



A thermodynamic approach to ferromagnetism and phase transitions

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ABSTRACT

The paper provides a modelling of the magnetization curve and of the ferromagnetic–paramagnetic transition within a continuum thermodynamic setting. The general model of the nonlinear, time dependent behaviour of ferromagnetic materials is accomplished by regarding the magnetization vector as an internal variable, namely as a vector field whose time evolution is a constitutive equation subject to the requirements of the second law of thermodynamics. The exchange interaction of the magnetization is modelled through a dependence of the free energy on the magnetization gradient. Consistent with the non-simple character of the material, the second law allows for a non-zero extra-entropy flux. A general three-dimensional scheme is elaborated which seems to be new in the literature. The three-dimensional setting is then established for stationary and homogeneous fields thus finding the collinearity and the corresponding form of the magnetic susceptibility. The whole evolution problem for the temperature and the magnetization is provided so that temperature-induced transition processes are allowed. The model accounts also for the dependence of the saturation magnetization on the temperature. Also for the sake of comparison with the existing literature, the evolution equations for the direction and the intensity of magnetization are derived. Known models, such as those of Landau–Lifshitz and Gilbert, are recovered as particular cases of saturated bodies. Next, the model is made more specific so as to account in detail for the saturation, the residual or spontaneous magnetization and the coercive field. First, the classical potential, which traces back to Ginzburg, and the Weiss model are revisited. The corresponding lack of the saturation effect or the description via implicit relations are emphasized. Hence, a new potential, with a logarithmic dependence on the magnetization, is investigated which provides the residual magnetization and the coercive field in an explicit way and satisfies expected properties of the residual magnetization as a function of the temperature.

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1. Introduction

Ferromagnetic materials exhibit a non-linear relation between the magnetic field \mathbf{H} and the magnetization \mathbf{M} . The non-linear relation provides the standard hysteretic phenomena. Nonlinearity and hysteresis occur below a characteristic temperature called the (magnetic) Curie temperature θ_c . Above the Curie temperature, the materials are paramagnetic in that the relation is linear with a coefficient, the magnetic susceptibility, which is inversely proportional to the difference $\theta - \theta_c$ between the current temperature θ and θ_c ; such a dependence is the content of the Curie–Weiss law. The fact that so different $\mathbf{M} - \mathbf{H}$ curves are parameterized by the temperature allows us to cast the passage from one curve to another within the scheme of phase transitions.

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There is a wide variety of approaches to the modelling of magnetization in ferromagnetic bodies. At the bottom, the models deal with the time evolution of the magnetization. We mention that this problem has been addressed in the works of Landau et al. [1,2] which model the time evolution of \mathbf{M} in a magnetically-saturated body under the action of a time-dependent field \mathbf{H} . Next, in the framework of micromagnetics and gyromagnetics, a similar phenomenological approach has been set up by Gilbert [3,4] and then improved by Brown [5] who replaced the field \mathbf{H} with an effective field \mathbf{H}_{eff} . By selecting an appropriate form of \mathbf{H}_{eff} , Mallinson [6] has derived a description of the switching effect in damped gyromagnetics. Recent papers exhibit more involved models to account for the evolution of domain walls in ferromagnets (see, e.g., [7–9]). Such models are well-motivated from micromagnetics but are essentially isothermal in character. Non-isothermal models are provided by Maugin [10] about magnetoelasticity through internal variables and next by Maugin and Fomethe [11] to model phase-transition fronts in deformable ferromagnets.

As a particular case, namely in stationary conditions, the evolution equation is expected to provide the $\mathbf{M} - \mathbf{H}$ relation of the magnetization curve. Upon the assumption that \mathbf{M} and \mathbf{H} have a common fixed alignment, the evolution equation was first set up in the form (see, e.g., [2,12])

$$\dot{\mathbf{M}} = \alpha_0 \mu_0 \mathbf{H} - \alpha_1 (\theta - \theta_c) \mathbf{M} + \alpha_2 \mathbf{M}^3 + \alpha_3 \Delta \mathbf{M}.$$

This non-isothermal model accounts for the transition, at the Curie temperature θ_c , between the paramagnetic and the ferromagnetic behaviours. Yet, the corresponding magnetization curve

$$\alpha_0 \mu_0 \mathbf{H} = \alpha_1 (\theta - \theta_c) \mathbf{M} - \alpha_2 \mathbf{M}^3,$$

does not allow for saturation in the ferromagnetic regime ($\theta < \theta_c$). In 1907, Weiss developed a theory of ferromagnetic domains structure known as *mean field theory*. In essence he arranged the Langevin potential in order to describe the non-isothermal paramagnetic–ferromagnetic transition and to account for the saturation phenomena [13]. Such a model is motivated by statistical physics but is one-dimensional in character.

The purpose of this paper is threefold. The first fold is to model the nonlinear, time dependent behaviour of ferromagnetic materials within a thermodynamic framework in three-dimensional setting. By analogy with [14], this is accomplished by regarding the magnetization \mathbf{M} as an internal variable, namely as a phase field whose time evolution is given by a constitutive equation subject to the requirements of the second law of thermodynamics. The exchange interaction of the magnetization is modelled through a dependence of the free energy on the gradient of \mathbf{M} . Consistent with the non-local character of the material, the second law allows for a non-zero extra entropy flux. A general form of the evolution equation is then derived and known models, such as those of Landau–Lifshitz, Gilbert and others, are recovered as particular cases of saturated bodies. The second fold is to improve the model of the $\mathbf{M} - \mathbf{H}$ relation in ferromagnetism so as to account in detail for the saturation, the residual or spontaneous magnetization and the coercive field. This part is realized by starting with the classical potential which traces back to Ginzburg and showing that a different behaviour (ferromagnetic–paramagnetic) occurs according as $\theta < \theta_c$ or $\theta > \theta_c$ but the saturation effect does not follow. The Weiss model is then examined whence it follows that transition and saturation are allowed though via implicit relations. A new singular potential, with a logarithmic dependence on the magnetization, is investigated which provides the residual magnetization and the coercive field in an explicit way and satisfies expected properties of the residual magnetization as a function of the temperature. The third fold is to provide the whole set of evolution equations, for the temperature and the order parameter, in the three-dimensional framework. As a result, our model describes temperature-induced reversible transitions between the paramagnetic and the ferromagnetic regimes. Hence, we can control the phase transition process by acting on the external heat source and the applied magnetic field. Moreover, the scheme is set up in a general way so as to allow also for the dependence of the spontaneous magnetization, relative to the saturation magnetization, on the temperature. These features seem to be new in the literature.

The advantage of the present approach is the unified, thermodynamically-consistent scheme of constitutive equations and evolution equations which in turn provide the magnetization curve. The pertinent equations prove to be characterized by the free energy as a thermodynamic potential. The scheme is three-dimensional in character and, also for the sake of comparison with the existing literature, the evolution equations for the direction and the intensity of \mathbf{M} are derived. It is of interest that all of the schemes appeared in the literature (e.g., time-dependent or stationary, Lagrangian) are recovered as particular cases. In particular, the evolution equation of the direction reduces to that of Landau and Lifshitz if the exchange and anisotropic interactions are neglected. Also, in static conditions, Brown's equation is obtained with a general form of the effective magnetic field. Next, as an application, the appropriate potential for crystals of iron is determined through the data for the residual magnetization versus θ/θ_c . Moreover, a one-parameter free energy is shown to provide a satisfactory description of both the ferromagnetic and the paramagnetic behaviour according as the temperature is below or above the Curie temperature.

2. Balance equations

An undeformable ferromagnetic material occupies the region $\Omega \subseteq \mathbb{R}^3$. The electric field \mathbf{E} , the magnetic induction \mathbf{B} , the electric displacement \mathbf{D} and the magnetic field \mathbf{H} satisfy Maxwell's equations, in the space-time domain $\Omega \times \mathbb{R}$,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \times \mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}, \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = \rho, \quad (2.2)$$

where \mathbf{J} is the current density and ρ is the charge density. The superposed dot denotes the time derivative and ∇ is the gradient operator. The balance of energy in electromagnetic materials is based on the view that $\mathbf{E} \times \mathbf{H}$ is the vector flux of energy of electromagnetic character. This view follows from Poynting's theorem which merely shows that

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{E} \cdot \mathbf{J} \quad (2.3)$$

is a consequence of Maxwell's equations.

Since the body is undeformable, then the balance of energy is taken in the form (see [15])

$$\dot{e} = -\nabla \cdot (\mathbf{E} \times \mathbf{H} + \mathbf{q}) + r,$$

where e is the internal energy density, \mathbf{q} is the heat flux and r is the heat supply, namely energy per unit volume and unit time provided by external sources. By means of the identity (2.3) we have

$$\dot{e} = \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{J} \cdot \mathbf{E} - \nabla \cdot \mathbf{q} + r. \quad (2.4)$$

The second law of thermodynamics is taken as the statement that the Clausius–Duhem inequality holds for any set of functions which satisfy Maxwell's equations (2.1) and (2.2) and the energy equation (2.4). Also because of possible nonlocal effects, the entropy flux is likely to be different from \mathbf{q}/θ , θ being the absolute temperature. Hence, letting η be the entropy density and \mathbf{k} the extra-entropy flux vector, we write the Clausius–Duhem inequality in the form

$$\dot{\eta} \geq -\nabla \cdot (\mathbf{q}/\theta) - \nabla \cdot \mathbf{k} + \frac{r}{\theta}. \quad (2.5)$$

The extra-entropy flux \mathbf{k} is required to satisfy the boundary condition

$$\int_{\partial\Omega} \mathbf{k} \cdot \mathbf{n} \, da = 0 \quad (2.6)$$

for the whole body (see [16]). This allows (2.5) to provide the standard global statement of the second law,

$$\frac{d}{dt} \int_{\Omega} \eta \, dv \geq \int_{\Omega} \frac{r}{\theta} \, dv - \int_{\partial\Omega} \frac{1}{\theta} \mathbf{q} \cdot \mathbf{n} \, da.$$

By (2.4) and (2.5) we have

$$\dot{e} - \theta \dot{\eta} - \mathbf{H} \cdot \dot{\mathbf{B}} - \mathbf{E} \cdot \dot{\mathbf{D}} - \mathbf{J} \cdot \mathbf{E} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta - \theta \nabla \cdot \mathbf{k} \leq 0.$$

For later convenience we consider the free energy density

$$\psi = e - \theta \eta.$$

Hence, the Clausius–Duhem inequality becomes

$$\dot{\psi} + \eta \dot{\theta} - \mathbf{H} \cdot \dot{\mathbf{B}} - \mathbf{E} \cdot \dot{\mathbf{D}} - \mathbf{J} \cdot \mathbf{E} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta - \theta \nabla \cdot \mathbf{k} \leq 0. \quad (2.7)$$

Having in mind a model for ferromagnetism, we disregard polarization and let

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \quad (2.8)$$

This assumption is consistent with the fact that polarization does not contribute to magnetization in bodies at rest (see e.g. [12], p. 83, and [17]).

Upon substitution of (2.8) in (2.7) we find that the Clausius–Duhem inequality takes the form

$$\dot{\psi} + \eta \dot{\theta} - \mu_0 \mathbf{H} \cdot \dot{\mathbf{H}} - \mu_0 \mathbf{H} \cdot \dot{\mathbf{M}} - \epsilon_0 \mathbf{E} \cdot \dot{\mathbf{E}} - \mathbf{J} \cdot \mathbf{E} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta - \theta \nabla \cdot \mathbf{k} \leq 0. \quad (2.9)$$

Restrictions placed by the inequality (2.9) are now evaluated for a rather general set of constitutive equations.

3. Thermodynamic restrictions

Let $\psi, \eta, \mathbf{q}, \mathbf{k}$ and $\dot{\mathbf{M}}$ be given by (constitutive) functions of the set of variables

$$\Gamma = (\theta, \mathbf{E}, \mathbf{H}, \mathbf{M}, \nabla \theta, \nabla \mathbf{M}).$$

The vector quantities \mathbf{q}, \mathbf{k} and the time derivative $\dot{\mathbf{M}}$ are allowed to depend also on the higher-order gradients

$$\nabla \nabla \theta, \nabla \nabla \mathbf{M}.$$

Maxwell's equations read

$$\mu_0(\dot{\mathbf{H}} + \dot{\mathbf{M}}) = -\nabla \times \mathbf{E}, \quad \nabla \times \mathbf{H} = \epsilon_0 \dot{\mathbf{E}} + \mathbf{J}, \quad (3.1)$$

$$\nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{H}, \quad \epsilon_0 \nabla \cdot \mathbf{E} = \rho. \quad (3.2)$$

Eq. (3.2) hold at any time t , as a consequence of (3.1), provided they hold at an initial time t_0 and the continuity equation

$$\nabla \cdot \mathbf{J} + \dot{\rho} = 0$$

holds. Hence we can take the values of $\dot{\mathbf{H}}, \dot{\mathbf{E}}$ as arbitrary whereas $\dot{\mathbf{M}}$ is provided by the pertinent constitutive equation,

$$\dot{\mathbf{M}} = \dot{\mathbf{M}}(\Gamma).$$

The function $\dot{\mathbf{M}}$ may be viewed as γ times the d'Alembertian inertia couple density, γ being the gyromagnetic ratio.

The space dependence of \mathbf{E} and \mathbf{H} is required to be appropriate so that $\nabla \times \mathbf{E}, \nabla \times \mathbf{H}$ satisfy Eq. (3.1) and $\nabla \cdot \dot{\mathbf{M}}, \nabla \cdot \dot{\mathbf{E}}$ satisfy

$$\nabla \cdot \dot{\mathbf{M}} + \nabla \cdot \dot{\mathbf{H}} = 0, \quad \epsilon_0 \nabla \cdot \dot{\mathbf{E}} + \nabla \cdot \mathbf{J} = 0.$$

The chain rule allows us to write the inequality (2.9) in the form

$$(\psi_\theta + \eta)\dot{\theta} + \psi_{\nabla\theta} \nabla \dot{\theta} + (\psi_{\mathbf{H}} - \mu_0 \mathbf{H}) \cdot \dot{\mathbf{H}} + (\psi_{\mathbf{M}} - \mu_0 \mathbf{H}) \cdot \dot{\mathbf{M}} + (\psi_{\mathbf{E}} - \epsilon_0 \mathbf{E}) \cdot \dot{\mathbf{E}} + \psi_{\nabla \mathbf{M}} \cdot \nabla \dot{\mathbf{M}} - \mathbf{J} \cdot \mathbf{E} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta - \theta \nabla \cdot \mathbf{k} \leq 0, \quad (3.3)$$

where the indices $\theta, \nabla\theta, \mathbf{H}, \mathbf{M}, \mathbf{E}, \nabla \mathbf{M}$ denote partial derivatives. The arbitrariness of $\dot{\theta}, \nabla \dot{\theta}$ and $\dot{\mathbf{H}}, \dot{\mathbf{E}}$ requires that

$$\eta = -\psi_\theta, \quad \psi_{\nabla\theta} = 0$$

and

$$\psi_{\mathbf{H}} = \mu_0 \mathbf{H}, \quad \psi_{\mathbf{E}} = \epsilon_0 \mathbf{E}.$$

As a consequence,

$$\psi = \frac{1}{2} \mu_0 \mathbf{H}^2 + \frac{1}{2} \epsilon_0 \mathbf{E}^2 + \Psi(\theta, \mathbf{M}, \nabla \mathbf{M}).$$

Upon some rearrangements, the inequality (3.3) becomes

$$(\Psi_{\mathbf{M}} - \mu_0 \mathbf{H} - \nabla \cdot \Psi_{\nabla \mathbf{M}}) \cdot \dot{\mathbf{M}} + \nabla \cdot (\Psi_{\nabla \mathbf{M}} \dot{\mathbf{M}} - \theta \mathbf{k}) + \mathbf{k} \cdot \nabla \theta - \mathbf{J} \cdot \mathbf{E} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \leq 0. \quad (3.4)$$

A simple scheme arises by letting

$$\theta \mathbf{k} = \Psi_{\nabla \mathbf{M}} \dot{\mathbf{M}} \quad (3.5)$$

and hence (2.6) requires that

$$\int_{\partial\Omega} \frac{1}{\theta} \Psi_{\nabla \mathbf{M}} \dot{\mathbf{M}} \cdot \mathbf{n} da = 0. \quad (3.6)$$

Look now at the corresponding conditions which guarantee the validity of (3.4). By (3.5) we have

$$(\Psi_{\mathbf{M}} - \mu_0 \mathbf{H} - \nabla \cdot \Psi_{\nabla \mathbf{M}}) \cdot \dot{\mathbf{M}} + \mathbf{k} \cdot \nabla \theta = [\theta(\hat{\Psi}_{\mathbf{M}} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}) - \mu_0 \mathbf{H}] \cdot \dot{\mathbf{M}},$$

where

$$\hat{\Psi} = \frac{\Psi}{\theta}, \quad \dot{\mathbf{M}} = \dot{\mathbf{M}}(\Gamma).$$

Hence (3.4) reduces to

$$[\theta(\hat{\Psi}_{\mathbf{M}} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}) - \mu_0 \mathbf{H}] \cdot \dot{\mathbf{M}} - \mathbf{J} \cdot \mathbf{E} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \leq 0. \quad (3.7)$$

The inequality (3.7) holds if

$$\mathbf{J} \cdot \mathbf{E} \geq 0, \quad \mathbf{q} \cdot \nabla \theta \leq 0, \quad (3.8)$$

and

$$[\theta(\hat{\Psi}_{\mathbf{M}} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}) - \mu_0 \mathbf{H}] \cdot \dot{\mathbf{M}} \leq 0. \quad (3.9)$$

The inequalities (3.8) are satisfied by Ohm's and Fourier's laws, namely

$$\mathbf{J} = \sigma \mathbf{E}, \quad \mathbf{q} = -\kappa \nabla \theta$$

with positive-valued functions σ and κ of Γ . The inequality (3.9) is a restriction on the constitutive equation for the time derivative $\dot{\mathbf{M}}$ and hence on the time evolution of the magnetic polarization \mathbf{M} . To save writing, we can express the inequality (3.9) as

$$\mathcal{N} \cdot \dot{\mathbf{M}} \leq 0, \quad (3.10)$$

where

$$\mathcal{N} := \delta_{\mathbf{M}} \hat{\Psi} - \mu_0 \mathbf{H}, \quad \delta_{\mathbf{M}} \hat{\Psi} := \theta (\hat{\Psi}_{\mathbf{M}} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}). \quad (3.11)$$

The time dependence of the magnetization \mathbf{M} and the $\mathbf{M} - \mathbf{H}$ relation are the subject of the next sections. The inequalities (3.9) and (3.10) are investigated to determine possible forms of the d'Alembertian inertia couple.

Remark 1. Henceforth, we let

$$(\Psi_{\nabla \mathbf{M}} \dot{\mathbf{M}}) \cdot \mathbf{n} = 0, \quad \text{on } \partial \Omega, \quad (3.12)$$

so that (3.6) is satisfied. Incidentally, there are approaches to magnetic modelling where the extra-entropy flux does not occur (see, e.g., [10]). Yet the boundary condition is placed just in the form (3.12).

Remark 2. In terms of the Gibbs free energy $\tilde{\Psi} = \Psi - \mu_0 \mathbf{H} \cdot \mathbf{M}$ we have

$$\mathcal{N} = \delta_{\mathbf{M}} \frac{\tilde{\Psi}}{\theta}.$$

Hence (3.9) becomes

$$\delta_{\mathbf{M}} \frac{\tilde{\Psi}}{\theta} \cdot \dot{\mathbf{M}} \leq 0.$$

4. Restrictions on the evolution of \mathbf{M}

We now have to find a function $\hat{\mathbf{M}}(\Gamma)$ compatible with (3.10). Letting

$$\mathbf{w} = \hat{\mathbf{M}} \quad (4.1)$$

we can write the inequality (3.10) as

$$\mathcal{N} \cdot \mathbf{w} \leq 0, \quad (4.2)$$

where \mathcal{N} and \mathbf{w} are parameterized by \mathbf{M} . Let

$$\mathbf{v} = \mathcal{N} + \epsilon \mathbf{M} \times \mathbf{w}, \quad \epsilon \in \mathbb{R}.$$

The inequality (4.2) is equivalent to

$$\mathbf{v} \cdot \mathbf{w} \leq 0 \quad (4.3)$$

and (4.3) holds if

$$\mathbf{w} = -\Lambda \mathbf{v} + \beta \mathbf{M} \times \mathbf{v} + \nu \mathbf{M} \times (\mathbf{M} \times \mathbf{v}),$$

where $\nu \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ and $\Lambda \in \text{Sym}^+$, Sym^+ being the space of positive semidefinite tensors. This is so because, upon substitution, we have

$$\mathbf{v} \cdot \mathbf{w} = -\mathbf{v} \cdot \Lambda \mathbf{v} - \nu |\mathbf{M} \times \mathbf{v}|^2.$$

Accordingly we can write the following statement.

Proposition 1. The inequality (4.2) holds if

$$\mathbf{w} = -\Lambda(\mathcal{N} + \epsilon \mathbf{M} \times \mathbf{w}) + \beta \mathbf{M} \times (\mathcal{N} + \epsilon \mathbf{M} \times \mathbf{w}) + \nu \mathbf{M} \times [\mathbf{M} \times (\mathcal{N} + \epsilon \mathbf{M} \times \mathbf{w})],$$

for every $\nu \in \mathbb{R}^+$, every $\beta, \epsilon \in \mathbb{R}$, and every $\Lambda \in \text{Sym}^+$.

By replacing \mathbf{w} via (4.1) and taking advantage of the identity

$$\mathbf{M} \times [\mathbf{M} \times (\mathbf{M} \times \dot{\mathbf{M}})] = -|\mathbf{M}|^2 \mathbf{M} \times \dot{\mathbf{M}},$$

we apply Proposition 1 to conclude that (3.10) holds if

$$\dot{\mathbf{M}} = -\epsilon(\Lambda + v|\mathbf{M}|^2\mathbf{1})\mathbf{M} \times \dot{\mathbf{M}} + \epsilon\beta\mathbf{M} \times (\mathbf{M} \times \dot{\mathbf{M}}) - \Lambda\mathcal{N} + \beta\mathbf{M} \times \mathcal{N} + v\mathbf{M} \times (\mathbf{M} \times \mathcal{N}), \quad (4.4)$$

where $\Lambda \in \text{Sym}^+$ and ϵ, β, v are functions of Γ subject only to $v \geq 0$. At this stage the rescaled free energy $\hat{\Psi}$ is any function of $\theta, \mathbf{M}, \nabla\mathbf{M}$. Hence, Eq. (4.4) is the most general three-dimensional evolution equation for \mathbf{M} compatible with thermodynamics.

We are now in a position to show that the evolution equation (4.4) generalizes both Landau–Lifshitz and Gilbert equations. Indeed, as we show in Proposition 2, (4.4) provides two equations which are more general. They reduce exactly to Landau–Lifshitz and Gilbert equations in saturation conditions.

Denote by \mathbf{m} the unit vector of \mathbf{M} , $\mathbf{m} = \mathbf{M}/|\mathbf{M}|$, and let

$$\Lambda = \alpha_1 \mathbf{m} \otimes \mathbf{m} + \alpha_2 (\mathbf{1} - \mathbf{m} \otimes \mathbf{m}), \quad \alpha_1, \alpha_2 > 0. \quad (4.5)$$

Of course, if $\alpha_1 = \alpha_2$ then Λ reduces to a scalar times the unit tensor $\mathbf{1}$. Indeed, for any vector \mathbf{v} the application of Λ as in (4.5) provides

$$\Lambda\mathbf{v} = \alpha_1 \mathbf{v}_{\parallel} + \alpha_2 \mathbf{v}_{\perp},$$

where \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} are the parallel and the perpendicular parts of \mathbf{v} , relative to the pertinent unit vector \mathbf{m} .

Since

$$(\mathbf{m} \otimes \mathbf{m})\mathbf{M} \times \dot{\mathbf{M}} = 0, \quad \mathbf{M} \times (\Lambda\mathbf{u}) = \alpha_2 \mathbf{M} \times \mathbf{u},$$

for any vector \mathbf{u} , then

$$\Lambda(\mathbf{M} \times \dot{\mathbf{M}}) = \alpha_2 \mathbf{M} \times \dot{\mathbf{M}}, \quad \mathbf{M} \times \Lambda\mathcal{N} = \alpha_2 \mathbf{M} \times \mathcal{N}.$$

Proposition 2. Let Λ be as in (4.5). The three-dimensional evolution equation (4.4) becomes

$$\dot{\mathbf{M}} = -\Lambda\mathcal{N} + \tau_1 \mathbf{M} \times \mathbf{M} \times \mathcal{N} - \tau_2 \mathbf{M} \times \dot{\mathbf{M}}, \quad (4.6)$$

or

$$\dot{\mathbf{M}} = -\tilde{\alpha}(\mathcal{N} \cdot \mathbf{M})\mathbf{M} + \tilde{\gamma}\mathbf{M} \times \mathcal{N} + \tilde{\lambda}\mathbf{M} \times (\mathbf{M} \times \mathcal{N}), \quad (4.7)$$

where $\tau_1, \tau_2, \tilde{\alpha}, \tilde{\gamma}$ and $\tilde{\lambda}$ are parameterized by the constants in (4.4) and by α_1, α_2 and $|\mathbf{M}|^2$.

Proof. Application of $\mathbf{M} \times$ to (4.4) provides the expression for $\mathbf{M} \times (\mathbf{M} \times \dot{\mathbf{M}})$. Substitution in (4.4) and some rearrangements allow us to write (4.4) in the more compact form (4.6), where

$$\tau_1 = v + \frac{\beta^2}{\alpha_2 + v|\mathbf{M}|^2}, \quad \tau_2 = \epsilon(\alpha_2 + v|\mathbf{M}|^2) + \beta \frac{1 + \epsilon\beta|\mathbf{M}|^2}{\alpha_2 + v|\mathbf{M}|^2}.$$

By (4.5) we can write (4.6) as

$$\dot{\mathbf{M}} = -\alpha_1 \mathcal{N}_{\parallel} - \alpha_2 \mathcal{N}_{\perp} + \tau_1 \mathbf{M} \times (\mathbf{M} \times \mathcal{N}_{\perp}) - \tau_2 \mathbf{M} \times \dot{\mathbf{M}}$$

whence

$$\dot{\mathbf{M}} = -\alpha_1 \mathcal{N}_{\parallel} - (\alpha_2 + \tau_1 |\mathbf{M}|^2) \mathcal{N}_{\perp} - \tau_2 \mathbf{M} \times \dot{\mathbf{M}}. \quad (4.8)$$

Vector multiplication of (4.8) by \mathbf{M} provides the expression for $\mathbf{M} \times \dot{\mathbf{M}}$. Substitution in (4.6) and some rearrangements yield

$$(1 + \tau_2^2 |\mathbf{M}|^2) \dot{\mathbf{M}} = -\alpha_1 \mathcal{N}_{\parallel} - (\alpha_2 + \tau_1 |\mathbf{M}|^2) \mathcal{N}_{\perp} + \tau_2 (\alpha_2 + \tau_1 |\mathbf{M}|^2) \mathbf{M} \times \mathcal{N}_{\perp} + \tau_2^2 (\mathbf{M} \cdot \dot{\mathbf{M}}) \mathbf{M}. \quad (4.9)$$

Inner multiplication by \mathbf{M} provides $\mathbf{M} \cdot \dot{\mathbf{M}}$. Hence (4.9) simplifies to

$$\dot{\mathbf{M}} = -\alpha_1 \mathcal{N}_{\parallel} - \kappa_1 \mathcal{N}_{\perp} + \kappa_2 \mathbf{M} \times \mathcal{N}_{\perp}, \quad (4.10)$$

where

$$\kappa_1 = \frac{\alpha_2 + \tau_1 |\mathbf{M}|^2}{1 + \tau_2^2 |\mathbf{M}|^2} > 0, \quad \kappa_2 = \tau_2 \kappa_1.$$

Now, because

$$\mathcal{N}_{\parallel} = \frac{1}{|\mathbf{M}|^2} (\mathcal{N} \cdot \mathbf{M}) \mathbf{M}, \quad \mathcal{N}_{\perp} = -\frac{1}{|\mathbf{M}|^2} \mathbf{M} \times (\mathbf{M} \times \mathcal{N}), \quad \mathbf{M} \times \mathcal{N}_{\perp} = \mathbf{M} \times \mathcal{N},$$

we can write (4.10) formally in terms of \mathcal{N} only as in (4.7), where $\tilde{\alpha}, \tilde{\lambda} > 0$ and $\tilde{\gamma} \in \mathbb{R}$ are given by

$$\tilde{\alpha} = \frac{\alpha_1}{|\mathbf{M}|^2}, \quad \tilde{\gamma} = \tau_2 \frac{\alpha_2 + \tau_1 |\mathbf{M}|^2}{1 + \tau_2^2 |\mathbf{M}|^2}, \quad \tilde{\lambda} = \frac{\alpha_2 + \tau_1 |\mathbf{M}|^2}{|\mathbf{M}|^2 (1 + \tau_2^2 |\mathbf{M}|^2)}. \quad \square$$

It is worth looking at the simplest case of (4.4) which follows by letting $\beta, \nu, \epsilon = 0$, or, equivalently, by letting $\tau_1, \tau_2 = 0$ in (4.6), namely

$$\dot{\mathbf{M}} = -\mathbf{A}\mathcal{N} = -\check{\alpha}(\mathcal{N} \cdot \mathbf{M})\mathbf{M} + \check{\lambda}_0 \mathbf{M} \times (\mathbf{M} \times \mathcal{N}), \quad (4.11)$$

where $\check{\lambda}_0 = \alpha_2/|\mathbf{M}|^2$. The same relation follows from (4.7) by letting $\hat{\gamma} = 0$ and $\check{\lambda} = \check{\lambda}_0$. As we see in Section 6, the form of \mathbf{A} is crucial to the splitting of (4.4) into two separate evolution equations, one governing the evolution of $|\mathbf{M}|$, the other the direction of \mathbf{M} . Henceforth we examine the role played by $\hat{\Psi}$ through \mathcal{N} .

Eq. (4.6) is now examined in the stationary regime to prove the following statement.

Proposition 3. *For any given \mathbf{H} , the stationary states for \mathbf{M} are solutions to the equation*

$$\theta(\hat{\Psi}_{\mathbf{M}} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}) = \mu_0 \mathbf{H}, \quad \hat{\Psi} = \frac{1}{\theta} \Psi(\theta, \mathbf{M}, \nabla \mathbf{M}), \quad (4.12)$$

subject to the constraint

$$\nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{H}$$

and the boundary condition

$$\hat{\Psi}_{\nabla \mathbf{M}} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

Proof. By (4.6), the stationary states solve the equation

$$0 = -\mathbf{A}\mathcal{N} + \tau_1 \mathbf{M} \times (\mathbf{M} \times \mathcal{N}),$$

which can be rewritten as

$$0 = \alpha_1 \mathcal{N}_{\parallel} + (\alpha_2 + \tau_1 |\mathbf{M}|^2) \mathcal{N}_{\perp}. \quad (4.13)$$

Since $\alpha_1, \alpha_2, \tau_1 > 0$ then (4.13) implies that $\mathcal{N}_{\perp}, \mathcal{N}_{\parallel} = 0$ and hence $\mathcal{N} = 0$. The equilibrium condition (4.12) follows from both (4.4) and (4.11). As a consequence, existence of one (or more) solutions depends on the convexity (or non-convexity) of Ψ with respect to \mathbf{M} . \square

Lemma 1. *If f is a function which depends on $\nabla \mathbf{M}$ through $\nabla \times \mathbf{M}$ then*

$$\nabla \cdot f_{\nabla \mathbf{M}} = -\nabla \times f_{\nabla \times \mathbf{M}}.$$

Proof. This identity follows by the observation that, in indicial notation,

$$\frac{\partial}{\partial M_{q,p}} = \frac{\partial}{\partial (\nabla \times \mathbf{M})_j} \frac{\partial (\nabla \times \mathbf{M})_j}{\partial M_{q,p}} = \epsilon_{pqj} \frac{\partial}{\partial (\nabla \times \mathbf{M})_j},$$

and hence

$$[(f_{\nabla \mathbf{M}})_{pq}]_{,p} = \epsilon_{pqj} \left[\frac{\partial f}{\partial (\nabla \times \mathbf{M})_j} \right]_{,p} = -(\nabla \times f_{\nabla \times \mathbf{M}})_q,$$

where $_{,p}$ denotes partial differentiation relative to the p -th coordinate. \square

Here we assume that Ψ depend on $\nabla \mathbf{M}$ only through $\nabla \times \mathbf{M}$ so that, with an abuse of notation, the additive free energy Ψ takes the form

$$\Psi = \Psi(\theta, \mathbf{M}, \nabla \times \mathbf{M}).$$

As a consequence, owing to Lemma 1, we have

$$\delta_{\mathbf{M}} \hat{\Psi} := \theta(\hat{\Psi}_{\mathbf{M}} + \nabla \times \hat{\Psi}_{\nabla \times \mathbf{M}}), \quad (4.14)$$

while the boundary condition (3.6) holds if

$$(\dot{\mathbf{M}} \times \hat{\Psi}_{\nabla \times \mathbf{M}}) \cdot \mathbf{n} = \dot{\mathbf{M}} \cdot (\hat{\Psi}_{\nabla \times \mathbf{M}} \times \mathbf{n}) = 0 \quad \text{on } \partial\Omega. \quad (4.15)$$

Remark 3. By virtue of Lemma 1, when $\hat{\Psi}$ depends on $\nabla \mathbf{M}$ only through $\nabla \times \mathbf{M}$ the stationary equation becomes

$$\theta(\hat{\Psi}_{\mathbf{M}} + \nabla \times \hat{\Psi}_{\nabla \times \mathbf{M}}) = \mu_0 \mathbf{H}$$

subject to the boundary condition

$$\hat{\Psi}_{\nabla \times \mathbf{M}} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

4.1. An example of free-energy function

Quite a general (rescaled) free-energy function $\hat{\Psi} = \Psi/\theta$ is given by

$$\hat{\Psi} = F(\theta, |\mathbf{M}|) + \frac{1}{2}[c_1 \mathbf{M} \cdot \nabla \times \mathbf{M} + c_2 |\nabla \times \mathbf{M}|^2 + c_3 |\mathbf{M} \cdot \mathbf{e}|^2],$$

where F is a non-convex function of $|\mathbf{M}|$, c_1 and c_2 are constants, c_3 is parameterized by θ and \mathbf{e} is a possibly-privileged unit direction. Hence we have

$$\delta_{\mathbf{M}} \hat{\Psi} = \theta \left[\frac{1}{|\mathbf{M}|} F_{|\mathbf{M}|} \mathbf{M} + c_1 \nabla \times \mathbf{M} + c_2 \nabla \times \nabla \times \mathbf{M} + c_3(\theta)(\mathbf{M} \cdot \mathbf{e})\mathbf{e} \right]. \quad (4.16)$$

If the material is isotropic then $c_3 = 0$. The term $\mathbf{M} \cdot \nabla \times \mathbf{M}$ in the free energy is considered for the sake of generality but does not seem to be motivated on physical grounds.

Landau et al. [2] regard Ψ as though $c_1, c_3 = 0$. Their position amounts to assuming that $\mathbf{\Lambda} = \alpha \mathbf{1}$ and that

$$F(\theta, |\mathbf{M}|) = \frac{1}{2}a_1(\theta - \theta_c)\mathbf{M}^2 + \frac{1}{4}a_2\mathbf{M}^4,$$

where a_1 and a_2 are allowed to depend on θ .

We now exhibit a three-dimensional setting and let Ψ depend on \mathbf{M} , through a logarithmic function of \mathbf{M}^2 . This dependence is motivated by an investigation of one-dimensional potentials which is shown in Section 8.

5. Evolution in the three-dimensional space

The whole evolution of the system is described by the equation for $\dot{\theta}$ (balance of energy) in addition to that for $\dot{\mathbf{M}}$. We now set up the corresponding scheme without any restriction on the amplitude and the direction of \mathbf{M} . Later on, we investigate the case when the system is saturated ($|\mathbf{M}| = \text{constant}$) or the direction of \mathbf{M} is fixed in space ($\mathbf{M}/|\mathbf{M}| = \text{constant}$). However, to avoid formal difficulties, we restrict attention to isotropic materials so that the free energy depends on $\nabla \mathbf{M}$ through $|\nabla \mathbf{M}|^2$. For definiteness, the pertinent coefficient is taken to be proportional to the temperature θ . Though with more involved formulae, the anisotropic case might be considered by following along the same lines.

Let the free energy Ψ be given by

$$\Psi = G(\theta, \mathbf{M}) + \frac{1}{2}\kappa\theta|\nabla \mathbf{M}|^2, \quad (5.1)$$

where

$$G(\theta, \mathbf{M}) = g(\theta) - b(u(\theta) + 1) \ln(1 - \mathbf{M}^2/M_s^2(\theta)) - b\mathbf{M}^2/M_s^2(\theta) \quad (5.2)$$

and, for simplicity, $u(\theta)$ takes the classical form $u = (\theta - \theta_c)/\theta_c$.

The function $M_s(\theta)$ models the dependence of M_s on θ as is the case for the relation

$$M_s(\theta) = M_s(0)(1 - B\theta^b) \quad (5.3)$$

which is often referred to as Bloch's law (see [18,19]). Here B is the Bloch constant and is obviously dependent on the material. The Bloch exponent b equals 3/2 for bulk materials but equals roughly 1/2 for some nanoparticle specimens [20].

We let $\mathbf{\Lambda} = \alpha \mathbf{1}$ so that the evolution equation (4.6) applies with $\alpha_1 = \alpha_2 = \alpha$. Moreover, let $\tau_1, \tau_2 = 0$, namely $\epsilon, \beta, \nu = 0$. Eq. (4.6) then reduces to

$$\dot{\mathbf{M}} = -\alpha(\delta_{\mathbf{M}} \hat{\Psi} - \mu_0 \mathbf{H}).$$

Hence, by (5.1) we have

$$\dot{\mathbf{M}} = \alpha\mu_0 \mathbf{H} - 2\alpha b \frac{(u + \mathbf{M}^2/M_s^2)}{M_s(1 - \mathbf{M}^2/M_s^2)} \frac{\mathbf{M}}{M_s} - \alpha\kappa\theta \Delta \mathbf{M}. \quad (5.4)$$

As a check on the validity of the evolution equation (5.4) we restrict attention to stationary ($\dot{\mathbf{M}} = 0$) and uniform ($\Delta \mathbf{M} = 0$) conditions. Eq. (5.4) gives

$$\mu_0 \mathbf{H} = 2b \frac{u + \mathbf{M}^2/M_s^2}{M_s(1 - \mathbf{M}^2/M_s^2)} \frac{\mathbf{M}}{M_s}. \quad (5.5)$$

As a consequence \mathbf{M} and \mathbf{H} are collinear. Letting M, H be the components in the common direction we find that the differential susceptibility

$$\chi = \frac{dM}{dH}$$

is a function of $M/M_s = \xi$ namely

$$\chi = \frac{\mu_0 M_s^2}{2b} \frac{(1 - \xi^2)^2}{u + (u + 3)\xi^2 - \xi^4}, \quad \xi \in [0, 1]. \quad (5.6)$$

If $\theta > \theta_c$ then $u > 0$. Hence, $\chi > 0$, in that $3\xi^2 > \xi^4$, and the material is paramagnetic. Also, for small values of ξ we have

$$\chi \simeq \frac{\mu_0 M_s^2}{2b} \frac{\theta_c}{\theta - \theta_c}, \quad (5.7)$$

whence the differential susceptibility χ varies with θ as $(\theta - \theta_c)^{-1}$, which is the content of the Curie–Weiss law (see [2]).

If $\theta < \theta_c$ then, by (5.7), $\chi < 0$ for small values of ξ whereas, by (5.6), $\chi > 0$ as ξ^2 approaches 1 or the body is almost saturated. This is the typical behaviour of ferromagnetic materials. Again, the differential susceptibility varies with θ as $(\theta - \theta_c)^{-1}$ for small values of ξ . If $\theta \rightarrow 0$ then $u \rightarrow -1$ and (5.4) reduces to

$$\dot{\mathbf{M}} = \alpha \mu_0 \mathbf{H} + \frac{2\alpha b}{M_s^2} \mathbf{M}.$$

Similar, though more involved, models and conclusions are obtained by letting u be given, e.g., as in Section 8.2.

It is of interest to look at the function $H(M)$, or $H(\xi)$, as $\theta > \theta_c$ or $\theta < \theta_c$. By (5.5) we have

$$\frac{\mu_0 M_s}{2b} H = \frac{(u + \xi^2)\xi}{1 - \xi^2}. \quad (5.8)$$

Fig. 1 shows the right-hand side of (5.8) as $\theta = 1.2\theta_c$ and $\theta = .8\theta_c$.

The evolution of the material is completed by accounting also for the balance of energy. Now, by (5.1) and (5.2) we have

$$\eta = -g' + \frac{b}{\theta_c} \ln(1 - \mathbf{M}^2/M_s^2) + 2 \frac{b\mathbf{M}^2}{M_s^3} \frac{u + \mathbf{M}^2/M_s^2}{1 - \mathbf{M}^2/M_s^2} M_s' - \frac{1}{2} \kappa |\nabla \mathbf{M}|^2,$$

$$e = \psi + \theta \eta = \frac{1}{2} (\mu_0 \mathbf{H}^2 + \epsilon_0 \mathbf{E}^2) + g - \theta g' + \ell(\theta, \mathbf{M}, M_s, M_s'),$$

where a prime stands for the derivative with respect to the temperature θ and

$$\ell(\theta, \mathbf{M}, M_s, M_s') = -b \frac{\mathbf{M}^2}{M_s^2} + 2b\theta \frac{\mathbf{M}^2}{M_s^3} \frac{[\theta - \theta_c(1 - \mathbf{M}^2/M_s^2)]\mathbf{M}^2}{\theta_c M_s^3 (1 - \mathbf{M}^2/M_s^2)} M_s'.$$

The balance of energy in the form (2.4) then yields

$$[-\theta g'' + \ell_\theta + \ell_{M_s} M_s' + \ell_{M_s'} M_s''] \dot{\theta} + (\ell_{\mathbf{M}} - \mu_0 \mathbf{H}) \cdot \dot{\mathbf{M}} - \mathbf{J} \cdot \mathbf{E} + \nabla \cdot \mathbf{q} - r = 0, \quad (5.9)$$

where \mathbf{J} and \mathbf{q} are to be viewed as given by the pertinent constitutive functions and r is a possible given function (heat supply). Eqs. (5.4) and (5.9) constitute the system of evolution equations for the two fields $\theta(\mathbf{x}, t)$, $\mathbf{M}(\mathbf{x}, t)$.

Remark 4. Ferrimagnetic materials, like ferromagnets, hold a spontaneous magnetization below the Curie temperature θ_c and are paramagnetic above θ_c . However, the amount of spontaneous magnetization in ferrimagnetic materials, such as ferrites and magnetic garnets, is smaller than in ferromagnets. This is so because a ferrimagnetic material consists of

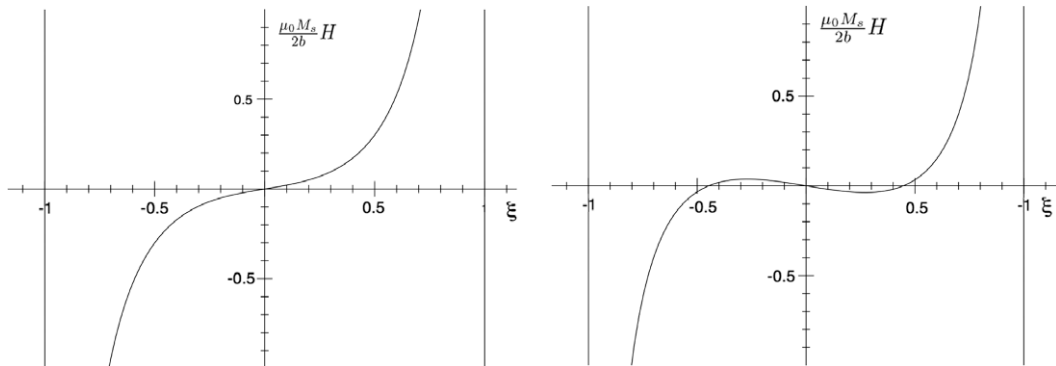


Fig. 1. On the left, plot of $(\mu_0 M_s / 2b) H$ versus ξ in the paramagnetic phase, $\theta = 1.2\theta_c$. On the right, in the ferromagnetic phase, $\theta = 0.8\theta_c$.

different sublattices with opposed but unequal magnetic moment, whereas in antiferromagnetic materials the magnetic moments of the two sublattices are equal and opposed. For instance, in ferrite the sublattices are given by two families of ions, Fe^{2+} and Fe^{3+} . In addition, in magnetic garnets there is a temperature below θ_c , called magnetic compensation point, at which the spontaneous magnetization vanishes. They also exhibit a third critical temperature corresponding to the angular momentum compensation point [21]. The modelling of such materials requires the occurrence of two distinct vector phase variables, say \mathbf{M}_1 and \mathbf{M}_2 , each obeying an evolution law like (4.11).

6. Evolution of direction and amplitude

Also for a more direct comparison with some models of magnetization in matter, we now examine two representations of the field \mathbf{M} . Let $M_s > 0$ be the saturation value of magnetization. We can write

$$\mathbf{M}(\mathbf{x}, t) = M_s \phi(\mathbf{x}, t) \mathbf{m}(\mathbf{x}, t), \quad \phi \in [0, 1], \quad |\mathbf{m}| = 1. \quad (6.1)$$

Hence, \mathbf{m} is the unit vector of \mathbf{M} . The saturation magnetization M_s is here regarded as a (temperature-independent) constant. As shown by measurements on nanoparticles (see [20]), the approximation is reasonable in many circumstances.

In saturation conditions,

$$\mathbf{M}(\mathbf{x}, t) = M_s \mathbf{m}(\mathbf{x}, t), \quad \phi = 1.$$

If the magnetization has a constant direction we can represent \mathbf{M} as

$$\mathbf{M}(\mathbf{x}, t) = M_s \xi(\mathbf{x}, t) \mathbf{e}, \quad \xi \in [-1, 1], \quad |\mathbf{e}| = 1,$$

where \mathbf{e} is the fixed unit vector. More generally, for a variable direction we can write

$$\mathbf{M}(\mathbf{x}, t) = M_s \xi(\mathbf{x}, t) \mathbf{r}(\mathbf{x}, t), \quad \xi \in [-1, 1], \quad \mathbf{r} \in \mathcal{U}, \quad (6.2)$$

where \mathcal{U} is a solid cone,

$$\mathcal{U} = \{\mathbf{r} : |\mathbf{r}| = 1, \mathbf{r} \in \mathcal{U} \Rightarrow -\mathbf{r} \notin \mathcal{U}\}.$$

If \mathcal{U} is the set of possible directions then the two representations are equivalent by letting

$$\phi = |\xi|, \quad \mathbf{r} = (\text{sgn } \xi) \mathbf{m}.$$

We now proceed by representing \mathbf{M} as $M_s |\xi| \mathbf{m}$, regarding ξ and \mathbf{m} as independent quantities and looking for the separate evolution equations

$$\dot{\mathbf{m}} = \hat{\mathbf{m}}(\Gamma), \quad \dot{\xi} = \hat{\xi}(\Gamma).$$

By the representation (6.2) we have

$$\dot{\mathbf{M}} = M_s (\dot{\xi} \mathbf{r} + |\xi| \dot{\mathbf{m}}), \quad \nabla \mathbf{M} = M_s (\nabla \xi \otimes \mathbf{r} + |\xi| \nabla \mathbf{m}), \quad |\xi| \dot{\mathbf{m}} = \xi \dot{\mathbf{r}}, \quad |\xi| \nabla \mathbf{m} = \xi \nabla \mathbf{r},$$

and hence

$$\begin{aligned} \dot{\Psi}_\xi &= M_s (\dot{\Psi}_\mathbf{M} \cdot \mathbf{r} + \dot{\Psi}_{\nabla \mathbf{M}} \cdot \mathbf{r}), & \dot{\Psi}_{\nabla \xi} &= M_s \dot{\Psi}_{\nabla \mathbf{M}} \mathbf{r}, \\ \dot{\Psi}_\mathbf{m} &= M_s (\dot{\Psi}_\mathbf{M} |\xi| + \dot{\Psi}_{\nabla \mathbf{M}} \cdot \nabla |\xi|), & \dot{\Psi}_{\nabla \mathbf{m}} &= M_s \dot{\Psi}_{\nabla \mathbf{M}} |\xi|, \end{aligned}$$

and

$$\dot{\mathbf{M}} \cdot \delta_\mathbf{M} \hat{\Psi} = M_s (\mathbf{r} \cdot \delta_\mathbf{M} \hat{\Psi}) \dot{\xi} + M_s (|\xi| \delta_\mathbf{M} \hat{\Psi}) \cdot \dot{\mathbf{m}}.$$

Now,

$$\begin{aligned} M_s \mathbf{r} \cdot \delta_\mathbf{M} \hat{\Psi} &= M_s \theta (\hat{\Psi}_\mathbf{M} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}) \cdot \mathbf{r} = \theta [\hat{\Psi}_\xi - \nabla \cdot (M_s \hat{\Psi}_{\nabla \mathbf{M}} \mathbf{r})] = \theta (\hat{\Psi}_\xi - \nabla \cdot \hat{\Psi}_{\nabla \xi}) =: \delta_\xi \hat{\Psi}, \\ M_s |\xi| \delta_\mathbf{M} \hat{\Psi} &= M_s \theta (\hat{\Psi}_\mathbf{M} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{M}}) |\xi| = \theta [\hat{\Psi}_\mathbf{m} - \nabla \cdot (M_s \hat{\Psi}_{\nabla \mathbf{M}} |\xi|)] = \theta (\hat{\Psi}_\mathbf{m} - \nabla \cdot \hat{\Psi}_{\nabla \mathbf{m}}) =: \delta_\mathbf{m} \hat{\Psi}. \end{aligned}$$

Hence,

$$\dot{\mathbf{M}} \cdot \delta_\mathbf{M} \hat{\Psi} = \delta_\xi \hat{\Psi} \dot{\xi} + \delta_\mathbf{m} \hat{\Psi} \cdot \dot{\mathbf{m}}$$

and (3.9) becomes

$$(\delta_\xi \hat{\Psi} - \mu_0 M_s \mathbf{H} \cdot \mathbf{r}) \dot{\xi} + (\delta_\mathbf{m} \hat{\Psi} - \mu_0 M_s |\xi| \mathbf{H}) \cdot \dot{\mathbf{m}} \leq 0. \quad (6.3)$$

Since $\xi \mathbf{r} = |\xi| \mathbf{m}$ then

$$\xi \delta_\xi \hat{\Psi} = \mathbf{m} \cdot \delta_\mathbf{m} \hat{\Psi}.$$

In view of [Proposition 1](#), a sufficient condition for the validity of (6.3) is that the evolution functions $\hat{\xi}$ and $\dot{\mathbf{m}}$ take the form

$$\hat{\xi} = -\omega(\delta_{\xi}\hat{\Psi} - \mu_0 M_s \mathbf{H} \cdot \mathbf{r}), \quad (6.4)$$

$$\dot{\mathbf{m}} = -\bar{\alpha}_1 \mathbf{N}_{\parallel} - \bar{\alpha}_2 \mathbf{N}_{\perp} + \bar{\beta} \mathbf{m} \times \mathbf{N} + \bar{\nu} \mathbf{m} \times (\mathbf{m} \times \mathbf{N}), \quad (6.5)$$

where the decomposition $\mathbf{N}_{\parallel}, \mathbf{N}_{\perp}$ is relative to \mathbf{m} and

$$\mathbf{N} = \mathcal{M} + \bar{\epsilon} \mathbf{m} \times \dot{\mathbf{m}}, \quad \mathcal{M} = \delta_{\mathbf{m}} \hat{\Psi} - \mu_0 M_s |\xi| \mathbf{H}. \quad (6.6)$$

Also, $\omega, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\nu}$ are non-negative valued functions of Γ and $\bar{\beta}, \bar{\epsilon}$ are real valued. Because $|\mathbf{m}| = 1$, $\dot{\mathbf{m}}$ satisfies the constraint

$$\dot{\mathbf{m}} \cdot \mathbf{m} = \dot{\mathbf{m}} \cdot \dot{\mathbf{m}} = 0$$

and hence it follows from (6.5) that

$$0 = \bar{\alpha}_1 \mathbf{N}_{\parallel} \cdot \mathbf{m}.$$

This condition holds by merely requiring that $\bar{\alpha}_1 = 0$. Hence, letting $\bar{\alpha}_2 = \bar{\alpha}$ we can write (6.5) in the form

$$\dot{\mathbf{m}} = -\bar{\alpha}[(\mathbf{1} - \mathbf{m} \otimes \mathbf{m})\mathcal{M} + \bar{\epsilon} \mathbf{m} \times \dot{\mathbf{m}}] + \mathbf{m} \times (\bar{\beta} \mathbf{N} + \bar{\nu} \mathbf{m} \times \mathbf{N}). \quad (6.7)$$

By means of (6.7) we can now prove the following statement.

Proposition 4. *If $\xi \neq 0$ then the evolution equation for \mathbf{m} can be expressed in terms of \mathcal{M} in the form*

$$\dot{\mathbf{m}} = \bar{\gamma} \mathbf{m} \times \mathcal{M} + \bar{\lambda} \mathbf{m} \times (\mathbf{m} \times \mathcal{M}), \quad (6.8)$$

where

$$\bar{\gamma} = \frac{\bar{\beta} + \bar{\epsilon}[\bar{\beta}^2 + (\bar{\alpha} + \bar{\nu})^2]}{(1 + \bar{\epsilon}\bar{\beta})^2 + \bar{\epsilon}^2(\bar{\alpha} + \bar{\nu})^2}, \quad \bar{\lambda} = \frac{\bar{\alpha} + \bar{\nu}}{(1 + \bar{\epsilon}\bar{\beta})^2 + \bar{\epsilon}^2(\bar{\alpha} + \bar{\nu})^2}.$$

Proof. Since $|\mathbf{m}| = 1$ then $\mathbf{m} \cdot \dot{\mathbf{m}} = 0$, $|\dot{\mathbf{m}}| = |\mathbf{m} \times \dot{\mathbf{m}}|$, and

$$\mathbf{m} \times (\mathbf{m} \times \dot{\mathbf{m}}) = -\dot{\mathbf{m}}.$$

Also,

$$(\mathbf{1} - \mathbf{m} \otimes \mathbf{m})\mathcal{M} = -\mathbf{m} \times (\mathbf{m} \times \mathcal{M}).$$

As a consequence, by (6.7) the evolution equation becomes

$$(1 + \bar{\epsilon}\bar{\beta})\dot{\mathbf{m}} = -\bar{\epsilon}(\bar{\alpha} + \bar{\nu})\mathbf{m} \times \dot{\mathbf{m}} + \bar{\beta}\mathbf{m} \times \mathcal{M} + (\bar{\alpha} + \bar{\nu})\mathbf{m} \times (\mathbf{m} \times \mathcal{M}). \quad (6.9)$$

Apply $\mathbf{m} \times$ to (6.9), derive the expression of $\mathbf{m} \times \dot{\mathbf{m}}$ and then replace it in (6.9). Upon some algebraic rearrangements we obtain (6.8). \square

As expected, the evolution equation (4.4) follows from (6.4) and (6.7) once we make appropriate identifications. They are given by

$$\omega = \frac{\alpha_1}{M_s^2}, \quad \bar{\alpha} = \frac{\alpha_2}{M_s^2 |\xi|^2}, \quad \bar{\beta} = \frac{\beta}{M_s |\xi|}, \quad \bar{\nu} = \nu, \quad \bar{\epsilon} = \epsilon M_s^3 |\xi|^3$$

while Λ has the form (4.5). At saturation ($|\xi| = 1$), the constancy of $\alpha_1, \alpha_2, \beta, \nu, \epsilon$ is equivalent to that of $\omega, \bar{\alpha}, \bar{\beta}, \bar{\nu}, \bar{\epsilon}$.

As we shall see in a moment, the term $\delta_{\mathbf{m}} \hat{\Psi}$, which occurs in \mathcal{M} , accounts for exchange and anisotropic interactions. If such effects are neglected then (6.8) reduces to

$$\dot{\mathbf{m}} = -\gamma_0 \mathbf{m} \times \mathbf{H} - \lambda_0 \mathbf{m} \times (\mathbf{m} \times \mathbf{H}), \quad (6.10)$$

where

$$\gamma_0 = \mu_0 M_s |\xi| \bar{\gamma}, \quad \lambda_0 = \mu_0 M_s |\xi| \bar{\lambda}.$$

Two particular cases are of interest, namely the *magnetic saturation*, which occurs if $|\xi| = 1$, and the *fixed alignment of \mathbf{M}* , which means that the magnetic direction $\mathbf{r} = \text{sgn } \xi \mathbf{m}$ is constant. They are now examined by allowing for non-isothermal conditions and assuming that the temperature is below the critical temperature θ_c .

6.1. Magnetic saturation

In magnetically-saturated bodies, $|\xi| = 1$, and hence

$$\mathcal{M} = \delta_{\mathbf{m}} \hat{\Psi} - \mu_0 M_s \mathbf{H}.$$

Also, $\delta_{\mathbf{m}} \hat{\Psi} = M_s \delta_{\mathbf{M}} \hat{\Psi}$ so that

$$\mathcal{M} = M_s(\delta_{\mathbf{M}} \hat{\Psi} - \mu_0 \mathbf{H}) =: -\mu_0 M_s \mathbf{H}_{\text{eff}},$$

which ascribes to

$$\mathbf{H}_{\text{eff}} = \mathbf{H} - \frac{1}{\mu_0} \delta_{\mathbf{M}} \hat{\Psi} \quad (6.11)$$

the role of *effective magnetic field*.

Let $\theta < \theta_c$ be constant in space and time. We look for equilibrium configurations by applying the evolution equation (6.8). If $\dot{\mathbf{M}} = 0$, that is $\dot{\mathbf{m}} = 0$, then we have

$$\bar{\gamma} \mathbf{m} \times \mathcal{M} + \bar{\lambda} \mathbf{m} \times (\mathbf{m} \times \mathcal{M}) = 0.$$

Hence the equilibrium condition becomes

$$\mu_0 M_s \mathbf{m} \times \mathbf{H}_{\text{eff}} = 0, \quad (6.12)$$

where \mathbf{H}_{eff} is the effective field given by (6.11). Eq. (6.12) traces back to Brown [5] and ascribes to $\delta_{\mathbf{M}} \hat{\Psi} / \mu_0$ the meaning of the field arising from exchange and anisotropic interactions. Indeed, the standard form of \mathbf{H}_{eff} ([5, p. 48])

$$\mathbf{H}_{\text{eff}} = \mathbf{H} - \frac{1}{\mu_0 M_s} f_{\mathbf{m}} + \frac{2}{\mu_0 M_s} \nabla \cdot (\mathbf{A} \nabla \mathbf{m}),$$

follows from (3.11) and (6.11) by letting

$$\Psi = f(\mathbf{M}) + \nabla \mathbf{M} \cdot \mathbf{A} \nabla \mathbf{M},$$

where \mathbf{A} is a symmetric fourth-order tensor.

Let f depend on \mathbf{M} in the anisotropic form

$$f(\mathbf{M}) = \frac{1}{2} \mu_0 \mathbf{M} \cdot \mathbf{Q} \mathbf{M}$$

where $\mathbf{Q} \in \text{Sym}^+$. Hence we have

$$\mathbf{H}_{\text{eff}} = \mathbf{H} - M_s \mathbf{Q} \mathbf{m} + \frac{2}{\mu_0} M_s \nabla \cdot (\mathbf{A} \nabla \mathbf{m}). \quad (6.13)$$

The last term, $(2/\mu_0) M_s \nabla \cdot (\mathbf{A} \nabla \mathbf{m})$, accounts for exchange interactions and penalizes magnetization inhomogeneities. If the exchange interaction is isotropic in space (for instance such is the case for a cubic cell, see [2]) then $\mathbf{A} = (\mu_0 A/2) \mathbf{1}$, where A is the so-called *exchange constant* which depends on the lattice geometry (body-centred, face-centred cubic crystals).

Denote by D_x, D_y, D_z and $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ the eigenvalues and the eigenvectors of \mathbf{Q} . Hence we can express f as a function of $M_x = M_s m_x, M_y = M_s m_y, M_z = M_s m_z$, the components of \mathbf{M} relative to the eigenvector basis, in the form

$$f(\mathbf{M}) = \frac{1}{2} \mu_0 M_s^2 (D_x m_x^2 + D_y m_y^2 + D_z m_z^2),$$

where the coefficients $D_x, D_y, D_z > 0$ characterize the anisotropy of the body. Of course, $m_x^2 + m_y^2 + m_z^2 = 1$. If D_x, D_y, D_z are distinct values then the energy term f accounts for biaxial anisotropy,

$$f(\mathbf{M}) = \frac{1}{2} \mu_0 M_s^2 [D_z + (D_x - D_z) m_x^2 + (D_y - D_z) m_y^2].$$

If, instead, $D_x = D_y = D_{\perp}$ and $D_z \neq D_{\perp}$ then f accounts for uniaxial anisotropy with easy direction \mathbf{e}_z ,

$$f(\mathbf{M}) = \frac{1}{2} \mu_0 M_s^2 [D_z + (D_{\perp} - D_z)(1 - m_z^2)].$$

In such a case the body is transversely isotropic (relative to the z-axis) and the effective field takes the form

$$\mathbf{H}_{\text{eff}} = \mathbf{H} - M_s (D_z m_z \mathbf{e}_z + D_{\perp} \mathbf{m}_{\perp}),$$

where $m_z = \mathbf{m} \cdot \mathbf{e}_z$ and $\mathbf{m}_{\perp} = \mathbf{m} - m_z \mathbf{e}_z$. In particular, for strongly anisotropic materials D_{\perp} is negligible with respect to D_z and hence

$$\mathbf{H}_{\text{eff}} \simeq \mathbf{H} - D_z (\mathbf{M} \cdot \mathbf{e}_z) \mathbf{e}_z. \quad (6.14)$$

In transversely-isotropic materials the evolution equation (6.8) accounts also for the damping switching [6], namely the switching of a component, say m_z , generated by the z-component of the magnetic field opposite to the initial value $m_z(0)$.

Finally, in isotropic bodies $D_x = D_y = D_z = D > 0$ and $\mathbf{A} = (\mu_0 A/2) \mathbf{1}$ so that f simplifies to the constant

$$f(\mathbf{M}) = \frac{1}{2} \mu_0 M_s^2 D$$

and the effective field reduces to

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + M_s A \Delta \mathbf{m} = \mathbf{H} + A \Delta \mathbf{M}.$$

Accordingly, when the body is magnetically isotropic and subject to uniform fields ($\nabla \mathbf{m} = 0$) the effective field reduces merely to the applied field \mathbf{H} .

Section 7 is devoted to the relation of these particular cases with known models appeared in the literature.

6.2. Fixed alignment

If the direction $\mathbf{e} = (\text{sgn } \xi) \mathbf{m}$ of \mathbf{M} is constant, say $\mathbf{r} = \mathbf{e}$, then $\mathbf{M} = M_s \xi \mathbf{e}$ and $\dot{\mathbf{M}} = M_s \dot{\xi} \mathbf{e}$. Accordingly, \mathbf{m} is constant except when $\text{sgn } \xi$ changes. Such is the case usually assumed for an isotropic magnet in a magnetic field \mathbf{H} with constant direction. Indeed, because $\delta_{\mathbf{m}} \hat{\Psi} = (\text{sgn } \xi) \delta_{\mathbf{r}} \hat{\Psi}$, by (6.8) we have

$$0 = \tilde{\gamma} \mathbf{r} \times (\delta_{\mathbf{r}} \hat{\Psi} - \mu_0 M_s \xi \mathbf{H}) + \bar{\lambda} \text{sgn } \xi \mathbf{r} \times [\mathbf{r} \times (\delta_{\mathbf{r}} \hat{\Psi} - \mu_0 M_s \xi \mathbf{H})]$$

whence

$$\mathbf{r} \times (\delta_{\mathbf{r}} \hat{\Psi} - \mu_0 M_s \xi \mathbf{H}) = 0.$$

As a consequence, when anisotropic and exchange interactions are neglected ($\delta_{\mathbf{r}} \hat{\Psi} = 0$) we have $\mathbf{r} \times \mathbf{H} = 0$, which means that \mathbf{H} and \mathbf{M} are parallel (one-dimensional setting). The same result holds also for magnets with strong uniaxial anisotropy provided that the direction of the applied field \mathbf{H} coincides with the easy direction of the body. If such is the case then it follows from (6.14) that \mathbf{H}_{eff} is approximately parallel to \mathbf{r} . In addition, by (6.4) we have

$$\dot{\xi} = -\omega(\delta_{\xi} \hat{\Psi} - \mu_0 \mathcal{H}), \quad \omega > 0, \quad (6.15)$$

where $\mathcal{H} = M_s \mathbf{H} \cdot \mathbf{r}$, which governs the evolution of the (magnetic intensity) component ξ . If, as a special case,

$$\hat{\Psi} = \frac{1}{2} a(\theta - \theta_c) |\xi|^2 + \frac{1}{4} b |\xi|^4$$

then (6.15) becomes

$$\dot{\xi} = -\omega[a(\theta - \theta_c)\xi + b\xi^3 - \mu_0 \mathcal{H}]$$

which is in fact the non-isothermal model by Landau et al. [2].

In Section 8 known models of free energy are shown to follow as particular cases and, moreover, a new free energy is established.

7. Relation to the Gilbert–Landau–Lifshitz model

It is natural to contrast (6.8) with known evolution equations of micromagnetics which apply to magnetically-saturated bodies. Landau and Lifshitz [1] started with the evolution equation

$$\dot{\boldsymbol{\mu}} = -\gamma \boldsymbol{\mu} \times \mathbf{H}$$

for a magnetic spin momentum $\boldsymbol{\mu}$ of an electron in a magnetic field \mathbf{H} , $\gamma > 0$ being the absolute value of the gyromagnetic ratio. In the continuum limit they wrote

$$\dot{\mathbf{M}} = -\gamma \mathbf{M} \times \mathbf{H}$$

where \mathbf{H} is the magnetic field in matter. Dissipation was modelled by adding a torque which pushes \mathbf{M} toward the field \mathbf{H} . The torque was taken in the form $-\Lambda \mathbf{M} \times (\mathbf{M} \times \mathbf{H})$, where $\Lambda > 0$, so that

$$\dot{\mathbf{M}} = -\gamma \mathbf{M} \times \mathbf{H} - \Lambda \mathbf{M} \times (\mathbf{M} \times \mathbf{H}) \quad (7.1)$$

Later on, Gilbert [3,4] wrote the evolution equation with a dissipative term $\mathcal{T} \mathbf{M} \times \dot{\mathbf{M}}$, $\mathcal{T} > 0$, in the form

$$\dot{\mathbf{M}} = -\gamma \mathbf{M} \times \mathbf{H} + \mathcal{T} \mathbf{M} \times \dot{\mathbf{M}}. \quad (7.2)$$

In both cases, taking the inner product with \mathbf{M} gives

$$\mathbf{M} \cdot \dot{\mathbf{M}} = 0, \quad (7.3)$$

which means that $|\mathbf{M}|$ is constant in time. Hence (7.1) and (7.2) apply to magnetically-saturated bodies. Since $|\mathbf{M}|$ is constant, we can replace \mathbf{M} with $M_s \mathbf{m}$ in (7.1), (7.2) to get

$$\dot{\mathbf{m}} = -\gamma \mathbf{m} \times \mathbf{H} - \lambda \mathbf{m} \times (\mathbf{m} \times \mathbf{H}), \quad (7.4)$$

$$\dot{\mathbf{m}} = -\tilde{\gamma} \mathbf{m} \times \mathbf{H} + \tau \mathbf{m} \times \dot{\mathbf{m}}, \quad (7.5)$$

where $\lambda = \Lambda M_s$ and $\tau = \mathcal{T} M_s$, τ being the dimensionless Gilbert damping constant. Eqs. (7.4) and (7.5) are formally equivalent. For instance, replacing $\dot{\mathbf{m}}$ in the right-hand side of (7.5) with the whole right-hand side we obtain

$$\dot{\mathbf{m}} = -\gamma^* \mathbf{m} \times \mathbf{H} - \lambda^* \mathbf{m} \times (\mathbf{m} \times \mathbf{H}), \quad (7.6)$$

where

$$\gamma^* = \frac{\hat{\gamma}}{1 + \tau^2}, \quad \lambda^* = \frac{\tau \hat{\gamma}}{1 + \tau^2}.$$

Incidentally, comparison of (7.5) and (7.6) shows that $\tau = \lambda/\gamma$ and $\hat{\gamma} = (\gamma^2 + \lambda^2)/\gamma$. This in turn is consistent with the fact that τ and λ are dissipation coefficients. If $\lambda \rightarrow 0$ then $\tau \rightarrow 0$ and $\hat{\gamma} \rightarrow \gamma$ so that (7.4) and (7.5) reduce to

$$\dot{\mathbf{m}} = -\gamma \mathbf{m} \times \mathbf{H}.$$

Remark 5. Let \mathbf{H} be constant. By (7.4) we have

$$\frac{d}{dt}(\mathbf{m} \cdot \mathbf{H}) = \lambda |\mathbf{m} \times \mathbf{H}|^2 \geq 0.$$

The component of \mathbf{m} along \mathbf{H} increases until \mathbf{m} and \mathbf{H} are collinear. Hence the Landau–Lifshitz equation (7.1), in the approximation of a constant magnetic field, models a magnetization \mathbf{M} , with constant $|\mathbf{M}|$, that tends to orient itself parallel to \mathbf{H} . In addition, letting \mathbf{e}_H be the unit vector of \mathbf{H} , so that $\mathbf{H} = |\mathbf{H}|\mathbf{e}_H$, we have

$$\frac{d}{dt}(\mathbf{m} \cdot \mathbf{e}_H) = \lambda |\mathbf{m} \times \mathbf{e}_H|^2 |\mathbf{H}|,$$

which shows that the rate $d(\mathbf{m} \cdot \mathbf{e}_H)/dt$ is proportional to $|\mathbf{H}|$.

Recently more involved evolution equations have been considered by replacing \mathbf{H} , in (7.5), with an effective magnetic field \mathbf{H}_{eff} . A generalization of the evolution equation (7.5) is performed in [8] (see also [9]) by replacing \mathbf{H} with

$$\mathbf{H}_{\text{eff}} = \zeta \Delta \mathbf{m} + \eta(\mathbf{e} \cdot \mathbf{m})\mathbf{e} + \mathbf{H}, \quad (7.7)$$

where \mathbf{e} is the unit vector of the easy axis of magnetization, \mathbf{H} is the sum of the stray field and of the external field (see also [7,11]). The model (7.7) is a particular case of (6.13) as it follows by letting $\mathbf{e} = \mathbf{e}_z$, $\eta = -M_s D_z$, $M_s \mathbf{A} = \frac{1}{2} \mu_0 \zeta \mathbf{1}$. In such a case the (Gilbert) evolution equation takes the form

$$\dot{\mathbf{m}} - \tau \mathbf{m} \times \dot{\mathbf{m}} = -\gamma \mathbf{m} \times [\zeta \Delta \mathbf{m} + \eta(\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \mathbf{H}], \quad (7.8)$$

which reduces to (6.12) in stationary conditions. As a check we see that $\mathbf{m} \cdot \dot{\mathbf{m}} = 0$. Hence we can write (7.8) also in the form

$$\dot{\mathbf{m}} = \gamma^* \{-\mathbf{m} \times [\zeta \Delta \mathbf{m} + \eta(\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \mathbf{H}] + \tau \mathbf{m} \times [\mathbf{m} \times (\zeta \Delta \mathbf{m} + \eta(\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \mathbf{H})]\}. \quad (7.9)$$

The evolution equations (7.4), (7.5) and (7.8) are particular cases of (6.8) and (6.10) and hence are compatible with thermodynamics. Eq. (7.4) coincides with (6.10) once the identifications $\lambda = \lambda_0$ and $\gamma = \gamma_0$ are made.

The same identifications hold for the Gilbert equation (7.6) with γ^* and λ^* in place of γ_0 and λ_0 . The generalized Gilbert equation, in the form (7.9), follows from (6.8) by letting the temperature θ be constant, the rescaled free energy take the form

$$\hat{\Psi} = \frac{\eta}{2\mu_0} |\mathbf{M} \cdot \mathbf{e}|^2 - \frac{\zeta}{2\mu_0} |\nabla \times \mathbf{M}|^2$$

and

$$\gamma^* = \mu_0 M_s |\zeta| \bar{\gamma}, \quad \tau = \bar{\lambda} / \bar{\gamma}.$$

8. Potentials for one-dimensional models

Henceforth we look for appropriate functions Ψ to model ferromagnetism and the associated phase transition. For simplicity we restrict attention to the one-dimensional case and apply (4.4) by letting $\beta, \gamma = 0$. Denote by $M, H \in \mathbb{R}$ the significant components of \mathbf{M}, \mathbf{H} . Eq. (4.4) for M , likewise (6.15) for $\xi = M/M_s$, then becomes

$$\dot{\xi} = -\alpha [\theta (\hat{\Psi}_\xi - \nabla \cdot \hat{\Psi}_{\nabla \xi}) - h], \quad (8.1)$$

where $h = \mu_0 H$. In stationary conditions, $\xi = \text{constant}$, we have

$$h = \theta (\hat{\Psi}_\xi - \nabla \cdot \hat{\Psi}_{\nabla \xi}). \quad (8.2)$$

Also we let

$$\Psi(\theta, \xi, \nabla \xi) = \frac{1}{2} c \theta |\nabla \xi|^2 + V(\xi; u),$$

where c is a positive constant and u is a suitable increasing function, of the temperature θ , which vanishes at the Curie point θ_c . The function V is often referred to as the potential. Hence (8.1) and (8.2) become

$$\dot{\xi} = -\alpha(V_\xi - h - c\theta\Delta\xi), \quad (8.3)$$

$$h = V_\xi - c\theta\Delta\xi. \quad (8.4)$$

This setting allows us to obtain an immediate connection with other approaches. For instance, a standard free energy applied in the literature (see, e.g., [12]) corresponds to letting $u = (\theta - \theta_c)/\theta_c$ and

$$V = \xi^2[a(\theta - \theta_c) + b\xi^2], \quad (8.5)$$

where a, b are positive constants. In such a case, Eqs. (8.3) and (8.4) become

$$\dot{\xi} = -2\alpha[a(\theta - \theta_c) + 2b\xi^2]\xi + \alpha h + \frac{1}{2}\alpha c\theta\Delta\xi,$$

and

$$h = -c\theta\Delta\xi + [2a(\theta - \theta_c) + 4b\xi^2]\xi. \quad (8.6)$$

The potential (8.5) traces back to Ginzburg [2]. The function

$$\tilde{\Phi} = V(\xi, u) - h\xi$$

is considered in [22], the potential $\tilde{\Phi}$ being identified with the Lagrangian density.

Equilibrium conditions are sought for homogeneous configurations, $\nabla\xi = 0$. Hence the equilibrium conditions are the stationary points of the potential. The potential (8.5), as well as similar ones in the literature, does not provide a reasonable set of equilibrium values and a satisfactory scheme for phase transition which occurs when $u = 0$. Indeed, it is a well-known drawback of the potential (8.5) that it does not allow for saturation. By (8.6), for large values of h in homogeneous configurations we have

$$\xi \simeq (4b)^{-1/3} h^{-2/3} h,$$

as though the material had a permittivity proportional to $h^{-2/3}$. This motivates the search for schemes where the saturation is allowed.

8.1. Weiss model of ferromagnetism

We now review the Weiss model of ferromagnetism (see, e.g., [23] and refs therein) and show how it allows for a potential $V_W(\theta, \xi)$.

Let \mathcal{L} be the Langevin function, on \mathbb{R} , defined by

$$\mathcal{L}(x) = \coth x - \frac{1}{x}.$$

The function \mathcal{L} is strictly increasing and odd and moreover $\mathcal{L}(x) \rightarrow \pm 1$ as $x \rightarrow \pm\infty$. As a consequence the inverse \mathcal{L}^{-1} maps $(-1, 1)$ into \mathbb{R} . Weiss model relates the magnetization M and the magnetic field H in the form

$$\frac{H + \beta M}{\mu\theta} = \mathcal{L}^{-1}(M/M_s) \quad (8.7)$$

where M_s is the maximum value of the magnetization, that is $M \in (-M_s, M_s)$, β is the molecular-field parameter, and $\mu = k/m$ is the ratio of Boltzmann's constant over the magnitude of the (atomic) magnetic moment. By (8.7) we have

$$H = \mu\theta\mathcal{L}^{-1}(M/M_s) - \beta M. \quad (8.8)$$

Letting $\xi = M/M_s$ we can write $H = \mathcal{H}(\xi)$, where \mathcal{H} is defined by

$$\mathcal{H}(\xi) = \mu\theta\mathcal{L}^{-1}(\xi) - \beta M_s \xi, \quad \xi \in (-1, 1).$$

If $\theta \rightarrow 0_+$ then

$$H = -\beta M,$$

which is the classical law of diamagnetism or paramagnetism according as $\beta > 0$ or $\beta < 0$. Now, ferromagnetism indicates that there is a temperature θ_c such that

$$\mathcal{H}'(0) > 0 \quad \text{as } \theta > \theta_c, \quad \mathcal{H}'(0) \in (-\beta M_s, 0) \quad \text{as } \theta \in (0, \theta_c).$$

Consistently, it is expected that $\mathcal{H}'(0) = 0$ at $\theta = \theta_c$, namely

$$\mu\theta_c(\mathcal{L}^{-1})'(0) - \beta M_s = 0. \quad (8.9)$$

The requirement (8.9) induces a relation between β , μ , and θ_c . In this connection, letting

$$z = \frac{H + \beta M}{\mu \theta}$$

we have

$$(\mathcal{L}^{-1})'(\xi) = \frac{1}{\mathcal{L}'(z(\xi))}.$$

Since $\mathcal{L}^{-1}(0) = 0$ then $z(0) = 0$. Moreover, $\mathcal{L}'(0) = 1/3$. As a consequence

$$(\mathcal{L}^{-1})'(0) = \frac{1}{\mathcal{L}'(0)} = 3.$$

Hence (8.9) provides

$$\theta_c = \frac{\beta M_s}{3\mu}.$$

This means that the parameter μ is related to the transition temperature θ_c by

$$\mu = \frac{\beta M_s}{3\theta_c}.$$

Upon substitution we can write Eq. (8.8) in the form

$$\frac{1}{\beta M_s} H = \frac{u+1}{3} \mathcal{L}^{-1}(\xi) - \xi \quad (8.10)$$

where $u = (\theta - \theta_c)/\theta_c$. Eq. (8.10) provides the value of the magnetic field H in terms of the magnetization M , in that $\xi = M/M_s$.

The equilibrium (8.10) may be viewed as the stationary condition for the potential V_W such that

$$V'_W(\xi) = \beta M_s \frac{u+1}{3} \mathcal{L}^{-1}(\xi) - \beta M_s \xi - H.$$

The integration gives

$$V_W(\xi) = \beta M_s \frac{u+1}{3} \Lambda(\xi) - \frac{1}{2} \beta M_s \xi^2 - H\xi, \quad (8.11)$$

where Λ is the integral of \mathcal{L}^{-1} .

To evaluate the residual magnetization M_r we can go back to (8.10) and require that the right-hand side vanishes at M_r . We find that $\xi_r = M_r/M_s$ satisfies

$$\xi_r = \mathcal{L}(\delta \xi_r), \quad \delta = \frac{3\theta_c}{\theta}. \quad (8.12)$$

To obtain the coercive field H_c , by (8.10) we look for the maximum of the right-hand side. We find that the coercive field H_c , at $\xi_c = M_c/M_s$, is given by

$$\mathcal{L}'(\mathcal{L}^{-1}(\xi_c)) = \frac{1}{\delta}, \quad H_c = \beta M_s \left[\frac{1}{\delta} \mathcal{L}^{-1}(\xi_c) - \xi_c \right]. \quad (8.13)$$

In essence, the advantage of the Weiss description is that the saturation effect is allowed and the condition $\xi_r \rightarrow 1$, as $\theta \rightarrow 0$, holds. However, both the residual magnetization M_r and the coercive field H_c are provided in an implicit way, through (8.12) and (8.13).

8.2. A logarithmic potential

Still we let M_s be the saturation value of the magnetization, $M \in (-M_s, M_s)$, and $\xi = M/M_s \in (-1, 1)$ while $h = \mu_0 H$. We denote by ξ_r the residual or remnant (relative) magnetization and by h_c the coercive field. By definition, ξ_r is the (positive) value of ξ at $h = 0$ whereas h_c is the local maximum of h .

We now examine the properties of the magnetization curve associated with the Lagrangian density

$$\tilde{\Phi} = -b(u+1) \ln(1 - \xi^2) - b\xi^2 - h\xi, \quad (8.14)$$

where b is a positive constant and u is a function of θ such that

$$u(\theta) \in (-1, 0) \quad \text{as} \quad \theta \in (0, \theta_c), \quad u(0) = -1, \quad u(\theta_c) = 0.$$

The potential

$$V_{\ln}(\xi) = \tilde{\Phi}(\xi) + h\xi = -b(u+1)\ln(1-\xi^2) - b\xi^2$$

is similar to the Weiss potential V_w in (8.11). In this sense $-b(u+1)\ln(1-\xi^2)$ replaces $\beta M_s \Lambda(u+1)/3$.

For definiteness we can take $u(\theta)$ in the form

$$u(\theta) = -\left[1 - \left(\frac{\theta}{\theta_c}\right)^q\right]^p \left[1 + a\left(\frac{\theta}{\theta_c}\right)^q\right]^p, \quad a \in (0, 1] \quad (8.15)$$

and $p, q \in \mathbb{Q}$. Since

$$u' = pq\theta^{q-1}\theta_c^{-q}[1 - a + 2a(\theta/\theta_c)^q]\{[1 - (\theta/\theta_c)^q][1 + a(\theta/\theta_c)^q]\}^{p-1}$$

it is apparent that u is a monotonic, increasing function of θ as $\theta \in [0, \theta_c]$. Furthermore, it is increasing as $\theta \in \mathbb{R}^+$ if $p = 1$. The standard form of u , namely $u = (\theta - \theta_c)/\theta_c$, is a particular case of (8.15) that corresponds to $a = 0, p = q = 1$ or to $a = p = 1, q = 1/2$.

In stationary conditions ($\dot{\xi} = 0$) Eq. (8.1) amounts to the vanishing of $\tilde{\Phi}_\xi$ whence

$$h(\xi) = 2b(u+1)\frac{\xi}{1-\xi^2} - 2b\xi. \quad (8.16)$$

Let $\theta \in (0, \theta_c)$ and hence $u \in (-1, 0)$. As $h = 0$, Eq. (8.16) provides the three solutions

$$\xi = 0, \quad \xi = -\sqrt{|u|}, \quad \xi = \sqrt{|u|}$$

and hence the residual magnetization ξ_r is given by

$$\xi_r = \sqrt{|u|}. \quad (8.17)$$

Look at the maximum and the minimum of $h(\xi)$. Observe that

$$\frac{1}{2b}h' = \frac{u + (u+3)\xi^2 - \xi^4}{(1-\xi^2)^2}.$$

Since $u \in (-1, 0)$ then h' vanishes if

$$\xi^4 - (3 - |u|)\xi^2 + |u| = 0.$$

There are then four solutions for ξ . Two of them, $\xi_i, -\xi_i$, such that

$$\xi_i^2 = \frac{1}{2}(3 - |u| - \sqrt{(3 - |u|)^2 - 4|u|}),$$

are found to belong to $(-1, 1)$. The remaining two solutions, $\xi_o, -\xi_o$, such that

$$\xi_o^2 = \frac{1}{2}(3 - |u| + \sqrt{(3 - |u|)^2 - 4|u|}),$$

are found to be outside $[-1, 1]$. Hence we have

$$h_c = \frac{2b\xi_i}{1-\xi_i^2}(\xi_i^2 - |u|). \quad (8.18)$$

Borrowing e.g. from [24], we say that the function

$$\xi_r(\theta) = \sqrt{|u(\theta)|}, \quad \theta \in (0, \theta_c),$$

is concave, monotonic decreasing and subject to $\xi_r(0) = 1, \xi_r(\theta_c) = 0$. Now, by (8.15) we see that $\xi_r(0) = 1, \xi_r(\theta_c) = 0$ and, moreover, $a \leq 1$ makes $\text{sgn } u$ constant. We now ascertain monotonicity and concavity of $\xi_r(\theta)$. Letting $x = \theta/\theta_c$ we have

$$\sqrt{|u|} = s(x) := [(1-x^q)(1+ax^q)]^{p/2}.$$

Upon evaluation of s' and s'' we conclude that monotonicity holds for $p, q \geq 0$ and that concavity holds if and only if $a = 1 - \epsilon$ and $q > 1, p \leq 2$, for small values of ϵ , or $a = 1$ and $q > 1/2, p \leq 2$.

Remark 6. If

$$p = 2, \quad q = \frac{3}{4},$$

we have

$$\xi_r = 1 + (a-1)(\theta/\theta_c)^{3/4} - a(\theta/\theta_c)^{3/2}.$$

Since $q = 3/4$ is allowed only if $a = 1$ then we have

$$\xi_r = 1 - (\theta/\theta_c)^{3/2}, \quad (8.19)$$

which is exactly Bloch's law (see (71.7) of [25] or [18]) for the dependence of the spontaneous magnetization on the temperature. The standard choice

$$u = (\theta - \theta_c)/\theta_c \quad (8.20)$$

corresponds to $a = 0, p = q = 1$ or to $a = p = 1, q = 1/2$, which are limit cases of (8.15).

Remark 7. At the absolute zero, $u = -1$ and the function $h(\xi)$ shows a crucial behaviour. For, as $u = -1$ we have

$$\xi_i^2 = \xi_0^2 = 1$$

and hence the magnetization curve $h(\xi)$ degenerates into three straight lines,

$$\xi = -1, \quad \xi = 1, \quad h = -2a\xi.$$

As $\theta > \theta_c$ we have $u > 0$ and hence, as expected, h vanishes only if $\xi = 0$.

In conclusion, the logarithmic potential V_{\ln} provides a good approximation of V_W around $\xi = 0$, allows for the saturation effect, yields the residual magnetization and the coercive field in an explicit way and provides a concave function $\xi_r(\theta)$.

8.3. The logarithmic potential for crystals of iron

For definiteness we look for the particular function u of (8.15) for crystals of iron. Although iron may be viewed as ferromagnetic, we apply the ferromagnetic model since no compensation point occurs.

Our purpose is to determine the parameters a, p, q by means of the experimental data for iron (see [24]) namely the curve of the residual magnetization ξ_r versus the absolute temperature θ .

Let $p, q > 0$. First we observe that by

$$s'(x) = \frac{1}{2}pq[1 + (a-1)x^q + ax^{2q}]^{(p-2)/2}x^{q-1}(a-1-2ax^q)$$

we have

$$s'(x) \rightarrow -\infty \quad \text{or} \quad s'(x) \rightarrow 0,$$

as $x \rightarrow 1_-$, according as $p < 2$ or $p > 2$. Because the experimental data show that $s(1_-) = -\infty$ we find a further reason for the condition $p < 2$.

Assume p, q are fixed, $p \in (0, 2)$. Since

$$s^{2/p}(x) = 1 - x^q + a(x^q - x^{2q}),$$

letting

$$y = s^{2/p}, \quad z(x) = 1 - x^q, \quad v(x) = x^q - x^{2q}$$

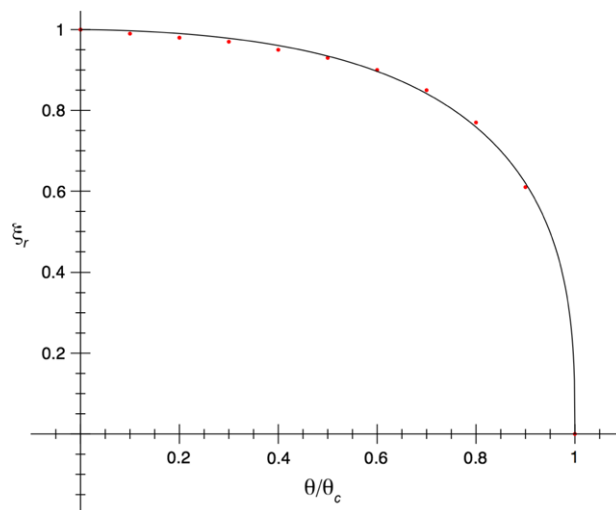


Fig. 2. The dots represent given values of $(\theta/\theta_c, \xi_r)$. The curve interpolates the dots.

we can write

$$y(x) = z(x) + av(x).$$

The linear dependence of y on a allows us to find a as the least squares solution.

Otherwise, once a set of points (x_i, s_i) are given, the joint derivation of the parameters a, p, q may be performed e.g. by the program PLOT (MacOSX). Fig. 2 shows the dots extracted from Bozorth [24]. The corresponding parameters turn out to be $p = 0.6838, q = 1.7298, a = 0.5777$. The curve gives the corresponding function $s(x)$.

As a check of consistency, if we let $p = 0.6838, q = 1.7298$ and evaluate the corresponding optimal value of a we find that $a = 0.568$. If, instead, we take $q = 1.7298, a = 0.5777$ then we find that $p = 0.688$. In both cases the values (of a or p) are very close to those given by the joint derivation.

9. Conclusions

The thermodynamic analysis of the modelling of magnetic materials is based on $\theta, \mathbf{E}, \mathbf{H}, \mathbf{M}, \nabla\theta, \nabla\mathbf{M}, \nabla\nabla\mathbf{M}$ as the set of independent variables and on the statement of the second law through an inequality involving an extra entropy flux \mathbf{k} . We have found that the second law inequality is satisfied if $\theta\mathbf{k} = \Psi_{\nabla\mathbf{M}}\mathbf{M}$ and \mathbf{M} is subject to (3.10).

The evolution equation is governed by the rescaled free energy $\Psi(\theta, \mathbf{M}, \nabla\mathbf{M})/\theta$. We find that the logarithmic function (8.14), with $a < b$, has two equilibrium solutions. The corresponding values of the order parameter is (± 1) times the residual magnetization. The choice (5.1) for Ψ where G has the logarithmic form (5.2) provides the model for the paramagnetic–ferromagnetic transition. Also, the Curie–Weiss law is obtained in the approximation of small values of \mathbf{M} .

Evolution equations arisen within micromagnetics have been considered, namely the Landau–Lifshitz equation (7.1), the Gilbert equation (7.2) and recent improvements (7.8). The generality of the present thermodynamic approach allows us to frame also the micromagnetic equations as evolution equations for \mathbf{M} , generated by an appropriate form of the free energy potential, as a particular form of (4.4).

It is a positive feature of the present approach that the same logarithmic potential (5.2) provides a satisfactory description of both the ferromagnetic ($\theta < \theta_c$) and the paramagnetic ($\theta > \theta_c$) behaviour and fits the curve of residual magnetization, as a function of the absolute temperature, for crystals of iron.

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