ON THE INTERSECTION OF HERMITIAN SURFACES

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ABSTRACT. We provide a description of the configuration arising from intersection of two Hermitian surfaces in $\mathrm{PG}(3,q)$, provided that the linear system they generate contains at least a degenerate variety.

1. INTRODUCTION

In [2], the seven point–line configurations arising from the intersection of two Hermitian curves are described and a classification of Hermitian pencils yielding a given configuration is provided. In [3] it has been shown that these configurations are projectively unique and their full collineation group has been determined. Given two Hermitian varieties $\mathcal{H}_1$ and $\mathcal{H}_2$, their intersection $\mathcal{E}$ is just the base locus of the $\mathrm{GF}(\sqrt{q})$–linear system they generate, namely

$$\Gamma(\mathcal{H}_1, \mathcal{H}_2) = \{\mathcal{H}_1 + \lambda \mathcal{H}_2 : \lambda \in \mathrm{GF}(\sqrt{q})\}.$$

In this paper we determine the size of such an intersection depending on the number of degenerate varieties in Γ and describe the actual point–line configurations arising in the 3–dimensional case, provided that Γ contains at least a degenerate surface.

2. INTERSECTION NUMBERS

The set of all singular points of a Hermitian variety $\mathcal{H}$ is a subspace $\mathrm{rad}\,\mathcal{H}$, the *radical* of $\mathcal{H}$. We recall that the *rank* of a Hermitian variety in $\mathrm{PG}(n,q)$ is the number $r = n + 1 - \dim \mathrm{rad}\,\mathcal{H}$.

Let now $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hermitian hypersurfaces of $\mathrm{PG}(n,q)$ and denote by r_i the number of varieties of rank i in the $\mathrm{GF}(\sqrt{q})$–pencil Γ they generate. We shall call the list $(r_1, \ldots, r_n)$ the *rank sequence* of Γ.

It has been observed in [2] that the cardinality of the base locus $\mathcal{E}$ of Γ depends only on its rank sequence. In fact the considerations provided in [2] about the 2–dimensional case may be generalised to arbitrary dimension n, as it has been done in [4].

Proposition 1. *The rank sequence* $(r_1, \ldots, r_n)$ *of a pencil* Γ *of Hermitian varieties in* $\mathrm{PG}(n,q)$ *satisfies the inequality*

$$\sum_{i=1}^{n} (n - i + 1) r_i \leq n + 1.$$

Since, see [1], the total number of points of the non–degenerate Hermitian hypersurface $\mathcal{H}$ of $\mathrm{PG}(n,q)$ is,

$$\mu(n,q) = [q^{(n+1)/2} + (-1)^n][q^{n/2} - (-1)^n]/(q-1)],$$

it is possible to formulate the following proposition.

Proposition 2. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two non–degenerate Hermitian varieties in $\mathrm{PG}(n,q)$, and let $(r_1, \ldots, r_n)$ be the rank sequence of the pencil $\Gamma(\mathcal{H}_1, \mathcal{H}_2)$. Then,*

$$|\mathcal{H}_1 \cap \mathcal{H}_2| = \eta_n(\Gamma, q) =$$

$$\frac{1}{\sqrt{q}(q-1)}\{(1-q^{n+1}) + \sum_{i=1}^{n} r_i[(q\sqrt{q}\mu(i-1,q)+1)(q^{n+1-i}-1)$$

$$- (q-1)\mu(n,q)]\} + (1+\frac{1}{\sqrt{q}})\mu(n,q).$$

Table 1 outlines the possible intersection sizes for any two non–degenerate Hermitian surfaces in $\mathrm{PG}(3,q)$. All cases are possible.

r_1	r_2	r_3	$\eta_3(\Gamma, q)$
0	0	0	$(q+1)^2$
0	0	1	$(q+\sqrt{q}+1)(q-\sqrt{q}+1)$
0	0	2	(q^2+1)
0	0	3	q^2-q+1
0	0	4	$(q-1)^2$
0	1	0	$q^2+q\sqrt{q}+q+1$
0	1	1	$q^2+q\sqrt{q}+1$
0	1	2	$(\sqrt{q}+1)(q\sqrt{q}-q+1)$
0	2	0	$(\sqrt{q}+1)(q\sqrt{q}+q-\sqrt{q}+1)$
1	0	0	$q\sqrt{q}+q+1$
1	0	1	$q\sqrt{q}+1$

TABLE 1. Possible intersection numbers for Hermitian surfaces: non–degenerate pencil.

In the rest of this paper we shall usually write just Γ for $\Gamma(\mathcal{H}_1, \mathcal{H}_2)$ when no ambiguity might arise and we shall denote by $\mathcal{E}$ the intersection $\mathcal{H}_1 \cap \mathcal{H}_2$.

3. DESCRIPTION OF THE CONFIGURATIONS

3.1. **Pencils with a degenerate surface of rank** 1. The simplest case to consider is when the linear system Γ contains a degenerate surface $\mathcal{C}$ of rank 1, that is a plane repeated $q+1$ times.

In this case, the intersection is either a degenerate or non–degenerate Hermitian, according as $\mathcal{C}$ is secant or tangent to all the other surfaces in the pencil. It follows directly from Table 1 that the former case occurs when Γ contains also a surface of rank 3, a *Hermitian cone*, whereas the latter is possible only if all the surfaces in $\Gamma \setminus \{\mathcal{C}\}$ are non–degenerate.

3.2. **Pencils whose degenerate surfaces have all rank** 2. A Hermitian surface $\mathcal{P}$ of rank 2 is a set of $\sqrt{q}+1$ planes through a line, the radical of $\mathcal{P}$. In the following propositions we distinguish several cases.

Proposition 3. *Suppose that Γ contains exactly one degenerate surface $\mathcal{P}$ of rank 2. Then, either 1, 2 or $(\sqrt{q}+1)$ components of $\mathcal{P}$ are degenerate Hermitian curves.*

Proof. The radical of $\mathcal{P}$ meets $\mathcal{E}$ in either 1, $\sqrt{q}+1$ or $q+1$ points. Let $n = |\mathrm{rad}\,\mathcal{P} \cap \mathcal{E}|$ and denote by v_2 the number of components of $\mathcal{P}$ which meet $\mathcal{E}$ in a degenerate Hermitian curve.

(1) $n = 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v_2(q\sqrt{q} + q) + (\sqrt{q} + 1 - v_2)q\sqrt{q} + 1;$$

hence, $v_2 = 1$.

(2) $n = \sqrt{q} + 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v_2(q\sqrt{q} + q - \sqrt{q}) + \sqrt{q}(\sqrt{q} + 1 - v_2)(q - 1) + \sqrt{q} + 1;$$

hence, $v_2 = 2$.

(3) $n = q + 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v_2(q\sqrt{q}) + q(\sqrt{q} + 1 - v_2)(\sqrt{q} - 1) + q + 1;$$

hence, $v_2 = \sqrt{q} + 1$.

$\square$

Observe that in general, if $\mathcal{P}$ and $\mathcal{P}'$ are any two degenerate surfaces in Γ, then $\mathcal{R} = \operatorname{rad}\mathcal{P} \cap \operatorname{rad}\mathcal{P}' = \emptyset$ since, otherwise, any point $V \in \mathcal{R}$ would be singular for all the surfaces of Γ. Suppose now that there are two distinct Hermitian surfaces $\mathcal{P}$ and $\mathcal{P}'$ both of rank 2 in Γ; the previous observation proves that $\operatorname{rad}\mathcal{P}$ and $\operatorname{rad}\mathcal{P}'$ have to be mutually skew. Furthermore, both $\operatorname{rad}\mathcal{P}$ and $\operatorname{rad}\mathcal{P}'$ meet any non–degenerate surface in Γ in $(\sqrt{q} + 1)$ points. Thus, we obtain the following proposition.

Proposition 4. *The intersection of two non degenerate Hermitian surfaces $\mathcal{H}_1$, $\mathcal{H}_2$ spawning a pencil with $r_2 = 2$ is the union of all generators of $\mathcal{H}_1$ which pass through two skew $(\sqrt{q} + 1)$– secants.*

3.3. Pencils whose degenerate surfaces have rank 2 and 3. Given a Hermitian cone $\mathcal{C}$ of vertex V and a non–degenerate Hermitian variety $\mathcal{H}$, we denote by $\Gamma'(\mathcal{C}, \mathcal{H})$ the $\mathrm{GF}(\sqrt{\mathrm{q}})$–linear system of Hermitian curves generated by $\mathcal{C}' = \mathcal{C} \cap \pi$ and $\mathcal{H}' = \mathcal{H} \cap \pi$, where π is the polar plane of V with respect to $\mathcal{H}$.

Lemma 5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct Hermitian cones of vertices respectively V_1 and V_2. Assume that the pencil $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ contains at least a non–singular surface and that $V_1 \notin \mathcal{E}$. Then, V_2 belongs to the polar plane of V_1 with respect to any non–degenerate Hermitian surface in Γ.*

Proof. Fix a non–degenerate Hermitian surface $\mathcal{H} \in \Gamma$ and let π be the polar plane of V_1 with respect to $\mathcal{H}$. Since $V_1 \notin \mathcal{H}$, the plane π cuts a non–singular Hermitian curve on $\mathcal{H}$. Suppose $V_2 \notin \pi$; then the line $V_1 V_2$ would meet $\mathcal{H}$ in $\sqrt{q} + 1$ points. On the other hand, $V_1 V_2$ meets $\mathcal{C}_1$ and $\mathcal{C}_2$ in either 1 or $q + 1$ points — a contradiction. It follows that $V_2 \in \pi$ and $|V_1 V_2 \cap \mathcal{E}| \leq 1$. $\square$

Lemma 6. *Take $\mathcal{C}$ to be a Hermitian cone of vertex V and $\mathcal{H}$ to be a non–degenerate Hermitian surface; let also π be the polar plane of V with respect to $\mathcal{H}$ and let $\Gamma = \Gamma(\mathcal{C}, \mathcal{H})$. Then,*
(1) *if $V \notin \mathcal{H}$,*

$$\eta_3(\Gamma, q) = q^2 + q\sqrt{q} + \sqrt{q} + 1 - \eta_2(\Gamma', q)\sqrt{q};$$

(2) *if $V \in \mathcal{H}$,*

$$\eta_3(\Gamma, q) = q^2 - q + |\pi \cap \mathcal{E}|.$$

Proof. Let $\mathcal{H}' = \mathcal{H} \cap \pi$ and $\mathcal{C}' = \mathcal{C} \cap \pi$. Take $h = \eta_2(\Gamma', q)$. Observe that any line through V tangent to $\mathcal{H}$ is of the form PV with $P \in \mathcal{H}'$. If $V \notin \mathcal{H}$, every line through V meets $\mathcal{H}$ in either 1 or $\sqrt{q} + 1$ points; on the other hand, exactly h generators of $\mathcal{C}$ are tangent to $\mathcal{H}$, whence it follows

$$\eta_3(\Gamma, q) = h + (q\sqrt{q} + 1 - h)(\sqrt{q} + 1).$$

Assume $V \in \mathcal{H}$. Hence, π is the tangent plane to $\mathcal{H}$ at V and $\mathcal{H}'$ consists of $\sqrt{q} + 1$ lines through V. However, $\mathcal{C}'$ consists of either 1 line or a degenerate Hermitian curve. In the former case we

would have $\eta_3(\Gamma, q) = q^2 + q + 1$, which is not possible. Hence, $\mathcal{C}'$ is the union of $\sqrt{q} + 1$ lines through V and

$$\eta_3(\Gamma, q) = \sqrt{q}(q\sqrt{q} - \sqrt{q}) + |\pi \cap \mathcal{E}|.$$

Observe that in this case, all the curves in the linear system Γ are degenerate. $\qquad\square$

Using the cardinality formula of Proposition 2, together with Lemma 6, it is possible to reconstruct the rank sequence of $\Gamma(\mathcal{C}, \mathcal{H})$ from the rank sequence of Γ'.

Lemma 7. *Assume Γ to contain at least one cone $\mathcal{C}$ of vertex $V \notin \mathcal{E}$, and let the rank sequence of Γ' be (r_1', r_2'). Then, the rank sequence of Γ is $(0, r_1', r_2' + 1)$.*

In fact, it is possible to describe in an accurate way the actual configuration $\mathcal{E} \cap \pi$ in terms of the seven classes of [2]. We shall denote such classes as in [3].

Proposition 8. *Let Γ be a $\mathrm{GF}(\sqrt{q})$-pencil of Hermitian surfaces with rank sequence $(0, r_1', r_2' + 1)$. Then, the base configuration $\mathcal{E}' = \mathcal{E} \cap \pi$ is uniquely determined.*

Proof. The cardinality of $\mathcal{E}'$ is determined by Lemma 7. Observe that 5 of the 7 classes of [2] are uniquely determined by their rank sequence. However, both class III and IV correspond to the same rank sequence $(r_1', r_2') = (0, 1)$. By Lemma 7, Γ has necessarily rank sequence $(0, 0, 2)$ and $|\mathcal{E}| = q^2 + 1$. Denote then by $\mathcal{C}_1$ and $\mathcal{C}_2$ the two distinct Hermitian cones of Γ and assume they have respectively vertices V_1 and V_2. There are two possibilities for $\mathcal{E}' = \mathcal{E} \cap \pi$:

(1) $\mathcal{E}'$ belongs to class III, that is $\mathcal{E}'$ consists of $\sqrt{q} - 1$ sublines, all disjoint, and 2 more points;
(2) $\mathcal{E}'$ belongs to class IV, that is $\mathcal{E}'$ consists of $\sqrt{q}$ sublines, which meet all in a point P.

The former case occurs if $V_2 \notin \mathcal{E}$. If $\mathcal{E}'$ belongs to class IV, then $V_2 = P \in \mathcal{E}$. $\qquad\square$

Proposition 9. *Let Γ be a non–degenerate linear system of Hermitian surfaces with $r_2(\Gamma) = r_3(\Gamma) = 1$. Then, $\mathcal{E}$ is either the union of $\sqrt{q} + 1$ non–degenerate Hermitian curves all with a point in common, or the union of $\sqrt{q}$ non–degenerate Hermitian curves and a degenerate Hermitian curve, all sharing a $(\sqrt{q} + 1)$–secant.*

Proof. Let $\mathcal{P}$ and $\mathcal{C}$ be respectively the only surface of rank 2 and the only Hermitian cone in Γ. Let also $L = \mathrm{rad}\,\mathcal{P}$ and V be the vertex of $\mathcal{C}$. Observe that $l = |L \cap \mathcal{E}| \in \{1, \sqrt{q} + 1\}$.

(1) $l = 1$. Let M be the point of intersection of $\mathcal{C}$ and L; clearly $M \neq V$. If $V \in \mathcal{E}$, then there is a component π of $\mathcal{P}$ such that $V \in \pi$. However, in this case $\mathcal{C} \cap \pi = PM$. However, this cannot be a plane section of a non–degenerate Hermitian surface; hence, it follows that $V \notin \mathcal{E}$. This being the case, all the $\sqrt{q} + 1$ sections cut on $\mathcal{C}$ by $\mathcal{P}$ are non–degenerate Hermitian curves having the point M in common.
(2) $l = \sqrt{q} + 1$: Let v_1 be the number of the components of $\mathcal{P}$ which meet $\mathcal{C}$ in a degenerate Hermitian curve. Observe that $v_1 \leq 1$ and equality occurs if and only if $V \in \mathcal{E}$. Since,

$$(q^2 + q\sqrt{q} + 1) = (\sqrt{q} + 1) + (\sqrt{q} + 1 - v_1)(q\sqrt{q} - \sqrt{q}) + v_1(q\sqrt{q} + q^2 - \sqrt{q}),$$

we get $v_1 = 1$ and $V \in \mathcal{E}$.

$\qquad\square$

Proposition 10. *Let Γ be a non–degenerate linear system of Hermitian surfaces with $r_2(\Gamma) = 1$ and $r_3(\Gamma) = 2$. Then, $\mathcal{E}$ is the union of $\sqrt{q} + 1$ non degenerate Hermitian curves, all sharing a chord.*

Proof. Let $\mathcal{P}$ be the only Hermitian surface or rank 2 in Γ and denote by $\mathcal{C}_1$, $\mathcal{C}_2$ be the two Hermitian cones. As before, denote by V_1 and V_2 the vertices of respectively $\mathcal{C}_1$ and $\mathcal{C}_2$ and define $L = \mathrm{rad}\,\mathcal{P}$ and $l = |L \cap \mathcal{E}|$. Either $l = 1$ or $l = \sqrt{q} + 1$. If it were $l = 1$, then we would obtain that $\mathcal{E}$ is the union of $\sqrt{q} + 1$ non–degenerate Hermitian curves, all with a point

in common. However, this is a contradiction because of Proposition 2. Assume then $l = \sqrt{q} + 1$ and denote by v_1 the number of components of $\mathcal{P}$ meeting $\mathcal{E}$ in a degenerate Hermitian curve. Then,

$$(\sqrt{q} + 1)(q\sqrt{q} - q + 1) = (\sqrt{q} + 1) + (\sqrt{q} + 1 - v_1)(q\sqrt{q} - \sqrt{q}) + v_1(q\sqrt{q} + q^2 - \sqrt{q}).$$

This is possible only if $v_1 = 0$, that is, $\mathcal{E}$ is the union of $\sqrt{q} + 1$ non–degenerate Hermitian curves, all sharing a chord. $\square$

3.4. Pencils whose degenerate surfaces have all rank 3.

All the pencils considered in this section contains at least a cone $\mathcal{C}$. We shall denote by s_1, s_2 and s_3 the number of generators of $\mathcal{C}$ meeting $\mathcal{E}$ in respectively $q + 1$, $\sqrt{q} + 1$ or 1 points.

Lemma 11. *Let Γ contain at least 3 distinct cones $\mathcal{C}_1 \ldots \mathcal{C}_3$ of vertices $V_1 \ldots V_3$. Then, either the vertices of all the cones in Γ are collinear or at most one of them is in $\mathcal{E}$.*

Proof. Suppose $V_1, V_2 \in \mathcal{E}$; then, $V_1 V_2 \subseteq \mathcal{E}$. However, for $V_1 V_2$ to be a subset of $\mathcal{E}$, it is necessary for it to be a generator of $\mathcal{C}_3$ also. It follows $V_3 \in V_1 V_2$. $\square$

Observe that when Γ contains at least two Hermitian cones, the number of lines in $\mathcal{E}$ is at most 1.

Proposition 12. *If Γ contains 4 Hermitian cones, then none of the vertices of such cones belongs to $\mathcal{E}$ and exactly $\sqrt{q}(q - \sqrt{q} - 2)$ generators of any cone meet $\mathcal{E}$ in $(\sqrt{q} + 1)$ points, the remaining $(\sqrt{q} + 1)^2$ being tangent lines.*

Proof. By Proposition 2, $|\mathcal{E}| = (q - 1)^2$. Suppose $V_1, V_2, V_3, V_4 \in \mathcal{E}$. Then,

$$(q - 1)^2 = (q + 1) + s_2\sqrt{q} + (q\sqrt{q} - s_2),$$

that is $q(q^2 - \sqrt{q} - 3) = s_2(\sqrt{q} - 1)$, a contradiction since $(\sqrt{q} - 1)$ does not divide $q^2 - \sqrt{q} - 3$. If $V_1 \in \mathcal{E}$ while $V_2, V_3, V_4 \notin \mathcal{E}$. Observe that a generator of $\mathcal{C}_1$ either meets a non–degenerate surface $\mathcal{H}$ is 1 or in $\sqrt{q} + 1$ points. Then,

$$(q - 1)^2 = s_2\sqrt{q} + 1,$$

which gives $s_2 = q\sqrt{q} - 2\sqrt{q}$, that is $2\sqrt{q} + 1$ generators of $\mathcal{C}$ meet $\mathcal{H}$ in V only and all these generators lie in the tangent plane to $\mathcal{H}$ at V. However, the number of generators of $\mathcal{C}$ on any plane is at most $\sqrt{q} + 1$, which gives a contradiction. It follows that $\mathcal{E}$ does not contain the vertex of any cone in Γ and

$$(q - 1)^2 = s_2(\sqrt{q} + 1) + (q\sqrt{q} + 1 - s_2),$$

that is, $s_2 = \sqrt{q}(q - \sqrt{q} - 2)$, which gives the result. $\square$

Proposition 13. *Suppose Γ to contain exactly 3 cones $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Then, there are two possibilities:*

(1) $V_1, V_2, V_3 \notin \mathcal{E}$: *then $\sqrt{q}(q - \sqrt{q} - 1)$ components of each cone are $(\sqrt{q} + 1)$–secants to any non–degenerate Hermitian surface in Γ;*

(2) $V_1 \in \mathcal{E}$ *but $V_2, V_3 \notin \mathcal{E}$: then $\sqrt{q}(q - 1)$ generators of the cone $\mathcal{C}_1$ meet $\mathcal{E}$ in $(\sqrt{q} + 1)$ points, the remaining intersecting $\mathcal{E}$ in V_1 only; the number of generators of the cones $\mathcal{C}_2$ and $\mathcal{C}_3$ meeting $\mathcal{E}$ in $(\sqrt{q} + 1)$ points is $q\sqrt{q} - q - \sqrt{q}$, the others meeting $\mathcal{E}$ in distinct points;*

Proof. The cardinality of $\mathcal{E}$ is $q^2 - q + 1$.

(1) Since $V_1 \notin \mathcal{E}$,

$$q^2 - q + 1 = s_2(\sqrt{q} + 1) + (q\sqrt{q} - s_2 + 1).$$

Hence, there are $\sqrt{q}(q - \sqrt{q} - 1)$ components of $\mathcal{C}_1$ are meeting $\mathcal{E}$ in $(\sqrt{q} + 1)$ points, the remaining $q + \sqrt{q} + 1$ being tangent to any surface.

(2) Since $V_1 \in \mathcal{E}$, each generator of $\mathcal{C}_1$ meets $\mathcal{E}$ in either 1 or $\sqrt{q}+1$ points. We get

$$q^2 - q + 1 = 1 + t\sqrt{q}.$$

It follows that $\sqrt{q}(q-1)$ generators through V meet $\mathcal{E}$ in $\sqrt{q}+1$ points. Consider now another cone $\mathcal{C}_2 \in \Gamma$. Since $V_2 \notin \mathcal{E}$, we get

$$q^2 - q + 1 = s_2(\sqrt{q}+1) + (q\sqrt{q}+1-s_2);$$

hence $s_2 = q\sqrt{q} - q - \sqrt{q}$.

Suppose now $V_1, V_2 \in \mathcal{E}$. Then, the line V_1V_2 is a generator of any surface $\mathcal{H} \in \Gamma$ and we have

$$q^2 - q + 1 = q + 1 + s_2\sqrt{q}.$$

It follows that $s_2 = q\sqrt{q} - 2\sqrt{q}$. This gives that there should be $2\sqrt{q}+1 > \sqrt{q}+1$ generators through V_1 meeting $\mathcal{H}$ in V_1 only — a contradiction. $\qquad\square$

Proposition 14. *Assume that the pencil Γ contains exactly two cones $\mathcal{C}_1$, $\mathcal{C}_2$ of respectively vertices V_1 and V_2. Then, one of the following possibilities holds:*
(1) *both $V_1, V_2 \in \mathcal{E}$; then, $\mathcal{E}$ contains the line V_1V_2; $q(\sqrt{q}-1)$ components of each cone meet $\mathcal{E}$ in $\sqrt{q}+1$ points.*
(2) *$V_1 \in \mathcal{E}$, while $V_2 \notin \mathcal{E}$; $\mathcal{E}$ does not contain any line; $q\sqrt{q}$ components of $\mathcal{C}_1$ and $q(\sqrt{q}-1)$ components of $\mathcal{C}_2$ meet $\mathcal{E}$ in $\sqrt{q}+1$ points.*
(3) *$V_1, V_2 \notin \mathcal{E}$ belong to $\mathcal{E}$; $q(\sqrt{q}-1)$ components of each cone meet $\mathcal{E}$ in $(q+1)$ points.*

Proof. Let V be the vertex of any cone $\mathcal{C}$ in Γ. Observe that for $V \notin \mathcal{E}$,

$$q^2 + 1 = s_2(\sqrt{q}+1) + (q\sqrt{q}+1-s_2);$$

hence, $s_2 = q\sqrt{q} - q$. On the other hand, if $V \in \mathcal{E}$

$$q^2 + 1 = s_1 q + s_2\sqrt{q} + 1;$$

hence, $s_2 = \sqrt{q}(q-s_1)$. The result now follows from $s_1 \leq 1$. $\qquad\square$

Proposition 15. *Assume that the only degenerate surface in the pencil Γ is a cone $\mathcal{C}$. Then, either*
(1) *$V \notin \mathcal{E}$ and $\sqrt{q}(q - \sqrt{q}+1)$ components of $\mathcal{C}$ are meet $\mathcal{E}$ in $\sqrt{q}+1$ points, or*
(2) *$V \in \mathcal{E}$ and $\mathcal{E}$ contains at least a line.*

Proof.
(1) If $V \notin \mathcal{E}$, then $s_1 = 0$ and

$$q^2 + q + 1 = s_2(\sqrt{q}+1) + (q\sqrt{q}+1-s_2);$$

hence, $s_2 = \sqrt{q}(q - \sqrt{q}+1)$.
(2) If $V \in \mathcal{E}$,

$$q^2 + q + 1 = s_1 q + s_2\sqrt{q} + 1;$$

hence, $s_2 = \sqrt{q}(q+1-s_1)$. Since the total number of components of $\mathcal{C}$ is $q\sqrt{q}+1$, it follows that $s_2 \leq q\sqrt{q}+1$ and $s_1 \geq 1$.

$\qquad\square$

Suppose that Γ contains exactly one Hermitian cone whose vertex belongs to $\mathcal{E}$, and let π be the tangent plane at P to a non–degenerate Hermitian surface in Γ. Clearly, all generators in $\mathcal{E}$ lie in π; furthermore $s_1 \in \{1, 2, \sqrt{q}+1\}$. Consequently, $s_2 \in \{q\sqrt{q}, q\sqrt{q} - \sqrt{q}, q\sqrt{q} - q\}$. We observe also that if $s_1 > 1$, then all non–degenerate Hermitian surfaces in Γ share the same tangent plane at P.

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