t. The function w is supposed to be known for all $t \leqq 0$, where it need not solve the differential equation. Accordingly, the initial datum has the form

$$w(t) = w_0(t), \quad t \leqq 0,$$

where w_0 is a given function defined on $(-\infty, 0]$.

A way to circumvent the intrinsic difficulties posed by the problem is to (try to) rephrase (1) as an ordinary differential equation in some abstract space, by introducing an auxiliary variable accounting for the past history of w, in order to be in a position to exploit the powerful machinery of the theory of dynamical systems. This strategy was devised by DAFERMOS [9], who, in the context of linear viscoelasticity, proposed to view w^t as an additional variable ruled by its own differential equation, so translating (1) into a differential system acting on an extended space accounting for the memory component.

However, when dealing with (1), what one can actually "measure" is the function $w(t)$ for $t \geqq 0$. The practical consequences are of some relevance, since for a concrete realization of (1) arising from a specific physical model, the problem of assigning the initial conditions is not only theoretical in nature. In particular, it might happen that two *different* initial past histories w_0 lead to the *same* $w(t)$ for

$t \geqq 0$. From the viewpoint of the dynamics, two such initial past histories, though different, are in fact indistinguishable. This observation suggests that, rather than the past history w^t, one should employ an alternative variable to describe the initial state of the system, satisfying the following natural minimality property:

two different initial states produce different evolutions $w(t)$ for $t \geqq 0$.

From the philosophical side, this means that the knowledge of $w(t)$ for all $t \geqq 0$ determines in a unique way the initial state of the problem, the only object that really influences the future dynamics.

Of course, the main task is then to determine, if possible, what is a minimal state associated to (1). Unfortunately, a universal strategy is out of reach, and the correct choice depends on the particular concrete realization of (1). Nonetheless, for a large class of equations with memory, where the memory contribution enters in the form of a convolution integral with a nonincreasing positive kernel, a general scheme seems to be applicable. In this paper, we discuss an abstract evolution equation with memory arising from linear viscoelasticity, presenting an approach which can be easily extended and adapted to many other differential models containing memory terms.

1.2. Plan of the paper

The goal of the next Section 2, of a more physical flavor, is twofold. First, we present an overview on materials with hereditary memory, dwelling on the attempts made through the years to construct mathematical models accounting for memory effects. Next, we illustrate the physical motivations leading to the concept of minimal state representation. In Section 3, we introduce an abstract linear evolution equation with memory in convolution form, which, besides its remarkable intrinsic interest, will serve as a prototype to develop the new approach highlighted in the previous sections. After some notation and preliminary assumptions (Section 4), we recall the history approach devised by Dafermos (Section 5), whereas, in Section 6, we make a heuristic derivation of the state framework, which will be given a suitable functional formulation in the subsequent Section 7 and Section 8. There, we prove the existence of a contraction semigroup, whose exponential stability is established in Section 9, within standard assumptions on the memory kernel. In Section 10 and Section 11, we discuss the link between the original equation and its translated version in the state framework. Finally, in Section 12, we compare the history and the state formulations, showing the advantages of the latter.

2. A physical introduction

2.1. The legacy of Boltzmann and Volterra

The problem of the correct modelling of materials with memory has always represented a major challenge to mathematicians. The origins of modern viscoelasticity and, more generally, of the so-called hereditary systems traditionally trace back to the works of Boltzmann and Volterra [1,2,48,49], who first introduced the notion of memory in connection with the analysis of elastic materials. The key assumption in the hereditary theory of elasticity can be stated in the following way. *For an elastic body occupying a certain region* $\mathcal{B} \subset \mathbb{R}^N$ *at rest, the deformation of the mechanical system at any point* $\boldsymbol{x} \in \mathcal{B}$ *is a function both of the instantaneous stress and of all the past stresses at* $\boldsymbol{x}$.

In other words, calling $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ the displacement vector at the point $\boldsymbol{x} \in \mathcal{B}$ at time $t \geqq 0$, the infinitesimal strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top}\right]$$

obeys a constitutive relation of the form

$$\boldsymbol{\varepsilon}(\boldsymbol{x}, t) = \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma}(\boldsymbol{x}, t), \boldsymbol{\sigma}^t(\boldsymbol{x}, \cdot)),$$

where

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) \quad \text{and} \quad \boldsymbol{\sigma}^t(\boldsymbol{x}, s) = \boldsymbol{\sigma}(\boldsymbol{x}, t - s),$$

with $s > 0$, are the stress tensor and its past history at $(\boldsymbol{x}, t)$, respectively. In the same fashion, the inverse relation can be considered; namely,

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \tilde{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}(\boldsymbol{x}, t), \boldsymbol{\varepsilon}^t(\boldsymbol{x}, \cdot)).$$

The above representations allow the appearance of discontinuities at time t; for instance, $\boldsymbol{\varepsilon}(\boldsymbol{x}, t)$ may differ from $\lim_{s\to 0} \boldsymbol{\varepsilon}^t(\boldsymbol{x}, s)$. In presence of an external force $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$, the related motion equation is given by

$$\partial_{tt}\boldsymbol{u}(\boldsymbol{x}, t) = \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) + \boldsymbol{f}(\boldsymbol{x}, t).$$

The concept of *heredity* was proposed by Boltzmann, essentially in the same form later developed by Volterra within a rigorous functional setting. However, a more careful analysis reveals some differences between the two approaches. Quoting [39], "...when speaking of Boltzmann and Volterra, we are facing two different scientific conceptions springing from two different traditions of classical mathematical physics".

Boltzmann's formulation is focused on hereditary elasticity, requiring a fading *initial* strain history $\boldsymbol{\varepsilon}^0(\boldsymbol{x}, \cdot) = \boldsymbol{\varepsilon}^t(\boldsymbol{x}, \cdot)_{|t=0}$ for every $\boldsymbol{x} \in \mathcal{B}$, that is,

$$\lim_{s\to\infty} \boldsymbol{\varepsilon}^0(\boldsymbol{x}, s) = \lim_{s\to\infty} \boldsymbol{\varepsilon}(\boldsymbol{x}, -s) = 0, \tag{2}$$

so that, for every fixed $(\boldsymbol{x}, t)$,

$$\boldsymbol{\varepsilon}(\boldsymbol{x}, t) = \int_{-\infty}^{t} \mathrm{d}\boldsymbol{\varepsilon}(\boldsymbol{x}, y),$$

and assuming the linear stress–strain constitutive relation at $(\boldsymbol{x}, t)$ in the Riemann–Stieltjes integral form

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \int_{-\infty}^{t} \mathbb{G}(\boldsymbol{x}, t - y)\mathrm{d}\boldsymbol{\varepsilon}(\boldsymbol{x}, y), \tag{3}$$

where

$$\mathbb{G} = \mathbb{G}(\boldsymbol{x}, s), \quad s > 0,$$

is a fourth order symmetric tensor (for every fixed s), nowadays called *Boltzmann function*. In particular, Boltzmann emphasized a peculiar behavior of viscoelastic solid materials, named *relaxation property*: if the solid is held at a constant strain [stress] starting from a given time $t_0 \geqq 0$, the stress [strain] tends (as $t \to \infty$) to a constant value which is "proportional" to the applied constant strain [stress]. Indeed, if

$$\boldsymbol{\varepsilon}(\boldsymbol{x}, t) = \boldsymbol{\varepsilon}(\boldsymbol{x}, t_0) = \boldsymbol{\varepsilon}_0(\boldsymbol{x}), \quad \forall t \geqq t_0, \tag{4}$$

it follows that

$$\lim_{t\to\infty} \boldsymbol{\sigma}(\boldsymbol{x}, t) = \lim_{t\to\infty} \int_{-\infty}^{t_0} \mathbb{G}(\boldsymbol{x}, t-y)\mathrm{d}\boldsymbol{\varepsilon}(\boldsymbol{x}, y) = \mathbb{G}_\infty(\boldsymbol{x})\boldsymbol{\varepsilon}_0(\boldsymbol{x}), \tag{5}$$

where the *relaxation modulus*

$$\mathbb{G}_\infty(\boldsymbol{x}) = \lim_{s\to\infty} \mathbb{G}(\boldsymbol{x}, s)$$

is assumed to be positive definite.

Conversely, the general theory devised by Volterra to describe the constitutive stress–strain relation is based on the Lebesgue representation of linear functionals in the history space. In this framework, he stated the fundamental postulates of the elastic hereditary action:

- the principle of invariable heredity,
- the principle of the closed cycle.

In its simpler linear version, the Volterra stress–strain constitutive relation reads

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \mathbb{G}_0(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}, t) + \int_{-\infty}^{t} \mathbb{G}'(\boldsymbol{x}, t-y)\boldsymbol{\varepsilon}(\boldsymbol{x}, y)\mathrm{d}y, \tag{6}$$

where

$$\mathbb{G}_0(\boldsymbol{x}) = \lim_{s\to 0} \mathbb{G}(\boldsymbol{x}, s)$$

and the *relaxation function* $\mathbb{G}'(\boldsymbol{x}, s)$ is the derivative with respect to s of the Boltzmann function $\mathbb{G}(\boldsymbol{x}, s)$. It is apparent that (6) can be formally obtained from (3) by means of an integration by parts, provided that (2) holds true. In which case, the Boltzmann and the Volterra constitutive relations are equivalent. It is also worth noting that if (4) is satisfied for some $t_0 \geqq 0$ and

$$\lim_{t\to\infty} \int_{-\infty}^{t_0} \mathbb{G}'(\boldsymbol{x}, t-y)\boldsymbol{\varepsilon}(\boldsymbol{x}, y)\mathrm{d}y = 0,$$

then, in light of (6), we recover the stress relaxation property (5), as

$$\lim_{t\to\infty} \boldsymbol{\sigma}(\boldsymbol{x}, t) = \mathbb{G}_0(\boldsymbol{x})\boldsymbol{\varepsilon}_0(\boldsymbol{x}) + \lim_{t\to\infty} \int_{t_0}^{t} \mathbb{G}'(\boldsymbol{x}, t-y)\boldsymbol{\varepsilon}_0(\boldsymbol{x})\mathrm{d}y = \mathbb{G}_\infty(\boldsymbol{x})\boldsymbol{\varepsilon}_0(\boldsymbol{x}).$$

The longterm memory appearing in (3) and (6) raised some criticism in the scientific community from the very beginning, due to the conceptual difficulty in accepting the idea of a past history defined on an infinite time interval (when even the age of the universe is finite!). Aiming to overcome such a philosophical objection, Volterra circumvented the problem in a simple and direct way, assuming that the past history vanishes before some time $t_c \leqq 0$ (say, the creation time). Hence, (6) is replaced by

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathbb{G}_0(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x},t) + \int_{t_c}^{t} \mathbb{G}'(\boldsymbol{x},t-y)\boldsymbol{\varepsilon}(\boldsymbol{x},y)\mathrm{d}y,$$

and the motion equation becomes the well-known *Volterra integro-differential equation*

$$\partial_{tt}\boldsymbol{u}(\boldsymbol{x},t) = \nabla\cdot\mathbb{G}_0(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t) + \nabla\cdot\int_{t_c}^{t} \mathbb{G}'(\boldsymbol{x},t-y)\nabla\boldsymbol{u}(\boldsymbol{x},y)\mathrm{d}y + \boldsymbol{f}(\boldsymbol{x},t),$$

which (besides appropriate boundary conditions) requires only the knowledge of the initial data $\boldsymbol{u}(\boldsymbol{x},t_c)$ and $\partial_t\boldsymbol{u}(\boldsymbol{x},t_c)$.

Remark 1. Here, we exploited the equality

$$\mathbb{F}\boldsymbol{S} = \mathbb{F}\boldsymbol{S}^\top,$$

which holds for any fourth order symmetric tensor $\mathbb{F}$ and any second order tensor $\boldsymbol{S}$.

2.2. *Further developments: the fading memory principle*

In the thirties, GRAFFI [30,31] applied Volterra's theory to electromagnetic materials with memory, successfully explaining certain nonlinear wave propagation phenomena occurring in the ionosphere. Nonetheless, the modern theory of materials with memory was developed after World War II, when the discovery of new materials (for example, viscoelastic polymers) gave a boost to experimental and theoretical research. In the sixties, many seminal papers appeared in the literature, dealing with both linear and nonlinear viscoelasticity [3,8,10,36,37,40,41]. In particular, the thermodynamics of materials with memory provided an interesting new field of investigation, mainly because some thermodynamic potentials, such as entropy and free energy, are not unique, even up to an additive constant, and their definition heavily depends on the choice of the history space (see, for instance, [5]).

During the same years, COLEMAN AND MIZEL [6,7] introduced a main novelty: the notion of *fading memory*. Precisely, they considered the Volterra constitutive stress–strain relation (6), with the further assumption that the values of the deformation history in the far past produce negligible effects on the value of the present stress. In other words, the memory of the material is fading in time. Incidentally, this also gave the ultimate answer to the philosophical question of a memory of infinite duration. The fading memory principle is mathematically stated by endowing the space $\mathfrak{E}$ of initial strain histories $\boldsymbol{\varepsilon}^0(\boldsymbol{x},\cdot)$ with a weighted L^2-norm

$$\|\boldsymbol{\varepsilon}^0\|^2_{\mathfrak{E}} = \int_{\mathcal{B}}\int_0^\infty h(s)|\boldsymbol{\varepsilon}^0(\boldsymbol{x},s)|^2 \mathrm{d}s\mathrm{d}\boldsymbol{x},$$

where the *influence function* h is positive, monotone decreasing, and controls the relaxation function $\mathbb{G}'$ in the following sense:

$$\int_{\mathcal{B}}\int_0^\infty h^{-1}(s)|\mathbb{G}'(\boldsymbol{x},s)|^2 \mathrm{d}s\mathrm{d}\boldsymbol{x} < \infty.$$

The theory of Coleman and Mizel encouraged many other relevant contributions in the field, and was the starting point of several improvements in viscoelasticity (see [11,24,29,33,47] and references therein). On the other hand, as pointed out in [25,26], the fading memory principle turns out to be unable to ensure the well-posedness of the full motion equation of linear viscoelasticity

$$\partial_{tt}\boldsymbol{u}(\boldsymbol{x},t) = \nabla\cdot\mathbb{G}_0(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t) + \nabla\cdot\int_0^\infty \mathbb{G}'(\boldsymbol{x},s)\nabla\boldsymbol{u}(\boldsymbol{x},t-s)\mathrm{d}s + \boldsymbol{f}(\boldsymbol{x},t), \quad (7)$$

with known initial data $\boldsymbol{u}(\boldsymbol{x},0)$, $\partial_t\boldsymbol{u}(\boldsymbol{x},0)$ and $\boldsymbol{u}(\boldsymbol{x},-s)$, for $s > 0$. Moreover, the arbitrariness of the influence function h (for a given $\mathbb{G}'$) reflects into the non-uniqueness of the history space norm topology.

In order to bypass these difficulties, Fabrizio and coauthors [12,16,23,24] moved in two directions. Firstly, they looked for more natural conditions on the relaxation function $\mathbb{G}'$, focusing on the restriction imposed by the second law of thermodynamics. Secondly, they tried to construct an intrinsically defined normed history space. To this end, in the spirit of Graffi's work [32], they suggested that any free energy functional endows the history space with a natural norm [17,18]. In this direction, starting from [3,10], many other papers proposed new analytic expressions of the maximum and minimum free energies [12,15,19,20,27]. The first and well-known expression of the Helmholtz potential in linear viscoelasticity is the so called *Graffi–Volterra free energy density*

$$\begin{aligned}\Psi_{\mathrm{G}}(\boldsymbol{x},t) &= \frac{1}{2}\mathbb{G}_\infty(\boldsymbol{x})|\boldsymbol{\varepsilon}(\boldsymbol{x},t)|^2 \\ &\quad -\frac{1}{2}\int_0^\infty \langle\mathbb{G}'(\boldsymbol{x},s)[\boldsymbol{\varepsilon}(\boldsymbol{x},t)-\boldsymbol{\varepsilon}^t(\boldsymbol{x},s)],\boldsymbol{\varepsilon}(\boldsymbol{x},t)-\boldsymbol{\varepsilon}^t(\boldsymbol{x},s)\rangle\mathrm{d}s,\end{aligned}$$

where $\mathbb{G}'$ is negative definite with s-derivative $\mathbb{G}''$ positive semidefinite. With this choice of the energy, the asymptotic (exponential) stability of the dynamical problem (7) with $\boldsymbol{f} = 0$ has been proved, under the further assumption that, for some $\delta > 0$, the fourth order symmetric tensor

$$\mathbb{G}''(\boldsymbol{x},s) + \delta\mathbb{G}'(\boldsymbol{x},s)$$

is positive semidefinite for (almost) every $s > 0$ (see [9,21,28,42,43,45]). Besides, the form of Ψ_{G} suggested the introduction of the new displacement history variable (see [9])

$$\boldsymbol{\eta}^t(\boldsymbol{x},s) = \boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t-s), \quad s > 0,$$

so that, with reference to (6),

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \mathbb{G}_\infty(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}, t) - \int_0^\infty \mathbb{G}'(\boldsymbol{x}, s)\nabla\boldsymbol{\eta}^t(\boldsymbol{x}, s)\mathrm{d}s.$$

Accordingly, the dynamical problem (7) translates into the system

$$\begin{cases} \partial_{tt}\boldsymbol{u}(\boldsymbol{x}, t) = \nabla \cdot \mathbb{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x}, t) - \nabla\cdot\displaystyle\int_0^\infty \mathbb{G}'(\boldsymbol{x}, s)\nabla\boldsymbol{\eta}^t(\boldsymbol{x}, s)\mathrm{d}s + \boldsymbol{f}(\boldsymbol{x}, t), \\ \partial_t\boldsymbol{\eta}^t(\boldsymbol{x}, s) = \partial_t\boldsymbol{u}(\boldsymbol{x}, t) - \partial_s\boldsymbol{\eta}^t(\boldsymbol{x}, s), \end{cases}$$

which requires the knowledge of the initial data

$$\boldsymbol{u}(\boldsymbol{x}, 0), \quad \partial_t\boldsymbol{u}(\boldsymbol{x}, 0), \quad \boldsymbol{\eta}^0(\boldsymbol{x}, s),$$

where the initial past history $\boldsymbol{\eta}^0(\boldsymbol{x}, s)$ is taken in the space $\mathfrak{H}$, dictated by Ψ_{G}, of all functions $\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{x}, s)$ such that

$$\|\boldsymbol{\eta}\|^2_{\mathfrak{H}} = -\int_{\mathcal{B}}\int_0^\infty \langle\mathbb{G}'(\boldsymbol{x}, s)\nabla\boldsymbol{\eta}(\boldsymbol{x}, s), \nabla\boldsymbol{\eta}(\boldsymbol{x}, s)\rangle\mathrm{d}s\mathrm{d}\boldsymbol{x} < \infty.$$

2.3. The concept of state

Unfortunately, a new difficulty arises in connection with the above energetic approach. Indeed, depending on the form of $\mathbb{G}'$, different (with respect to the almost everywhere equivalence relation) initial past histories $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \mathfrak{H}$ could produce the same solution to the motion problem (7) (clearly, with the same initial data $\boldsymbol{u}(\boldsymbol{x}, 0)$ and $\partial_t\boldsymbol{u}(\boldsymbol{x}, 0)$). This is the case when

$$\int_{\mathcal{B}}\int_0^\infty \mathbb{G}'(\boldsymbol{x}, s+\tau)\nabla\left[\boldsymbol{\eta}_1(\boldsymbol{x}, s) - \boldsymbol{\eta}_2(\boldsymbol{x}, s)\right]\mathrm{d}s\mathrm{d}\boldsymbol{x} = 0, \quad \forall\tau > 0. \tag{8}$$

Remark 2. As a consequence, there is no way to reconstruct the initial past history $\boldsymbol{\eta}^0(\boldsymbol{x}, s)$ of a given material considered at the initial time $t = 0$, neither from the knowledge of the actual state of the system, nor by assuming to know in advance the future dynamics.

NOLL [44] tried to solve the problem by collecting all equivalent histories, in the sense of (8), into the same equivalence class, named *state* of the material with memory. Nevertheless, any two different histories in the same equivalence class satisfy the relation

$$\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathfrak{H}} \neq 0,$$

which implies that the history space $\mathfrak{H}$ is *not* a state space (unless each equivalence class is a singleton) and Ψ_{G} is not a state function, as required by thermodynamics. Further efforts have been made to endow the state space of materials with memory with a suitable "quotient" topology, which is typically generated by an uncountable family of seminorms [13,34]. In this direction, however, there is no hope of recovering a natural norm on the state space. Indeed, the main obstacle consists in handling a space where each element is a set containing an infinite number of functions (histories).

A different and more fruitful line of investigation was devised in [14] (see also [27]), through the introduction of the notion of a *minimal state*. Drawing the inspiration from the equivalence relation (8), the authors called the minimal state of the system at time t a function of the variable $\tau > 0$

$$\begin{aligned}\boldsymbol{\zeta}^t(\boldsymbol{x},\tau) &= -\int_0^\infty \mathbb{G}'(\boldsymbol{x},\tau+s)\,\nabla\boldsymbol{\eta}^t(\boldsymbol{x},s)\mathrm{d}s\\ &= -\int_0^\infty \mathbb{G}'(\boldsymbol{x},\tau+s)\,[\boldsymbol{\varepsilon}(\boldsymbol{x},t)-\boldsymbol{\varepsilon}(\boldsymbol{x},t-s)]\,\mathrm{d}s.\end{aligned}$$

With this position, the stress–strain relation takes the compact form

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathbb{G}_\infty(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x},t) + \boldsymbol{\zeta}^t(\boldsymbol{x},0). \tag{9}$$

The difficulties of the previous approaches are circumvented: the minimal state space is a function space endowed with a natural weighted L^2-norm arising from the free energy functional

$$\Psi_{\mathrm{F}}(\boldsymbol{x},t) = \frac{1}{2}\mathbb{G}_\infty(\boldsymbol{x})|\boldsymbol{\varepsilon}(\boldsymbol{x},t)|^2 - \frac{1}{2}\int_0^\infty \langle(\mathbb{G}')^{-1}(\boldsymbol{x},\tau)\partial_\tau\boldsymbol{\zeta}^t(\boldsymbol{x},\tau),\,\partial_\tau\boldsymbol{\zeta}^t(\boldsymbol{x},\tau)\rangle\mathrm{d}\tau,$$

which involves the minimal state representation.

Remark 3. As a matter of fact, the history and the state frameworks are comparable, and the latter is more general (see Section 12 for details). In particular, as devised in [22] (see also Lemma 7), it can be shown that

$$\Psi_{\mathrm{F}}(\boldsymbol{x},t) \leqq \Psi_{\mathrm{G}}(\boldsymbol{x},t).$$

2.4. The problem of initial conditions

The classical approach to problems with memory requires the knowledge of the past history of $\boldsymbol{u}$ at time $t = 0$, playing the role of an initial datum of the problem. This raises a strong theoretical objection: as mentioned in Remark 2, it is physically impossible to establish the past history of $\boldsymbol{u}$ up to time $-\infty$ from measurements of the material at the actual time, or even assuming the dynamics known for all $t > 0$. On the other hand, in the state formulation one needs to know the *initial* state function

$$\boldsymbol{\zeta}^0(\boldsymbol{x},\tau) = \boldsymbol{\zeta}^t(\boldsymbol{x},\tau)_{|t=0},$$

which, as we will see, is the same as knowing the answer of the stress subject to a constant process in the time interval $(0,\infty)$; namely, the answer in the future. At first glance, this appears even more conceptually ambiguous and technically difficult than recovering the past history of $\boldsymbol{u}$. We will show that this is not so. To this end, let us rewrite equation (7) in the form

$$\partial_{tt}\boldsymbol{u}(\boldsymbol{x},t) = \nabla\cdot\mathbb{G}_0(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t) \\ +\nabla\cdot\int_0^t \mathbb{G}'(\boldsymbol{x},s)\nabla\boldsymbol{u}(\boldsymbol{x},t-s)\mathrm{d}s - \boldsymbol{F}_0(\boldsymbol{x},t) + \boldsymbol{f}(\boldsymbol{x},t), \quad (10)$$

having set

$$\boldsymbol{F}_0(\boldsymbol{x},t) = -\nabla\cdot\int_0^\infty \mathbb{G}'(\boldsymbol{x},t+s)\nabla\boldsymbol{u}(\boldsymbol{x},-s)\mathrm{d}s \\ = \nabla\cdot\Big[\mathbb{G}(\boldsymbol{x},t)\nabla\boldsymbol{u}(\boldsymbol{x},0) - \mathbb{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},0) - \boldsymbol{\zeta}^0(\boldsymbol{x},t)\Big].$$

Under, say, Dirichlet boundary conditions, and having a given assignment of the initial values $\boldsymbol{u}(\boldsymbol{x},0)$ and $\partial_t\boldsymbol{u}(\boldsymbol{x},0)$, the problem is well-posed, whenever $\boldsymbol{F}_0$ is available (that is, whenever $\boldsymbol{\zeta}^0$ is available).

Remark 4. The function $\boldsymbol{F}_0$ is not affected by the choice of the initial data, nor by the presence of the forcing term $\boldsymbol{f}$. Moreover, if the initial past history of $\boldsymbol{u}$ is given, the above relation allows us to reconstruct $\boldsymbol{F}_0$. In this respect, the picture is at least no worse than before.

Remark 5. It is important to point out that, whereas in the previous approach the *whole* past history of $\boldsymbol{u}$ is required, here, in order to solve the equation up to any given time $T > 0$, the values of $\boldsymbol{F}_0(\boldsymbol{x},t)$ are needed *only* for $t \in [0,T]$.

Agreed that the hypothesis of an assigned initial past history of $\boldsymbol{u}$ is inconsistent, we describe an operative method to construct the function $\boldsymbol{F}_0$ by means of direct measurement, moving from the observations that materials with memory (such as polymers) are built by means of specific industrial procedures. Assume that a given material, after its artificial generation at time $t = 0$, undergoes a process in such a way that

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},0), \quad \forall t > 0.$$

In which case, the equality

$$\boldsymbol{\zeta}^t(\boldsymbol{x},0) = \boldsymbol{\zeta}^0(\boldsymbol{x},t)$$

holds. Indeed,

$$\boldsymbol{\zeta}^t(\boldsymbol{x},0) = -\int_0^\infty \mathbb{G}'(\boldsymbol{x},s)\,[\boldsymbol{\varepsilon}(\boldsymbol{x},0) - \boldsymbol{\varepsilon}(\boldsymbol{x},t-s)]\,\mathrm{d}s \\ = -\int_t^\infty \mathbb{G}'(\boldsymbol{x},s)\,[\boldsymbol{\varepsilon}(\boldsymbol{x},0) - \boldsymbol{\varepsilon}(\boldsymbol{x},t-s)]\,\mathrm{d}s \\ = -\int_0^\infty \mathbb{G}'(\boldsymbol{x},t+s)\,[\boldsymbol{\varepsilon}(\boldsymbol{x},0) - \boldsymbol{\varepsilon}(\boldsymbol{x},-s)]\,\mathrm{d}s = \boldsymbol{\zeta}^0(\boldsymbol{x},t).$$

Thus, (9) entails the relation

$$\boldsymbol{\zeta}^0(\boldsymbol{x},t) = \boldsymbol{\sigma}(\boldsymbol{x},t) - \mathbb{G}_\infty(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x},0),$$

meaning that $\boldsymbol{\zeta}^0$, and in turn $\boldsymbol{F}_0$, can be obtained by measuring the stress $\boldsymbol{\sigma}(\boldsymbol{x},t)$, for all times $t > 0$, of a process frozen at the displacement field $\boldsymbol{u}(\boldsymbol{x},0)$.

Next, we consider the equation of motion (10), relative to a material generated by the same procedure, but delayed by a time $t_{\mathrm{d}} > 0$. For this equation, the corresponding function $\boldsymbol{F}_0(\boldsymbol{x},t)$ is now available.

Remark 6. As a matter of fact, $\boldsymbol{F}_0(\boldsymbol{x},t)$ is not *simultaneously* available for all $t > 0$. However, since the first process (which constructs $\boldsymbol{F}_0$) keeps going, at any given time $T > 0$, referred to the initial time $t = 0$ of the problem under consideration, we have the explicit expression of $\boldsymbol{F}_0(\boldsymbol{x},t)$ for all $t \in [0, t_{\mathrm{d}} + T]$, which is even more than needed to solve (10) on $[0,T]$.

3. An abstract equation with memory

We now turn to the mathematical aspects of the problem, developing the state approach for an abstract model equation.

Let H be a separable real Hilbert space, and let A be a selfadjoint strictly positive linear operator on H with compact inverse, defined on a dense domain $\mathcal{D}(A) \subset H$. For $t > 0$, we consider the abstract homogeneous linear differential equation with memory of the second order in time

$$\partial_{tt}u(t) + A\left[\alpha u(t) - \int_0^\ell \mu(s)u(t-s)\mathrm{d}s\right] = 0. \tag{11}$$

Here, $\alpha > 0$, $\ell \in (0,\infty]$ and the *memory kernel*

$$\mu : \Omega = (0,\ell) \to (0,\infty)$$

is a (strictly positive) nonincreasing summable function of total mass

$$\int_0^\ell \mu(s)\mathrm{d}s \in (0,\alpha),$$

satisfying the condition (automatically fulfilled if $\ell = \infty$)

$$\lim_{s\to\ell} \mu(s) = 0.$$

The dissipativity of the system is entirely contained in the convolution term, which accounts for the *delay* effects: precisely, finite delay if $\ell < \infty$, infinite delay if $\ell = \infty$. The equation is supplemented with the initial conditions given at the time $t = 0$

$$\begin{cases} u(0) = u_0, \\ \partial_t u(0) = v_0, \\ u(-s)_{|s\in\Omega} = \phi_0(s), \end{cases} \tag{12}$$

where u_0, v_0 and the function ϕ_0, defined on Ω, are prescribed data.

Remark 7. A concrete realization of the abstract equation (11) is obtained by setting $\Omega = \mathbb{R}^+$, $H = [L^2(\mathcal{B})]^N$, where $\mathcal{B} \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\mathcal{B}$, and

$$A = -\Delta \quad \text{with} \quad \mathcal{D}(A) = [H^2(\mathcal{B})]^N \cap [H_0^1(\mathcal{B})]^N.$$

In that case, setting

$$\alpha - \int_0^s \mu(\sigma)\mathrm{d}\sigma = G(s)$$

and

$$u(t) = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \mathcal{B},$$

the equation reads

$$\begin{cases} \partial_{tt}\boldsymbol{u}(\boldsymbol{x}, t) - \Delta\left[G(0)\boldsymbol{u}(\boldsymbol{x}, t) + \displaystyle\int_0^\infty G'(s)\boldsymbol{u}(\boldsymbol{x}, t - s)\mathrm{d}s\right] = 0, \\ \boldsymbol{u}(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\mathcal{B}} = 0, \end{cases}$$

and rules the evolution of the relative displacement field $\boldsymbol{u}$ in a homogeneous isotropic linearly viscoelastic solid occupying a volume $\mathcal{B}$ at rest [24,47].

Putting $\mu(s) = 0$ if $s > \ell$, and defining

$$F_0(t) = \int_0^\ell \mu(t + s)\phi_0(s)\mathrm{d}s, \tag{13}$$

equation (11) takes the form

$$\partial_{tt}u(t) + A\left[\alpha u(t) - \int_0^t \mu(s)u(t - s)\mathrm{d}s - F_0(t)\right] = 0. \tag{14}$$

Introducing the Hilbert space

$$V = \mathcal{D}(A^{1/2}),$$

with the standard inner product and norm

$$\langle u_1, u_2\rangle_V = \langle A^{1/2}u_1, A^{1/2}u_2\rangle_H, \quad \|u\|_V = \|A^{1/2}u\|_H,$$

we stipulate the following definition of (weak) solution.

Definition 1. Let $u_0 \in V$, $v_0 \in H$ and $\phi_0 : \Omega \to V$ be such that the corresponding function F_0 given by (13) fulfills

$$F_0 \in L^1_{\mathrm{loc}}([0, \infty); V).$$

A function

$$u \in C([0, \infty), V) \cap C^1([0, \infty), H)$$

is said to be a solution to the Cauchy problem (11)–(12) if

$$u(0) = u_0, \quad \partial_t u(0) = v_0,$$

and the equality

$$\langle \partial_{tt} u(t), w \rangle + \alpha \langle u(t), w \rangle_V - \int_0^t \mu(s) \langle u(t-s), w \rangle_V \mathrm{d}s - \langle F_0(t), w \rangle_V = 0$$

holds for every $w \in V$ and almost every $t > 0$, where $\langle \cdot, \cdot \rangle$ denotes duality.

4. Notation and assumptions

4.1. Notation

The symbols $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$ stand for the inner product and the norm on a generic Hilbert space X, respectively. In particular, for the spaces H and V, we have the well-known norm relations

$$\|u\|_V = \|A^{1/2} u\|_H \geqq \sqrt{\lambda_1}\, \|u\|_H, \quad \forall u \in V,$$

where $\lambda_1 > 0$ is the first eigenvalue of A. We denote by

$$V^* = \mathcal{D}(A^{-1/2})$$

the dual space of V, and by $\langle \cdot, \cdot \rangle$ the duality product between V^* and V. We also recall the equality

$$\|w\|_{V^*} = \|A^{-1/2} w\|_H, \quad \forall w \in V^*.$$

For a nonnegative (measurable) function ω on $\Omega = (0, \ell)$ and for $p = 1, 2$, we define the weighted L^p-space of X-valued functions

$$L^p_\omega(\Omega; X) = \left\{ \psi : \Omega \to X \; : \; \int_0^\ell \omega(s) \|\psi(s)\|^p_X \mathrm{d}s < \infty \right\},$$

normed by

$$\|\psi\|_{L^p_\omega(\Omega; X)} = \left(\int_0^\ell \omega(s) \|\psi(s)\|^p_X \mathrm{d}s \right)^{1/p}.$$

If $p = 2$, this is a Hilbert space endowed with the inner product

$$\langle \psi_1, \psi_2 \rangle_{L^2_\omega(\Omega; X)} = \int_0^\ell \omega(s) \langle \psi_1(s), \psi_2(s) \rangle_X \mathrm{d}s.$$

Finally, given a generic function $\psi : \Omega \to X$, we denote by $D\psi$ its distributional derivative.

A word of warning. In order to simplify the notation, if ψ is any function on Ω, we agree to interpret $\psi(s) = 0$ whenever $s \notin \Omega$ (in particular, if $\ell < \infty$, whenever $s > \ell$).

4.2. Assumptions on the memory kernel

As anticipated above, $\mu : \Omega \to (0, \infty)$ is nonincreasing and summable. Setting

$$M(s) = \int_s^\ell \mu(\sigma)\mathrm{d}\sigma = \int_0^\ell \mu(s+\sigma)\mathrm{d}\sigma,$$

we require that $M(0) < \alpha$.

Remark 8. Note that $M(s) > 0$ for every $s \in [0, \ell)$, and $\lim_{s\to\ell} M(s) = 0$.

For simplicity, we will take

$$\alpha - M(0) = 1. \tag{15}$$

In addition, we suppose that μ is absolutely continuous on every closed interval contained in Ω. In particular, μ is differentiable almost everywhere in Ω and $\mu' \leqq 0$. Finally, μ is assumed to be continuous at $s = \ell$, with $\mu(\ell) = 0$, if $\ell < \infty$. Conversely, if $\ell = \infty$, as μ is nonincreasing and summable, we automatically have that $\mu(s) \to 0$ as $s \to \infty$. In fact, we could consider more general kernels as well, allowing μ to have a finite or even a countable number of jumps (cf. [4,45]). However, in this work, we will restrict ourselves to the continuous case, in order not to introduce further technical difficulties.

5. The history approach

An alternative way to look at the equation is to work in the so-called history space framework, devised by DAFERMOS in his pioneering paper [9], by considering the *history* variable

$$\eta^t(s) = u(t) - u(t-s), \quad t \geqq 0, s \in \Omega,$$

which, formally, fulfills the problem

$$\begin{cases} \partial_t \eta^t(s) = -\partial_s \eta^t(s) + \partial_t u(t), \\ \eta^t(0) = 0, \\ \eta^0(s) = u_0 - \phi_0(s). \end{cases}$$

To set the idea in a precise context, let us introduce the *history space*

$$\mathcal{M} = L^2_\mu(\Omega; V),$$

along with the strongly continuous semigroup $R(t)$ of right translations on $\mathcal{M}$, namely,

$$(R(t)\eta)(s) = \begin{cases} 0 & 0 < s \leqq t, \\ \eta(s-t) & s > t, \end{cases}$$

whose infinitesimal generator is the linear operator T defined as in (cf. [35,45])

$$T\eta = -D\eta, \quad \mathcal{D}(T) = \{\eta \in \mathcal{M} \,:\, D\eta \in \mathcal{M},\ \eta(0) = 0\},$$

where $\eta(0) = \lim_{s\to 0} \eta(s)$ in V. Then, recalling (15), equation (11) translates into the differential system in the two variables $u = u(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} \partial_{tt} u(t) + A\left[u(t) + \int_0^\ell \mu(s)\eta^t(s)\mathrm{d}s\right] = 0, \\ \partial_t \eta^t = T\eta^t + \partial_t u(t). \end{cases} \tag{16}$$

Accordingly, the initial conditions (12) turn into

$$\begin{cases} u(0) = u_0, \\ \partial_t u(0) = v_0, \\ \eta^0 = \eta_0, \end{cases} \tag{17}$$

where

$$\eta_0(s) = u_0 - \phi_0(s). \tag{18}$$

Introducing the *extended history space*

$$\mathfrak{M} = V \times H \times \mathcal{M},$$

normed by

$$\|(u, v, \eta)\|^2_{\mathfrak{M}} = \|u\|^2_V + \|v\|^2_H + \|\eta\|^2_{\mathcal{M}},$$

problem (16)–(17) generates a contraction semigroup $\Sigma(t)$ on $\mathfrak{M}$ (see [21,35,45]), such that, for every $(u_0, v_0, \eta_0) \in \mathfrak{M}$,

$$\Sigma(t)(u_0, v_0, \eta_0) = (u(t), \partial_t u(t), \eta^t).$$

Moreover, η^t has the explicit representation

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \leqq t, \\ \eta_0(s-t) + u(t) - u_0 & s > t. \end{cases} \tag{19}$$

Concerning the relation between (16)–(17) and the original problem (11)–(12), the following result holds [35].

Proposition 1. *Let* $(u_0, v_0, \eta_0) \in \mathfrak{M}$. *Then, the first component* $u(t)$ *of the solution* $\Sigma(t)(u_0, v_0, \eta_0)$ *solves* (11)–(12) *with*

$$F_0(t) = \int_0^\ell \mu(t+s)\,\{u_0 - \eta_0(s)\}\,\mathrm{d}s.$$

It is easy to see that $\eta_0 \in \mathcal{M}$ implies that $F_0 \in L^\infty(\mathbb{R}^+; V)$.

6. The state approach

An essential drawback of the history approach is that, for given initial data u_0 and v_0, two different initial histories may lead to the same solution $u(t)$, for $t \geqq 0$. Somehow, this is not surprising, since what really enters in the definition of a solution to (11)–(12), rather than ϕ_0 (which, by (18), is related to the initial history η_0), is the function F_0, defined in (13) and appearing in equation (14). Thus, from the dynamical viewpoint, two initial data, ϕ_{01} and ϕ_{02}, should be considered by all means *equivalent* when the corresponding function F_{01} and F_{02} coincide, due to the impossibility of distinguishing their effects in the future. On this basis, it seems natural to devise a scheme where, rather than ϕ_0, the function F_0 appears as the *actual* initial datum accounting for the past history of u.

In order to translate this insight into a consistent mathematical theory, it is quite helpful to see first what happens at a *formal level*. To this aim, for $t \geqq 0$ and $\tau \in \Omega$, we introduce the (minimal) *state* variable

$$\zeta^t(\tau) = \int_0^\ell \mu(\tau + s)\,\{u(t) - u(t-s)\}\,\mathrm{d}s,$$

which fulfills the problem

$$\begin{cases} \partial_t \zeta^t(\tau) = \partial_\tau \zeta^t(\tau) + M(\tau)\partial_t u(t), \\ \zeta^t(\ell) = 0, \\ \zeta^0(\tau) = \zeta_0(\tau), \end{cases}$$

having set

$$\zeta_0(\tau) = \int_0^\ell \mu(\tau + s)\{u_0 - \phi_0(s)\}\mathrm{d}s = M(\tau)u_0 - F_0(\tau).$$

Accordingly, in light of (15), equation (11) takes the form

$$\partial_{tt} u(t) + A\left[u(t) + \zeta^t(0)\right] = 0,$$

where

$$\zeta^t(0) = \lim_{\tau \to 0} \zeta^t(\tau) = \int_0^\ell \mu(s)\,\{u(t) - u(t-s)\}\,\mathrm{d}s.$$

Rather than ζ^t, it seems more convenient to consider as a state the new variable

$$\xi^t(\tau) = -\partial_\tau \zeta^t(\tau) = -\int_0^\ell \mu'(\tau + s)\,\{u(t) - u(t-s)\}\,\mathrm{d}s,$$

which, in turn, fulfills the problem

$$\begin{cases} \partial_t \xi^t(\tau) = \partial_\tau \xi^t(\tau) + \mu(\tau)\partial_t u(t), \\ \xi^0(\tau) = \xi_0(\tau), \end{cases}$$

where the initial datum ξ_0 reads

$$\xi_0(\tau) = -\int_0^\ell \mu'(\tau+s)\{u_0 - \phi_0(s)\}\,\mathrm{d}s = \mu(\tau)u_0 + \int_0^\ell \mu'(\tau+s)\phi_0(s)\mathrm{d}s.$$

If $\ell < \infty$, we also have the "boundary" condition

$$\xi^t(\ell) = 0,$$

which comes from the very definition of ξ^t. Since $\zeta^t(\ell) = 0$, we find the relation

$$\int_{\tau_0}^\ell \xi^t(\tau)\mathrm{d}\tau = \zeta^t(\tau_0), \quad \forall \tau_0 \in \Omega. \tag{20}$$

In particular, in the limit $\tau_0 \to 0$,

$$\int_0^\ell \xi^t(\tau)\mathrm{d}\tau = \zeta^t(0).$$

Therefore, (11)–(12) is (formally) translated into the system

$$\begin{cases} \partial_{tt}u(t) + A\left[u(t) + \displaystyle\int_0^\ell \xi^t(\tau)\mathrm{d}\tau\right] = 0, \\ \partial_t \xi^t(\tau) = \partial_\tau \xi^t(\tau) + \mu(\tau)\partial_t u(t), \end{cases} \tag{21}$$

with initial conditions

$$\begin{cases} u(0) = u_0, \\ \partial_t u(0) = v_0, \\ \xi^0(\tau) = \xi_0(\tau). \end{cases} \tag{22}$$

Remark 9. Observe that the nonlocal character of (11) is not present in (21) any longer, since it is hidden in the new variable ξ^t.

At this point, to complete the project, two major issues need to be addressed:

- Firstly, we have to write (21)–(22) as a differential equation in a suitable functional space, providing an existence and uniqueness result.
- Secondly, we have to establish a correspondence (not only formal) between the solutions to (21)–(22) and the solutions to the original problem (11)–(12).

7. The state space

Aiming to set (21)–(22) into a proper functional framework, the first step is interpreting in a correct way the derivative ∂_τ appearing in the second equation of (21). To do that, introducing the new memory kernel

$$\nu(\tau) = \frac{1}{\mu(\tau)} : \Omega \to [0, \infty),$$

we define the *state space*

$$\mathcal{V} = L^2_\nu(\Omega; V),$$

whose norm is related to the free energy functional Ψ_F discussed in Section 2.3. Due to the assumptions on μ, putting

$$\nu(0) = \lim_{\tau\to 0} \nu(\tau),$$

the function ν is nondecreasing and continuous on $\overline{\Omega}$ (the closure of Ω), with nonnegative derivative (defined almost everywhere)

$$\nu'(\tau) = -\frac{\mu'(\tau)}{[\mu(\tau)]^2},$$

and fulfills

$$\lim_{\tau\to\ell} \nu(\tau) = \infty.$$

The following simple lemma will be needed in the sequel.

Lemma 1. *Let $\xi \in \mathcal{V}$. Then, $\xi \in L^1(\Omega; V)$ and*

$$\int_\tau^\ell \|\xi(s)\|_V \mathrm{d}s \leqq \sqrt{M(\tau)}\, \|\xi\|_{\mathcal{V}}, \quad \forall \tau \in \Omega.$$

As a byproduct, the map $t \mapsto \int_t^\ell \xi(\tau)\mathrm{d}\tau$ belongs to $C([0,\infty), V)$ and vanishes at infinity.

Proof. Using the Hölder inequality,

$$\begin{aligned}\int_\tau^\ell \|\xi(s)\|_V \mathrm{d}s &= \int_\tau^\ell \sqrt{\mu(s)}\,\sqrt{\nu(s)}\, \|\xi(s)\|_V \mathrm{d}s \\ &\leqq \left(\int_\tau^\ell \mu(s)\mathrm{d}s\right)^{1/2} \left(\int_\tau^\ell \nu(s)\|\xi(s)\|_V^2 \mathrm{d}s\right)^{1/2} \\ &\leqq \sqrt{M(\tau)}\, \|\xi\|_{\mathcal{V}},\end{aligned}$$

as claimed. □

Next, we consider the strongly continuous semigroup $L(t)$ of left translations on $\mathcal{V}$, given by

$$(L(t)\xi)(\tau) = \xi(t+\tau).$$

It is standard matter to verify that the infinitesimal generator of $L(t)$ is the linear operator P on $\mathcal{V}$ with domain

$$\mathcal{D}(P) = \{\xi \in \mathcal{V} \,:\, D\xi \in \mathcal{V},\ \xi(\ell) = 0\},$$

where $\xi(\ell) = \lim_{\tau\to\ell} \xi(\tau)$ in V, acting as

$$P\xi = D\xi, \quad \forall \xi \in \mathcal{D}(P).$$

Remark 10. From Lemma 1, we infer the implication

$$\xi, D\xi \in \mathcal{V} \quad \Rightarrow \quad \xi \in H^1(\Omega; V) \subset C(\overline{\Omega}, V).$$

In particular, the condition $\xi(\ell) = 0$ is automatically satisfied whenever $\ell = \infty$.

Lemma 2. *For every $\xi \in \mathcal{D}(P)$, we have the equality*

$$2\langle P\xi, \xi\rangle_{\mathcal{V}} = -\nu(0)\|\xi(0)\|_V^2 - \int_0^\ell \nu'(\tau)\|\xi(\tau)\|_V^2 \mathrm{d}\tau \leqq 0. \tag{23}$$

Proof. We begin to prove the existence of a sequence $\ell_n \uparrow \ell$ such that

$$\nu(\ell_n)\|\xi(\ell_n)\|_V^2 \to 0.$$

If $\ell = \infty$, this is a direct consequence of the summability of $\nu\|\xi\|_V^2$. Conversely, if $\ell < \infty$, for every $\tau < \ell$ we have (recalling that $\xi(\ell) = 0$)

$$\nu(\tau)\|\xi(\tau)\|_V^2 \leqq \left(\int_\tau^\ell \sqrt{\nu(s)}\, \|D\xi(s)\|_V \mathrm{d}s\right)^2 \leqq (\ell - \tau)\|D\xi\|_{\mathcal{V}}^2.$$

Then,

$$\begin{aligned} 2\langle P\xi, \xi\rangle_{\mathcal{V}} &= \lim_{n\to\infty} \int_0^{\ell_n} \nu(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\|\xi(\tau)\|_V^2 \mathrm{d}\tau \\ &= -\nu(0)\|\xi(0)\|_V^2 - \lim_{n\to\infty} \int_0^{\ell_n} \nu'(\tau)\|\xi(\tau)\|_V^2 \mathrm{d}\tau, \end{aligned}$$

and since the limit exists finite, we obtain (23). □

8. The semigroup in the extended state space

We are now in a position to formulate (21)–(22) as an abstract evolution equation on a suitable Hilbert space. To this end, we define the *extended state space*

$$\mathfrak{A} = V \times H \times \mathcal{V},$$

normed by

$$\|(u, v, \xi)\|_{\mathfrak{A}}^2 = \|u\|_V^2 + \|v\|_H^2 + \|\xi\|_{\mathcal{V}}^2,$$

and the linear operator $\mathbb{A}$ on $\mathfrak{A}$, with domain

$$\mathcal{D}(\mathbb{A}) = \left\{(u, v, \xi) \in \mathfrak{A} \,:\, v \in V,\ u + \textstyle\int_0^\ell \xi(\tau)\mathrm{d}\tau \in \mathcal{D}(A),\ \xi \in \mathcal{D}(P)\right\},$$

acting as

$$\mathbb{A}(u, v, \xi) = \left(v, -A\left[u + \textstyle\int_0^\ell \xi(\tau)\mathrm{d}\tau\right], P\xi + \mu v\right).$$

Introducing the 3-component vectors

$$Z(t) = (u(t), v(t), \xi^t) \quad \text{and} \quad z = (u_0, v_0, \xi_0) \in \mathfrak{A},$$

we view (21)–(22) as the Cauchy problem in $\mathfrak{A}$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} Z(t) = \mathbb{A} Z(t), \\ Z(0) = z. \end{cases} \tag{24}$$

The following result establishes the existence and uniqueness of a (mild) solution

$$Z \in C([0, \infty), \mathfrak{A}).$$

Theorem 1. *Problem* (24) *generates a contraction semigroup* $S(t) = \mathrm{e}^{t\mathbb{A}}$ *on* $\mathfrak{A}$ *such that*

$$Z(t) = S(t)z, \quad \forall t \geqq 0.$$

Moreover, the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|S(t)z\|^2_{\mathfrak{A}} = -\nu(0)\|\xi^t(0)\|^2_V - \int_0^\ell \nu'(\tau)\|\xi^t(\tau)\|^2_V \mathrm{d}\tau \tag{25}$$

holds for every $z \in \mathcal{D}(\mathbb{A})$.

Proof. On account of the classical Lumer–Phillips theorem [46], we know that $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup on $\mathfrak{A}$ provided that

(i) the inequality $\langle \mathbb{A}z, z\rangle_{\mathfrak{A}} \leqq 0$ holds for every $z \in \mathcal{D}(\mathbb{A})$; and
(ii) the map $\mathbb{I} - \mathbb{A} : \mathcal{D}(\mathbb{A}) \to \mathfrak{A}$ is onto.

Concerning point (i), from (23) we see at once that

$$\langle \mathbb{A}z, z\rangle_{\mathfrak{A}} = \langle P\xi, \xi\rangle_{\mathcal{V}} \leqq 0,$$

for every $z = (u, v, \xi) \in \mathcal{D}(\mathbb{A})$.

In order to prove (ii), let $z_\star = (u_\star, v_\star, \xi_\star) \in \mathfrak{A}$ be given. We look for a solution $z = (u, v, \xi) \in \mathcal{D}(\mathbb{A})$ to the equation

$$(\mathbb{I} - \mathbb{A})z = z_\star,$$

which, written in components, reads

$$\begin{cases} u - v = u_\star, \\ v + A\left[u + \displaystyle\int_0^\ell \xi(\tau)\mathrm{d}\tau\right] = v_\star, \\ \xi(\tau) - D\xi(\tau) - \mu(\tau)v = \xi_\star(\tau). \end{cases} \tag{26}$$

Given a function g on Ω (extended on the whole real line by setting $g(s) = 0$ if $s \notin \Omega$) and denoting

$$\mathcal{E}(s) = \mathrm{e}^s \chi_{(-\infty,0]}(s),$$

we consider the convolution product in $\mathbb{R}$ of $\mathcal{E}$ and g at the point $\tau \in \Omega$

$$(\mathcal{E} * g)(\tau) = \int_{\mathbb{R}} \mathcal{E}(\tau - s) g(s) \mathrm{d}s = \int_{\tau}^{\ell} \mathrm{e}^{\tau - s} g(s) \mathrm{d}s.$$

It is well known (see, for example, [38]) that

$$g \in L^2(\Omega) \quad \Rightarrow \quad \mathcal{E} * g \in L^2(\Omega)$$

and

$$\|\mathcal{E} * g\|_{L^2(\Omega)} \leqq \|g\|_{L^2(\Omega)}.$$

For any fixed $v \in V$, we define the function

$$\xi(\tau) = v(\mathcal{E} * \mu)(\tau) + (\mathcal{E} * \xi_\star)(\tau). \tag{27}$$

We begin to show that $\xi \in \mathcal{V}$. Indeed,

$$\begin{aligned}
\int_0^\ell \nu(\tau) \|\xi(\tau)\|_V^2 \mathrm{d}\tau &\leqq 2\|v\|_V^2 \int_0^\ell \nu(\tau) |(\mathcal{E} * \mu)(\tau)|^2 \mathrm{d}\tau \\
&\quad + 2 \int_0^\ell \nu(\tau) \|(\mathcal{E} * \xi_\star)(\tau)\|_V^2 \mathrm{d}\tau \\
&\leqq 2\|v\|_V^2 \int_0^\ell \left(\int_\tau^\ell \mathrm{e}^{\tau - s} \sqrt{\nu(\tau)}\, \mu(s) \mathrm{d}s \right)^2 \mathrm{d}\tau \\
&\quad + 2 \int_0^\ell \left(\int_\tau^\ell \mathrm{e}^{\tau - s} \sqrt{\nu(\tau)}\, \|\xi_\star(s)\|_V \mathrm{d}s \right)^2 \mathrm{d}\tau \\
&\leqq 2\|v\|_V^2 \|\mathcal{E} * \sqrt{\mu}\,\|_{L^2(\Omega)}^2 + 2\|\mathcal{E} * (\sqrt{\nu}\, \|\xi_\star\|_V)\|_{L^2(\Omega)}^2 \\
&\leqq 2M(0)\|v\|_V^2 + 2\|\xi_\star\|_{\mathcal{V}}^2.
\end{aligned}$$

Moreover,

$$\|\xi(\tau)\|_V \leqq \int_\tau^\ell \mathrm{e}^{\tau - s} \left\{ \mu(s)\|v\|_V + \|\xi_\star(s)\|_V \right\} \mathrm{d}s.$$

Since (cf. Lemma 1)

$$s \mapsto \mu(s)\|v\|_V + \|\xi_\star(s)\|_V \in L^1(\Omega),$$

we conclude that

$$\lim_{\tau \to \ell} \|\xi(\tau)\|_V = 0.$$

Taking the distributional derivative in both sides of (27), it is apparent that such a ξ satisfies the third equation of (26). Moreover, by comparison, it is readily seen that $D\xi \in \mathcal{V}$. In summary, $\xi \in \mathcal{D}(P)$ (for any given $v \in V$) and fulfills the third equation of (26). At this point, we plug ξ into the second equation of (26), reading u from the first one. Noting that

$$\gamma = 1 + \int_0^\ell \left(\int_\tau^\ell \mathrm{e}^{\tau - s} \mu(s) \mathrm{d}s \right) \mathrm{d}\tau = 1 + \int_0^\ell \mu(s) \left\{ 1 - \mathrm{e}^{-s} \right\} \mathrm{d}s > 0,$$

we obtain

$$v + \gamma A v = v_\star - A(u_\star + w), \tag{28}$$

having set

$$w = \int_0^\ell (\mathcal{E} * \xi_\star)(\tau) \mathrm{d}\tau.$$

The elliptic equation (28) admits a (unique) solution $v \in V$, provided that its right-hand side belongs to V^*, which immediately follows from $w \in V$. Indeed, using the Hölder inequality,

$$\begin{aligned} \|w\|_V &\leqq \int_0^\ell \|(\mathcal{E} * \xi_\star)(\tau)\|_V \mathrm{d}\tau \\ &\leqq \int_0^\ell (\mathcal{E} * \|\xi_\star\|_V)(\tau) \mathrm{d}\tau \\ &\leqq \int_0^\ell \sqrt{\mu(\tau)} \left(\mathcal{E} * (\sqrt{\nu}\, \|\xi_\star\|_V) \right) (\tau) \mathrm{d}\tau \\ &\leqq \sqrt{M(0)}\, \|\xi_\star\|_{\mathcal{V}}. \end{aligned}$$

Finally, by comparison, we learn that

$$u + \int_0^\ell \xi(\tau) \mathrm{d}\tau = A^{-1}(v_\star - v) \in \mathcal{D}(A).$$

This completes the proof of point (ii).

We now appeal to a general result of the theory of linear semigroups [46]. Namely, if $z \in \mathcal{D}(\mathbb{A})$, then

$$S(t)z \in \mathcal{D}(\mathbb{A}), \quad \forall t \geqq 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \|S(t)z\|_{\mathfrak{A}}^2 = 2\langle \mathbb{A} S(t)z, S(t)z \rangle_{\mathfrak{A}}.$$

On the other hand, since

$$\langle \mathbb{A} S(t)z, S(t)z \rangle_{\mathfrak{A}} = \langle P\xi^t, \xi^t \rangle_{\mathcal{V}},$$

the energy equality (25) follows from Lemma 2. □

Corollary 1. *The third component ξ^t of the solution $S(t)z$ (the state) has the explicit representation formula*

$$\xi^t(\tau) = \xi_0(t+\tau) + \mu(\tau)u(t) - \mu(t+\tau)u_0 + \int_0^t \mu'(\tau+s)u(t-s)\mathrm{d}s, \quad (29)$$

which is valid for every $z = (u_0, v_0, \xi_0) \in \mathfrak{A}$.

Proof. Assume first that z lies in a more regular space, so that

$$\partial_t u \in L^1_{\mathrm{loc}}([0,\infty); V).$$

Then, ξ^t satisfies the nonhomogeneous Cauchy problem in $\mathcal{V}$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\xi^t = P\xi^t + \mu\partial_t u(t), \\ \xi^0 = \xi_0. \end{cases}$$

Applying the variation of constants to the semigroup $L(t) = \mathrm{e}^{tP}$ (see [46]), we obtain

$$\xi^t(\tau) = \xi_0(t+\tau) + \int_0^t \mu(t+\tau-s)\partial_t u(s)\mathrm{d}s.$$

The desired conclusion (29) is drawn integrating by parts. Using a standard approximation argument, the representation formula holds for all $z \in \mathfrak{A}$. □

Remark 11. In the above corollary, the continuity properties of μ play a crucial role when integration by parts occurs. Nonetheless, if μ has jumps, it is still possible to find a representation formula, which contains extra terms accounting for the jumps of μ.

Remark 12. The state variable ξ^t is *minimal* in the following sense: if the triplet $(u(t), \partial_t u(t), \xi^t)$ is a solution to (24) with $u(t) = 0$ for every $t \geqq 0$, then ξ^t is identically zero. Indeed, on account of (24) and (29),

$$\xi^t(\tau) = \xi_0(t+\tau), \quad \forall t \geqq 0,$$

and

$$0 = \int_0^\ell \xi^t(\tau)\mathrm{d}\tau = \int_0^\ell \xi_0(t+\tau)\mathrm{d}\tau = \int_t^\ell \xi_0(\tau)\mathrm{d}\tau, \quad \forall t \geqq 0,$$

which implies that $\xi_0 = 0$ and, in turn, $\xi^t = 0$.

9. Exponential stability

9.1. Statement of the result

We prove the exponential stability of the semigroup $S(t)$ on $\mathfrak{A}$, assuming in addition that μ satisfies

$$\mu'(s) + \delta\mu(s) \leqq 0, \tag{30}$$

for some $\delta > 0$ and almost every $s \in \Omega$.

Theorem 2. *Let μ satisfy (30). Then, there exist $K > 1$ and $\omega > 0$ such that*

$$\|S(t)z\|_{\mathfrak{A}} \leqq K\|z\|_{\mathfrak{A}}\,\mathrm{e}^{-\omega t}, \tag{31}$$

for every $z \in \mathfrak{A}$.

Before proceeding to the proof, some comments are in order. Condition (30) is quite popular in the literature; indeed, it has been employed by several authors to prove the exponential decay of semigroups related to various equations with memory in the history space framework (for example, in connection with the present equation, [21,28,42,43]). On the other hand, the recent paper [45] shows that the exponential decay for such semigroups can be obtained under the weaker condition

$$\mu(\sigma + s) \leqq C\mathrm{e}^{-\delta\sigma}\mu(s), \tag{32}$$

for some $C \geqq 1$, every $\sigma \geqq 0$ and almost every $s \in \Omega$, assuming that the set where $\mu' = 0$ is not too large (in a suitable sense). It is apparent that (32) and (30) coincide if $C = 1$. However, if $C > 1$, then (32) is much more general. For instance, it is always satisfied when $\ell < \infty$ (provided that μ fulfills the general assumptions of Section 4). On the contrary, (30) does not allow μ to have flat zones, or even horizontal inflection points. As shown in [4], condition (32) is actually *necessary* for the exponential decay in the history space framework. This is true also in the state framework.

Proposition 2. *Assume that the semigroup $S(t)$ on $\mathfrak{A}$ is exponentially stable. Then, μ fulfills (32).*

We omit the proof of the proposition, which can be obtained along the lines of [4], showing that the exponential stability of $S(t)$ implies the exponential stability of the left-translation semigroup $L(t)$ on $\mathcal{V}$.

We finally point out that, although we stated the theorem using (30), the result is still true under the more general hypotheses of [45] (but a much more complicated proof is needed).

9.2. Proof of Theorem 2

Appealing to the continuity of $S(t)$, it is enough to prove inequality (31) for all $z \in \mathcal{D}(\mathbb{A})$. Fix then

$$z = (u_0, v_0, \xi_0) \in \mathcal{D}(\mathbb{A}),$$

and denote

$$S(t)z = (u(t), v(t), \xi^t) \in \mathcal{D}(\mathbb{A}).$$

Introducing the energy

$$E(t) = \frac{1}{2}\|S(t)z\|^2_{\mathfrak{A}},$$

and writing (30) in terms of ν as

$$\nu'(\tau) \geqq \delta\nu(\tau),$$

on account of (25) we derive the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqq -\frac{1}{2}\int_0^\ell \nu'(\tau)\|\xi^t(\tau)\|^2_V \mathrm{d}\tau \leqq -\frac{\delta}{2}\|\xi^t\|^2_{\mathcal{V}}. \tag{33}$$

For an arbitrary $\beta \in \Omega$, we define the (absolutely continuous) function $\rho : \Omega \to [0, 1]$

$$\rho(\tau) = \begin{cases} \beta^{-1}\tau & \tau \leqq \beta, \\ 1 & \tau > \beta, \end{cases}$$

and we consider the further functionals

$$\Phi_1(t) = -\int_0^\ell \rho(\tau)\langle v(t), \xi^t(\tau)\rangle_H \mathrm{d}\tau,$$
$$\Phi_2(t) = \langle v(t), u(t)\rangle_H.$$

Recalling Lemma 1,

$$\int_0^\ell \|\xi^t(\tau)\|_V \mathrm{d}\tau \leqq \sqrt{M(0)}\,\|\xi^t\|_{\mathcal{V}}. \tag{34}$$

Thus, from the continuous embedding $V \subset H$,

$$|\Phi_\imath(t)| \leqq c_0 E(t), \quad \imath = 1, 2, \tag{35}$$

for some $c_0 > 0$ independent of the choice of $z \in \mathcal{D}(\mathbb{A})$.

Lemma 3. *There is* $c_1 > 0$ *independent of* z *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t) \leqq M(\beta)\left\{\frac{1}{12}\|u(t)\|^2_V - \frac{1}{2}\|v(t)\|^2_H + c_1\|\xi^t\|^2_{\mathcal{V}}\right\}. \tag{36}$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 = -\int_0^\ell \rho(\tau)\langle \partial_t v, \xi(\tau)\rangle_H \mathrm{d}\tau - \int_0^\ell \rho(\tau)\langle v, \partial_t \xi(\tau)\rangle_H \mathrm{d}\tau.$$

We now estimate the two terms of the right-hand side, exploiting the equations of (24) and the integral control (34). For the first one,

$$\begin{aligned}
&-\int_0^\ell \rho(\tau)\langle \partial_t v, \xi(\tau)\rangle_H \mathrm{d}\tau \\
&\quad = \int_0^\ell \rho(\tau)\langle u, \xi(\tau)\rangle_V \mathrm{d}\tau + \int_0^\ell \rho(\tau)\left(\int_0^\ell \langle \xi(\tau'), \xi(\tau)\rangle_V \mathrm{d}\tau'\right)\mathrm{d}\tau \\
&\quad \leqq \|u\|_V \int_0^\ell \|\xi(\tau)\|_V \mathrm{d}\tau + \left(\int_0^\ell \|\xi(\tau)\|_V \mathrm{d}\tau\right)^2 \\
&\quad \leqq \sqrt{M(0)}\,\|u\|_V\|\xi\|_{\mathcal{V}} + M(0)\|\xi\|_{\mathcal{V}}^2 \\
&\quad \leqq \frac{1}{12}M(\beta)\|u\|_V^2 + M(0)\left(1+\frac{3}{M(\beta)}\right)\|\xi\|_{\mathcal{V}}^2.
\end{aligned}$$

Concerning the second term, we preliminarily observe that, since $\xi \in \mathcal{D}(P)$ for all times, we have (cf. Remark 10)

$$\sup_{\tau\in\Omega}\|\xi(\tau)\|_V < \infty \quad \text{and} \quad \|\xi(\ell)\|_V = 0.$$

Thus, an integration by parts gives

$$-\int_0^\ell \rho(\tau)\langle v, P\xi(\tau)\rangle_H \mathrm{d}\tau = -\int_0^\ell \rho(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\langle v, \xi(\tau)\rangle_H \mathrm{d}\tau = \frac{1}{\beta}\int_0^\beta \langle v, \xi(\tau)\rangle_H \mathrm{d}\tau.$$

Hence,

$$\begin{aligned}
-\int_0^\ell \rho(\tau)\langle v, \partial_t\xi(\tau)\rangle_H \mathrm{d}\tau &= -\left(\int_0^\ell \rho(\tau)\mu(\tau)\mathrm{d}\tau\right)\|v\|_H^2 + \frac{1}{\beta}\int_0^\beta \langle v, \xi(\tau)\rangle_H \mathrm{d}\tau \\
&\leqq -M(\beta)\|v\|_H^2 + \frac{1}{\beta\sqrt{\lambda_1}}\|v\|_H \int_0^\ell \|\xi(\tau)\|_V \mathrm{d}\tau \\
&\leqq -M(\beta)\|v\|_H^2 + \frac{\sqrt{M(0)}}{\beta\sqrt{\lambda_1}}\|v\|_H\|\xi\|_{\mathcal{V}} \\
&\leqq -\frac{1}{2}M(\beta)\|v\|_H^2 + \frac{M(0)}{2\beta^2\lambda_1 M(\beta)}\|\xi\|_{\mathcal{V}}^2.
\end{aligned}$$

Collecting the above inequalities, the conclusion follows. □

Lemma 4. *The functional $\Phi_2(t)$ fulfills the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(t) \leqq -\frac{3}{4}\|u(t)\|_V^2 + \|v(t)\|_H^2 + M(0)\|\xi^t\|_{\mathcal{V}}^2. \tag{37}$$

Proof. By virtue of (24) and (34),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2 &= -\|u\|_V^2 + \|v\|_H^2 - \int_0^\ell \langle u, \xi(\tau)\rangle_V \mathrm{d}\tau \\ &\leqq -\|u\|_V^2 + \|v\|_H^2 + \sqrt{M(0)}\,\|u\|_V\|\xi\|_{\mathcal{V}} \\ &\leqq -\frac{3}{4}\|u\|_V^2 + \|v\|_H^2 + M(0)\|\xi\|_{\mathcal{V}}^2,\end{aligned}$$

as claimed. □

At this point, we define the functional

$$\Phi(t) = \frac{3}{M(\beta)}\Phi_1(t) + \Phi_2(t),$$

which, due to (36) and (37), satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + E(t) \leqq c_2\|\xi^t\|_{\mathcal{V}}^2, \tag{38}$$

for some $c_2 > 0$ independent of z. Besides, in light of (35),

$$|\Phi(t)| \leqq c_3 E(t), \tag{39}$$

with $c_3 = c_0(3/M(\beta) + 1)$. Finally, we fix

$$\varepsilon = \min\left\{\frac{\delta}{2c_2}, \frac{1}{2c_3}\right\}$$

and we set

$$\Psi(t) = E(t) + \varepsilon\Phi(t).$$

Note that, by (39),

$$\frac{1}{2}E(t) \leqq \Psi(t) \leqq \frac{3}{2}E(t),$$

and in turn, by (33) and (38),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) + 2\omega\Psi(t) \leqq 0,$$

with $\omega = \varepsilon/3$. Therefore, the standard Gronwall lemma yields

$$\|S(t)z\|_{\mathfrak{A}}^2 = 2E(t) \leqq 4\Psi(t) \leqq 4\Psi(0)\mathrm{e}^{-2\omega t} \leqq 6E(0)\mathrm{e}^{-2\omega t} = 3\|z\|_{\mathfrak{A}}^2\mathrm{e}^{-2\omega t}.$$

The proof of Theorem 2 is completed.

Remark 13. Observe that the proof of Theorem 2 is carried out employing *only* energy functionals, and it makes no use of linear semigroup techniques. Thus, the same energy functionals can be exploited to analyze semilinear versions of the problem (for instance, to prove the existence of absorbing sets and global attractors).

10. The original equation revisited

Somehow, this novel state approach urges us to consider the original problem from a different perspective. Indeed, as we saw in Section 6, the solutions to (11)–(12) are determined not only by u_0 and v_0, but by the knowledge of the function F_0, and not by the particular form of the initial past history ϕ_0. Therefore, with reference to Definition 1, we introduce the class of *admissible past history functions*

$$\mathcal{A} = \left\{ \phi : \Omega \to V \ : \ t \mapsto \int_0^\ell \mu(t+s)\phi(s)\mathrm{d}s \in L^1_{\mathrm{loc}}([0,\infty); V) \right\},$$

and we define the linear map

$$\Lambda : \mathcal{A} \to L^1_{\mathrm{loc}}([0,\infty); V)$$

as

$$\phi \mapsto \Lambda\phi(t) = \int_0^\ell \mu(t+s)\phi(s)\mathrm{d}s.$$

Note that $\Lambda\phi(t) = 0$ if $t \geqq \ell$. Accordingly, we define the class of *state functions*

$$\mathcal{S} = \Lambda\mathcal{A}.$$

Clearly (and this is really the point), the map Λ may not be injective, meaning that different $\phi \in \mathcal{A}$ may lead to the same element of $\mathcal{S}$.

Coming back to Definition 1, the assumption on F_0 can now be rephrased as

$$F_0 = \Lambda\phi_0 \quad \text{with} \quad \phi_0 \in \mathcal{A},$$

and we can reformulate the definition of solution to (11) in the following more convenient (and certainly more physical) way.

Definition 2. Let the triplet

$$(u_0, v_0, F_0) \in V \times H \times \mathcal{S}$$

be given. A function

$$u \in C([0,\infty), V) \cap C^1([0,\infty), H)$$

is said to be a solution to equation (11) with *initial state* (u_0, v_0, F_0) if

$$u(0) = u_0, \quad \partial_t u(0) = v_0,$$

and the equality

$$\langle \partial_{tt} u(t), w\rangle + \alpha\langle u(t), w\rangle_V - \int_0^t \mu(s)\langle u(t-s), w\rangle_V \mathrm{d}s - \langle F_0(t), w\rangle_V = 0$$

holds for every $w \in V$ and almost every $t > 0$.

In this definition, the initial datum ϕ_0 has completely disappeared, since the state function F_0 contains all the necessary information on the past history of the variable u needed to capture the future dynamics of the equation. Hence, we removed the (unphysical) ambiguity caused by two different initial histories leading to the same state function, which, as we saw, is what really enters into the definition of a solution.

Remark 14. We point out that the function $F_0(t)$ is not influenced by the dynamics for $t \geqq 0$, nor by the presence of a possible external force. As a matter of fact, if the initial past history ϕ_0 is known, then F_0 is uniquely determined by (13). On the other hand, even if the particular ϕ_0 leading to F_0 is unknown, in principle, F_0 can still be determined (cf. Section 2.4).

The remainder of this section is devoted to investigating the properties of the space $\mathcal{S}$. We begin with a lemma, which provides a precise formulation of the formal equality (20), devised in Section 6.

Lemma 5. *Whenever $\phi \in \mathcal{A}$, the map*

$$\tau \mapsto \int_0^\ell \mu'(\tau + s)\|\phi(s)\|_V \mathrm{d}s$$

belongs to $L^1(t, \infty)$ for every $t > 0$, and the equality

$$\Lambda\phi(t) = -\int_t^\ell \left(\int_0^\ell \mu'(\tau + s)\phi(s)\mathrm{d}s\right)\mathrm{d}\tau \tag{40}$$

holds for every $t > 0$. Moreover, if $\phi \in L^1_\mu(\Omega; V)$, then $\phi \in \mathcal{A}$ and (40) holds for every $t \geqq 0$.

Proof. Let $\phi \in \mathcal{A}$ be given. For every fixed $t > 0$,

$$\Lambda\phi(t_0) \in V, \quad \text{for some } t_0 \leqq t.$$

Since μ is a nonincreasing function and $\Lambda\phi(t_0)$ is a Bochner integral, this is the same as saying that

$$\int_0^\ell \mu(t + s)\|\phi(s)\|_V \mathrm{d}s \leqq \int_0^\ell \mu(t_0 + s)\|\phi(s)\|_V \mathrm{d}s < \infty.$$

Exploiting the equality

$$\mu(t + s) = -\int_t^\ell \mu'(\tau + s)\mathrm{d}\tau,$$

and exchanging the order of integration, we conclude that

$$\int_0^\ell \mu(t + s)\|\phi(s)\|_V \mathrm{d}s = -\int_t^\ell \left(\int_0^\ell \mu'(\tau + s)\|\phi(s)\|_V \mathrm{d}s\right)\mathrm{d}\tau < \infty.$$

Hence,

$$\tau \mapsto \int_0^\ell \mu'(\tau + s)\|\phi(s)\|_V \mathrm{d}s \in L^1(t, \infty),$$

and (40) follows from the Fubini theorem. Concerning the last assertion, just note that $\phi \in L^1_\mu(\Omega; V)$ if and only if $\Lambda\phi(0) \in V$. □

Remark 15. As a straightforward consequence of the lemma,

$$\mathcal{S} \subset C_0([t, \infty), V), \quad \forall t > 0,$$

where C_0 denotes the space of continuous functions vanishing at infinity.

Given $F \in \mathcal{S}$, it is then interesting to see what happens to $F(t)$ in the limit $t \to 0$. Three mutually disjoint situations may occur:

(i) $\lim_{t\to 0} F(t)$ exists in V;
(ii) $F \in L^\infty(\mathbb{R}^+; V)$ but $\lim_{t\to 0} F(t)$ does not exist in V;
(iii) $\|F(t)\|_V$ is unbounded in a neighborhood of $t = 0$.

As we will see, (i) is the most interesting case in view of our scope. For this reason, we introduce the further space

$$\mathcal{S}_0 = \left\{ F \in \mathcal{S} \,:\, \exists \lim_{t\to 0} F(t) \text{ in } V \right\}.$$

In light of Remark 15, it is apparent that

$$\mathcal{S}_0 \subset C_0([0, \infty), V).$$

We preliminarily observe that if $F = \Lambda\phi$ with $\phi \in L^1_\mu(\Omega; V)$, then Lemma 5 yields at once

$$\lim_{t\to 0} F(t) = \Lambda\phi(0) \quad \text{in } V,$$

so that $F \in \mathcal{S}_0$. However, the picture can be more complicated. Indeed, it may happen that $F \in \mathcal{S}_0$ but $\Lambda\phi(0)$ is not defined for *any* $\phi \in \Lambda^{-1}F$, as the following example shows.

Example 1. Consider the kernel

$$\mu(s) = 1 - s, \quad \Omega = (0, 1).$$

Given any nonzero vector $u \in V$, set

$$F(t) = \left[(1 - t)\sin 1 - \int_t^1 \sin\frac{1}{x}\mathrm{d}x \right] \chi_{[0,1]}(t)\, u,$$

which clearly satisfies

$$\lim_{t\to 0} F(t) = \left[\sin 1 - \int_0^1 \sin\frac{1}{x}\mathrm{d}x \right] u.$$

Then, $F = \Lambda\phi$ with

$$\phi(s) = -\left[\frac{1}{(1-s)^2}\cos\frac{1}{1-s}\right]u,$$

but $\Lambda\phi(0)$ is not defined, since $\phi \notin L^1_\mu(\Omega; V)$. To complete the argument, we show that, for this particular kernel, the linear map Λ is injective. Indeed, let $\tilde{\phi} \in \mathcal{A}$ be such that

$$0 = \Lambda\tilde{\phi}(t) = \int_0^{1-t}(1-t-s)\tilde{\phi}(s)\mathrm{d}s.$$

The above equality readily implies that $\tilde{\phi} = 0$.

Let us provide examples also for (ii) and (iii). Again, $u \in V$ is any nonzero vector.

Example 2. With μ as in the previous example, set

$$F(t) = \left[\sin\frac{1}{t} - \sin 1 + t\cos 1 - \cos 1\right]\chi_{[0,1]}(t)\,u.$$

Then, $F = \Lambda\phi$ with

$$\phi(s) = \left[\frac{2}{(1-s)^3}\cos\frac{1}{1-s} - \frac{1}{(1-s)^4}\sin\frac{1}{1-s}\right]u.$$

Note that $\|F\|_V \in L^\infty(\mathbb{R}^+)$ but $\lim_{t\to 0} F(t)$ does not exist in V.

Example 3. Consider the kernel

$$\mu(s) = \sqrt{\frac{1-s}{s}}, \quad \Omega = (0, 1),$$

and set

$$F(t) = \left[\int_0^{1-t}\sqrt{\frac{1-t-s}{s(t+s)}}\,\mathrm{d}s\right]\chi_{[0,1]}(t)\,u.$$

Then, $F = \Lambda\phi$ with

$$\phi(s) = \frac{1}{\sqrt{s}}\,u.$$

It is easily verified that $\|F\|_V$ is summable on $\mathbb{R}^+$ but

$$\lim_{t\to 0}\|F(t)\|_V = \infty.$$

Remark 16. A related (and very challenging) question is the following: given a function $F \in C_0([t,\infty), V)$ for every $t > 0$, find (easy to handle) conditions ensuring that $F \in \mathcal{S}$. In fact, the answer seems to be strongly dependent on the particular choice of the kernel. For instance, with μ as in Example 1, $F \in \mathcal{S}$ if and only if $F(t) = DF(t) = 0$ for $t \geqq 1$, $F \in L^1(\mathbb{R}^+; V)$, and DF is absolutely continuous from $[t, 1]$ into V for all $t > 0$. In which case, $F = \Lambda\phi$ with $\phi(s) = D^2F(1-s)$. On the other hand, if $\mu(s) = \mathrm{e}^{-s}$, then $F \in \mathcal{S}$ if and only if $F(t) = \mathrm{e}^{-t}u$ with $u \in V$ (cf. the upcoming Example 6).

11. Proper states: recovering the original equation

The purpose of this section is to establish the link between (24) and the original equation (11), up to now only formal. To this end, we have to recall the particular form of the initial datum ξ_0, obtained in a somewhat heuristic way in Section 6. This gives a clue that not all the states are apt to describe the behavior of the original equation, but only certain particular states having a well defined structure.

Definition 3. A vector $\xi \in \mathcal{V}$ is said to be a *proper state* if

$$\xi(\tau) = DF(\tau),$$

for some $F \in \mathcal{S}$. We denote by $\mathcal{P}$ the normed subspace of $\mathcal{V}$ (with the norm inherited by $\mathcal{V}$) of proper states.

For any given kernel μ, an immediate example of proper state is

$$\xi(\tau) = \mu(\tau)u, \quad u \in V.$$

Indeed, $\xi = DF$ with

$$F(t) = -M(t)u = -\int_0^\ell \mu(t+s)u\,\mathrm{d}s.$$

Lemma 6. *Let $\xi \in \mathcal{P}$. Then, there exists a unique $F \in \mathcal{S}$ such that $\xi = DF$. Besides, F belongs to $\mathcal{S}_0$. Moreover, for every $\phi \in \mathcal{A}$ such that $F = \Lambda\phi$, it follows that*

$$\xi(\tau) = \int_0^\ell \mu'(\tau+s)\phi(s)\mathrm{d}s.$$

Conversely, if $\xi \in \mathcal{V}$ has the above representation for some $\phi \in \mathcal{A}$, then $\xi \in \mathcal{P}$ and

$$\xi(\tau) = D\Lambda\phi(\tau).$$

Proof. Let $F \in \mathcal{S}$ be such that $\xi = DF$. From Lemma 1, $DF \in L^1(\Omega; V)$. Hence, the map

$$t \mapsto -\int_t^\ell DF(\tau)\mathrm{d}\tau = F(t)$$

belongs to $C_0([0,\infty), V)$. Therefore, $F \in \mathcal{S}_0$, and it is apparent that F is uniquely determined by DF. The remaining assertions follow by (40). □

Here is a concrete application of the lemma.

Example 4. Let μ and u as in Example 3, and define

$$F(t) = \left[\int_0^{1-t} \sqrt{\frac{1-t-s}{s(t+s)}}\,\sin\frac{1}{s}\,\mathrm{d}s\right] \chi_{[0,1]}(t)\,u.$$

Then, $F = \Lambda\phi$ with

$$\phi(s) = \left[\frac{1}{\sqrt{s}} \sin \frac{1}{s}\right] u.$$

Setting

$$\xi(\tau) = \int_0^\ell \mu'(\tau + s)\phi(s)\mathrm{d}s = -\left[\frac{1}{2}\int_0^{1-\tau} \frac{1}{\sqrt{s(1-\tau-s)}\,(\tau+s)^{3/2}} \sin\frac{1}{s}\,\mathrm{d}s\right] u,$$

it is not hard to verify that $\xi \in \mathcal{V}$. From Lemma 6, we conclude that $\xi = DF \in \mathcal{P}$. Note that, as expected, $F \in \mathcal{S}_0$. Indeed,

$$\lim_{t\to 0} F(t) = \varkappa u \quad \text{in } V,$$

with

$$\varkappa = \lim_{N\to\infty} \int_1^N \frac{1}{x}\sqrt{\frac{x-1}{x}}\,\sin x\,\mathrm{d}x \sim 0.28.$$

In particular, Lemma 6 says that the map

$$\Gamma : \mathcal{P} \to \mathcal{S}$$

defined as

$$\Gamma\xi(t) = -\int_t^\ell \xi(\tau)\mathrm{d}\tau$$

is injective. Since

$$\Gamma\mathcal{P} \subset \mathcal{S}_0,$$

and the inclusion $\mathcal{S}_0 \subset \mathcal{S}$ can be strict, the map Γ is not, in general, onto. In fact, the inclusion $\Gamma\mathcal{P} \subset \mathcal{S}_0$ can be strict either.

Example 5. Let

$$\mu(s) = 1 - s, \quad \Omega = (0, 1).$$

Given any nonzero vector $u \in V$, consider the function

$$F(t) = \left(\sqrt{t} - 1\right)^2 \chi_{[0,1]}(t)\,u.$$

Then, $F = \Lambda\phi$ with

$$\phi(s) = \left[\frac{1}{2(1-s)^{3/2}}\right] u.$$

We conclude that $F \in \mathcal{S}_0$. On the other hand,

$$DF(\tau) = \left[\frac{\sqrt{\tau} - 1}{\sqrt{\tau}}\right] u,$$

which does not belong to $\mathcal{V}$.

We now have all the ingredients to state the main result of the section.

Theorem 3. *Let* $(u_0, v_0, F_0) \in V \times H \times \mathcal{S}$. *Assume in addition that*

$$F_0 \in \Gamma\mathcal{P}.$$

Then, a function u is a solution to (11) with initial state (u_0, v_0, F_0) (according to Definition 2) if and only if

$$(u(t), \partial_t u(t), \xi^t) = S(t)(u_0, v_0, \xi_0),$$

with ξ^t as in (29) with

$$\xi_0(\tau) = \mu(\tau)u_0 + DF_0(\tau).$$

Conversely, if u is a solution to (11) with initial state (u_0, v_0, F_0) and $F_0 \notin \Gamma\mathcal{P}$, then there is no corresponding solution in the extended state space.

Proof. Since $u \in C([0,\infty), V)$, arguing as in the proof of Lemma 5, the equality

$$\int_0^\ell \left(\int_0^t \mu'(\tau + s)u(t-s)\mathrm{d}s \right) \mathrm{d}\tau = -\int_0^t \mu(s)u(t-s)\mathrm{d}s$$

holds for every $t > 0$. Thus, using (29), keeping in mind the particular form of ξ_0 and the fact that $F_0 \in \mathcal{S}_0$, we readily get

$$\int_0^\ell \xi^t(\tau)\mathrm{d}\tau = M(0)u(t) - \int_0^t \mu(s)u(t-s)\mathrm{d}s - F_0(t). \tag{41}$$

This equality, in light of (14), (15) and (24), proves the first statement.

To prove the converse, assume that $u(t)$ is, at the same time, a solution to (11) with initial state (u_0, v_0, F_0) and equal to the first component of $S(t)(u_0, v_0, \xi_0)$, for some $\xi_0 \in \mathcal{V}$. We reach the conclusion by showing that $F_0 \in \Gamma\mathcal{P}$. Indeed, now calling ξ^t the third component of $S(t)(u_0, v_0, \xi_0)$, from (14), (15) and (24), we again obtain (41). Since, by (29),

$$\int_0^\ell \xi^t(\tau)\mathrm{d}\tau = \int_t^\ell \xi_0(\tau)\mathrm{d}\tau + M(0)u(t) - M(t)u_0 - \int_0^t \mu(s)u(t-s)\mathrm{d}s,$$

we conclude that

$$\int_t^\ell [\mu(\tau)u_0 - \xi_0(\tau)]\,\mathrm{d}\tau = M(t)u_0 - \int_t^\ell \xi_0(\tau)\mathrm{d}\tau = F_0(t).$$

Hence,

$$-\mu(\tau)u_0 + \xi_0(\tau) = DF_0(\tau),$$

meaning that $\xi_0 - \mu u_0 \in \mathcal{P}$ and $F_0 = \Gamma(\xi_0 - \mu u_0)$. □

Remark 17. Since we have an existence and uniqueness result in the extended state space, Theorem 3 provides an existence and uniqueness result for (11), according to Definition 2, whenever we restrict ourselves to initial states with $F_0 \in \Gamma\mathcal{P}$.

However, there are situations where the equality $\mathcal{S} = \Gamma\mathcal{P}$ holds, as in the case of the exponential kernel.

Example 6. For $a > 0$ and $\kappa > 0$, consider the kernel

$$\mu(s) = a\mathrm{e}^{-\kappa s}, \quad \Omega = \mathbb{R}^+.$$

Since

$$\mu(t+s) = \mathrm{e}^{-\kappa t}\mu(s),$$

it is apparent that

$$\mathcal{S} = \mathcal{S}_0 = \left\{F(t) = \mathrm{e}^{-\kappa t}u \ \text{ with } \ u \in V\right\}.$$

In turn,

$$\mathcal{P} = \left\{\xi(\tau) = \mathrm{e}^{-\kappa\tau}u \ \text{ with } \ u \in V\right\}.$$

Clearly, $\mathcal{S} = \Gamma\mathcal{P}$.

Remark 18. Incidentally, the above example sheds light on another important issue: there exist states which are not proper states; in other words, the inclusion $\mathcal{P} \subset \mathcal{V}$ is strict (and not even dense).

In summary, there might be state functions of the original approach that have no corresponding (proper) states. Conversely, only the proper states describe the original problem. In this respect, the state approach is a more general model which is also able to describe within the formalism of semigroups a certain class of Volterra equations with nonautonomous forcing terms.

Nonetheless, if we start from a proper state, it is reasonable to expect that the evolution remains confined within the space of proper states. To this end, let us define the *extended proper state space* as

$$\mathfrak{A}_p = V \times H \times \mathcal{P},$$

which is a normed subspace of $\mathfrak{A}$.

Proposition 3. *If $z \in \mathfrak{A}_p$, it follows that $S(t)z \in \mathfrak{A}_p$.*

Proof. Let $z = (u_0, v_0, \xi_0) \in \mathfrak{A}_p$. Then, $\xi_0 = DF$ for some $F \in \mathcal{S}$. In turn, $F = \Lambda\phi$ for some $\phi \in \mathcal{A}$. Denoting as usual $S(t)z = (u(t), v(t), \xi^t)$, and setting

$$\psi^t(s) = u(t-s)\chi_{(0,t)}(s) + u_0\chi_{(t,\ell)}(s) - u(t)$$

and

$$\phi^t(s) = \phi(s-t)\chi_{(t,\ell)}(s),$$

the representation formula (29) can be equivalently written as

$$\xi^t(\tau) = \int_0^\ell \mu'(\tau+s)\left\{\psi^t(s) + \phi^t(s)\right\}\mathrm{d}s.$$

By Lemma 6, in order to prove that $\xi^t \in \mathcal{P}$, we are left to show that $\psi^t + \phi^t \in \mathcal{A}$. Indeed, since $\|S(t)z\|_{\mathfrak{A}} \leqq \|z\|_{\mathfrak{A}}$,

$$\int_0^\ell \mu(s)\|\psi^t(s)\|_V \mathrm{d}s \leqq 3M(0)\|z\|_{\mathfrak{A}}.$$

Therefore, $\psi^t \in L^1_\mu(\Omega; V) \subset \mathcal{A}$. Concerning ϕ^t, we have

$$\int_0^\ell \mu(s)\phi^t(s)\mathrm{d}s = \Lambda\phi(t) \in V,$$

which yields $\phi^t \in L^1_\mu(\Omega; V) \subset \mathcal{A}$. □

In particular, from Theorem 1 and Theorem 2, we have the following corollary.

Corollary 2. *The restriction*

$$S_p(t) = S(t)_{|\mathfrak{A}_p} : \mathfrak{A}_p \to \mathfrak{A}_p$$

is a contraction semigroup on $\mathfrak{A}_p$. *Assuming, also, condition* (30), *the semigroup* $S_p(t)$ *is exponentially stable.*

One might ask whether $\mathcal{P}$ (and in turn $\mathfrak{A}_p$) is a Banach space. This is true, for instance, for the exponential kernel of Example 6. However, in general, the answer is negative.

Example 7. Take the kernel

$$\mu(s) = 1 - s, \quad \Omega = (0, 1).$$

Let $\mathfrak{C} : [0, 1] \to [0, 1]$ denote the famous Vitali–Cantor–Lebesgue singular function, and let $\mathfrak{C}_n$ be the usual approximating sequence of absolutely continuous functions (cf. [38]). Consider the sequence

$$F_n(t) = \left[\int_0^{1-t} \mathfrak{C}_n(s)\mathrm{d}s\right] \chi_{[0,1]}(t)\, u,$$

where $u \in V$ is any nonzero vector. Then, $F_n = \Lambda\phi_n$ with

$$\phi_n(s) = \mathfrak{C}_n'(s)\, u.$$

Setting

$$\xi_n(\tau) = DF_n(\tau) = -\mathfrak{C}_n(1-\tau)\, u,$$

it is readily seen that $\xi_n \in \mathcal{V}$, and, consequently, $\xi_n \in \mathcal{P}$. It also apparent that

$$\lim_{n\to\infty} \xi_n = \xi \quad \text{in } \mathcal{V},$$

where

$$\xi(\tau) = -\mathfrak{C}(1-\tau)\, u.$$

However, $\xi \notin \mathcal{P}$. Indeed, if not so, the function

$$F(t) = \Gamma\xi(t) = \left[\int_0^{1-t} \mathfrak{C}(s)\mathrm{d}s\right] \chi_{[0,1]}(t)\, u$$

belongs to $\mathcal{S}$. Hence, there is $\phi \in \mathcal{A}$ such that

$$\Lambda\phi(t) = \int_0^{1-t} \left(\int_0^s \phi(\sigma)\mathrm{d}\sigma\right) \mathrm{d}s = \left[\int_0^{1-t} \mathfrak{C}(s)\mathrm{d}s\right] u.$$

But this implies that

$$\phi(s) = \mathfrak{C}'(s)\, u = 0$$

for almost every $s \in (0, 1)$. Thus, $F = 0$ and $\xi = DF = 0$, leading to a contradiction.

12. State versus history

We finally turn to the main issue that motivated this work: the comparison between the past history and the state approaches. We begin by showing that each element of $\mathcal{M}$ gives rise to a proper state, defining the linear map

$$\Pi : \mathcal{M} \to \mathcal{P}$$

as

$$\Pi\eta(\tau) = -\int_0^\ell \mu'(\tau + s)\eta(s)\mathrm{d}s.$$

Lemma 7. *Let $\eta \in \mathcal{M}$. Then, the vector*

$$\Pi\eta(\tau) = -\int_0^\ell \mu'(\tau + s)\eta(s)\mathrm{d}s$$

belongs to $\mathcal{P}$. Moreover,

$$\|\Pi\eta\|_{\mathcal{V}} \leqq \|\eta\|_{\mathcal{M}}.$$

Proof. Let $\eta \in \mathcal{M}$. Then,

$$\begin{aligned}
\|\Pi\eta(\tau)\|_V^2 &\leqq \left(\int_0^\ell -\mu'(\tau + s)\|\eta(s)\|_V \mathrm{d}s\right)^2 \\
&\leqq \int_0^\ell -\mu'(\tau + s)\mathrm{d}s \int_0^\ell -\mu'(\tau + s)\|\eta(s)\|_V^2 \mathrm{d}s \\
&= \mu(\tau) \int_0^\ell -\mu'(\tau + s)\|\eta(s)\|_V^2 \mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Pi\eta\|_{\mathcal{V}}^2 &\leqq \int_0^\ell \mathrm{d}\tau \int_0^\ell -\mu'(\tau+s)\|\eta(s)\|_V^2 \mathrm{d}s \\ &= \int_0^\ell \left(\int_0^\ell -\mu'(\tau+s)\mathrm{d}\tau\right)\|\eta(s)\|_V^2 \mathrm{d}s \\ &= \|\eta\|_{\mathcal{M}}^2.\end{aligned}$$

Thus $\Pi\eta \in \mathcal{V}$, and the norm inequality stated above holds (in fact, equality for $\eta(s) = u$, with $u \in V$). Since $\mathcal{M} \subset L^1_\mu(\Omega; V)$, because of the straightforward estimate

$$\int_0^\ell \mu(s)\|\eta(s)\|_V \mathrm{d}s \leqq \sqrt{M(0)}\,\|\eta\|_{\mathcal{M}},$$

and $L^1_\mu(\Omega; V) \subset \mathcal{A}$, it follows from Lemma 6 that $\Pi\eta$ is a proper state. □

Rephrasing the lemma, $\Pi \in L(\mathcal{M}, \mathcal{V})$; that is, Π is a bounded linear operator from $\mathcal{M}$ into $\mathcal{V}$. Moreover

$$\|\Pi\|_{L(\mathcal{M},\mathcal{V})} = 1.$$

We now clarify the correspondence between $\eta \in \mathcal{M}$ and its related proper state $\Pi\eta$. Letting

$$\bar{z} = (u_0, v_0, \eta_0) \in \mathfrak{M}, \quad z = (u_0, v_0, \Pi\eta_0) \in \mathfrak{A}_p,$$

and denoting

$$\Sigma(t)\bar{z} = (\bar{u}(t), \partial_t\bar{u}(t), \bar{\eta}^t), \quad S_p(t)z = (u(t), \partial_t u(t), \xi^t),$$

we have the following result.

Proposition 4. *The equalities*

$$u(t) = \bar{u}(t) \quad \textit{and} \quad \xi^t = \Pi\bar{\eta}^t$$

hold for every $t \geqq 0$.

Proof. Introduce the function (cf. (19))

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \leqq t, \\ \eta_0(s-t) + u(t) - u_0 & s > t, \end{cases}$$

which solves the Cauchy problem in $\mathcal{M}$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\eta^t = T\eta^t + \partial_t u(t), \\ \eta^0 = \eta_0. \end{cases} \tag{42}$$

The representation formula (29) for ξ^t furnishes

$$\begin{aligned}\xi^t(\tau) &= \Pi\eta_0(t+\tau) + \mu(\tau)u(t) - \mu(t+\tau)u_0 + \int_0^t \mu'(\tau+s)u(t-s)\mathrm{d}s \\ &= \Pi\eta^t(\tau).\end{aligned}$$

Thus, exploiting (40),

$$\int_0^\ell \xi^t(\tau)\mathrm{d}\tau = \int_0^\ell \Pi\eta^t(\tau)\mathrm{d}\tau = \Lambda\eta^t(0) = \int_0^\ell \mu(s)\eta^t(s)\mathrm{d}s,$$

and, consequently,

$$\partial_{tt}u + A\left[u + \int_0^\ell \mu(s)\eta^t(s)\mathrm{d}s\right] = \partial_{tt}u + A\left[u + \int_0^\ell \xi^t(\tau)\mathrm{d}\tau\right] = 0. \quad (43)$$

Since

$$u(0) = u_0 \quad \text{and} \quad \partial_t u(0) = v_0,$$

collecting (42)–(43) we conclude that

$$(u(t), \partial_t u(t), \eta^t) = \Sigma(t)\bar{z} = (\bar{u}(t), \partial_t\bar{u}(t), \bar{\eta}^t).$$

This finishes the proof. □

Nonetheless, in general, the map $\Pi : \mathcal{M} \to \mathcal{P}$ is not injective. This means that two *different* initial histories may entail the *same* initial proper state, so leading to the same dynamics in the future.

Example 8. Let $N \in \mathbb{N}$. Given $a_n > 0$ and $\kappa_N > \cdots > \kappa_1 > 0$, consider the kernel

$$\mu(s) = \sum_{n=1}^N a_n \mathrm{e}^{-\kappa_n s}, \quad \Omega = \mathbb{R}^+.$$

For $x_m \in \mathbb{R}$ to be determined later, define

$$\eta_0(s) = u, \quad \eta_N(s) = \left[\sum_{m=1}^N x_m s^m\right] u,$$

where $u \in V$ is a fixed nonzero vector. Clearly, $\eta_0, \eta_N \in \mathcal{M}$ and $\eta_0 \neq \eta_N$. Besides,

$$\Pi\eta_0(\tau) = \left[\sum_{n=1}^N a_n \mathrm{e}^{-\kappa_n\tau}\right] u$$

and

$$\Pi\eta_N(\tau) = \left[\sum_{n=1}^N a_n J_n \mathrm{e}^{-\kappa_n\tau}\right] u,$$

having set

$$J_n = \kappa_n \sum_{m=1}^{N} x_m \int_0^\infty s^m \mathrm{e}^{-\kappa_n s} \mathrm{d}s = \sum_{m=1}^{N} b_{nm} x_m,$$

with

$$b_{nm} = \frac{m!}{\kappa_n^m}.$$

The determinant of the matrix $\mathbb{B} = \{b_{nm}\}$ is given by

$$\det(\mathbb{B}) = \prod_{1 \leqq n \leqq N} \frac{n!}{\kappa_n} \prod_{1 \leqq m < n \leqq N} \left(\frac{1}{\kappa\, n} - \frac{1}{\kappa\, m} \right) \neq 0.$$

Therefore $\mathbb{B}$ is nonsingular, and we can choose $\boldsymbol{x} = [x_1, \ldots, x_N]^\top$ to be the (unique) solution to the linear system

$$\mathbb{B}\boldsymbol{x} = [1, \ldots, 1]^\top.$$

In which case, $J_n = 1$ for all n, so that the equality $\Pi\eta_0 = \Pi\eta_N$ holds true.

However, for the kernel of Example 8, one can verify that Π maps $\mathcal{M}$ *onto* $\mathcal{P}$. Thus, every proper state is realized by a history from $\mathcal{M}$. On the contrary, the next example describes a situation where the map Π is injective on $\mathcal{M}$, but $\Pi\mathcal{M}$ is strictly contained in $\mathcal{P}$, meaning that all different histories in $\mathcal{M}$ lead to different proper states, but there are proper states which do not come from histories. We first need a definition and some preliminary results.

Definition 4. A positive sequence $\{\kappa_n\}$, $n \in \mathbb{N}$, is called a *Müntz sequence* if $\kappa_n \uparrow \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{\kappa_n} = \infty.$$

Given a function $g \in L^1_{\mathrm{loc}}([0, \infty))$ such that $s \mapsto \mathrm{e}^{-\lambda s} g(s) \in L^1(\mathbb{R}^+)$, for some $\lambda > 0$, we denote its (real) Laplace transform by

$$\mathcal{L}g(x) = \int_0^\infty \mathrm{e}^{-xs} g(s) \mathrm{d}s.$$

A celebrated result due to C. Müntz says that if $\{\kappa_n\}$ is a Müntz sequence belonging to the domain of $\mathcal{L}g$ and

$$\mathcal{L}g(\kappa_n) = 0, \quad \forall n \in \mathbb{N},$$

then g is identically zero (see [50]).

The following lemma is standard. A three-line proof is included for the reader's convenience.

Lemma 8. *Let $\kappa_n > 0$ be strictly increasing, and let $\beta_n \in \mathbb{R}$ be the general term of an absolutely convergent series. Consider the function $h : [0, \infty) \to \mathbb{R}$ defined as*

$$h(t) = \sum_{n=1}^{\infty} \beta_n \mathrm{e}^{-\kappa_n t}.$$

Then, h is identically zero if and only if $\beta_n = 0$ for every n.

Proof. One implication is trivial. If $h \equiv 0$, we have the equality

$$0 = \int_0^t \mathrm{e}^{\kappa_1 \tau} h(\tau)\mathrm{d}\tau = \beta_1 t + \sum_{n=2}^{\infty} \frac{\beta_n}{\kappa_n - \kappa_1} \Big(1 - \mathrm{e}^{-(\kappa_n - \kappa_1)t}\Big), \quad \forall t \geqq 0.$$

The uniform boundedness of the series forces $\beta_1 = 0$. Iterate the argument for all n. □

We are now ready to provide the aforementioned example.

Example 9. Consider the kernel

$$\mu(s) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\kappa_n s}, \quad \Omega = \mathbb{R}^+,$$

with $\kappa_n > 0$ strictly increasing and $a_n > 0$ such that

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Such a μ is summable on $\mathbb{R}^+$. We first observe that if $g \in L^1_\mu(\mathbb{R}^+)$, then $g \in L^1_{\mathrm{loc}}([0, \infty))$ and $\{\kappa_n\}$ belongs to the domain of $\mathcal{L}g$. Let us extend in the obvious way the map Π to the domain

$$\mathcal{M}_\star = \left\{\eta \in L^1_\mu(\Omega; V) \,:\, \tau \mapsto -\int_0^\ell \mu'(\tau + s)\eta(s)\mathrm{d}s \in \mathcal{V}\right\}$$

(we keep calling Π such an extension). Note that $\mathcal{M} \subset \mathcal{M}_\star \subset \mathcal{A}$, and from Lemma 6 we learn that $\Pi\mathcal{M}_\star \subset \mathcal{P}$. Given $\eta \in \mathcal{M}_\star$ and $w \in V^*$, we consider the duality product

$$g_w(s) = \langle \eta(s), w\rangle \in L^1_\mu(\mathbb{R}^+).$$

In view of (40),

$$\Pi\eta = 0 \quad \Leftrightarrow \quad \Lambda\eta = 0.$$

But $\Lambda\eta = 0$ if and only if

$$\sum_{n=1}^{\infty} \beta_n(w)\mathrm{e}^{-\kappa_n t} = 0, \quad \forall t \geqq 0, \, \forall w \in V^*,$$

having set $\beta_n(w) = a_n\, \mathcal{L}g_w(\kappa_n)$. Moreover,

$$\sum_{n=1}^{\infty} |\beta_n(w)| \leqq \sum_{n=1}^{\infty} a_n \int_0^{\infty} \mathrm{e}^{-\kappa_n s} |g_w(s)| \mathrm{d}s = \|g_w\|_{L^1_\mu(\mathbb{R}^+)} < \infty.$$

Hence, from Lemma 8, the above equality is true if and only if

$$\mathcal{L}g_w(\kappa_n) = 0, \quad \forall n \in \mathbb{N},\ \forall w \in V^*.$$

Therefore, if $\{\kappa_n\}$ is a Müntz sequence,

$$\Pi\eta = 0 \quad \Leftrightarrow \quad g_w = 0,\ \forall w \in V^* \quad \Leftrightarrow \quad \eta = 0.$$

In which case, the map Π is injective on $\mathcal{M}_\star$. Accordingly, to conclude that $\Pi\mathcal{M}$ is a proper subset of $\mathcal{P}$ we have to show that the inclusion $\mathcal{M} \subset \mathcal{M}_\star$ is strict. This is obtained, for instance, by looking at the elements

$$\eta(s) = \mathrm{e}^{\sigma\kappa_1 s} u,$$

where $\sigma \in [\frac{1}{2}, 1)$ and $u \in V$ is any nonzero vector. The details are left to the reader.

References

1. BOLTZMANN, L.: Zur Theorie der elastischen Nachwirkung. *Wien. Ber.* **70**, 275–306 (1874)
2. BOLTZMANN, L.: Zur Theorie der elastischen Nachwirkung. *Wied. Ann.* **5**, 430–432 (1878)
3. BREUER, S., ONAT, E.T.: On recoverable work in linear viscoelasticity. *Z. Angew. Math. Phys.* **15**, 13–21 (1964)
4. CHEPYZHOV, V.V., PATA, V.: Some remarks on stability of semigroups arising from linear viscoelasticity. *Asymptot. Anal.* **50**, 269–291 (2006)
5. COLEMAN, B.D.: Thermodynamics of materials with memory. *Arch. Rational Mech. Anal.* **17**, 1–45 (1964)
6. COLEMAN, B.D., MIZEL, V.J.: Norms and semi-groups in the theory of fading memory. *Arch. Rational Mech. Anal.* **23**, 87–123 (1967)
7. COLEMAN, B.D., MIZEL, V.J.: On the general theory of fading memory. *Arch. Rational Mech. Anal.* **29**, 18–31 (1968)
8. COLEMAN, B.D., NOLL, W.: Foundations of linear viscoelasticity. *Rev. Modern Phys.* **33**, 239–249 (1961)
9. DAFERMOS, C.M.: Asymptotic stability in viscoelasticity. *Arch. Rational Mech. Anal.* **37**, 297–308 (1970)
10. DAY, W.A.: Reversibility, recoverable work and free energy in linear viscoelasticity. *Quart. J. Mech. Appl. Math.* **23**, 1–15 (1970)
11. DAY, W.A.: *The Thermodynamics of Simple Materials with Fading Memory*. Springer, New York, 1972
12. DEL PIERO, G., DESERI, L.: Monotonic, completely monotonic and exponential relaxation functions in linear viscoelasticity. *Quart. Appl. Math.* **53**, 273–300 (1995)
13. DEL PIERO, G., DESERI, L.: On the concepts of state and free energy in linear viscoelasticity. *Arch. Rational Mech. Anal.* **138**, 1–35 (1997)
14. DESERI, L., FABRIZIO, M., GOLDEN, M.J.: The concept of minimal state in viscoelasticity: new free energies an applications to PDEs. *Arch. Rational Mech. Anal.* **181**, 43–96 (2006)

15. Deseri, L., Gentili, G., Golden, M.J.: An explicit formula for the minimum free energy in linear viscoelasticity. *J. Elasticity* **54**, 141–185 (1999)
16. Fabrizio, M., Giorgi, C., Morro, A.: Minimum principles, convexity, and thermodynamics in linear viscoelasticity. *Continuum Mech. Thermodyn.* **1**, 197–211 (1989)
17. Fabrizio, M., Giorgi, C., Morro, A.: Free energies and dissipation properties for systems with memory. *Arch. Rational Mech. Anal.* **125**, 341–373 (1994)
18. Fabrizio, M., Giorgi, C., Morro, A.: Internal dissipation, relaxation property and free energy in materials with fading memory. *J. Elasticity* **40**, 107–122 (1995)
19. Fabrizio, M., Golden, M.J.: Maximum and minimum free energies for a linear viscoelastic material. *Quart. Appl. Math.* **60**, 341–381 (2002)
20. Fabrizio, M., Golden, M.J.: Minimum free energies for materials with finite memory. *J. Elasticity* **72**, 121–143 (2003)
21. Fabrizio, M., Lazzari, B.: On the existence and asymptotic stability of solutions for linear viscoelastic solids. *Arch. Rational Mech. Anal.* **116**, 139–152 (1991)
22. Fabrizio, M., Lazzari, B.: Stability and free energies in linear viscoelasticity. *Matematiche (Catania)* **62**, 175–198 (2007)
23. Fabrizio, M., Morro, A.: Viscoelastic relaxation functions compatible with thermodynamics. *J. Elasticity* **19**, 63–75 (1988)
24. Fabrizio, M., Morro, A.: *Mathematical Problems in Linear Viscoelasticity*. SIAM Studies in Applied Mathematics No. 12. SIAM, Philadelphia, 1992
25. Fichera, G.: *Analytic Problems of Hereditary Phenomena in Materials with Memory*. Corso CIME, Bressanone, 1977, pp. 111–169. Liguori, Napoli, 1979
26. Fichera, G.: Avere una memoria tenace crea gravi problemi. *Arch. Rational Mech. Anal.* **70**, 101–112 (1979)
27. Gentili, G.: Maximum recoverable work, minimum free energy and state space in linear viscoelasticity. *Quart. Appl. Math.* **60**, 152–182 (2002)
28. Giorgi, C., Muñoz Rivera, J.E., Pata, V.: Global attractors for a semilinear hyperbolic equation in viscoelasticity. *J. Math. Anal. Appl.* **260**, 83–99 (2001)
29. Golden, J.M., Graham, G.A.C.: *Boundary Value Problems in Linear Viscoelasticity*. Springer, New York, 1988
30. Graffi, D.: Sui problemi della eredità lineare. *Nuovo Cimento* **5**, 53–71 (1928)
31. Graffi, D.: Sopra alcuni fenomeni ereditari dell'elettrologia. *Rend. Istit. Lombardo Sc. Lett.* **68–69**, 124–139 (1936)
32. Graffi, D.: Sull'espressione analitica di alcune grandezze termodinamiche nei materiali con memoria. *Rend. Sem. Mat. Univ. Padova* **68**, 17–29 (1982)
33. Graffi, D.: On the fading memory. *Appl. Anal.* **15**, 295–311 (1983)
34. Graffi, D., Fabrizio, M.: Sulla nozione di stato per materiali viscoelastici di tipo "rate". *Atti Accad. Lincei Rend. Fis.* **83**, 201–208 (1989)
35. Grasselli, M., Pata, V.: Uniform attractors of nonautonomous systems with memory. In: Lorenzi, A., Ruf, B. (eds.) *Evolution Equations, Semigroups and Functional Analysis*, pp. 155–178. Progr. Nonlinear Differential Equations Appl. No. 50. Birkhäuser, Boston, 2002
36. Green, A.E., Rivlin, R.S.: The mechanics of nonlinear materials with memory. *Arch. Rational Mech. Anal.* **1**, 1–21 (1957-58)
37. Gurtin, M.E., Sternberg, E.: On the linear theory of viscoelasticity. *Arch. Rational Mech. Anal.* **11**, 291–356 (1962)
38. Hewitt, E., Stromberg, K.: *Real and Abstract Analysis*. Springer-Verlag, New York, 1965
39. Ianniello, M.G., Israel, G.: Boltzmann's concept of "Nachwirkung" and the "mechanics of heredity". In: Battimelli, G., Ianniello, M.G., Kresten, O. (eds.) *Proceedings of the International Symposium on Ludwig Boltzmann*, Rome, February 9–11, 1989, pp. 113–133. Verlag der Österreichischen Akademie der Wissenschaften, Wien, 1993
40. König, H., Meixner, J.: Lineare Systeme und lineare Transformationen. *Math. Nachr.* **19**, 265–322 (1958)

41. LEITMAN, M.J., FISHER, G.M.C.: The linear theory of viscoelasticity. In: Flügge, S. (ed.) *Handbuch der Physik*, vol. VIa/3, pp. 1–123. Springer, Berlin, 1973
42. LIU, Z., ZHENG, S.: On the exponential stability of linear viscoelasticity and thermoviscoelasticity. *Quart. Appl. Math.* **54**, 21–31 (1996)
43. MUÑOZ RIVERA, J.E.: Asymptotic behaviour in linear viscoelasticity. *Quart. Appl. Math.* **52**, 629–648 (1994)
44. NOLL, W.: A new mathematical theory of simple materials. *Arch. Rational Mech. Anal.* **48**, 1–50 (1972)
45. PATA, V.: Exponential stability in linear viscoelasticity. *Quart. Appl. Math.* **64**, 499–513 (2006)
46. PAZY, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983
47. RENARDY, M., HRUSA, W.J., NOHEL, J.A.: *Mathematical Problems in Viscoelasticity*. Wiley, New York, 1987
48. VOLTERRA, V.: Sur les équations intégro-différentielles et leurs applications. *Acta Math.* **35**, 295–356 (1912)
49. VOLTERRA, V.: *Leçons sur les fonctions de lignes*. Gauthier-Villars, Paris, 1913
50. WIDDER, D.V.: *The Laplace Transform*. Princeton University Press, Princeton, 1941

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